Some Generic Constructions of Generalized Plateaued Functions†

Jiaxin Wang*, Fang-Wei Fu

Abstract

Plateaued functions as an extension of bent functions play a significant role in cryptography, coding theory, sequences and combinatorics. In 2019, Hodžić *et al*. [IEEE TIT 65(9): 5865-5879, 2019] designed Boolean plateaued functions in spectral domain and provided some efficient construction methods in spectral domain. However, in their constructions, the Walsh support of Boolean s-plateaued functions in n variables, when written as a matrix of order $2^{n-s} \times n$, contains at least $n-s$ columns corresponding to affine functions on $\mathbb{F}_2^{n-s}$. In this paper, we study generalized s-plateaued functions from V_n to $\mathbb{Z}_{p^k}$ where p is an odd prime and $k \geq 1$ or $p=2, k \geq 2$ and $n+s$ is even. Firstly, inspired by the work of Hodžić *et al*., we give a complete characterization of generalized plateaued functions with affine Walsh support and provide some construction methods of generalized plateaued functions with (non)-affine Walsh support in spectral domain. In our constructions of generalized s-plateaued functions with non-affine Walsh support, the Walsh support can contain strictly less than $n-s$ columns corresponding to affine functions and our construction methods are also applicable to Boolean plateaued functions. Secondly, we provide a generalized indirect sum construction method of generalized plateaued functions, which can also be used to construct (non)-weakly regular generalized bent functions. In particular, we show that the canonical way to construct Generalized Maiorana-McFarland bent functions is a special case of the generalized indirect sum construction method and we illustrate that the generalized indirect sum construction method can be used to construct bent functions not in the complete Generalized Maiorana-McFarland class. Furthermore, based on this construction method, we give constructions of plateaued functions in the subclass *WRP* of the class of weakly regular plateaued functions and vectorial plateaued functions.

Jiaxin Wang and Fang-Wei Fu are with Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, P.R.China, Emails: wjiaxin@mail.nankai.edu.cn, fwfu@nankai.edu.cn.

†This research is supported by the National Key Research and Development Program of China (Grant No. 2018YFA0704703), the National Natural Science Foundation of China (Grant No. 61971243), the Natural Science Foundation of Tianjin (20JCZDJC00610), the Fundamental Research Funds for the Central Universities of China (Nankai University), and the Nankai Zhide Foundation.

*Corresponding author

Manuscript submitted March 18, 2021

Index Terms

Plateaued functions; generalized plateaued functions; Walsh transform; bent functions; generalized bent functions; generalized indirect sum construction

I. Introduction

Boolean bent functions introduced by Rothaus [31] play an important role in cryptography, coding theory, sequences and combinatorics. In 1985, Kumar *et al*. [15] generalized Boolean bent functions to bent functions over finite fields of odd characteristic. Due to the importance of bent functions, they have been studied extensively. There is an exhaustive survey [5] and a book [20] for bent functions and generalized bent functions. Recently, the notion of generalized bent functions from V_n to $\mathbb{Z}_{2^k}$ has been generalized to generalized bent functions from V_n to $\mathbb{Z}_{p^k}$ where p is a prime [28]. For more characterizations and constructions of generalized bent functions from V_n to $\mathbb{Z}_{p^k}$, we refer to [10], [11], [17], [18], [19], [21], [28], [32], [33]. Also note that by Theorem 16 of [28], one can construct some generalized bent functions by the constructed infinite families of p-ary weakly regular bent functions in [30].

In 1993, Carlet [4] introduced the definition of Boolean partially bent functions which is an extension of Boolean bent functions. As an extension of Boolean partially bent functions, Zheng and Zhang [34] introduced the definition of Boolean plateaued functions. Boolean plateaued functions have many good cryptographic properties. The notions of Boolean partially bent functions and Boolean plateaued functions have been extended to partially bent functions and plateaued functions over finite fields of odd characteristic (see [6], [7]). Apart from the desirable cryptographic properties, plateaued functions play a significant role in coding theory, sequences and combinatorics (see e.g. [1], [22], [25], [26], [29]). In [27], Mesnager *et al*. extended the usual notion of plateaued functions to generalized plateaued functions, which includes the notion of generalized bent functions.

For the generic framework of (generalized) plateaued functions, there has been some progress [2], [14], [23], [24], [27]. However, there are not many efficient generic constructions. In [13], Hodžić *et al*. designed Boolean plateaued functions in spectral domain. Designing plateaued functions in spectral domain is based on the fact that any function from V_n to $\mathbb{Z}_{p^k}$ where p is a prime, k is a positive integer and its Walsh spectrum are mutually determined. In this paper, we study generalized s-plateaued functions from V_n to $\mathbb{Z}_{p^k}$ where p is an odd prime and $k \geq 1$ or $p = 2, k \geq 2$ and $n + s$ is even. Firstly, inspired by the work of Hodžić *et al.*, we give a complete characterization of generalized plateaued functions with affine Walsh support and provide some construction methods of generalized plateaued functions with (non)-affine Walsh support in spectral domain. As pointed out in [13], for the constructions in spectral domain given in [13], the Walsh support of Boolean s-plateaued functions in n variables, when written as a matrix, contains at least $n - s$ columns corresponding to affine functions on $\mathbb{F}_2^{n-s}$. And they proposed an open problem to provide constructions of Boolean s-plateaued functions whose Walsh support, when written as a matrix, contains strictly less than $n-s$ columns corresponding to affine functions. In our constructions of generalized s-plateaued functions with non-affine Walsh support, the Walsh support, when written as a matrix, can contain strictly less than $n - s$ columns corresponding to affine functions and our construction methods are also applicable to Boolean plateaued functions. Secondly, we provide a generalized indirect sum construction method of generalized plateaued functions, which can also be used to construct (non)-weakly regular generalized bent functions. In particular, we show that the canonical way to construct Generalized Maiorana-McFarland bent functions is a special case of the generalized indirect sum construction method and we illustrate that the generalized indirect sum construction method can be used to construct bent functions not in the complete Generalized Maiorana-McFarland class. Furthermore, based on this construction method, we give constructions of plateaued functions in the subclass *WRP* of the class of weakly regular plateaued functions and vectorial plateaued functions.

The rest of the paper is organized as follows. In Section 2, we introduce the needed definitions and results related to generalized plateaued functions. In Section 3.1, we give a necessary and sufficient condition of constructing generalized plateaued functions in spectral domain and give a useful corollary. In Section 3.2, we give a complete characterization of generalized plateaued functions whose Walsh support is an affine subspace. In Section 3.3, we provide some generic construction methods of generalized plateaued functions with (non)-affine Walsh support. In Section 4.1, we give a generalized indirect sum construction method of generalized plateaued functions. In Section 4.2, we give some applications of the generalized indirect sum construction method. In Section 5, we make a conclusion.

II. Preliminaries

For any complex number $z=a+b\sqrt{-1}$, let $|z|=\sqrt{a^2+b^2}$. For any finite set S, let $|S|$ denote the size of S. Throughout this paper, let $\mathbb{Z}_{p^k}$ be the ring of integers modulo p^k, $\zeta_{p^k}=e^{\frac{2\pi\sqrt{-1}}{p^k}}$ be the complex primitive p^k-th root of unity, $\mathbb{F}_p^n$ be the vector space of the n-tuples over $\mathbb{F}_p$, $\mathbb{F}_{p^n}$ be the finite field with p^n elements and V_n be an n-dimensional vector space over $\mathbb{F}_p$ where p is a prime and k, n are positive integers. The classical representations of V_n are $\mathbb{F}_p^n$ and $\mathbb{F}_{p^n}$. For $a, b \in V_n$, let $\langle a, b\rangle$ denote a (nondegenerate) inner product in V_n. When $a=(a_1,\ldots,a_n), b=(b_1,\ldots,b_n) \in \mathbb{F}_p^n$, let $\langle a, b\rangle = a\cdot b=\sum_{i=1}^n a_i b_i$. When $a, b \in \mathbb{F}_{p^n}$, let $\langle a,b\rangle = Tr_1^n(ab)$ where $Tr_1^n(\cdot)$ is the absolute trace function. When $V_n=V_{n_1}\times\cdots\times V_{n_s} (n=\sum_{i=1}^s n_i)$, let $\langle a,b\rangle=\sum_{i=1}^s\langle a_i,b_i\rangle$ where $a=(a_1,\ldots,a_s), b=(b_1,\ldots,b_s)\in V_n$. Let $GL(n,\mathbb{F}_p)$ denote the group formed by all invertible matrices over $\mathbb{F}_p$ of size $n\times n$.

A function f from V_n to $\mathbb{Z}_{p^k}$ is called a generalized p-ary function, or simply p-ary function when $k=1$. A p-ary function $L: V_n \to \mathbb{F}_p$ is called a linear function if $L(ax+by) = aL(x)+bL(y)$ for any $a,b\in\mathbb{F}_p$ and $x,y\in V_n$. All linear functions from V_n to $\mathbb{F}_p$ form an n-dimensional linear space $\mathcal{L}_n$ and $\{\langle\alpha_i,x\rangle, 1\le i\le n\}$ is a basis of $\mathcal{L}_n$ where $\{\alpha_i, 1\le i\le n\}$ is a basis of V_n. If p-ary function $A: V_n\to\mathbb{F}_p$ is the sum of a linear function and a constant, then A is called an affine function.

The Walsh transform of generalized p-ary function $f: V_n\to\mathbb{Z}_{p^k}$ is the function W_f from V_n to $\mathbb{Z}[\zeta_{p^k}]$:

$$W_f(a)=\sum_{x\in V_n}\zeta_{p^k}^{f(x)}\zeta_p^{-\langle a,x\rangle}\ where\ a\in V_n. \tag{1}$$

And f can be recovered by the inverse transform

$$\zeta_{p^k}^{f(x)}=\frac{1}{p^n}\sum_{a\in V_n}W_f(a)\zeta_p^{\langle a,x\rangle}\ where\ x\in V_n. \tag{2}$$

The multiset $\{W_f(a), a\in V_n\}$ is called the Walsh spectrum of f. The set $S_f=\{a\in V_n: W_f(a)\neq 0\}$ is called the Walsh support of f. Functions $f_1,\ldots,f_m$ are called pairwise disjoint spectra functions if $S_{f_i}\cap S_{f_j}=\emptyset$ for any $i\neq j$.

A generalized p-ary function $f: V_n\to\mathbb{Z}_{p^k}$ is called a generalized p-ary s-plateaued function, or simply p-ary s-plateaued function when $k=1$ if $|W_f(a)|=p^{\frac{n+s}{2}}$ or 0 for any $a\in V_n$. If $s=0$, the generalized p-ary 0-plateaued function f is just the generalized p-ary bent function and $S_f=V_n$.

For generalized s-plateaued functions $f : V_n \to \mathbb{Z}_{p^k}$, there is a basic property: $|S_f| = p^{n-s}$, which is obtained by Parseval identity $\sum_{x\in V_n} |W_f(x)|^2 = p^{2n}$. In [27], Mesnager *et al*. have shown the Walsh transform of a generalized p-ary s-plateaued function $f : V_n \to \mathbb{Z}_{p^k}$ satisfies that for any $a \in S_f$, when $p = 2$ and $n + s$ is even, $W_f(a) = 2^{\frac{n+s}{2}} \zeta_{2^k}^{f^*(a)}$ and when p is an odd prime,

$$W_f(a) = \begin{cases} \pm p^{\frac{n+s}{2}} \zeta_{p^k}^{f^*(a)} & if\ n+s\ is\ even\ or\ p \equiv 1(mod\ 4), \\ \pm\sqrt{-1} p^{\frac{n+s}{2}} \zeta_{p^k}^{f^*(a)} & if\ n+s\ is\ odd\ and\ p \equiv 3(mod\ 4) \end{cases}$$

where f^* is a function from S_f to $\mathbb{Z}_{p^k}$. We call f^* the dual function of f.

In the sequel, if $f : V_n \to \mathbb{Z}_{p^k}$ is a generalized s-plateaued function with dual function f^*, define function μ_f as

$$\mu_f(a) = p^{-\frac{n+s}{2}} \zeta_{p^k}^{-f^*(a)} W_f(a), a \in S_f. \tag{3}$$

If $p \equiv 1(mod\ 4)$ or $p \equiv 3(mod\ 4)$ and $n + s$ is even, then μ_f is a function from S_f to $\{\pm 1\}$. If $p \equiv 3(mod\ 4)$ and $n + s$ is odd, then μ_f is a function from S_f to $\{\pm\sqrt{-1}\}$. If $p = 2$ and $n + s$ is even, then $\mu_f(x) = 1, x \in S_f$. For generalized bent function $f : V_n \to \mathbb{Z}_{p^k}$, that is, generalized 0-plateaued function, if μ_f is a constant function, then f is called weakly regular, otherwise f is called non-weakly regular. In particular, if $\mu_f(x) = 1, x \in V_n$, f is called regular. In [22], Mesnager *et al*. introduced the notion of (non)-weakly regular plateaued functions in odd characteristic. For an s-plateaued function $f : V_n \to \mathbb{F}_p$, if μ_f is a constant function, then f is called weakly regular, otherwise f is called non-weakly regular. In particular, if $\mu_f(x) = 1, x \in S_f$, f is called regular.

If $f : V_n \to \mathbb{Z}_{p^k}$ is a generalized n-plateaued function, then $|S_f| = 1$ and it is easy to obtain $f(x) = p^{k-1}\langle a, x\rangle + b$ for some $a \in V_n, b \in \mathbb{Z}_{p^k}$ by the inverse transform (2). In this paper, we study generalized s-plateaued functions $f : V_n \to \mathbb{Z}_{p^k}$ where $0 \le s < n$, p is an odd prime and $k \ge 1$ or $p = 2$, $k \ge 2$ and $n + s$ is even.

III. Constructing generalized plateaued functions in spectral domain

In this section, we provide some generic construction methods of generalized s-plateaued functions in spectral domain where $s \ge 1$.

To this end, we fix some notation unless otherwise stated. Let m be an arbitrary positive integer. Define the notation of lexicographic order $\prec$: $a \prec b$ if $\sum_{i=1}^{m} p^{m-i} a_i < \sum_{i=1}^{m} p^{m-i} b_i$

where $a=(a_1,\ldots,a_m), b=(b_1,\ldots,b_m)\in\mathbb{F}_p^m$. Define

$$v_i=\sum_{j=1}^{m} v_{i,j}\alpha_j, 0\le i\le p^m-1, \tag{4}$$

where $\{\alpha_1,\ldots,\alpha_m\}$ is some fixed basis of V_m over $\mathbb{F}_p$ and $\{(v_{0,1},\ldots,v_{0,m}),\ldots,(v_{p^m-1,1},\ldots,v_{p^m-1,m})\}$ is the lexicographic order of $\mathbb{F}_p^m$. When $V_m=\mathbb{F}_p^m$, we let $\alpha_1=(1,0,\ldots,0,0)\in\mathbb{F}_p^m,\ldots,\ \alpha_m=(0,0,\ldots,0,1)\in\mathbb{F}_p^m$, that is, $\{v_0,\ldots,v_{p^m-1}\}$ denotes the lexicographic order of $\mathbb{F}_p^m$. For a p-ary function $f:V_m\to\mathbb{F}_p$, define its true table

$$T_f=(f(v_0),\ldots,f(v_{p^m-1}))^T \tag{5}$$

where M^T denotes the transpose of matrix M. For two matrices $A=(a_1,\ldots,\ a_{n_1})$ and $B=(b_1,\ldots,b_{n_2})$ where n_1,n_2 are positive integers and $a_i(1\le i\le n_1)$, $b_j(1\le j\le n_2)$ are column vectors of the same size, let $\wr$ denotes column concatenations of A and B, that is, $A\wr B=(a_1,\ldots,a_{n_1},b_1,\ldots,b_{n_2})$.

A. A Necessary and Sufficient Condition

In this subsection, inspired by [13], we provide a necessary and sufficient condition of constructing generalized plateaued functions in spectral domain and provide a corollary which plays an important role in generic constructions.

Suppose $S\subseteq\mathbb{F}_p^n$ with size p^m is ordered as $S=\{w_0,w_1,\ldots,w_{p^m-1}\}$. For any $a\in\mathbb{F}_p^n$, define ψ_a from V_m to $\mathbb{F}_p$:

$$\psi_a(v_i)=a\cdot w_i, 0\le i\le p^m-1, \tag{6}$$

where v_i is defined by (4).

Under notation as above we have the following proposition:

Proposition 1. *Let p be a prime. Let $n,k,s(<n)$ be positive integers and $k\ge 2$, $n+s$ be even for $p=2$. Let S be a subset of $\mathbb{F}_p^n$ with size p^{n-s} and be ordered as $S=\{w_0,w_1,\ldots,w_{p^{n-s}-1}\}$. Let d be a function from V_{n-s} to $\mathbb{Z}_{p^k}$. Let μ be a function from V_{n-s} to $\{\pm 1\}$ if $p\equiv 1(mod\ 4)$ or $p\equiv 3(mod\ 4)$ and $n+s$ is even, μ be a function from V_{n-s} to $\{\pm\sqrt{-1}\}$ if $p\equiv 3(mod\ 4)$ and $n+s$ is odd and $\mu(x)=1, x\in V_{n-s}$ if $p=2$ and $n+s$ is even. Define function $W:\mathbb{F}_p^n\to\mathbb{Z}[\zeta_{p^k}]$ as*

$$W(a)=\begin{cases}\mu(v_i)p^{\frac{n+s}{2}}\zeta_{p^k}^{d(v_i)} & if\ \exists\ 0\le i\le p^{n-s}-1\ s.t.\ a=w_i,\\ 0 & otherwise.\end{cases} \tag{7}$$

Then $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ if and only if $(p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k} = 1$ for any $a \in \mathbb{F}_p^n$, where ψ_a is defined by (6).

Proof: First by the well-known fact that $\sqrt{p} \in \mathbb{Z}[\zeta_p]$ if $p \equiv 1(mod\ 4)$ and $\sqrt{-1}\sqrt{p} \in \mathbb{Z}[\zeta_p]$ if $p \equiv 3(mod\ 4)$, it is easy to see that the function W defined by (7) is a function from $\mathbb{F}_p^n$ to $\mathbb{Z}[\zeta_{p^k}]$.

If $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$, by the inverse transform (2) we have

$$\begin{aligned}
\zeta_{p^k}^{f(a)} &= \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} W(x)\zeta_p^{a\cdot x} \\
&= \frac{1}{p^n} \sum_{x \in S} W(x)\zeta_p^{a\cdot x} \\
&= \frac{1}{p^n} \sum_{i=0}^{p^{n-s}-1} \mu(v_i) p^{\frac{n+s}{2}} \zeta_{p^k}^{d(v_i)} \zeta_p^{a\cdot w_i} \\
&= p^{\frac{s-n}{2}} \sum_{i=0}^{p^{n-s}-1} \mu(v_i)\zeta_{p^k}^{d(v_i)+p^{k-1}\psi_a(v_i)} \\
&= p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)},
\end{aligned}$$

hence $(p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k} = 1$ for any $a \in \mathbb{F}_p^n$.

Conversely, suppose $(p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k} = 1$ for any $a \in \mathbb{F}_p^n$. Since all roots of $x^{p^k} = 1$ are $\zeta_{p^k}^0, \zeta_{p^k}^1, \ldots, \zeta_{p^k}^{p^k-1}$, there is a unique generalized function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ such that $p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)} = \zeta_{p^k}^{f(a)}$. Then function W is the Walsh transform of f. Indeed,

$$\begin{aligned}
W_f(a) &= \sum_{x \in \mathbb{F}_p^n} \zeta_{p^k}^{f(x)} \zeta_p^{-a\cdot x} \\
&= \sum_{x \in \mathbb{F}_p^n} p^{\frac{s-n}{2}} \sum_{y \in V_{n-s}} \mu(y)\zeta_{p^k}^{d(y)+p^{k-1}\psi_x(y)} \zeta_p^{-a\cdot x} \\
&= \sum_{x \in \mathbb{F}_p^n} p^{\frac{s-n}{2}} \sum_{i=0}^{p^{n-s}-1} \mu(v_i)\zeta_{p^k}^{d(v_i)+p^{k-1}x\cdot w_i} \zeta_p^{-a\cdot x} \\
&= p^{\frac{s-n}{2}} \sum_{i=0}^{p^{n-s}-1} \mu(v_i)\zeta_{p^k}^{d(v_i)} \sum_{x \in \mathbb{F}_p^n} \zeta_p^{(w_i-a)\cdot x}.
\end{aligned}$$

If $a \notin S = \{w_0, w_1, \dots, w_{p^{n-s}-1}\}$, then $W_f(a) = 0$ since $\sum_{x \in \mathbb{F}_p^n} \zeta_p^{(w_i - a)\cdot x} = 0$ for any $0 \leq i \leq p^{n-s} - 1$. If $a = w_i$ for some $0 \leq i \leq p^{n-s} - 1$, then $W_f(a) = p^{\frac{n+s}{2}} \mu(v_i) \zeta_{p^k}^{d(v_i)}$ since $\sum_{x \in \mathbb{F}_p^n} \zeta_p^{(w_i - a)\cdot x} = p^n$ and for any other $j \neq i$ we have $\sum_{x \in \mathbb{F}_p^n} \zeta_p^{(w_j - a)\cdot x} = 0$. Hence, $W_f(a) = W(a)$ for any $a \in \mathbb{F}_p^n$ and $S_f = S$, $|W_f(a)| = p^{\frac{n+s}{2}}$ for any $a \in S_f$, that is, W is the Walsh transform of f and f is a generalized s-plateaued function. ■

Remark 1. *Proposition 1 provides a necessary and sufficient condition of constructing generalized plateaued functions in spectral domain. If the condition of Proposition 1 is satisfied, then one can obtain function f by the inverse transform (2).*

By Proposition 1 (with the same notation), if the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$, then obviously

$$| \sum_{x \in V_{n-s}} \mu(x) \zeta_{p^k}^{d(x) + p^{k-1} \psi_a(x)} | = p^{\frac{n-s}{2}} \ for \ any \ a \in \mathbb{F}_p^n. \tag{8}$$

We show that the inverse is true when $p = 2$ and $n+s$ is even and is not necessarily true when p is an odd prime. We give an analysis by using Lemma 24 of [27]. Suppose (8) holds. For any $a \in \mathbb{F}_p^n$, let $h_a = \sum_{x \in V_{n-s}} \mu(x) \zeta_{p^k}^{d(x) + p^{k-1}\psi_a(x)}$. Let $\mathcal{O}_K = \mathbb{Z}[\zeta_{p^k}]$ denote the ring of integers in cyclotomic field $K = \mathbb{Q}(\zeta_{p^k})$. Let W_K denote the group of unity of K, then $W_K = \{\zeta_{2^k}^i : 0 \leq i \leq 2^k - 1\}$ if $p = 2$ and $W_K = \{\pm \zeta_{p^k}^i : 0 \leq i \leq p^k - 1\}$ if p is an odd prime. Let $p^* = \left(\frac{-1}{p}\right) p$ if p is an odd prime where $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$ denotes the Legendre symbol and $p^* = 2$ if $p = 2$.

(1) When $p = 2$ and $n + s$ is even or $p \equiv 1 (mod\ 4)$ or $p \equiv 3 (mod\ 4)$ and $n + s$ is even, we have $h_a \in \mathcal{O}_K$ since $\mu(x) \in \{\pm 1\}$ for any $x \in V_{n-s}$. Then by Lemma 24 of [27], we have $\frac{h_a}{\sqrt{p^*}^{n-s}} \in W_K$, hence $\frac{h_a}{\sqrt{p}^{n-s}} \in W_K$ since $\sqrt{p^*}^{n-s} \in \{\pm \sqrt{p}^{n-s}\}$ if $p = 2$ and $n + s$ is even or $p \equiv 1 (mod\ 4)$ or $p \equiv 3 (mod\ 4)$ and $n + s$ is even.

(2) When $p \equiv 3 (mod\ 4)$ and $n + s$ is odd, we have $\sqrt{-1} h_a \in \mathcal{O}_K$ since $\mu(x) \in \{\pm \sqrt{-1}\}$ for any $x \in V_{n-s}$. Then by Lemma 24 of [27], we have $\frac{\sqrt{-1} h_a}{\sqrt{p^*}^{n-s}} \in W_K$, hence $\frac{\sqrt{-1} h_a}{\sqrt{-1}\sqrt{p}^{n-s}} \in W_K$ since $\sqrt{p^*}^{n-s} \in \{\pm \sqrt{-1} \sqrt{p}^{n-s}\}$ if $p \equiv 3 (mod\ 4)$ and $n + s$ is odd, that is, $\frac{h_a}{\sqrt{p}^{n-s}} \in W_K$.

Hence, one can see that $|h_a| = p^{\frac{n-s}{2}}$ is equivalent to $(p^{\frac{s-n}{2}} h_a)^{p^k} = 1$ if $p = 2$ and $n + s$ is even and $|h_a| = p^{\frac{n-s}{2}}$ is equivalent to $(p^{\frac{s-n}{2}} h_a)^{2p^k} = 1$ if p is an odd prime. When p is an odd prime, there is still a gap with the condition of Proposition 1. For example, let $p = 3, k = 2, n = 3, s = 1$, ordered $S = \{0\} \times \mathbb{F}_3^2 = \{(0,0,0), (0,0,1), (0,0,2), \dots, (0,2,2)\}$ and $\mu(x_1, x_2) = -1, (x_1, x_2) \in \mathbb{F}_3^2$, $d(x_1, x_2) = 3x_1 x_2, (x_1, x_2) \in \mathbb{F}_3^2$. Then one can verify that for any $a \in \mathbb{F}_3^3$, $|h_a| = 3$ and $(3^{-1}h_a)^9 = -1$.

From the above analysis, we can obtain the following result:

Proposition 2. *With the same notation as Proposition 1.*

(1) When $p = 2$ *and* $n+s$ *is even, the function* $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ *defined by (7) is the Walsh transform of a generalized* s*-plateaued function* $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ *if and only if* $|\sum_{x \in V_{n-s}} \zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)}| = p^{\frac{n-s}{2}}$ *for any* $a \in \mathbb{F}_p^n$.

(2) When p *is an odd prime, the function* $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ *defined by (7) is the Walsh transform of a generalized* s*-plateaued function* $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ *if and only if* $|\sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)}| = p^{\frac{n-s}{2}}$ *and* $(p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k} \neq -1$ *for any* $a \in \mathbb{F}_p^n$.

Now we provide a corollary which plays an important role in generic constructions.

Corollary 1. *With the same notation as Proposition 1. For any* $a \in \mathbb{F}_p^n$, *define* $g_a(x) = d(x) + p^{k-1}\psi_a(x),\ x \in V_{n-s}$. *If for any* $a \in \mathbb{F}_p^n$, $g_a : V_{n-s} \to \mathbb{Z}_{p^k}$ *is a generalized bent function and there exists a constant* u *independent of* a *such that* $\mu_{g_a}(x) = u, x \in V_{n-s}$ *where* μ_{g_a} *is defined by (3), let* $\mu(x) = u^{-1}, x \in V_{n-s}$. *Then the function* $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ *defined by (7) is the Walsh transform of a generalized* s*-plateaued function* $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$. *Furthermore,* $f(a) = g_a^*(0), a \in \mathbb{F}_p^n$ *where* g_a^* *is the dual function of* g_a.

Proof: If for any $a \in \mathbb{F}_p^n$, $g_a : V_{n-s} \to \mathbb{Z}_{p^k}$ is a generalized bent function and there exists a constant u independent of a such that $\mu_{g_a}(x) = u, x \in V_{n-s}$ where μ_{g_a} is defined by (3) and $\mu(x) = u^{-1}, x \in V_{n-s}$, then function μ satisfy the condition of Proposition 1 and

$$\begin{aligned}
\sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)} &= \sum_{x \in V_{n-s}} u^{-1}\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)} \\
&= u^{-1} \sum_{x \in V_{n-s}} \zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)} \\
&= u^{-1} W_{g_a}(0) \\
&= u^{-1} \cdot u p^{\frac{n-s}{2}} \zeta_{p^k}^{g_a^*(0)} \\
&= p^{\frac{n-s}{2}} \zeta_{p^k}^{g_a^*(0)},
\end{aligned}$$

where g_a^* is the dual function of g_a. So $(p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k} = 1$. Hence by Proposition 1, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \rightarrow \mathbb{Z}_{p^k}$ and furthermore, from the proof of Proposition 1, we have $f(a) = g_a^*(0)$ for any $a \in \mathbb{F}_p^n$. ■

B. Characterization of Generalized Plateaued Functions with Affine Walsh Support in Spectral Domain

In this subsection, we give a complete characterization of generalized plateaued functions whose Walsh support is an affine subspace in spectral domain, which generalizes the case of Boolean plateaued functions [13].

To get the theorem of this subsection, we need a lemma, which is a generalization of the results in the proof of Lemma 3.1 of [12].

Lemma 1. *Let p be a prime. Suppose $E \subseteq \mathbb{F}_p^n$ is an m-dimensional linear subspace over $\mathbb{F}_p$ and $E = \{e_0, e_1, \ldots, e_{p^m-1}\}$ is the lexicographic order of E. Then $\{e_{p^0}, e_{p^1}, \ldots, e_{p^{m-1}}\}$ is a basis of E and $e_i = v_i R$ for any $0 \leq i \leq p^m - 1$ where R is the matrix whose row vectors are $e_{p^{m-1}}, e_{p^{m-2}}, \ldots, e_{p^0}$ and $\{v_0, \ldots, v_{p^m-1}\}$ is the lexicographic order of $\mathbb{F}_p^m$.*

Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a basis of E over $\mathbb{F}_p$. For the matrix whose row vectors are $\alpha_1, \alpha_2, \ldots, \alpha_m$, by using elementary row operations, we can get the row echelon matrix

$$R = \begin{pmatrix} e^{(m-1)} \\ e^{(m-2)} \\ \ldots \\ e^{(1)} \\ e^{(0)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \ldots & 0 & 1 & * & \ldots & * & 0 & * & \ldots & * & \ldots & 0 & * & \ldots & * & 0 & * & \ldots & * \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & * & \ldots & * & \ldots & 0 & * & \ldots & * & 0 & * & \ldots & * \\ & & & & & & & & & \ldots\ldots & & & & & & & & & & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 1 & * & \ldots & * & 0 & * & \ldots & * \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & * & \ldots & * \end{pmatrix},$$

where $*$ denotes some elements in $\mathbb{F}_p$, the first nonzero element in each row is one from left to right and these ones belong to different columns and the other elements in the same column are zero. Furthermore, if the first nonzero element of i-th row is in the k_i-th column $(0 \leq i \leq m-1)$, then $0 \leq k_0 < \cdots < k_{m-1} \leq n - 1$.

If $(i_0, i_1, \ldots, i_{m-1}) \in \mathbb{F}_p^m, (i'_0, i'_1, \ldots, i'_{m-1}) \in \mathbb{F}_p^m$ with $(i_0, i_1, \ldots, i_{m-1}) \prec (i'_0, i'_1, \ldots, i'_{m-1})$, that is, there exists $0 \leq j_0 \leq m-1$ such that $i_j = i'_j$ for any $j < j_0$ and $i_{j_0} < i'_{j_0}$. Let $s = (s_0, \ldots, s_{n-1}) = \sum_{j=0}^{m-1} i_j e^{(m-1-j)}, s' = (s'_0, \ldots, s'_{n-1}) = \sum_{j=0}^{m-1} i'_j e^{(m-1-j)}$. By the properties of $e^{(i)}(0 \leq i \leq m-1)$, one can get $s_j = s'_j$ for any $j < k_{j_0}$ and $s_{k_{j_0}} < s'_{k_{j_0}}$, that is, $s \prec s'$. Hence, the lexicographic order of $(i_0, \ldots, i_{m-1}) \in \mathbb{F}_p^m$ determines the lexicographic order of $\sum_{j=0}^{m-1} i_j e^{(m-1-j)}$. So for $i = \sum_{j=0}^{m-1} i_j p^{m-1-j}$, we have $e_i = \sum_{j=0}^{m-1} i_j e^{(m-1-j)}$ where $\{e_0, \ldots, e_{p^{n-s}-1}\}$ is the lexicographic order of E. For any $0 \leq j \leq m-1$, let $i = p^j$, then $e_i = e^{(j)}$. Hence, $\{e_{p^0}, e_{p^1}, \ldots, e_{p^{m-1}}\}$ is a basis of E and $e_i = v_i R$ for any $0 \leq i \leq p^m - 1$ where $\{v_0, \ldots, v_{p^m-1}\}$ is the lexicographic order of $\mathbb{F}_p^m$. ■

Now we give a complete characterization of generalized plateaued functions with affine Walsh support in spectral domain by using Lemma 1.

Theorem 1. *With the same notation as Proposition 1. Let ordered $S = \{w_0, w_1, \ldots, w_{p^{n-s}-1}\}$ where $w_i = t + e_i M$ for any $0 \leq i \leq p^{n-s} - 1$, $t \in \mathbb{F}_p^n$, $M \in GL(n, \mathbb{F}_p)$ and $\{e_0, e_1, \ldots, e_{p^{n-s}-1}\}$ is the lexicographic order of an $(n-s)$-dimensional linear subspace $E \subseteq \mathbb{F}_p^n$. Let d be a function from $\mathbb{F}_p^{n-s}$ to $\mathbb{Z}_{p^k}$. Then the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ if and only if d is the dual function of some generalized bent function g and $\mu = \mu_g$ where μ_g is defined by (3). Furthermore, if d is the dual function of some generalized bent function g and $\mu = \mu_g$, then $f(x) = g(xM^T R^T) + p^{k-1} x \cdot t$, $x \in \mathbb{F}_p^n$ where R is the matrix whose row vectors are $e_{p^{n-s-1}}, e_{p^{n-s-2}}, \ldots, e_{p^0}$.*

Proof: Since E is a linear subspace, then by Lemma 1, for any $a \in \mathbb{F}_p^n$ and any $0 \leq i \leq p^{n-s} - 1$, we have $\psi_a(v_i) = a \cdot w_i = a \cdot (t + e_i M) = a \cdot t + aM^T \cdot e_i = a \cdot t + aM^T \cdot (v_i R) = a \cdot t + aM^T R^T \cdot v_i$.

If d is the dual function of some generalized bent function g and $\mu = \mu_g$, that is, $W_g(b) = \mu(b) p^{\frac{n-s}{2}} \zeta_{p^k}^{d(b)}$ for any $b \in \mathbb{F}_p^{n-s}$, then we have

$$\begin{aligned}\sum_{x \in \mathbb{F}_p^{n-s}} \mu(x) \zeta_{p^k}^{d(x) + p^{k-1} \psi_a(x)} &= \sum_{x \in \mathbb{F}_p^{n-s}} \mu(x) \zeta_{p^k}^{d(x)} \zeta_p^{a \cdot t + aM^T R^T \cdot x} \\ &= \zeta_p^{a \cdot t} p^{\frac{n-s}{2}} \zeta_{p^k}^{g(aM^T R^T)}\end{aligned}$$

where the second equation is obtained by the inverse transform. So for any $a \in \mathbb{F}_p^n$, $(p^{\frac{s-n}{2}} \sum_{x \in \mathbb{F}_p^{n-s}} \mu(x) \zeta_{p^k}^{d(x) + p^{k-1} \psi_a(x)})^{p^k} = 1$. By Proposition 1 and its proof, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$

defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ and $f(x) = g(xM^T R^T) + p^{k-1} x \cdot t, x \in \mathbb{F}_p^n$.

Conversely, if the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$, by the proof of Proposition 1 we have

$$p^{\frac{s-n}{2}} \sum_{x \in \mathbb{F}_p^{n-s}} \mu(x) \zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)} = \zeta_{p^k}^{f(a)}.$$

Then

$$p^{\frac{s-n}{2}} \sum_{x \in \mathbb{F}_p^{n-s}} \mu(x) \zeta_{p^k}^{d(x)} \zeta_p^{aM^T R^T \cdot x} = \zeta_{p^k}^{f(a) - p^{k-1} a \cdot t}. \tag{9}$$

For any $y \in \mathbb{F}_p^{n-s}$, since R is row full rank and M is invertible, we have $Rank(RM) = Rank((RM, y^T)) = n-s$. Hence, for any $y \in \mathbb{F}_p^{n-s}$, there exists $a_y \in \mathbb{F}_p^n$ such that $a_y M^T R^T = y$. When $a_y M^T R^T = b_y M^T R^T = y$, by (9) we have $f(a_y) - p^{k-1} a_y \cdot t = f(b_y) - p^{k-1} b_y \cdot t$. Define

$$g : \mathbb{F}_p^{n-s} \to \mathbb{Z}_{p^k}$$
$$g(y) = f(a_y) - p^{k-1} a_y \cdot t,$$

where $a_y \in \mathbb{F}_p^n$ satisfies $a_y M^T R^T = y$.

Then for any $b \in \mathbb{F}_p^{n-s}$,

$$\begin{aligned}
W_g(b) &= \sum_{y \in \mathbb{F}_p^{n-s}} \zeta_{p^k}^{g(y)} \zeta_p^{-b \cdot y} \\
&= \sum_{y \in \mathbb{F}_p^{n-s}} \zeta_{p^k}^{f(a_y) - p^{k-1} a_y \cdot t} \zeta_p^{-b \cdot y} \\
&= \sum_{y \in \mathbb{F}_p^{n-s}} p^{\frac{s-n}{2}} \sum_{x \in \mathbb{F}_p^{n-s}} \mu(x) \zeta_{p^k}^{d(x)} \zeta_p^{a_y M^T R^T \cdot x} \zeta_p^{-b \cdot y} \\
&= p^{\frac{s-n}{2}} \sum_{x \in \mathbb{F}_p^{n-s}} \mu(x) \zeta_{p^k}^{d(x)} \sum_{y \in \mathbb{F}_p^{n-s}} \zeta_p^{y \cdot (x-b)} \\
&= p^{\frac{n-s}{2}} \mu(b) \zeta_{p^k}^{d(b)},
\end{aligned}$$

that is, $g : \mathbb{F}_p^{n-s} \to \mathbb{Z}_{p^k}$ is a generalized bent function and d is the dual function of g and $\mu_g = \mu$.

■

Remark 2. *It is known that plateaued functions with affine Walsh support correspond to partially bent functions. A function $f : V_n \to \mathbb{F}_p$ is called a partially bent function if for any $a \in V_n$, $f(x+a) - f(x), x \in V_n$ is either balanced or constant. When p is an odd prime and*

$k=1$, *Theorem 1 gives a completely characterization of* p*-ary partially bent functions, which generalizes the case of Boolean partially bent functions [13].*

We give two examples of generalized plateaued functions with affine Walsh support by using Theorem 1.

Example 1. *Let* $p=3, k=1, n=4, s=1$. *Let* $d:\mathbb{F}_3^3 \to \mathbb{F}_3$ *be defined as* $d(x_1,x_2,x_3)=x_1x_3+2x_2^2+2x_3^2$, *then* d *is the dual function of weakly regular bent function* $g(x_1,x_2,x_3)=2x_1^2+2x_1x_3+x_2^2$ *with* $\mu_g(x_1,x_2,x_3)=\sqrt{-1},(x_1,x_2,x_3)\in\mathbb{F}_3^3$. *Let* $\mu(x_1,x_2,x_3)=\sqrt{-1},(x_1,x_2,x_3)\in\mathbb{F}_3^3$. *Let* $S=\{w_0,\dots,w_{26}\}$ *where* $w_i=(2,0,0,0)+e_iM$, $E=\{e_0,\dots,e_{26}\}=<(0,0,1,1),(0,1,0,0),(1,0,0,0)>$, $M=\begin{pmatrix}0&0&1&1\\0&1&0&0\\0&0&0&1\\1&0&0&2\end{pmatrix}$. *Then the constructed weakly regular* 1*-plateaued function* $f:\mathbb{F}_3^4\to\mathbb{F}_3$ *by Theorem 1 is* $f(x_1,x_2,x_3,x_4)=g(x_3+x_4,x_2,x_1)+2x_1=2x_1x_3+2x_1x_4+x_2^2+2x_3^2+x_3x_4+2x_4^2+2x_1$.

Example 2. *Let* $p=2, k=3, n=4, s=2$. *Let* $d:\mathbb{F}_2^2\to\mathbb{Z}_8$ *be defined as* $d(x_1,x_2)=4x_1x_2+x_2$, *then* d *is the dual function of generalized bent function* $g(x_1,x_2)=4x_1x_2+x_1$ *with* $\mu_g(x_1,x_2)=1,(x_1,x_2)\in\mathbb{F}_2^2$. *Let* $\mu(x_1,x_2)=1,(x_1,x_2)\in\mathbb{F}_2^2$. *Let* $S=\{w_0,\dots,w_3\}$ *where* $w_i=(0,1,1,0)+e_i$ *and* $E=\{e_0,\dots,e_3\}=<(0,0,1,1),(1,1,0,1)>$. *Then the constructed generalized* 2*-plateaued function* $f:\mathbb{F}_2^4\to\mathbb{Z}_8$ *by Theorem 1 is* $f(x_1,x_2,x_3,x_4)=g(x_1+x_2+x_4,x_3+x_4)+4(x_2+x_3)=((x_1+x_2+x_4)\,mod\,2)+4(x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4+x_2+x_3+x_4)$.

One can construct pairwise disjoint spectra generalized p-ary s-plateaued functions $f_0,f_1,\dots,f_{p^s-1}$ by Theorem 1. When E is an $(n-s)$-dimensional subspace of $\mathbb{F}_p^n$, we have $\mathbb{F}_p^n=\cup_{e_i'\in E'}(e_i'+E)$ where $E'\oplus E=\mathbb{F}_p^n$ ($\oplus$ denotes direct sum) and $e_i'\in E'$ for $0\le i\le p^s-1$. Let $S_i=e_i'+E$ for any $0\le i\le p^s-1$, then $S_i\cap S_j=\emptyset$ if $i\ne j$ and one can construct generalized p-ary s-plateaued functions $f_i(0\le i\le p^s-1)$ with S_i as Walsh support by using Theorem 1 and some known generalized bent functions as building blocks. By using pairwise disjoint spectra generalized plateaued functions as building blocks, one can get the following construction method of generalized bent functions which is an extension of Theorem 2 of [7].

Proposition 3. *Let* p *be a prime,* $n, s(\le n), k$ *be positive integers. Let* $f_y(y\in\mathbb{F}_p^s):\mathbb{F}_p^n\to\mathbb{Z}_{p^k}$

be pairwise disjoint spectra generalized s-plateaued functions. Let W and U be n-dimensional and s-dimensional subspaces of $\mathbb{F}_p^{n+s}$ respectively and satisfy $\mathbb{F}_p^{n+s} = W \oplus U$. Define

$$F(xM + \pi(y)) = f_y(x), x \in \mathbb{F}_p^n, y \in \mathbb{F}_p^s,$$

where M is a matrix whose row vectors form a basis of W and π is a bijection from $\mathbb{F}_p^s$ to U. Then F is a generalized bent function from $\mathbb{F}_p^{n+s}$ to $\mathbb{Z}_{p^k}$.

Proof: For any $z \in \mathbb{F}_p^{n+s}$, there exist unique $w_z \in W, u_z \in U$ such that $z = w_z + u_z$ since $\mathbb{F}_p^{n+s} = W \oplus U$. It is easy to see that the function L defined by $L(x) = xM, x \in \mathbb{F}_p^n$ is a bijection from $\mathbb{F}_p^n$ to W. As $L : \mathbb{F}_p^n \to W$ and $\pi : \mathbb{F}_p^s \to U$ are both bijections, there exist unique $x_z \in \mathbb{F}_p^n, y_z \in \mathbb{F}_p^s$ such that $z = x_z M + \pi(y_z)$. Hence F is a function from $\mathbb{F}_p^{n+s}$ to $\mathbb{Z}_{p^k}$. For any $a \in \mathbb{F}_p^{n+s}$, we have

$$\begin{aligned} W_F(a) &= \sum_{z \in \mathbb{F}_p^{n+s}} \zeta_{p^k}^{F(z)} \zeta_p^{-a \cdot z} \\ &= \sum_{x \in \mathbb{F}_p^n} \sum_{y \in \mathbb{F}_p^s} \zeta_{p^k}^{F(xM+\pi(y))} \zeta_p^{-a \cdot (xM + \pi(y))} \\ &= \sum_{x \in \mathbb{F}_p^n} \sum_{y \in \mathbb{F}_p^s} \zeta_{p^k}^{f_y(x)} \zeta_p^{-aM^T \cdot x - a \cdot \pi(y)} \\ &= \sum_{y \in \mathbb{F}_p^s} \zeta_p^{-a \cdot \pi(y)} \sum_{x \in \mathbb{F}_p^n} W_{f_y}(aM^T). \end{aligned}$$

Since $f_y, y \in \mathbb{F}_p^s$ are pairwise disjoint spectra generalized s-plateaued functions, we have $|S_{f_y}| = p^{n-s}$ and $S_{f_y} \cap S_{f_{y'}} = \emptyset$ for any $y \neq y'$, which yields that $S_{f_y}, y \in \mathbb{F}_p^s$ is a partition of $\mathbb{F}_p^n$. Hence for any $a \in \mathbb{F}_p^{n+s}$, there exists a unique $y_a \in \mathbb{F}_p^s$ such that $aM^T \in S_{f_{y_a}}$ and $|W_F(a)| = |\zeta_p^{-a \cdot \pi(y_a)} W_{f_{y_a}}(aM^T)| = p^{\frac{n+s}{2}}$, that is, F is a generalized bent function. ■

When $k = 1$, $W = \mathbb{F}_p^n \times \{0_s\}$, $U = \{0_n\} \times \mathbb{F}_p^s$, M is the matrix whose row vectors are $(1, 0, \ldots, 0, 0, \ldots, 0), (0, 1, \ldots, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1, 0, \ldots, 0)$ and $\pi(y) = (0_n, y), y \in \mathbb{F}_p^s$ where 0_n denotes the zero vector of $\mathbb{F}_p^n$, Proposition 3 reduces to Theorem 2 of [7]. We give an example to illustrate Proposition 3.

Example 3. *Let $p = 2, n = 5, s = 1, k = 3$. Let $f_0, f_1 : \mathbb{F}_2^5 \to \mathbb{Z}_{2^3}$ be defined as $f_0(x_1, \ldots, x_5) = 4(x_1x_3 + x_2x_4) + 2x_3 + x_3x_4$, $f_1(x_1, \ldots, x_5) = 4(x_1x_3 + x_2x_4 + x_5) + 2x_1x_2 + x_1$. Then f_0, f_1 are disjoint spectra generalized 1-plateaued functions. Let $W = \mathbb{F}_2^5 \times \{0\}$, $U = \{0_5\} \times \mathbb{F}_2$, M is the matrix whose row vectors are $(1, 0, \ldots, 0, 0), \ldots, (0, 0, \ldots, 1, 0)$ and $\pi(y) =$*

$(0,\dots,0,y), y \in \mathbb{F}_2$. *Then the constructed generalized bent function* $F : \mathbb{F}_2^6 \to \mathbb{Z}_{2^3}$ *by Proposition 3 is* $F(x_1,\dots,x_6) = f_{x_6}(x_1,\dots,x_5) = 4(x_1x_3+x_2x_4+x_5x_6)+2((x_1x_2x_6+x_3(1+x_6))\ mod\ 2)+((x_3x_4(1+x_6)+x_1x_6)\ mod\ 2)$.

C. Some Generic Construction Methods of Generalized Plateaued Functions with (Non)-Affine Walsh Support in Spectral Domain

In this subsection, we provide some generic construction methods of generalized plateaued functions with (non)-affine Walsh support in spectral domain.

With the same notation as Proposition 1. If $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ is a generalized s-plateaued function constructed in spectral domain, by the proof of Proposition 1, we have $S_f = S$ where ordered $S = \{w_0,\dots,w_{p^{n-s}-1}\}$. It is easy to see that the matrix form of S_f whose row vectors are $w_0,\dots,w_{p^{n-s}-1}$ can be written as

$$S_f = T_{\psi_{a_1}} \wr \cdots \wr T_{\psi_{a_n}} \tag{10}$$

where $\{a_1,\dots,a_n\}$ is the canonical basis of $\mathbb{F}_p^n$, that is, $a_1 = (1,0,0,\dots,0,0), a_2 = (0,1,0,\dots,0,0),\dots,a_n = (0,0,0,\dots,0,1)$, $\psi_{a_i} : V_{n-s} \to \mathbb{F}_p$ is defined by (6) and $T_{\psi_{a_i}}$ defined by (5) is the true table of ψ_{a_i}. If ψ_{a_i} is an affine function, we say that the i-th column of (ordered) S_f corresponds to an affine function. Note that if f is constructed by Theorem 1, then every column of S_f corresponds to an affine function by Lemma 1.

In [13], Hodžić *et al*. designed Boolean plateaued functions with (non)-affine Walsh support in spectral domain. As pointed out in [13], for the constructions in spectral domain given in [13], the Walsh support of Boolean s-plateaued functions in n variables, when written as a matrix of form (10), contains at least $n-s$ columns corresponding to affine functions on $\mathbb{F}_2^{n-s}$. They proposed an open problem to provide constructions of Boolean plateaued functions whose Walsh support, when written as a matrix of form (10), contains strictly less than $n-s$ columns corresponding to affine functions. In our constructions of generalized s-plateaued functions with non-affine Walsh support, the Walsh support, when written as a matrix of form (10), can contain strictly less than $n-s$ columns corresponding to affine functions and our construction methods are also applicable to Boolean plateaued functions.

In the first generic construction method, we will utilize an important class of generalized bent functions $f : \mathbb{F}_{p^n} \times \mathbb{F}_{p^n} \to \mathbb{Z}_{p^k}$ defined as

$$f(x_1, x_2) = p^{k-1} Tr_1^n(\alpha x_1 \pi(x_2)) + g(x_2), (x_1, x_2) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^n}$$

where $\alpha \in \mathbb{F}_{p^n}^*$, π is a permutation over $\mathbb{F}_{p^n}$ and g is an arbitrary function from $\mathbb{F}_{p^n}$ to $\mathbb{Z}_{p^k}$, which is a generalization of the well-known Maiorana-McFarland bent functions. It is easy to obtain its dual function $f^*(x_1, x_2) = -p^{k-1} Tr_1^n(x_2 \pi^{-1}(\alpha^{-1} x_1)) + g(\pi^{-1}(\alpha^{-1} x_1))$ and $\mu_f(x_1, x_2) = 1, (x_1, x_2) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^n}$.

For the sake of simplicity, we give the functions needed in the following theorem. Let $n, s(< n)$ be positive integers with $n-s = 2m$, $\{\alpha_1, \dots, \alpha_m\}$ be a basis of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$, π be a permutation over $\mathbb{F}_{p^m}$ and $L_1, \dots, L_{n-s} : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p$ be linearly independent linear functions. Define $d : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{Z}_{p^k}$ as

$$d(x_1, x_2) = p^{k-1} Tr_1^m(\alpha_1 x_1 \pi(x_2)) + g(x_2), \tag{11}$$

where g is an arbitrary function from $\mathbb{F}_{p^m}$ to $\mathbb{Z}_{p^k}$. Define $t_i : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p, 1 \le i \le s$ as

$$t_i(x_1, x_2) = \begin{cases} Tr_1^m(\beta_i x_1 \pi(x_2)) + g_i(x_2) + A_i(x_1, x_2) \ if \ m \ge 2, \\ g_i(x_2) + A_i(x_1, x_2) \ if \ m = 1 \end{cases} \tag{12}$$

where $\beta_i = \sum_{j=2}^m c_{i,j} \alpha_j$ with $c_{i,j} \in \mathbb{F}_p$, g_i is an arbitrary function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ and A_i is an arbitrary affine function from $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ to $\mathbb{F}_p$. Define $h_j : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p, 1 \le j \le n-s$ as

$$h_j = \begin{cases} \sum_{i=1}^{s} d_{j,i} t_i + L_j + b_j \ if \ I = \emptyset, \\ \sum_{i \notin I} d_{j,i} t_i + F_j(t_{i_1}, \dots, t_{i_{|I|}}) + L_j + b_j \ if \ I \neq \emptyset \end{cases} \tag{13}$$

where $I = \{1 \le i \le s : t_i(x_1, x_2) \ only \ depends \ on \ variable \ x_2\}$ and denote I by $\{i_1, \dots, i_{|I|}\}$ if $I \neq \emptyset$, $d_{j,i}, b_j \in \mathbb{F}_p$ and F_j is an arbitrary function from $\mathbb{F}_p^{|I|}$ to $\mathbb{F}_p$.

Theorem 2. *With the same notation as Proposition 1. Let $n - s = 2m$ be an even positive integer. Let $d : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{Z}_{p^k}$ be defined by (11). Let $\mu(x_1, x_2) = 1, (x_1, x_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$. Let the matrix form of $S = \{w_0, \dots, w_{p^{n-s}-1}\} \subseteq \mathbb{F}_p^n$ be defined by*

$$S = \begin{pmatrix} w_0 \\ w_1 \\ \dots \\ w_{p^{n-s}-1} \end{pmatrix} = T_{t_1} \wr \cdots \wr T_{t_s} \wr T_{h_1} \wr \cdots \wr T_{h_{n-s}},$$

where $t_i(1 \leq i \leq s)$ *are defined by (12) and* $h_j(1 \leq j \leq n-s)$ *are defined by (13). Then the function* $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ *defined by (7) is the Walsh transform of a generalized* s*-plateaued function* $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$*.*

Proof: First we show that the size of S is equal to p^{n-s}, that is to prove

$$(t_1(x), \ldots, h_{n-s}(x)) = (t_1(x'), \ldots, h_{n-s}(x'))\ if\ and\ only\ if\ x = x'$$

where $x = (x_1, x_2), x' = (x_1', x_2') \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$. If $(t_1(x), \ldots, h_{n-s}(x)) = (t_1(x'), \ldots, h_{n-s}(x'))$, then by the definitions of $h_j(1 \leq j \leq n-s)$, it is easy to see that $L_j(x) = L_j(x')$ for any $1 \leq j \leq n-s$. Since $L_1, \ldots, L_{n-s}$ are linearly independent linear functions, it is easy to see that $x = x'$. Hence we have $|S| = p^{n-s}$.

For any $a \in \mathbb{F}_p^n$ and $0 \leq i \leq p^{n-s}-1$, $\psi_a(v_i) = a \cdot w_i = a \cdot (t_1(v_i), \ldots, t_s(v_i), h_1(v_i), \ldots, h_{n-s}(v_i))$. When $m \geq 2$, by the constructions of $t_i, h_j(1 \leq i \leq s, 1 \leq j \leq n-s)$, we have $\psi_a(x_1, x_2) = Tr_1^m(\alpha_a x_1 \pi(x_2)) + g_a(x_2) + A_a(x_1, x_2)$ where $\alpha_a \in \mathbb{F}_{p^m}$ is some linear combination of $\alpha_2, \ldots, \alpha_m$, g_a is some function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ and $A_a : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p$ is some affine function. Then $d(x_1, x_2) + p^{k-1}\psi_a(x_1, x_2) = p^{k-1}Tr_1^m((\alpha_1 + \alpha_a)x_1\pi(x_2)) + (g(x_2) + p^{k-1}g_a(x_2)) + p^{k-1}A_a(x_1, x_2)$. Since $\alpha_1, \ldots, \alpha_m$ are linearly independent and $\alpha_a \in \mathbb{F}_{p^m}$ is some linear combination of $\alpha_2, \ldots, \alpha_m$, we have $\alpha_1 + \alpha_a \neq 0$. Note that if $h : V_n \to \mathbb{Z}_{p^k}$ is a weakly regular generalized bent function and $A : V_n \to \mathbb{F}_p$ is an arbitrary affine function, then $h + p^{k-1}A$ is also a weakly regular generalized bent function and $\mu_{h+p^{k-1}A} = \mu_h$. Hence, $d + p^{k-1}\psi_a$ is a weakly regular generalized bent function and $\mu_{d+p^{k-1}\psi_a}(x_1, x_2) = 1, (x_1, x_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ for any $a \in \mathbb{F}_p^n$. By Corollary 1, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$. When $m = 1$, by similar arguments, we have the same conclusion. ■

We give an example of generalized plateaued function by using Theorem 2.

Example 4. *Let* $p = 3, k = 2, n = 7, s = 3$. *Let* z *be the primitive element of* $\mathbb{F}_{3^2}$ *with* $z^2 + 2z + 2 = 0$. *Let* $d(x_1, x_2) = 3Tr_1^2(zx_1x_2) + 2(Tr_1^2(x_2))^2$, $\mu(x_1, x_2) = 1$, $t_1(x_1, x_2) = Tr_1^2(x_1x_2)$, $t_2(x_1, x_2) = Tr_1^2(x_2^2)$, $t_3(x_1, x_2) = Tr_1^2(zx_2^2)$, $h_1 = t_1 + Tr_1^2(x_1)$, $h_2 = t_2^2 + Tr_1^2(zx_1)$, $h_3 = t_3^2 + Tr_1^2(x_2)$, $h_4 = t_2 + t_3 + Tr_1^2(zx_2)$, $(x_1, x_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. *Then by the Walsh inverse transform or by computing* $(d + 3\psi_a)^*(0)$, *we can obtain generalized* 3*-plateaued function* $f(b_1, \ldots, b_3, a_1, \ldots, a_4) = 2(((b_1+a_1)^2a_2 + (2(b_1+a_1)+1)(a_1+a_2)) mod\ 3)^2 + 3((b_1+a_1)^2((b_2+$

$a_4)(2a_1^2+2a_1a_2)+(b_3+a_4)(a_1^2+a_2^2)+2a_1^2a_2+2a_1a_2^2+a_1a_4+a_2a_3+a_2a_4+a_2)+(b_1+a_1)((b_2+a_4)(2a_1^2+a_1a_2+a_2^2)+(b_3+a_4)(2a_1^2+2a_1a_2)+2a_1^2a_2+2a_1a_2^2+2a_1a_3+2a_1a_4+2a_2a_3+a_2a_4+a_2)+2a_1^2a_2^2a_3+(b_2+a_4)(a_1a_2+2a_2^2)+(b_3+a_4)(2a_1^2+a_1a_2+a_2^2)+a_1^2a_2+a_1^2a_3+a_1a_2^2+a_2^2a_3+a_1a_3+a_1a_4+a_2a_3+2a_2a_4+a_2)$ *from* $\mathbb{F}_3^7$ *to* $\mathbb{Z}_{3^2}$. *Since* $t_i(1 \leq i \leq 3), h_j(1 \leq j \leq 4)$ *are all non-affine functions and the matrix form of* S_f *defined by (10) is* $S_f = T_{t_1} \wr \cdots \wr T_{t_3} \wr T_{h_1} \wr \cdots \wr T_{h_4}$, *every column of* S_f *corresponds to a non-affine function.*

When $k = 1$, Theorem 2 can be seen as an extension of Theorem 4.1 of [13] in the sense of equivalence. And it can also be applied to construct Boolean plateaued functions whose Walsh support, when written as a matrix of form (10), contains strictly less than $n - s$ columns corresponding to affine functions. We give an example of Boolean plateaued function which satisfies that every column of the matrix form of S_f defined by (10) corresponds to a non-affine function and the function has no nonzero linear structure. For a Boolean function $f : V_n \to \mathbb{F}_2$, if $f(x) + f(x + a)$ is a constant function, then a is called a linear structure of f.

Example 5. *Let* $p = 2$, $k = 1$, $n = 10$, $s = 4$. *Let* z *be the primitive element of* $\mathbb{F}_{2^3}$ *with* $z^3+z+1 = 0$. *Let* $d(x_1, x_2) = Tr_1^3(z^2x_1x_2)$, $\mu(x_1, x_2) = 1$, $t_1(x_1, x_2) = Tr_1^3(x_1x_2)$, $t_2(x_1, x_2) = Tr_1^3(zx_1x_2)$, $t_3(x_1, x_2) = Tr_1^3(x_2^3)$, $t_4(x_1, x_2) = Tr_1^3(zx_2^3)$, $h_1 = t_1+Tr_1^3(x_1)$, $h_2 = t_1+Tr_1^3(zx_1)$, $h_3 = t_2 + Tr_1^3(z^2x_1)$, $h_4 = t_2 + Tr_1^3(x_2)$, $h_5 = t_3t_4 + Tr_1^3(zx_2)$, $h_6 = t_3t_4 + Tr_1^3(z^2x_2)$. *Then by the Walsh inverse transform or by computing* $(d + \psi_a)^*(0)$, *we can obtain Boolean* 4*-plateaued function* $f(b_1, \ldots, b_4, a_1, \ldots, a_6) = (b_1 + a_1 + a_2 + 1)(b_3(a_1a_3 + a_2a_3 + a_1) + b_4(a_1a_2 + a_1a_3 + a_2a_3 + a_1 + a_3) + (a_1a_2 + a_1a_3)(a_5 + a_6) + a_1a_4 + a_2a_6 + a_3a_5) + ((b_1 + a_1 + a_2)(b_2 + a_3 + a_4) + 1)(a_1a_5 + a_2a_5 + a_3a_4 + a_3a_5) + (b_1 + b_2 + a_1 + a_2 + a_3 + a_4 + 1)(b_3(a_1a_3 + a_2 + a_3) + b_4(a_1a_3 + a_2a_3 + a_1) + a_1a_2(a_5 + a_6) + a_1a_5 + a_2a_4 + a_2a_5 + a_3a_5 + a_3a_6) + b_3(a_1a_2 + a_2a_3 + a_1 + a_2) + b_4(a_1a_3 + a_2 + a_3) + (a_2a_3 + a_1 + a_2 + a_3)(a_5 + a_6)$. *Since* $t_i(1 \leq i \leq 4), h_j(1 \leq j \leq 6)$ *are all non-affine functions and the matrix form of* S_f *defined by (10) is* $S_f = T_{t_1} \wr \cdots \wr T_{t_4} \wr T_{h_1} \wr \cdots \wr T_{h_6}$, *every column of* S_f *corresponds to a non-affine function. And one can verify that* S_f *contains a basis of* $\mathbb{F}_2^{10}$ *and* $(0, \ldots, 0) \in S_f$, *hence by Corollary 3.1 of [13],* f *has no nonzero linear structure.*

In the second generic construction method, we take advantage of the good properties of general generalized bent functions. Let $t \geq 2$ be an integer. Let $f(x) = \sum_{i=0}^{t-1} p^{t-1-i} f_i(x)$ with

$f_i : V_n \to \mathbb{F}_p, 0 \le i \le t-1$ be a generalized bent function from V_n to $\mathbb{Z}_{p^t}$ where p is an odd prime or $p = 2$ and n is even. Let k be a positive integer. Then by Corollary 7 of [27], for any function $G : \mathbb{F}_p^{t-1} \to \mathbb{Z}_{p^k}$, the function $p^{k-1} f_0 + G(f_1, \ldots, f_{t-1})$ is a generalized bent function from V_n to $\mathbb{Z}_{p^k}$ with $\mu_{p^{k-1} f_0 + G(f_1,\ldots,f_{t-1})} = \mu_f$.

For the sake of simplicity, we give the functions needed in the following theorem. Let $n, s(< n)$ be positive integers with $n-s$ even if $p = 2$, $L_1, \ldots, L_{n-s} : V_{n-s} \to \mathbb{F}_p$ be linearly independent linear functions and $g = \sum_{i=0}^{t-1} p^{t-1-i} g_i$ with $g_i : V_{n-s} \to \mathbb{F}_p, 0 \le i \le t-1$ be a weakly regular generalized bent function from V_{n-s} to $\mathbb{Z}_{p^t}$ where $t \ge 2$. Define $d : V_{n-s} \to \mathbb{Z}_{p^k}$ as

$$d(x) = p^{k-1} g_0(x) + G(g_1(x), \ldots, g_{t-1}(x)), \tag{14}$$

where G is an arbitrary function from $\mathbb{F}_p^{t-1}$ to $\mathbb{Z}_{p^k}$. Define

$$\mu(x) = \mu_g(x)^{-1}, x \in V_{n-s} \tag{15}$$

where μ_g is defined by (3). Note that μ_g is a constant function since g is weakly regular. Define $t_i : V_{n-s} \to \mathbb{F}_p, 1 \le i \le s$ as

$$t_i(x) = F_i(g_1(x), \ldots, g_{t-1}(x)), \tag{16}$$

where F_i is an arbitrary function from $\mathbb{F}_p^{t-1}$ to $\mathbb{F}_p$. Define $h_j : V_{n-s} \to \mathbb{F}_p, 1 \le j \le n-s$ as

$$h_j(x) = H_j(t_1(x), \ldots, t_s(x)) + L_j(x) + b_j, \tag{17}$$

where H_j is an arbitrary function from $\mathbb{F}_p^s$ to $\mathbb{F}_p$ and $b_j \in \mathbb{F}_p$.

Theorem 3. *With the same notation as Proposition 1. Let $d : V_{n-s} \to \mathbb{Z}_{p^k}$ be defined by (14). Let μ be defined by (15). Let the matrix form of $S = \{w_0, \ldots, w_{p^{n-s}-1}\} \subseteq \mathbb{F}_p^n$ be defined by*

$$S = \begin{pmatrix} w_0 \\ w_1 \\ \cdots \\ w_{p^{n-s}-1} \end{pmatrix} = T_{t_1} \wr \cdots \wr T_{t_s} \wr T_{h_1} \wr \cdots \wr T_{h_{n-s}},$$

where $t_i (1 \le i \le s)$ are defined by (16) and $h_j (1 \le j \le n-s)$ are defined by (17). Then the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$.

Proof: First we show that the size of S is equal to p^{n-s}, that is to prove

$$(t_1(x),\ldots,h_{n-s}(x))=(t_1(x'),\ldots,h_{n-s}(x'))\ if\ and\ only\ if\ x=x'.$$

If $(t_1(x),\ldots,h_{n-s}(x))=(t_1(x'),\ldots,\ h_{n-s}(x'))$, then by the definitions of $h_j(1\leq j\leq n-s)$, it is easy to see that $L_j(x)=L_j(x')$ for any $1\leq j\leq n-s$. Since $L_1,\ldots,L_{n-s}$ are linearly independent linear functions, it is easy to see that $x=x'$. Hence we have $|S|=p^{n-s}$.

For any $a\in\mathbb{F}_p^n$ and $0\leq i\leq p^{n-s}-1$, $\psi_a(v_i)=a\cdot w_i=a\cdot(t_1(v_i),\ldots,t_s(v_i),h_1(v_i),\ldots,h_{n-s}(v_i))$. By the constructions of $t_i,h_j(1\leq i\leq s,1\leq j\leq n-s)$, we have $\psi_a(x)=G_a(g_1(x),\ldots,g_{t-1}(x))+A_a(x)$ where G_a is some function from $\mathbb{F}_p^{t-1}$ to $\mathbb{F}_p$ and $A_a:V_{n-s}\rightarrow\mathbb{F}_p$ is some affine function. Then $d(x)+p^{k-1}\psi_a(x)=p^{k-1}g_0(x)+(G(g_1(x),\ldots,g_{t-1}(x))+p^{k-1}G_a(g_1(x),\ \ldots,g_{t-1}(x)))+p^{k-1}A_a(x)$. Note that if $h:V_n\rightarrow\mathbb{Z}_{p^k}$ is a weakly regular generalized bent function and $A:V_n\rightarrow\mathbb{F}_p$ is an arbitrary affine function, then $h+p^{k-1}A$ is also a weakly regular generalized bent function and $\mu_{h+p^{k-1}A}=\mu_h$. Hence, $d+p^{k-1}\psi_a$ is a weakly regular generalized bent function and $\mu_{d+p^{k-1}\psi_a}=\mu_g$ for any $a\in\mathbb{F}_p^n$. By Corollary 1, the function $W:\mathbb{F}_p^n\rightarrow\mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f:\mathbb{F}_p^n\rightarrow\mathbb{Z}_{p^k}$. ■

We give an example of generalized plateaued function by using Theorem 3.

Example 6. *Let* $p=5,k=3,n=4,s=1,t=2$. *Let* z *be the primitive element of* $\mathbb{F}_{5^3}$ *with* $z^3+3z+3=0$. *Let* $g(x)=5g_0(x)+g_1(x),x\in\mathbb{F}_{5^3}$ *where* $g_0(x)=Tr_1^3(2x^2)$, $g_1(x)=Tr_1^3(z^{16}x)$. *Then by Theorem 16 of [28] and Corollary 3 of [30],* g *is a weakly regular generalized bent function with* $\mu_g(x)=-1,x\in\mathbb{F}_{5^3}$. *Let* $d(x)=25g_0(x)+g_1^4(x)$, $\mu(x)=-1$, $t_1(x)=g_1^3(x)$, $h_1(x)=t_1^2(x)+Tr_1^3(x)$, $h_2(x)=t_1^4(x)+Tr_1^3(zx)$, $h_3(x)=t_1(x)+Tr_1^3(z^2x)$, $x\in\mathbb{F}_{5^3}$. *Then by the Walsh inverse transform or by computing* $(d+25\psi_a)^*(0)$, *we can obtain generalized* 1*-plateaued function* $f(b_1,a_1,a_2,a_3)=((a_1-a_3)\ mod\ 5)^4+25(a_2(a_1-a_3)^4+(b_1+a_3)(a_1-a_3)^3+a_1(a_1-a_3)^2-a_1^2-a_1a_3+2a_2^2+a_2a_3-a_3^2)$ *from* $\mathbb{F}_5^4$ *to* $\mathbb{Z}_{5^3}$. *Since* $t_1,h_j(1\leq j\leq 3)$ *are all non-affine functions and the matrix form of* S_f *defined by (10) is* $S_f=T_{t_1}\wr T_{h_1}\wr\cdots\wr T_{h_3}$, *every column of* S_f *corresponds to a non-affine function.*

Theorem 3 is also applicable to Boolean plateaued functions. We give an example of Boolean plateaued function which satisfies that every column of the matrix form of S_f defined by (10) corresponds to a non-affine function and the function has no nonzero linear structure.

Example 7. *Let $p = 2, k = 1, n = 8, s = 2, t = 3$. Let $g(x_1, \ldots, x_6) = 4g_0(x_1, \ldots, x_6) + 2g_1(x_1, \ldots, x_6) + g_2(x_1, \ldots, x_6), (x_1, \ldots, x_6) \in \mathbb{F}_2^6$ where $g_0(x_1, \ldots, x_6) = x_1x_3 + x_2x_4 + x_5x_6$, $g_1(x_1, \ldots, x_6) = x_1x_2x_6 + x_3(x_6 + 1)$, $g_2(x_1, \ldots, x_6) = x_3x_4(x_6 + 1) + x_1x_6$. Then g is the generalized Boolean bent function constructed in Example 3. Let $d = g_0$, $\mu = 1$, $t_1 = g_1$, $t_2 = g_2$, $h_1 = t_1t_2 + x_1$, $h_2 = t_1 + x_2$, $h_3 = t_1 + x_3$, $h_4 = t_2 + x_4$, $h_5 = t_2 + x_5$, $h_6 = t_1 + t_2 + x_6$. Then by the Walsh inverse transform or by computing $(d + \psi_a)^*(0)$, we can obtain Boolean 2-plateaued function $f(b_1, b_2, a_1, \ldots, a_6) = a_1a_3 + a_2a_4 + a_5a_6 + a_1(a_5 + 1)(b_2a_2 + a_2a_4 + a_2a_6 + b_1 + a_3 + a_6) + a_3a_5(b_1a_4 + a_1a_4 + a_2a_4 + a_4a_6 + b_2 + a_6 + 1)$. Since $t_i (1 \leq i \leq 2), h_j (1 \leq j \leq 6)$ are all non-affine functions and the matrix form of S_f defined by (10) is $S_f = T_{t_1} \wr T_{t_2} \wr T_{h_1} \wr \cdots \wr T_{h_6}$, every column of S_f corresponds to a non-affine function. And one can verify that S_f contains a basis of $\mathbb{F}_2^8$ and $(0, \ldots, 0) \in S_f$, hence by Corollary 3.1 of [13], f has no nonzero linear structure.*

The third generic construction method is used to construct p-ary plateaued functions, that is, $k = 1$. In the following theorem, we utilize vectorial bent functions. Let $m \geq 2$ be an integer. A function $f = (f_1, \ldots, f_m) : V_n \to \mathbb{F}_p^m$ is called a vectorial bent function if for any nonzero vector $(a_1, \ldots, a_m) \in \mathbb{F}_p^m$, $\sum_{i=1}^m a_i f_i(x), x \in V_n$ is a bent function. The following theorem can be seen as a generalization of Theorem 4.3 of [13] in the sense of equivalence. And it can also be applied to construct Boolean plateaued functions whose Walsh support, when written as a matrix of form (10), contains strictly less than $n - s$ columns corresponding to affine functions.

For the sake of simplicity, we give the functions needed in the following theorem. Let $g = (g_1, \ldots, g_m)$ be a vectorial bent function from V_{n-s} to $\mathbb{F}_p^m$ where $m \geq 2$ and there exists a constant u such that $\mu_{g_1 + \sum_{i=2}^m c_i g_i}(x) = u, x \in V_{n-s}$ for any $(c_2, \ldots, c_m) \in \mathbb{F}_p^{m-1}$ where $\mu_{g_1 + \sum_{i=2}^m c_i g_i}$ is defined by (3). Let $L_1, \ldots, L_{n-s} : V_{n-s} \to \mathbb{F}_p$ be linearly independent linear functions. Define $d : V_{n-s} \to \mathbb{F}_p$ as

$$d(x) = g_1(x). \tag{18}$$

Define μ as

$$\mu(x) = u^{-1}, x \in V_{n-s}. \tag{19}$$

Define $t_i : V_{n-s} \to \mathbb{F}_p, 1 \leq i \leq s$ as

$$t_i(x) = \sum_{j=2}^m c_{i,j} g_j(x) + A_i(x), \tag{20}$$

where $c_{i,j} \in \mathbb{F}_p$, A_i is an arbitrary affine function from V_{n-s} to $\mathbb{F}_p$. Define $h_j : V_{n-s} \to \mathbb{F}_p, 1 \leq j \leq n-s$ as

$$h_j(x) = \sum_{i=1}^{s} d_{j,i} t_i(x) + L_j(x) + b_j, \tag{21}$$

where $d_{j,i}, b_j \in \mathbb{F}_p$.

Theorem 4. *With the same notation as Proposition 1. Let $d : V_{n-s} \to \mathbb{F}_p$ be defined by (18). Let μ be defined by (19). Let the matrix form of $S = \{w_0, \ldots, w_{p^{n-s}-1}\} \subseteq \mathbb{F}_p^n$ be defined by*

$$S = \begin{pmatrix} w_0 \\ w_1 \\ \cdots \\ w_{p^{n-s}-1} \end{pmatrix} = T_{t_1} \wr \cdots \wr T_{t_s} \wr T_{h_1} \wr \cdots \wr T_{h_{n-s}},$$

where $t_i (1 \leq i \leq s)$ are defined by (20) and $h_j (1 \leq j \leq n-s)$ are defined by (21). Then the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_p]$ defined by (7) is the Walsh transform of an s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{F}_p$.

Proof: First we show that the size of S is equal to p^{n-s}, that is to prove

$$(t_1(x), \ldots, h_{n-s}(x)) = (t_1(x'), \ldots, h_{n-s}(x'))\ if\ and\ only\ if\ x = x'.$$

If $(t_1(x), \ldots, h_{n-s}(x)) = (t_1(x'), \ldots, h_{n-s}(x'))$, then by the definitions of $h_j (1 \leq j \leq n-s)$, it is easy to see that $L_j(x) = L_j(x')$ for any $1 \leq j \leq n-s$. Since $L_1, \ldots, L_{n-s}$ are linearly independent linear functions, it is easy to see that $x = x'$. Hence we have $|S| = p^{n-s}$.

For any $a \in \mathbb{F}_p^n$ and $0 \leq i \leq p^{n-s}-1$, $\psi_a(v_i) = a \cdot w_i = a \cdot (t_1(v_i), \ldots, t_s(v_i), h_1(v_i), \ldots, h_{n-s}(v_i))$. By the constructions of $t_i, h_j (1 \leq i \leq s, 1 \leq j \leq n-s)$, we have $\psi_a(x) = L_a(g_2(x), \ldots, g_m(x)) + A_a(x)$ where L_a is some linear function from $\mathbb{F}_p^{m-1}$ to $\mathbb{F}_p$ and $A_a : V_{n-s} \to \mathbb{F}_p$ is some affine function. Then $d(x) + \psi_a(x) = g_1(x) + L_a(g_2(x), \ldots, g_m(x)) + A_a(x)$ is a weakly regular bent function and $\mu_{d+\psi_a}(x) = u, x \in V_{n-s}$ for any $a \in \mathbb{F}_p^n$. By Corollary 1, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_p]$ defined by (7) is the Walsh transform of an s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{F}_p$. ■

IV. Generalized indirect sum construction method of generalized plateaued functions and its applications

In this section, we provide a generalized indirect sum construction method of generalized s-plateaued functions where $s \geq 0$. In particular, we show that the canonical way to construct Generalized Maiorana-McFarland bent functions is a special case of the generalized indirect sum construction method and we illustrate that the generalized indirect sum construction method can be used to construct bent functions not in the complete Generalized Maiorana-McFarland class. Furthermore, we give some applications of the generalized indirect sum construction method.

A. Generalized Indirect Sum Construction Method

In this subsection, we give a generalized indirect sum construction method of generalized plateaued functions, which is an extension of indirect sum construction method of Boolean case [3].

Theorem 5. *Let p be a prime. Let k, t, r, m be positive integers, $s(\leq r)$ be a non-negative integer and m be even for $p = 2$, $r + s$ be even for $p = 2, k = 1$. Let $f_i(i \in \mathbb{F}_p^t) : V_r \to \mathbb{Z}_{p^k}$ be generalized s-plateaued functions. Let $g_i(0 \leq i \leq t) : V_m \to \mathbb{F}_p$ be bent functions which satisfy that for any $j = (j_1, \ldots, j_t) \in \mathbb{F}_p^t$, $G_j \triangleq (1 - j_1 - \cdots - j_t)g_0 + j_1 g_1 + \cdots + j_t g_t$ is a bent function and $G_j^* = (1 - j_1 - \cdots - j_t)g_0^* + j_1 g_1^* + \cdots + j_t g_t^*$ and $\mu_{G_j} = u$ where μ_{G_j} is defined by (3) and u is a function from V_m to $\{\pm 1, \pm\sqrt{-1}\}$ independent of j. Let $g : \mathbb{F}_p^t \to \mathbb{Z}_{p^k}$ be an arbitrary function. Then $h(x, y) = f_{(g_0(y)-g_1(y),\ldots,g_0(y)-g_t(y))}(x) + p^{k-1}g_0(y) + g(g_0(y) - g_1(y), \ldots, g_0(y) - g_t(y)), (x, y) \in V_r \times V_m$ is a generalized s-plateaued function from $V_r \times V_m$ to $\mathbb{Z}_{p^k}$.*

Proof: For any $(a, b) \in V_r \times V_m$, we have

$$
\begin{aligned}
&W_h(a,b)\\
&= \sum_{x \in V_r, y \in V_m} \zeta_{p^k}^{f_{(g_0(y)-g_1(y),\ldots,g_0(y)-g_t(y))}(x)+g(g_0(y)-g_1(y),\ldots,g_0(y)-g_t(y))} \zeta_p^{g_0(y)-\langle a,x\rangle-\langle b,y\rangle}\\
&= \sum_{i_1,\ldots,i_t \in \mathbb{F}_p} \sum_{y: g_0(y)-g_j(y)=i_j, 1\leq j\leq t} \sum_{x \in V_r} \zeta_{p^k}^{f_{(i_1,\ldots,i_t)}(x)+g(i_1,\ldots,i_t)} \zeta_p^{g_0(y)-\langle a,x\rangle-\langle b,y\rangle}\\
&= p^{-t} \sum_{i_1,\ldots,i_t \in \mathbb{F}_p} \zeta_{p^k}^{g(i_1,\ldots,i_t)} W_{f_{(i_1,\ldots,i_t)}}(a) \sum_{y \in V_m} \zeta_p^{g_0(y)-\langle b,y\rangle} \sum_{j_1 \in \mathbb{F}_p} \zeta_p^{(i_1-(g_0-g_1)(y))j_1} \cdots \sum_{j_t \in \mathbb{F}_p} \zeta_p^{(i_t-(g_0-g_t)(y))j_t}
\end{aligned}
$$

$$
\begin{aligned}
&= p^{-t} \sum_{i_1,\ldots,i_t \in \mathbb{F}_p} \zeta_{p^k}^{g(i_1,\ldots,i_t)} W_{f_{(i_1,\ldots,i_t)}}(a) \sum_{j_1,\ldots,j_t \in \mathbb{F}_p} \zeta_p^{i_1 j_1 + \cdots + i_t j_t} W_{G_{(j_1,\ldots,j_t)}}(b) \\
&= u(b) p^{\frac{m}{2}} p^{-t} \sum_{i_1,\ldots,i_t \in \mathbb{F}_p} \zeta_{p^k}^{g(i_1,\ldots,i_t)} W_{f_{(i_1,\ldots,i_t)}}(a) \sum_{j_1,\ldots,j_t \in \mathbb{F}_p} \zeta_p^{i_1 j_1 + \cdots + i_t j_t} \zeta_p^{(1-j_1-\cdots-j_t) g_0^*(b) + j_1 g_1^*(b) + \cdots + j_t g_t^*(b)} \\
&= u(b) p^{\frac{m}{2}} p^{-t} \zeta_p^{g_0^*(b)} \sum_{i_1,\ldots,i_t \in \mathbb{F}_p} \zeta_{p^k}^{g(i_1,\ldots,i_t)} W_{f_{(i_1,\ldots,i_t)}}(a) \sum_{j_1 \in \mathbb{F}_p} \zeta_p^{(g_1^*(b) - g_0^*(b) + i_1) j_1} \cdots \sum_{j_t \in \mathbb{F}_p} \zeta_p^{(g_t^*(b) - g_0^*(b) + i_t) j_t} \\
&= u(b) p^{\frac{m}{2}} \zeta_p^{g_0^*(b)} \zeta_{p^k}^{g(g_0^*(b) - g_1^*(b), \ldots, g_0^*(b) - g_t^*(b))} W_{f_{(g_0^*(b) - g_1^*(b), \ldots, g_0^*(b) - g_t^*(b))}}(a),
\end{aligned}
\tag{22}
$$

where the fifth equation is obtained by the properties of bent functions $g_i (0 \leq i \leq t)$, which satisfy that for any $j_1, \ldots, j_t \in \mathbb{F}_p$, $G_{(j_1,\ldots,j_t)} \triangleq (1 - j_1 - \cdots - j_t) g_0 + j_1 g_1 + \cdots + j_t g_t$ is a bent function and $G^*_{(j_1,\ldots,j_t)} = (1 - j_1 - \cdots - j_t) g_0^* + j_1 g_1^* + \cdots + j_t g_t^*$ and $\mu_{G_{(j_1,\ldots,j_t)}} = u$ where u is a function from V_m to $\{\pm 1, \pm\sqrt{-1}\}$ independent of $(j_1, \ldots, j_t)$.

Hence, by (22), it is easy to see that $h : V_r \times V_m \rightarrow \mathbb{Z}_{p^k}$ is a generalized s-plateaued function if $f_i, i \in \mathbb{F}_p^t$ are generalized s-plateaued functions from V_r to $\mathbb{Z}_{p^k}$. ■

If $s = 0$, then Theorem 5 can be used to construct (non)-weakly regular generalized bent functions and the dual function can be given. The following corollary is an immediate consequence of Theorem 5 and its proof.

Corollary 2. *If $s = 0$, then the function $h : V_r \times V_m \rightarrow \mathbb{Z}_{p^k}$ constructed by Theorem 5 is a generalized bent function and its dual function $h^*(x, y) = f^*_{(g_0^*(y) - g_1^*(y), \ldots, g_0^*(y) - g_t^*(y))}(x) + p^{k-1} g_0^*(y) + g(g_0^*(y) - g_1^*(y), \ldots, g_0^*(y) - g_t^*(y))$. Furthermore, h is non-weakly regular if any one of the following conditions holds:*

(1) There exists $i \in \mathbb{F}_p^t$ such that f_i is non-weakly regular and $|\{b \in V_m : (g_0^*(b) - g_1^*(b), \ldots, g_0^*(b) - g_t^*(b)) = i\}| \geq 1$;

(2) u is a constant function and there exist $i_1 \neq i_2 \in \mathbb{F}_p^t$ such that f_{i_1}, f_{i_2} are weakly regular with $\mu_{f_{i_1}} \neq \mu_{f_{i_2}}$ and $|\{b \in V_m : (g_0^*(b) - g_1^*(b), \ldots, g_0^*(b) - g_t^*(b)) = i_j\}| \geq 1$ for $j = 1, 2$;

(3) u is not a constant function and $\mu_{f_i} = c, i \in \mathbb{F}_p^t$ where c is a constant function independent of i.

Now we illustrate that why we call Theorem 5 generalized indirect sum construction method. Note that when $p = 2$ and $t = 1$, it is easy to verify that any Boolean bent functions g_0, g_1 satisfy the condition of Theorem 5. Let $p = 2, k = t = 1$, $f_0, f_1 : V_r \rightarrow \mathbb{F}_2$ be Boolean plateaued functions, $g_0, g_1 : V_m \rightarrow \mathbb{F}_2$ be Boolean bent functions and $g = 0$, the plateaued function constructed by Theorem 5 is $h(x,y) = f_{g_0(y)+g_1(y)}(x) + g_0(y) = g_0(y) + f_0(x) + (f_0(x) + f_1(x))(g_0(y)+g_1(y))$. It is just the famous indirect sum construction [3]. Hence, Theorem 5 can be seen as an extension of indirect sum construction of Boolean case. If $g_i(0 \leq i \leq t)$ are bent functions satisfying $g_i = g_0 - c_i, 1 \leq i \leq t$ where $c_i(1 \leq i \leq t)$ are constants, then $g_i(0 \leq i \leq t)$ satisfy the condition of Theorem 5. In this case $h(x,y) = f_{(c_1,\ldots,c_t)}(x)+p^{k-1}g_0(y)+g(c_1,\ldots,c_t)$, which belongs to direct sum construction. We call it a trivial case. When p is an odd prime or $t \geq 2$, except the trivial case, the condition of Theorem 5 for $g_i(0 \leq i \leq t)$ is not trivial.

In [8], the authors defined a class of bent functions

$$F(x,y) = f_y(x), (x,y) \in \mathbb{F}_p^m \times \mathbb{F}_p^s$$

where p is an odd prime, m, s are positive integers with $s \leq m$ and $f_y, y \in \mathbb{F}_p^s$ are pairwise disjoint spectra partially bent functions with s-dimensional linear kernel, which are called Generalized Maiorana-McFarland bent functions. We will show that the canonical way to construct Generalized Maiorana-McFarland bent functions given in (6) of [8] can be obtained by Theorem 5. Let $g_i(0 \leq i \leq t)$ be Maiorana-McFarland bent functions from $\mathbb{F}_p^m \times \mathbb{F}_p^m$ to $\mathbb{F}_p$ defined as $g_0(y_1,y_2) = y_1 \cdot \pi(y_2)$, $g_i(y_1,y_2) = g_0(y_1,y_2) + h_i(y_2), 1 \leq i \leq t$ where π is a permutation over $\mathbb{F}_p^m$ and for any $1 \leq i \leq t$, h_i is an arbitrary function from $\mathbb{F}_p^m$ to $\mathbb{F}_p$. Then it is easy to verify that $g_i(0 \leq i \leq t)$ satisfy the condition of Theorem 5. When p is an odd prime, $k = 1, m = t, s = 0$, $g = 0$, $h_i(y_2) = -y_{2,i}, 1 \leq i \leq t$ where $y_2 = (y_{2,1},\ldots,y_{2,t}) \in \mathbb{F}_p^t$ and $f_i(i \in \mathbb{F}_p^t)$ are bent functions, the bent function constructed by Theorem 5 is $h(x,y_1,y_2) = f_{y_2}(x)+y_1\cdot\pi(y_2), (x,y_1,y_2) \in \mathbb{F}_p^r \times \mathbb{F}_p^t \times \mathbb{F}_p^t$. It is just the canonical way to construct Generalized Maiorana-McFarland bent functions given in (6) of [8]. By Theorem 2 and its proof of [8], any bent function in the complete Generalized Maiorana-McFarland class (that is, equivalent to a Generalized Maiorana-McFarland bent function) is equivalent to a Maiorana-McFarland bent function or a bent function of the form (6) of [8]. Hence, any bent function in the complete Generalized Maiorana-McFarland class and not in the Maiorana-McFarland class is equivalent to a bent function which can be constructed by the generalized indirect sum construction.

Now we provide another construction for $g_i(0 \leq i \leq t)$ to satisfy the condition of Theorem 5.

For any $0 \leq i \leq t$, let

$$g_i(y_1,y_2) = Tr_1^m(\alpha_i G(y_1 y_2^{p^m-2})),\ (y_1,y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$$

where $m \geq t+1$, G is a permutation over $\mathbb{F}_{p^m}$ with $G(0)=0$ and $\alpha_0, \alpha_1, \ldots, \alpha_t \in \mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$. Then $g_i (0 \leq i \leq t)$ are in bent function class PS_{ap} which is a subclass of the famous class of partial spread bent functions (see [9], [16]) and satisfy the condition of Theorem 5.

Indeed, if a PS_{ap} bent function $g(y_1, y_2) = Tr_1^m(\alpha G(y_1 y_2^{p^m-2})), (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ where m is a positive integer, $\alpha \in \mathbb{F}_{p^m}^*$ and G is a permutation over $\mathbb{F}_{p^m}$ with $G(0)=0$, then $g^*(y_1, y_2) = Tr_1^m(\alpha G(-y_1^{p^m-2} y_2))$ and $\mu_g(y_1, y_2) = 1, (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$. Since $m \geq t+1$, there exist linearly independent elements $\alpha_0, \ldots, \alpha_t \in \mathbb{F}_{p^m}$. As $\alpha_0, \ldots, \alpha_t \in \mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$, $\alpha_j \triangleq (1-j_1-\cdots-j_t)\alpha_0 + j_1\alpha_1 + \cdots + j_t\alpha_t \neq 0$ for any $j = (j_1, \ldots, j_t) \in \mathbb{F}_p^t$. Then for any $j = (j_1, \ldots, j_t) \in \mathbb{F}_p^t$, $G_j(y_1, y_2) \triangleq (1-j_1-\cdots-j_t)g_0(y_1, y_2) + j_1 g_1(y_1, y_2) + \cdots + j_t g_t(y_1, y_2) = Tr_1^m(\alpha_j G(y_1 y_2^{p^m-2}))$ is a bent function with $\mu_{G_j}(y_1, y_2) = 1$ and $(1-j_1-\cdots-j_t)g_0^*(y_1, y_2) + j_1 g_1^*(y_1, y_2) + j_t g_t^*(y_1, y_2) = (1-j_1-\cdots-j_t)Tr_1^m(\alpha_0 G(-y_1^{p^m-2} y_2)) + j_1 Tr_1^m(\alpha_1 G(-y_1^{p^m-2} y_2)) + \cdots + j_t Tr_1^m(\alpha_t G(-y_1^{p^m-2} y_2)) = Tr_1^m(\alpha_j G(-y_1^{p^m-2} y_2)) = G_j^*(y_1, y_2)$, that is, $g_i (0 \leq i \leq t)$ satisfy the condition of Theorem 5.

As the above $g_i (0 \leq i \leq t)$ satisfy the condition of Theorem 5, we obtain the following corollary from Theorem 5.

Corollary 3. *Let p be a prime. Let k, t, r, m be positive integers with $m \geq t+1$, $s (\leq r)$ be a non-negative integer and $r+s$ be even for $p=2, k=1$. Let $f_i (i \in \mathbb{F}_p^t) : V_r \rightarrow \mathbb{Z}_{p^k}$ be generalized s-plateaued functions. Let $g_i (0 \leq i \leq t) : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \rightarrow \mathbb{F}_p$ be defined as $g_i(y) = Tr_1^m(\alpha_i G(y_1 y_2^{p^m-2})), y = (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ where G is a permutation over $\mathbb{F}_{p^m}$ with $G(0)=0$ and $\alpha_0, \alpha_1, \ldots, \alpha_t \in \mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$. Let $g : \mathbb{F}_p^t \rightarrow \mathbb{Z}_{p^k}$ be an arbitrary function. Then $h(x, y) = f_{(g_0(y)-g_1(y), \ldots, g_0(y)-g_t(y))}(x) + p^{k-1} g_0(y) + g(g_0(y) - g_1(y), \ldots, g_0(y) - g_t(y)), (x, y) = (x, y_1, y_2) \in V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ is a generalized s-plateaued function from $V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ to $\mathbb{Z}_{p^k}$.*

We give two examples by using Corollary 3. The second example gives a non-weakly regular bent function which is not in the complete Generalized Maiorana-McFarland class.

Example 8. *Let $p=7$, $k=2$, $t=1$, $r=3$, $m=2$, $s=1$. Let $f_0(x_1, x_2, x_3) = 7(x_1^2 + x_2^2)$, $f_1(x_1, x_2, x_3) = 7(x_1^2 + 3x_2^2)$, $f_2(x_1, x_2, x_3) = 7(x_1^2 + 2x_3^2)$, $f_3(x_1, x_2, x_3) = 7(x_1^2 + 5x_3^2)$, $f_4(x_1, x_2, x_3) = 7(x_2^2 + 4x_3^2)$, $f_5(x_1, x_2, x_3) = 7(x_2^2 + 6x_3^2)$, $f_6(x_1, x_2, x_3) = 7(x_1^2 + 3x_2^2 + x_3)$. Then*

$f_0, \ldots, f_6 : \mathbb{F}_7^3 \rightarrow \mathbb{Z}_{7^2}$ are generalized 1-plateaued functions. Let z be the primitive element of $\mathbb{F}_{7^2}$ with $z^2+6z+3=0$. Let $g_0(y_1,y_2) = Tr_1^2(y_1 y_2^{47}), g_1(y_1,y_2) = Tr_1^2(z y_1 y_2^{47}), (y_1,y_2) \in \mathbb{F}_{7^2} \times \mathbb{F}_{7^2}$. Let $g : \mathbb{F}_7 \rightarrow \mathbb{Z}_{7^2}$ be defined as $g(x) = x^5 + 2x^3$. Then the function $h : \mathbb{F}_7^3 \times \mathbb{F}_{7^2} \times \mathbb{F}_{7^2} \rightarrow \mathbb{Z}_{7^2}$ constructed by Corollary 3 is a generalized 1-plateaued function and one can verify that the Walsh support is not an affine subspace.

Example 9. *Let $p=3$, $k=1$, $t=1$, $r=4$, $m=2$, $s=0$. Let ξ be the primitive element of $\mathbb{F}_{3^4}$ with $\xi^4+2\xi^3+2=0$. Let z be the primitive element of $\mathbb{F}_{3^2}$ with $z^2+z+2=0$. Let $f_0(x) = Tr_1^4(x^{34}+x^2)$, $f_1(x) = Tr_1^4(x^2)$, $f_2(x) = Tr_1^4(\xi x^2)$. Then $f_0, f_1, f_2 : \mathbb{F}_{3^4} \rightarrow \mathbb{F}_3$ are weakly regular bent functions with $\mu_{f_0}(x) = \mu_{f_1}(x) = -1, \mu_{f_2}(x) = 1, x \in \mathbb{F}_{3^4}$. Let $g_0(y_1,y_2) = Tr_1^2(y_1 y_2^7)$, $g_1(y_1,y_2) = Tr_1^2(z y_1 y_2^7)$, $(y_1,y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. Let $g(x) = 0, x \in \mathbb{F}_3$. Then the function $h : \mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2} \rightarrow \mathbb{F}_3$ constructed by Corollary 3 is a non-weakly regular bent function. And we will prove in Appendix that it is not in the complete Generalized Maiorana-McFarland class.*

Remark 3. *The generalized indirect sum construction method can also be applied to construct generalized s-plateaued functions from V_n to $\mathbb{Z}_{2^k}$ where $k \geq 2$ and $n+s$ is odd if using generalized s-plateaued functions from V_r to $\mathbb{Z}_{2^k}$ where $k \geq 2$ and $r+s$ is odd as building blocks.*

B. Applications

In this subsection, we give some applications of Corollary 3 in constructing plateaued functions in the subclass *WRP* of the class of weakly regular plateaued functions and vectorial plateaued functions.

In [25], Mesnager *et al*. introduced the notion of class *WRP*, which is a subclass of the class of weakly regular plateaued functions and plays an important role in constructing minimal linear codes and strong regular graphs (see [25], [26]).

Let p be an odd prime. Let $f : V_n \rightarrow \mathbb{F}_p$ be an unbalanced weakly regular s-plateaued function. If $f(0)=0$ and there exists an even positive integer h with $gcd(h-1, p-1) = 1$ such that $f(ax) = a^h f(x), x \in V_n$ for any $a \in \mathbb{F}_p^*$, then f belongs to the class *WRP*. Note that all quadratic functions are plateaued functions. By Theorem 1, it is easy to verify that any quadratic function without affine term is unbalanced. Therefore, all quadratic functions without affine term are in the class *WRP* and $h=2$. We will give a construction of non-quadratic plateaued functions in the class *WRP* by using Corollary 3.

Let p be an odd prime and m be an even positive integer. Let $f:\mathbb{F}_p^m \to \mathbb{F}_p$ be a partial spread bent function (see [16]). Then by Theorem 3.3 and Theorem 3.6 of [16], it is easy to see that for any $a\in\mathbb{F}_p^*$, $f(ax)=f(x)$. Let t,r be positive integers, s be a non-negative integer and $r-s$ be an even positive integer. For any $i\in\mathbb{F}_p^t$, let $b_i:\mathbb{F}_p^{r-s}\to\mathbb{F}_p$ be a partial spread bent function, $M_i\in GL(r,\mathbb{F}_p)$, $E_i\subseteq\mathbb{F}_p^r$ be an $(r-s)$-dimensional subspace and R_i be the corresponding matrix defined by Lemma 1. Define

$$f_i(x)=b_i(xM_i^T R_i^T), x\in\mathbb{F}_p^r, i\in\mathbb{F}_p^t. \tag{23}$$

Then for any $i\in\mathbb{F}_p^t$, f_i is an s-plateaued function with $\mu_{f_i}(x)=1, x\in\mathbb{F}_p^r$ by Theorem 1. And it is easy to see that $f_i(ax)=f_i(x), x\in\mathbb{F}_p^r$ for any $a\in\mathbb{F}_p^*$.

Now we give a construction of non-quadratic plateaued functions in the class *WRP* by using Corollary 3.

Theorem 6. *Let p be an odd prime and $k=1$. Let t,r,m be positive integers with $m\geq t+1$. Let $s(\leq r)$ be a non-negative integer. Let $g_j(0\leq j\leq t):\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}\to\mathbb{F}_p$ be defined as $g_j(y)=Tr_1^m(\alpha_j G(y_1 y_2^{p^m-2})), y=(y_1,y_2)\in\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}$ where G is a permutation over $\mathbb{F}_{p^m}$ with $G(0)=0$ and $\alpha_0,\alpha_1,\dots,\alpha_t\in\mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$.*

- *Case $p=3$: Let $f_i(i\in\mathbb{F}_p^t):V_r\to\mathbb{F}_p$ be weakly regular s-plateaued functions satisfying $\mu_{f_i}(x)=u, x\in V_r, i\in\mathbb{F}_p^t$ where μ_{f_i} is defined by (3) and u is some constant independent of i, $f_i(ax)=a^2 f_i(x), x\in V_r, i\in\mathbb{F}_p^t$ for any $a\in\mathbb{F}_p^*$ and $0\in S_{f_{(0,\dots,0)}}$. Let $g:\mathbb{F}_p^t\to\mathbb{F}_p$ be an arbitrary function with $g(0,\dots,0)=-f_{(0,\dots,0)}(0)$.*
- *Case $p\geq 5$: Let $r-s$ be an even positive integer. Let $f_i, i\in\mathbb{F}_p^t$ be defined as (23). Let $g:\mathbb{F}_p^t\to\mathbb{F}_p$ be an arbitrary function with $g(0,\dots,0)=-f_{(0,\dots,0)}(0)$.*

Then the function $h:V_r\times\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}\to\mathbb{F}_p$ constructed by Corollary 3 is a weakly regular s-plateaued function and in the class WRP.

Proof: By Corollary 3, h is an s-plateaued function. And by the proof of Theorem 5, it is easy to see that h is weakly regular and the Walsh support of h is $S_h=\cup_{y\in\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}} S_{f_{(g_0^*(y)-g_1^*(y),\dots,g_0^*(y)-g_t^*(y))}}\times\{y\}$. Since $g_0^*(0,0)-g_j^*(0,0)=0, 1\leq j\leq t$ and $0\in S_{f_{(0,\dots,0)}}$, we have $(0,0,0)\in S_h$, that is, h is unbalanced. By $g(0,\dots,0)=-f_{(0,\dots,0)}(0)$, $h(0,0,0)=$

$f_{(g_0(0,0)-g_1(0,0),\ldots,g_0(0,0)-g_t(0,0))}(0)+g_0(0,0)+g(g_0(0,0)-g_1(0,0),\ldots,g_0(0,0)-g_t(0,0)) = f_{(0,\ldots,0)}$ $(0)+0+g(0,\ldots,0)=0$. As $f_i(ax)=f_i(x), x\in V_r, i\in\mathbb{F}_p^t$, $g_j(ay)=g_j(y), y\in\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}, 0\leq j\leq t$ for any $a\in\mathbb{F}_p^*$, the weakly regular plateaued function h constructed by Corollary 3 satisfies $h(ax,ay)=h(x,y)=a^{p-1}h(x,y),(x,y)\in V_r\times\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}$ for any $a\in\mathbb{F}_p^*$. Note that $p-1$ is even and $gcd(p-2,p-1)=1$. By definition, h is in the *WRP* class. ■

We give an example of non-quadratic plateaued function in the *WRP* class by using Theorem 6.

Example 10. *Let* $p=3,t=1,r=2,m=2,s=1$. *Let* z *be the primitive element of* $\mathbb{F}_{3^2}$ *with* $z^2+2z+2=0$. *Let* $g_0(y_1,y_2)=Tr_1^2(y_1y_2^7), g_1(y_1,y_2)=Tr_1^2(zy_1y_2^7),(y_1,y_2)\in\mathbb{F}_{3^2}\times\mathbb{F}_{3^2}$. *Let* $f_0(x_1,x_2)=x_1^2, f_1(x_1,x_2)=x_2^2, f_2(x_1,x_2)=x_1^2+x_1x_2+x_2^2,(x_1,x_2)\in\mathbb{F}_3^2$. *Then* $f_i, i\in\mathbb{F}_3$ *are* 1*-plateaued functions with* $\mu_{f_i}(x_1,x_2)=\sqrt{-1},(x_1,x_2)\in\mathbb{F}_3^2$ *and* $f_i(ax_1,ax_2)=a^2f_i(x_1,x_2),(x_1,x_2)\in\mathbb{F}_3^2$ *for any* $a\in\mathbb{F}_3^*$ *and* $(0,0)\in S_{f_0}$. *Let* $g=0$. *Then the function* h *constructed by Theorem 6 is* $h(x_1,x_2,y_1,y_2)=Tr_1^2(y_1y_2^7)+x_1^2+(Tr_1^2((1-z)y_1y_2^7))^2(x_1^2+2x_1x_2+x_2^2)+(Tr_1^2((1-z)y_1y_2^7))(x_1^2+x_1x_2),(x_1,x_2,y_1,y_2)\in\mathbb{F}_3^2\times\mathbb{F}_{3^2}\times\mathbb{F}_{3^2}$, *which is a non-quadratic weakly regular* 1*-plateaued function and in the WRP class. And one can verify that the Walsh support of* h *is not an affine subspace, that is,* h *is not a partially bent function.*

The second application of Corollary 3 is about vectorial plateaued functions. Let $f=(f_1,\ldots,f_m)$ be a vectorial function from V_n to $\mathbb{F}_p^m$. Then f is said to be a vectorial plateaued function if for any nonzero vector $(c_1,\ldots,c_m)\in\mathbb{F}_p^m$, $\sum_{i=1}^m c_if_i$ is a plateaued function from V_n to $\mathbb{F}_p$. Now we give a construction of vectorial plateaued functions.

Theorem 7. *Let* p *be a prime. Let* $r\geq 1, m\geq 3, 0\leq s\leq r$ *be integers and* $r+s$ *be even for* $p=2$. *Let* $\{\alpha_0,\ldots,\alpha_{m-1}\}$ *be a basis of* $\mathbb{F}_{p^m}$ *over* $\mathbb{F}_p$. *Let* $f_0,\ldots,f_{p-1}:V_r\to\mathbb{F}_p$ *be* s*-plateaued functions. Let* G *be a permutation over* $\mathbb{F}_{p^m}$ *with* $G(0)=0$. *Let* $h_i(x,y_1,y_2)=f_{Tr_1^m(\alpha_0G(y_1y_2^{p^m-2}))}(x)+Tr_1^m(\alpha_iG(y_1y_2^{p^m-2})),(x,y_1,y_2)\in V_r\times\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}, 1\leq i\leq m-1$. *Then vectorial function* $H=(h_1,\ldots,h_{m-1})$ *is a vectorial plateaued function from* $V_r\times\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}$ *to* $\mathbb{F}_p^{m-1}$.

Proof: First we observe that if $\alpha,\beta\in\mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$, then function $h(x,y_1,y_2)=f_{Tr_1^m(\beta G(y_1y_2^{p^m-2}))}(x)+Tr_1^m(\alpha G(y_1y_2^{p^m-2})),(x,y_1,y_2)\in V_r\times\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}$ is an s-plateaued function where $f_0,\ldots,f_{p-1}$ are s-plateaued functions and G is a permutation over

$\mathbb{F}_{p^m}$ with $G(0)=0$. Indeed, we have $h(x,y_1,y_2)=f_{g_0(y_1,y_2)-g_1(y_1,y_2)}(x)+g_0(y_1,y_2)$ where $g_0(y_1,y_2)=Tr_1^m(\alpha G(y_1y_2^{p^m-2})), g_1(y_1,y_2)=Tr_1^m((\alpha-\beta)G(y_1y_2^{p^m-2}))$. By Corollary 3, h is an s-plateaued function since $\alpha,\alpha-\beta$ are linearly independent over $\mathbb{F}_p$.

For any nonzero vector $a=(a_1,\ldots,a_{m-1})\in\mathbb{F}_p^{m-1}$, if $\bar{a}\triangleq\sum_{i=1}^{m-1}a_i\neq 0$, then $\sum_{i=1}^{m-1}a_ih_i(x,y_1,y_2)=\bar{a}f_{Tr_1^m(\alpha_0G(y_1y_2^{p^m-2}))}(x)+Tr_1^m(\alpha_aG(y_1y_2^{p^m-2}))$ where $\alpha_a=\sum_{i=1}^{m-1}a_i\alpha_i$. By Theorem 1 of [7], we have $\bar{a}f_0,\ldots,\bar{a}f_{p-1}$ are s-plateaued functions. Since $\bar{a}f_0,\ldots,\bar{a}f_{p-1}$ are s-plateaued functions and α_0,α_a are linearly independent, we have $\sum_{i=1}^{m-1}a_ih_i$ is an s-plateaued function.

For any nonzero vector $a=(a_1,\ldots,a_{m-1})\in\mathbb{F}_p^{m-1}$, if $\bar{a}\triangleq\sum_{i=1}^{m-1}a_i=0$, then $\sum_{i=1}^{m-1}a_ih_i(x,y_1,y_2)=Tr_1^m(\alpha_aG(y_1y_2^{p^m-2}))$ where $\alpha_a=\sum_{i=1}^{m-1}a_i\alpha_i$. Since $\alpha_a\neq 0$, it is easy to see that $\sum_{i=1}^{m-1}a_ih_i(x,y_1,y_2)$ is an r-plateaued function.

Hence, for any nonzero vector $a=(a_1,\ldots,a_{m-1})\in\mathbb{F}_p^{m-1}$, $\sum_{i=1}^{m-1}a_ih_i$ is a plateaued function, that is, $H=(h_1,\ldots,h_{m-1})$ is a vectorial plateaued function. ■

We give an example of vectorial plateaued function by using Theorem 7.

Example 11. *Let* $p=3,r=3,m=4,s=0$. *Let* $f_0(x)=Tr_1^3(x^2),f_1(x)=Tr_1^3(\xi x^2),f_2(x)=Tr_1^3(\xi^2x^2),x\in\mathbb{F}_{3^3}$ *where* ξ *is a primitive element of* $\mathbb{F}_{3^3}$. *Then* $f_i(0\leq i\leq 2)$ *are weakly regular bent functions with* $\mu_{f_0}(x)=\mu_{f_2}(x)=-\sqrt{-1},\mu_{f_1}(x)=\sqrt{-1},x\in\mathbb{F}_{3^3}$. *Let* $h_i(x,y_1,y_2)=f_{Tr_1^4(y_1y_2^{79})}(x)+Tr_1^4(z^iy_1y_2^{79}),(x,y_1,y_2)\in\mathbb{F}_{3^3}\times\mathbb{F}_{3^4}\times\mathbb{F}_{3^4},i=1,2,3$ *where* z *is a primitive element of* $\mathbb{F}_{3^4}$. *Then* $H=(h_1,h_2,h_3)$ *is a vectorial plateaued function. And one can verify that* H *contains non-weakly regular plateaued component functions and weakly regular plateaued component functions.*

Remark 4. *Let* $H=(h_1,\ldots,h_{m-1})$ *be the constructed vectorial plateaued function by Theorem 7. Define* $g_i=h_{i+1},0\leq i\leq m-2$. *Then one can verify that* $g_i(0\leq i\leq m-2)$ *satisfy the condition of Theorem 5.*

V. CONCLUSION

In this paper, we mainly study generalized s-plateaued functions from V_n to $\mathbb{Z}_{p^k}$ where p is an odd prime and $k\geq 1$ or $p=2,k\geq 2$ and $n+s$ is even. Firstly, inspired by [13], we give a complete characterization of generalized plateaued functions with affine Walsh support in spectral domain (Theorem 1) and provide some generic construction methods of generalized plateaued functions with (non)-affine Walsh support in spectral domain (Theorem 2, Theorem 3, Theorem

4). Compared with the constructions in spectral domain in [13], in our constructions of Theorem 2, Theorem 3 and Theorem 4, the Walsh support can contain strictly less than $n-s$ columns corresponding to affine functions. And we give two examples of Boolean plateaued functions, which satisfy that every column of S_f corresponds to a non-affine function and the functions have no nonzero linear structure. Secondly, we give a generalized indirect sum construction method (Theorem 5), which can also be used to construct (non)-weakly regular generalized bent functions (Corollary 2). In particular, we show that the canonical way to construct Generalized Maiorana-McFarland bent functions is a special case of the generalized indirect sum construction method. And we illustrate that the generalized indirect sum construction method can be used to construct bent functions not in the complete Generalized Maiorana-McFarland class (Example 9 and Appendix). Furthermore, we give some applications of the generalized indirect sum construction method (Theorem 6, Theorem 7). In this paper, we do not study generalized s-plateaued functions from V_n to $\mathbb{Z}_{p^k}$ where $p=2, k \geq 2$ and $n+s$ is odd in spectral domain. For this case, although similar theoretical characterizations can be given, we cannot provide efficient construction methods in spectral domain for this case. In the following research, we will pay attention to this case.

APPENDIX

We prove that the bent function constructed in Example 9 is not in the complete Generalized Maiorana-McFarland class.

Recall that the bent function constructed in Example 9 is $h(x, y_1, y_2) = f_{g_0(y_1,y_2)-g_1(y_1,y_2)}(x) + g_0(y_1, y_2) = f_0(x) + g_0(y_1, y_2) + (g_0(y_1, y_2) - g_1(y_1, y_2))^2(-f_0(x) - f_1(x) - f_2(x)) + (g_0(y_1, y_2) - g_1(y_1, y_2))(2f_1(x) + f_2(x)), (x, y_1, y_2) \in \mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$ where $f_0(x) = Tr_1^4(x^{34} + x^2)$, $f_1(x) = Tr_1^4(x^2)$, $f_2(x) = Tr_1^4(\xi x^2)$, $g_0(y_1, y_2) = Tr_1^2(y_1 y_2^7)$, $g_1(y_1, y_2) = Tr_1^2(z y_1 y_2^7)$ and ξ is the primitive element of $\mathbb{F}_{3^4}$ with $\xi^4+2\xi^3+2 = 0$, z is the primitive element of $\mathbb{F}_{3^2}$ with $z^2+z+2 = 0$.

By Theorem 2 of [8], if h is in the complete Generalized Maiorana-McFarland class, then for an integer $1 \leq s \leq 4$ there exists an s-dimensional subspace V of $\mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$ such that the second order derivative

$$D_a D_c h(x, y_1, y_2) = 0 \tag{24}$$

for any $a = (a_0, a_1, a_2), c = (c_0, c_1, c_2) \in V, (x, y_1, y_2) \in \mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. Define $\bar{g_i}(y) = g_i(y_1, y_2), i = 0, 1$ and $\bar{h}(x, y) = f_{\bar{g_0}(y)-\bar{g_1}(y)}(x) + \bar{g_0}(y)$ where $y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in$

$\mathbb{F}_3^4, (y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}, y_1 = y_{1,1} + y_{1,2}z, y_2 = y_{2,1} + y_{2,2}z$. Then $\bar{h}$ is a non-weakly regular bent function from $\mathbb{F}_{3^4} \times \mathbb{F}_3^4$ to $\mathbb{F}_3$. By simple calculation we have $\bar{g_0}(y) - \bar{g_1}(y) = (y_{1,1}+y_{1,2})y_{2,1}^2 y_{2,2} + (2y_{1,1}+y_{1,2})y_{2,1}y_{2,2}^2 + 2y_{1,1}y_{2,2} + 2y_{1,2}y_{2,1}$, $(\bar{g_0}(y) - \bar{g_1}(y))^2 = y_{1,1}^2 y_{2,2}^2 + y_{1,2}^2 y_{2,1}^2 + y_{1,1}y_{1,2}y_{2,1}y_{2,2}$ where $y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in \mathbb{F}_3^4$.

Suppose (24) holds. Then

$$D_{\bar{a}} D_{\bar{c}} \bar{h}(x, y) = 0 \tag{25}$$

for any $a = (a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}), c = (c_0, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}) \in \bar{V}, (x, y) \in \mathbb{F}_{3^4} \times \mathbb{F}_3^4$ where $\bar{V} = \{(a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) \in \mathbb{F}_{3^4} \times \mathbb{F}_3^4 : (a_0, a_{1,1}+a_{1,2}z, a_{2,1}+a_{2,2}z) \in V\}, y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in \mathbb{F}_3^4$. As $\{30 \cdot 3^i \ (mod\ (3^4 - 1)) : i \geq 0\} = \{10, 30\}$ and $\begin{pmatrix} 34 \\ 10 \end{pmatrix} \equiv 0\ (mod\ 3)$, $\begin{pmatrix} 34 \\ 30 \end{pmatrix} \equiv 2\ (mod\ 3)$, $D_{\bar{a}} D_{\bar{c}} \bar{h}$ contains $-y_{1,1}^2 y_{2,2}^2 Tr_1^4(2((a_0+c_0)^4 - a_0^4 - c_0^4)x^{30})$. Then by (25), $(a_0+c_0)^4 - a_0^4 - c_0^4 = 0$ for any $\bar{a} = (a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}), \bar{c} = (c_0, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}) \in \bar{V}$. If there exists $a_0 \neq 0$ such that $\bar{a} = (a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) \in \bar{V}$, let $\bar{c} = \bar{a}$, then $c_0 = a_0 \neq 0$ and $(a_0+c_0)^4 - a_0^4 - c_0^4 = 2a_0^4 \neq 0$, which is a contradiction. Hence $\bar{V} \subseteq \{0\} \times \mathbb{F}_3^4$, that is, $V \subseteq \{0\} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. For any fixed $(0, a_1, a_2)$, $(0, c_1, c_2) \in V$ and $(y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, let $d_0 = D_{(a_1,a_2)} D_{(c_1,c_2)} g_0(y_1, y_2)$, $d_1 = D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2))$, $d_2 = D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2))^2$. By $D_{(0,a_1,a_2)} D_{(0,c_1,c_2)} h(x, y_1, y_2) = D_{(a_1,a_2)} D_{(c_1,c_2)} g_0(y_1, y_2) + (-f_0(x) - f_1(x) - f_2(x)) D_{(a_1,a_2)} D_{(c_1,c_2)} (g_0(y_1, y_2) - g_1(y_1, y_2))^2 + (2f_1(x) + f_2(x)) D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2)) = 0$ for any $(0, a_1, a_2), (0, c_1, c_2) \in V, (x, y_1, y_2) \in \mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, for any fixed $(0, a_1, a_2), (0, c_1, c_2) \in V$ and $(y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, we have $-d_2 f_0(x) + (2d_1 - d_2) f_1(x) + (d_1 - d_2) f_2(x) = -d_0, x \in \mathbb{F}_{3^4}$. By $f_0(0) = f_1(0) = f_2(0) = 0$, we have $d_0 = 0$. By $i + j\xi \neq 0$ for any $i, j \in \mathbb{F}_3$ and the algebraic degree of f_0 is 4, the algebraic degree of f_1 and f_2 is 2, we have f_0, f_1, f_2 are linearly independent, hence $d_1 = d_2 = 0$. Therefore, (24) holds if and only if for any $(0, a_1, a_2), (0, c_1, c_2) \in V, (y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$,

$$D_{(a_1,a_2)} D_{(c_1,c_2)} g_0(y_1, y_2) = 0 \tag{26}$$

and

$$D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2)) = 0 \tag{27}$$

and

$$D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2))^2 = 0. \tag{28}$$

By (26), (27) and the fact that $\{1, 1-z\}$ is a basis of $\mathbb{F}_{3^2}$ over $\mathbb{F}_3$, we have for any fixed $(0, a_1, a_2), (0, c_1, c_2) \in V$ and $(y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, $Tr_1^2(((y_1+a_1+c_1)(y_2+a_2+c_2)^7 - (y_1+a_1)(y_2+a_2)^7 - (y_1+c_1)(y_2+c_2)^7 + y_1y_2^7)x) = 0, x \in \mathbb{F}_{3^2}$, which yields $(y_1+a_1+c_1)(y_2+a_2+c_2)^7 - (y_1+a_1)(y_2+a_2)^7 - (y_1+c_1)(y_2+c_2)^7 + y_1y_2^7 = 0$ for any $(0, a_1, a_2), (0, c_1, c_2) \in V$ and $(y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. We claim $V \subseteq \{0\} \times \mathbb{F}_{3^2} \times \{0\}$. If there exists $a_2 \neq 0$ such that $a = (0, a_1, a_2) \in V$, let $c = a$. Then $c_2 = a_2 \neq 0$ and the coefficient of $y_1y_2^3$ is $C_7^3((a_2+c_2)^4 - a_2^4 - c_2^4) = a_2^4 \neq 0$, which is a contradiction. Hence $V \subseteq \{0\} \times \mathbb{F}_{3^2} \times \{0\}$, that is, $\bar{V} \subseteq \{0\} \times \mathbb{F}_3^2 \times \{(0,0)\}$. By (28), we have $D_{(a_{1,1},a_{1,2},0,0)}D_{(c_{1,1},c_{1,2},0,0)}(\bar{g_0}(y) - \bar{g_1}(y))^2 = 0$ for any $(0, a_{1,1}, a_{1,2}, 0, 0), (0, c_{1,1}, c_{1,2}, 0, 0) \in \bar{V}, y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in \mathbb{F}_3^4$. By simple calculation, we have $2a_{1,1}c_{1,1}y_{2,2}^2 + 2a_{1,2}c_{1,2}y_{2,1}^2 + (a_{1,1}c_{1,2} + a_{1,2}c_{1,1})y_{2,1}y_{2,2} = 0$, which yields $a_{1,1}c_{1,1} = a_{1,2}c_{1,2} = a_{1,1}c_{1,2} + a_{1,2}c_{1,1} = 0$ for any $(0, a_{1,1}, a_{1,2}, 0, 0), (0, c_{1,1}, c_{1,2}, 0, 0) \in \bar{V}$. If there exists $(a_{1,1}, a_{1,2}) \neq (0,0)$ such that $\bar{a} = (0, a_{1,1}, a_{1,2}, 0, 0) \in \bar{V}$, let $\bar{c} = \bar{a}$, then $a_{1,1}c_{1,1} = a_{1,1}^2 \neq 0$ or $a_{1,2}c_{1,2} = a_{1,2}^2 \neq 0$ since $(a_{1,1}, a_{1,2}) \neq (0,0)$, which is a contradiction. Hence, $\bar{V} = \{(0,0,0,0,0)\}$, that is, $V = \{(0,0,0)\}$. By Theorem 2 of [8], h is not in the complete Generalized Maiorana-McFarland class.

REFERENCES

[1] S. Boztaş, F. Özbudak and E. Tekin, Explicit full correlation distribution of sequence families using plateaued functions, *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2858-2875, 2018.

[2] C. Carlet, Boolean and vectorial plateaued functions and APN functions, *IEEE Trans. Inf. Theory*, vol. 61, no. 11, pp. 6272-6289, 2015.

[3] C. Carlet, On the secondary constructions of resilient and bent functions, In Proceedings of the Workshop on Coding, Cryptography and Combinatorics 2003, published by Birkhäuser Verlag, pp. 3-28, 2004.

[4] C. Carlet, Partially-bent functions, *Des. Codes Cryptogr.*, vol. 3, no. 2, pp. 135-145, 1993.

[5] C. Carlet and S. Mesnager, Four decades of research on bent functions, *Des. Codes Cryptogr.*, vol. 78, no. 1, pp. 5-50, 2016.

[6] A. Çeşmelioğlu, G. McGuire and W. Meidl, A construction of weakly and non-weakly regular bent functions, *J. Combinat. Theory A*, vol. 119, no. 2, pp. 420-429, 2012.

[7] A. Çeşmelioğlu and W. Meidl, A construction of bent functions from plateaued functions, *Des. Codes Cryptogr.*, vol. 66, nos. 1-3, pp. 231-242, 2013.

[8] A. Çeşmelioğlu, W. Meidl and A. Pott, Generalized Maiorana-McFarland class and normality of *p*-ary bent functions, *Finite Fields Appl.*, vol. 24, pp. 105-117, 2013.

[9] J. F. Dillon, Elementary Hadamard difference sets, Ph. D. Thesis, University of Maryland, 1974.

[10] S. Hodžić, W. Meidl and E. Pasalic, Full characterization of generalized bent functions as (semi)-bent spaces, their dual, and the Gray image, *IEEE Trans. Inf. Theory*, vol. 64, no. 7, pp. 5432-5440, 2018.

[11] S. Hodžić and E. Pasalic, Construction methods for generalized bent functions, *Discret. Appl. Math.*, vol. 238, pp. 14-23, 2018.

[12] S. Hodžić, E. Pasalic and Y. Wei, A general framework for secondary constructions of bent and plateaued functions, *Des. Codes Cryptogr.*, vol. 88, no. 10, pp. 2007-2035, 2020.

[13] S. Hodžić, E. Pasalic, Y. Wei and F. Zhang, Designing plateaued Boolean functions in spectral domain and their classification, *IEEE Trans. Inf. Theory*, vol. 65, no. 9, pp. 5865-5879, 2019.

[14] J. Y. Hyun, J. Lee and Y. Lee, Explicit criteria for construction of plateaued functions, *IEEE Trans. Inf. Theory*, vol. 62, no. 12, pp. 7555-7565, 2016.

[15] P. V. Kumar, R. A. Scholtz and L. R. Welch, Generalized bent functions and their properties, *J. Combinat. Theory A*, vol. 40, no. 1, pp. 90-107, 1985.

[16] P. Lisoněk and H. Y. Lu, Bent functions on partial spreads, *Des. Codes Cryptogr.*, vol. 73, no. 1, pp. 209-216, 2014.

[17] T. Martinsen, W. Meidl, S. Mesnager and P. Stănică, Decomposing generalized bent and hyperbent functions, *IEEE Trans. Inf. Theory*, vol. 63, no. 12, pp. 7804-7812, 2017.

[18] T. Martinsen, W. Meidl and P. Stănică, Partial spread and vectorial generalized bent functions, *Des. Codes Cryptogr.*, vol. 85, no. 1, pp. 1-13, 2017.

[19] W. Meidl and A. Pott, Generalized bent functions into $\mathbb{Z}_{p^k}$ from the partial spread and the Maiorana-McFarland class, *Cryptogr. Commun.*, vol. 11, no. 6, pp. 1233-1245, 2019.

[20] S. Mesnager, Bent Functions-Fundamentals and Results, Springer, Switzerland, 2016.

[21] S. Mesnager, On generalized hyper-bent functions, *Cryptogr. Commun.*, vol. 12, no. 3, pp. 455-468, 2020.

[22] S. Mesnager, F. Özbudak and A. Sınak, Linear codes from weakly regular plateaued functions and their secret sharing schemes, *Des. Codes Cryptogr.*, vol. 87, nos. 2-3, pp. 463-480, 2019.

[23] S. Mesnager, F. Özbudak and A. Sınak, On the p-ary (cubic) bent and plateaued (vectorial) functions, *Des. Codes Cryptogr.*, vol. 86, no. 8, pp. 1865-1892, 2018.

[24] S. Mesnager, F. Özbudak and A. Sınak, Results on characterizations of plateaued functions in arbitrary characteristic, Cryptography and information security in the Balkans, BalkanCrytSec 2015, Koper, Slovenia, Revised Selected Papers. In: Pasalic E., Knudsen L.R.(eds.) LNCS 9540, pp. 17-30, Springer, Berlin, 2016.

[25] S. Mesnager and A. Sınak, Several classes of minimal linear codes with few weights from weakly regular plateaued functions, *IEEE Trans. Inf. Theory*, vol. 66, no. 4, pp. 2296-2310, 2020.

[26] S. Mesnager and A. Sınak, Strongly regular graphs from weakly regular plateaued functions, 2019 Ninth International Workshop on Signal Design and its Applications in Communications (IWSDA), Dongguan, China, pp. 1-5, 2019.

[27] S. Mesnager, C. Tang and Y. Qi, Generalized plateaued functions and admissible (plateaued) functions, *IEEE Trans. Inf. Theory*, vol. 63, no. 10, pp. 6139-6148, 2017.

[28] S. Mesnager, C. Tang, Y. Qi, L. Wang, B. Wu and K. Feng, Further results on generalized bent functions and their complete characterization, *IEEE Trans. Inf. Theory*, vol. 64, no. 7, pp. 5441-5452, 2018.

[29] O. Olmez, Plateaued functions and one-and-half difference sets, *Des. Codes Cryptogr.*, vol. 76, no. 3, pp. 537-549, 2015.

[30] Y. Qi, C. Tang, Z. Zhou and C. Fan, Several infinite families of p-ary weakly regular bent functions, *Adv. Math. Commun.*, vol. 12, no. 2, pp. 303-315, 2018.

[31] O. S. Rothaus, On bent functions, *J. Combinat. Theory A*, vol. 20, no. 3, pp. 300-305, 1976.

[32] P. Stănică, T. Martinsen, S. Gangopadhyay and B. K. Singh, Bent and generalized bent Boolean functions, *Des. Codes Cryptogr.*, vol. 69, no. 1, pp. 77-94, 2013.

[33] C. Tang, C. Xiang, Y. Qi and K. Feng, Complete characterization of generalized bent and 2^k-bent Boolean functions, *IEEE Trans. Inf. Theory*, vol. 63, no. 7, pp. 4668-4674, 2017.

[34] Y. Zheng and X. M. Zhang, On plateaued functions, *IEEE Trans. Inf. Theory*, vol. 47, no. 3, pp. 1215-1223, 2001.