Construction of minimal linear codes with few weights from weakly regular plateaued functions

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Abstract. The construction of linear (minimal) codes from functions over finite fields has been greatly studied in the literature since determining the parameters of linear codes based on functions is rather easy due to the nice structures of functions. In this paper, we derive 3-weight and 4-weight linear codes from weakly regular plateaued unbalanced functions in the recent construction method of linear codes over the odd characteristic finite fields. The Hamming weights and their weight distributions for proposed codes are determined by using the Walsh transform values and Walsh distribution of the employed functions, respectively. We next derive projective 3-weight punctured codes with good parameters from the constructed codes. These punctured codes may be almost optimal due to the Griesmer bound, and they can be employed to design association schemes. We lastly show that all constructed codes are minimal, which approves that they can be employed to design high democratic secret sharing schemes.

Keywords: Linear code · minimal code · weight distribution · weakly regular plateaued function · unbalanced function

1 Introduction

There are many construction methods for linear codes, one of them is derived from functions over finite fields. Constructing linear codes from functions is a popular research topic in the literature although considerable progress has been done in this direction. A great number of linear codes have been obtained from popular cryptographic functions such as quadratic functions [6,7,10,11,25,29], (weakly regular) bent functions [6,7,19,24,26,29], (almost) perfect nonlinear functions [4,16,27] and (weakly regular) plateaued functions [6,20,21,23]. Two generic (say, *first* and *second*) construction methods of linear codes from functions can be isolated from the others in the literature. In the past two decades, several linear codes with excellent parameters have been derived from cryptographic functions based on the first generic construction method (*e.g.* [4,7,19,20]) and the second generic construction method (*e.g.* [7,11,24,25,29]). Recently, weakly regular plateaued (especially, bent) functions have been employed to design linear (minimal) codes with a few weights over the odd characteristic finite fields ([19,20,21,23,24,26]). In this paper, motivated by [15,26], we use some unbalanced weakly regular plateaued functions so that we can get minimal linear codes with new parameters. It is worth noting that a very nice survey [17] written by Li and Mesnager is devoted to the construction methods of linear codes from cryptographic functions over finite fields.

The rest of this paper is organized as follows. In Section 2, we set the main notation and give some properties of weakly regular plateaued functions. In Section 3, we introduce the parameters of 3-weight and 4-weight linear codes derived from these functions over finite fields. We also propose punctured codes for the constructed codes. We hereby obtain projective 3-weight codes with flexible parameters. We finally highlight that our codes are minimal codes.

2 Preliminaries

For a set T, its size is shown by $\#T$, and $T^\star = T \setminus \{0\}$. The magnitude of a complex number $z \in \mathbb{C}$ is denoted by $|z|$. The finite field with q elements is represented by $\mathbb{F}_q$, where $q = p^n$ for a positive integer n and an odd prime p. The trace of $\alpha \in \mathbb{F}_q$ over $\mathbb{F}_p$ is defined as $\mathrm{Tr}^n(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{n-1}}$. The set of all *non-squares* and *squares* in $\mathbb{F}_p^\star$ are represented by NSQ and SQ, respectively. The *quadratic character* of $\mathbb{F}_p^\star$ is η_0, and for simplicity we write $p^* = \eta_0(-1)p$, which is frequently used in the sequel.

A cyclotomic field $\mathbb{Q}(\xi_p)$ can be obtained from the rational field $\mathbb{Q}$ by joining the *complex primitive p-th root of unity* ξ_p. The field $\mathbb{Q}(\xi_p)$ is the splitting field of the polynomial $x^p - 1$, and so the field $\mathbb{Q}(\xi_p)/\mathbb{Q}$ is a Galois extension of degree $p-1$. Here, a field basis for an extension $\mathbb{Q}(\xi_p)/\mathbb{Q}$ is the subset $\{1, \xi_p, \xi_p^2, \ldots, \xi_p^{p-2}\}$ of the cyclotomic field $\mathbb{Q}(\xi_p)$. The Galois group $Gal(\mathbb{Q}(\xi_p)/\mathbb{Q})$ is described as the set $\{\sigma_c \colon c \in \mathbb{F}_p^\star\}$, where σ_c is the automorphism of $\mathbb{Q}(\xi_p)$ defined as $\sigma_c(\xi_p) = \xi_p^c$. The cyclotomic field $\mathbb{Q}(\xi_p)$ has a unique quadratic subfield $\mathbb{Q}(\sqrt{p^*})$, and its Galois group $Gal(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}) = \{1, \sigma_\gamma\}$ for some $\gamma \in NSQ$. For $a, c \in \mathbb{F}_p^\star$, we clearly have $\sigma_c(\xi_p^a) = \xi_p^{ca}$ and $\sigma_c(\sqrt{p^*}^n) = {\eta_0}^n(c)\sqrt{p^*}^n$. The following lemma is frequently used in the subsequent proofs.

Lemma 1. *[18]*

i.) $\sum_{c \in \mathbb{F}_p^\star} \eta_0(c) = 0$,

ii.) $\sum_{c \in \mathbb{F}_p^\star} \xi_p^{ca} = -1$ *for every* $a \in \mathbb{F}_p^\star$,

iii.) $\sum_{c \in \mathbb{F}_p^\star} \eta_0(c)\xi_p^c = \sqrt{p^*}$.

2.1 Linear codes

A linear $[n, k, d]$ code $\mathcal{C}$ over $\mathbb{F}_p$ is a subspace with k-dimension of vector space $\mathbb{F}_p^n$. Here, n is the length of $\mathcal{C}$, k is its dimension and d is its minimum Hamming distance. For a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{F}_p^n$, its Hamming weight $W_H(\mathbf{v})$ is the size of its support described as $\mathrm{supp}(\mathbf{v}) = \{1 \leq i \leq n \colon v_i \neq 0\}$. We remark that d is the smallest Hamming weight of the nonzero elements (codewords) of $\mathcal{C}$. The *dual code* of $\mathcal{C}$ is defined to be the set

$$\mathcal{C}^\perp = \{(u_1, \ldots, u_n) \in \mathbb{F}_p^n \colon u_1v_1 + \cdots + u_nv_n = 0 \text{ for all } (v_1, \ldots, v_n) \in \mathcal{C}\},$$

which is represented by $[n, n-k, d^\perp]$ over $\mathbb{F}_p$, where $d^\perp$ is the minimum Hamming distance of $\mathcal{C}^\perp$. The *weight distribution* of $\mathcal{C}$ is given by $(1, A_1, \ldots, A_n)$ and its *weight enumerator* is the polynomial $1 + A_1y + \cdots + A_ny^n$, where A_ω is the number of nonzero codewords with weight ω in $\mathcal{C}$. As a result, we say that $\mathcal{C}$ is an *l-weight linear code* if the number of nonzero A_ω in $\{A_i\}_{i \geq 1}$ is equal to l, where l is an integer with $1 \leq l \leq n$. The first two Pless power moments are given as

$$\sum_{i=0}^{n} A_i = p^k \quad \text{and} \quad \sum_{i=0}^{n} iA_i = n(p-1)p^{k-1} - A_1^\perp p^{k-1},$$

where $A_1^\perp$ is the number of codewords with weight 1 in $\mathcal{C}^\perp$. For the proposed codes in this paper, $A_1^\perp = 0$ since their defining sets do not cover the element $(0, 0)$.

2.2 Weakly regular plateaued functions

Let $f : \mathbb{F}_q \longrightarrow \mathbb{F}_p$ be a p-ary function. The *Walsh transform* of f is a complex valued function defined as

$$\mathcal{W}_f(\beta) = \sum_{x\in\mathbb{F}_q} {\xi_p}^{f(x)-\mathrm{Tr}^n(\beta x)}, \quad \beta \in \mathbb{F}_q.$$

A function f is called *balanced* over $\mathbb{F}_p$ if f gets all elements of $\mathbb{F}_p$ with the same number of pre-images; or else, f is said to be *unbalanced*. Note that f is balanced iff $\mathcal{W}_f(0) = 0$.

A function f is said to be a *bent* function if $|\mathcal{W}_f(\beta)|^2 = p^n$ for every $\beta \in \mathbb{F}_q$ (see [22] for Boolean bent and [14] for p-ary bent). In addition, f is said to be *s-plateaued* if $|\mathcal{W}_f(\beta)|^2 \in \{0, p^{n+s}\}$ for every $\beta \in \mathbb{F}_q$, with $0 \le s \le n$, (see [28] for Boolean plateaued and [5] for p-ary plateaued). For an s-plateaued f, its *Walsh support* is described as the set

$$\mathcal{S}_f = \{\beta \in \mathbb{F}_q \colon |\mathcal{W}_f(\beta)|^2 = p^{n+s}\}.$$

From the Parseval identity, we have $\#\mathcal{S}_f = p^{n-s}$, and the explicit Walsh distribution of a plateaued function is given as follows.

Lemma 2. *Let f be an s-plateaued function. For $\beta \in \mathbb{F}_q$, $|\mathcal{W}_f(\beta)|^2$ takes the values p^{n+s} and 0 for the times p^{n-s} and $p^n - p^{n-s}$, respectively.*

Mesnager et al. [20] have described the notion of weakly regular plateaued functions. An s-plateaued f is called *weakly regular* if we have

$$\mathcal{W}_f(\beta) \in \left\{0, up^{\frac{n+s}{2}}\xi_p^{f^\star(\beta)}\right\},$$

where $u \in \{\pm 1, \pm i\}$, $f^\star$ is a p-ary function over $\mathbb{F}_q$ with $f^\star(\beta) = 0$ for every $\beta \in \mathbb{F}_q \setminus \mathcal{S}_f$; otherwise, f is called *non-weakly regular*. We remark that a weakly regular 0-plateaued is the weakly regular bent function.

The following lemma is very useful to compute the Hamming weights of codes.

Lemma 3. *[20] Let f be a weakly regular s-plateaued function. Then, we have*

$$\mathcal{W}_f(\beta) = \epsilon_f \sqrt{p^*}^{\,n+s} \xi_p^{f^\star(\beta)}$$

for every $\beta \in \mathcal{S}_f$, where $\epsilon_f = \pm 1$ is the sign of $\mathcal{W}_f$ and $f^\star$ is a p-ary function over $\mathcal{S}_f$.

The following two lemmas are needed to determine the weight distributions of proposed codes.

Lemma 4. *[21] Let f be a weakly regular s-plateaued function. For $x \in \mathbb{F}_q$,*

$$\sum_{\beta\in\mathcal{S}_f} \xi_p^{f^\star(\beta)+\mathrm{Tr}^n(\beta x)} = \epsilon_f \eta_0^n(-1)\sqrt{p^*}^{\,n-s}\xi_p^{f(x)},$$

where $\epsilon_f = \pm 1$ is the sign of $\mathcal{W}_f$ and $f^\star$ is a p-ary function over $\mathcal{S}_f$.

Lemma 5. *[21] Let f be a weakly regular s-plateaued function with $0 \leq s < n$, and it has $\mathcal{W}_f(\beta) = \epsilon_f \sqrt{p^*}^{n+s} \xi_p^{f^\star(\beta)}$ for every $\beta \in \mathcal{S}_f$, where $\epsilon_f = \pm 1$ is the sign of $\mathcal{W}_f$ and $f^\star$ is a p-ary function over $\mathcal{S}_f$. For $j \in \mathbb{F}_p$, define $\mathcal{N}_{f^\star}(j) = \#\{\beta \in \mathcal{S}_f \colon f^\star(\beta) = j\}$. Then we have*

$$\mathcal{N}_{f^\star}(j) = \begin{cases} p^{n-s-1} + \epsilon_f \eta_0^{n+1}(-1)(p-1)\sqrt{p^*}^{n-s-2}, & \text{if } j = 0, \\ p^{n-s-1} - \epsilon_f \eta_0^{n+1}(-1)\sqrt{p^*}^{n-s-2}, & \text{if not,} \end{cases}$$

when $n - s$ is even; otherwise,

$$\mathcal{N}_{f^\star}(j) = \begin{cases} p^{n-s-1}, & \text{if } j = 0, \\ p^{n-s-1} + \epsilon_f \eta_0^{n}(-1)\sqrt{p^*}^{n-s-1}, & \text{if } j \in SQ, \\ p^{n-s-1} - \epsilon_f \eta_0^{n}(-1)\sqrt{p^*}^{n-s-1}, & \text{if } j \in NSQ. \end{cases}$$

Mesnager et al. [21] have very recently introduced the subclass *WRP* of weakly regular plateaued functions over the odd characteristic finite fields. For an integer s_f with $0 \leq s_f \leq n$, *WRP* defines the set of weakly regular s_f-plateaued unbalanced functions $f : \mathbb{F}_q \to \mathbb{F}_p$ that satisfy two homogeneous conditions:

- $f(0) = 0$, and
- $f(ax) = a^{k_f} f(x)$ for all $x \in \mathbb{F}_q$ and $a \in \mathbb{F}_p^\star$, where k_f is an even positive integer with $\gcd(k_f - 1, p - 1) = 1$.

In this paper, to construct linear codes with flexible parameters, we use a large class *WRP* of functions in the recent construction method of [13,15,26]. The class *WRP* is a non-trivial and non-empty set of functions since for example, all quadratic unbalanced functions belong to this class.

We finalize this section by proposing the following results that are used in the subsequent proofs.

Proposition 1. *[21] If $f \in WRP$, then $f^\star(0) = 0$ and $f^\star(a\beta) = a^{l_f} f^\star(\beta)$ for all $a \in \mathbb{F}_p^\star$ and $\beta \in \mathcal{S}_f$, where l_f is an even positive integer with $\gcd(l_f - 1, p - 1) = 1$.*

Lemma 6. *[21] If $f \in WRP$, then for every $\beta \in \mathcal{S}_f$ (resp., $\beta \in \mathbb{F}_q \setminus \mathcal{S}_f$), we have $z\beta \in \mathcal{S}_f$ (resp., $z\beta \in \mathbb{F}_q \setminus \mathcal{S}_f$) for every $z \in \mathbb{F}_p^\star$.*

3 Linear codes derived from weakly regular plateaued unbalanced functions over $\mathbb{F}_p$

In this section, weakly regular plateaued unbalanced functions are employed to obtain minimal linear codes in the second generic construction method.

3.1 The construction method of linear codes from functions

For a long time, cryptographic functions have been extensively used to design linear codes with few weights in coding theory. Constructing linear codes from functions including quadratic, almost bent, (almost) perfect nonlinear, (weakly regular) bent and plateaued functions is a highly interesting research topic in the literature. Remarkably, determining the parameters of the codes derived from these functions is rather easy due to the nice structure of these functions although it is generally a difficult problem in coding theory.

Two construction methods of linear codes from functions are generic in the sense that several classes of known codes could be obtained from these construction methods. We below define two generic construction methods of linear codes from functions. For a polynomial $F(x)$ on $\mathbb{F}_q$, the *first* generic construction of linear codes is given by

$$\mathcal{C}(F) = \{(\mathrm{Tr}^n(aF(x) + bx))_{x\in\mathbb{F}_q^\star} \colon a, b \in \mathbb{F}_q\}$$

with length $(q-1)$ and dimension at most $2n$. For a subset $D = \{d_1, \ldots, d_m\} \subseteq \mathbb{F}_q$, the *second* generic construction based on D is defined as

$$\mathcal{C}_D = \{(\mathrm{Tr}^n(ad_1), \ldots, \mathrm{Tr}^n(ad_m)) \colon a \in \mathbb{F}_q\} \tag{1}$$

with length m and dimension at most n. The set D is called the *defining set* of $\mathcal{C}_D$, and the quality of $\mathcal{C}_D$ depends on the choice of D. The construction method of the form (1) has been initially studied by Ding et al. [8,9], and many linear codes have been proposed in [6,7,8,9,10,11]. Furthermore, new linear codes have been obtained from some cryptographic functions in this construction method (see *e.g.* [21,23,25,24,29]). Motivated by the method of the form (1), for a subset $\mathcal{D} = \{(x_1, y_1), \ldots, (x_m, y_m)\} \subseteq \mathbb{F}_q^2$, Li et al. [15] have defined the following linear code

$$\mathcal{C}_{\mathcal{D}} = \{\mathbf{c}_{(a,b)} = (\mathrm{Tr}^n(ax_1 + by_1), \ldots, \mathrm{Tr}^n(ax_m + by_m)) \colon a, b \in \mathbb{F}_q\}, \tag{2}$$

whose length is m and dimension at most $2n$. They have then constructed some linear codes by using the set $\mathcal{D} = \{(x, y) \in \mathbb{F}_q^2 \setminus \{(0, 0)\} \colon \mathrm{Tr}^n(x^k + y^l) = 0\}$, where $k, l \in \{1, 2, p^{n/2} + 1\}$. Recently, Jian et al. [13] have constructed further linear codes of the form (2) by using the set $\mathcal{D} = \{(x, y) \in \mathbb{F}_q^2 \setminus \{(0, 0)\} \colon \mathrm{Tr}^n(x^k + y^{p^u+1}) = 0\}$, where $k \in \{1, 2\}$. Very recently, Wu et al. [26] have constructed new linear codes of the form (2) based on the set

$$\mathcal{D} = \{(x, y) \in \mathbb{F}_q^2 \setminus \{(0, 0)\} \colon f(x) + g(y) = 0\} \subset \mathbb{F}_q^2, \tag{3}$$

where f and g are two weakly regular bent functions from $\mathbb{F}_q$ to $\mathbb{F}_p$. Motivated by the works [13,15,26], we in this paper construct minimal linear codes of the form (2) based on the set $\mathcal{D}$ of the form (3) for the following two cases:

1) $f(x) = \mathrm{Tr}^n(x)$ and $g(y) \in WRP$,
2) both $f(x) \in WRP$ and $g(y) \in WRP$.

Let f and g be two p-ary functions from $\mathbb{F}_q$ to $\mathbb{F}_p$, and let $\mathcal{D}$ be the set of the form (3). From the definition of the code $\mathcal{C}_{\mathcal{D}}$ of the form (2), we define

$$\mathcal{N}(a, b) = \#\{(x, y) \in \mathbb{F}_q^2 \setminus \{(0, 0)\} \colon f(x) + g(y) = 0 \text{ and } \mathrm{Tr}^n(ax + by) = 0\} \tag{4}$$

and hence, the Hamming weight of the nonzero codeword $\mathbf{c}_{(a,b)}$ is given by $W_H(\mathbf{c}_{(a,b)}) = \#\mathcal{D} - \mathcal{N}(a, b)$ for every $(a, b) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}$, and we clearly have $W_H(\mathbf{c}_{(0,0)}) = 0$.

3.2 Three-weight linear codes derived from $\mathrm{Tr}^n(x) + g(y) \in WRP$

In this subsection, we construct the linear code $\mathcal{C}_{\mathcal{D}}$ of the form (2) based on the set

$$\mathcal{D} = \{(x, y) \in \mathbb{F}_q^2 \setminus \{(0,0)\}\colon \mathrm{Tr}^n(x) + g(y) = 0\}, \tag{5}$$

when $g(y) \in WRP$ is an s_g-plateaued function with $0 \leq s_g < n$. From the orthogonality of exponential sums, we can derive its size $\#\mathcal{D} = p^{2n-1} - 1$, which is the length of the code $\mathcal{C}_{\mathcal{D}}$. To find the Hamming weights in $\mathcal{C}_{\mathcal{D}}$, for $(a, b) \in (\mathbb{F}_q^2)^\star$ we define

$$\mathcal{N}(a, b) = \#\{(x, y) \in \mathcal{D}\colon \mathrm{Tr}^n(ax + by) = 0\}. \tag{6}$$

We can derive the following lemma from the proof of [26, Lemma 5].

Lemma 7. *[26] Let $\mathcal{N}(a, b)$ be defined as in (6) for $(a, b) \in (\mathbb{F}_q^2)^\star$. Then, we have*

$$\mathcal{N}(a, b) = \begin{cases} p^{2n-2} - 1, & \text{if } a \in \mathbb{F}_q \setminus \mathbb{F}_p^\star, \\ p^{2n-2} - 1 + \frac{A(a,b)}{p^2}, & \text{if } a \in \mathbb{F}_p^\star, \end{cases}$$

for which the value $A(a, b)$ can be expressed as

$$A(a, b) = p^n \sum_{z_1 \in \mathbb{F}_p^\star} \sigma_{z_1} \left(\sum_{y \in \mathbb{F}_q} \xi_p^{g(y) - \mathrm{Tr}^n(a^{-1}by)} \right) \tag{7}$$

where a^{-1} is the multiplicative inverse of $a \in \mathbb{F}_p^\star$.

The following lemma calculates the value $A(a, b)$ by using the Walsh spectrum of the employed plateaued function.

Lemma 8. *Let $g \in WRP$ and $\mathcal{S}_g$ be its Walsh support. Let $A(a, b)$ be defined as in (7) for $a \in \mathbb{F}_p^\star$ and $b \in \mathbb{F}_q$. Then, for every $a^{-1}b \in \mathbb{F}_q \setminus \mathcal{S}_g$ we have $A(a, b) = 0$, and for every $a^{-1}b \in \mathcal{S}_g$*

$$A(a, b) = \begin{cases} \epsilon_g (p-1) p^n \sqrt{p^*}^{\,n+s_g}, & \text{if } g^\star(a^{-1}b) = 0, \\ -\epsilon_g p^n \sqrt{p^*}^{\,n+s_g}, & \text{if not,} \end{cases}$$

when $n + s_g$ is even; otherwise,

$$A(a, b) = \begin{cases} 0, & \text{if } g^\star(a^{-1}b) = 0, \\ \epsilon_g p^n \sqrt{p^*}^{\,n+s_g+1}, & \text{if } g^\star(a^{-1}b) \in SQ, \\ -\epsilon_g p^n \sqrt{p^*}^{\,n+s_g+1}, & \text{if } g^\star(a^{-1}b) \in NSQ. \end{cases}$$

Proof. When $a^{-1}b \in \mathbb{F}_q \setminus \mathcal{S}_g$, we clearly get $A(a, b) = 0$. When $a^{-1}b \in \mathcal{S}_g$, we have

$$\begin{aligned} A(a, b) = p^n \sum_{z_1 \in \mathbb{F}_p^\star} \sigma_{z_1} \left(\mathcal{W}_g \left(a^{-1}b \right) \right) &= p^n \sum_{z_1 \in \mathbb{F}_p^\star} \sigma_{z_1} \left(\epsilon_g \sqrt{p^*}^{\,n+s_g} \xi_p^{g^\star(a^{-1}b)} \right) \\ &= \epsilon_g p^n \sqrt{p^*}^{\,n+s_g} \sum_{z_1 \in \mathbb{F}_p^\star} \eta_0^{n+s_g}(z_1) \xi_p^{z_1 g^\star(a^{-1}b)}, \end{aligned}$$

where Lemma 3 is used in the second equality. The proof is hence complete by Lemma 1. □

The following lemma helps to determine the weights of codewords in $\mathcal{C}_{\mathcal{D}}$.

Lemma 9. *Let $g \in WRP$ and $\mathcal{S}_g$ be its Walsh support. Let $\mathcal{N}(a,b)$ be defined as in (6) for $(a,b) \in (\mathbb{F}_q^2)^\star$. Then, we have $\mathcal{N}(a,b) = p^{2n-2} - 1$ if $a \in \mathbb{F}_q \setminus \mathbb{F}_p^\star$ or if $a^{-1}b \notin \mathcal{S}_g$ where $a \in \mathbb{F}_p^\star$. For every $a^{-1}b \in \mathcal{S}_g$ where $a \in \mathbb{F}_p^\star$, we have*

$$\mathcal{N}(a,b) = \begin{cases} p^{2n-2} - 1 + \epsilon_g(p-1)p^{n-2}\sqrt{p^*}^{\,n+s_g}, & \text{if } g^\star(a^{-1}b) = 0, \\ p^{2n-2} - 1 - \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g}, & \text{if not,} \end{cases}$$

when $n + s_g$ is even; otherwise,

$$\mathcal{N}(a,b) = \begin{cases} p^{2n-2} - 1, & \text{if } g^\star(a^{-1}b) = 0, \\ p^{2n-2} - 1 + \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g+1}, & \text{if } g^\star(a^{-1}b) \in SQ, \\ p^{2n-2} - 1 - \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g+1}, & \text{if } g^\star(a^{-1}b) \in NSQ. \end{cases}$$

Proof. The proof follows from the combination of Lemmas 7 and 8. □

The following theorem proposes the code $\mathcal{C}_\mathcal{D}$ of the form (2) based on the set $\mathcal{D}$ of the form 5.

Theorem 1. *Let $g \in WRP$ with $0 \le s_g < n$ and $\mathcal{S}_g$ be its Walsh support. Let $\mathcal{D}$ be defined as in (5). Then, the code $\mathcal{C}_\mathcal{D}$ of the form (2) is a 3-weight linear $[p^{2n-1} - 1, 2n]$ code over $\mathbb{F}_p$. All parameters are listed in Tables 1 and 2 when $n + s_g$ is even and odd, respectively.*

Proof. From the definition of $\mathcal{C}_\mathcal{D}$, its length is the size of $\mathcal{D}$, and for every $(a,b) \in (\mathbb{F}_q^2)^\star$, its Hamming weight $W_H(\mathbf{c}_{(a,b)}) = \#\mathcal{D} - \mathcal{N}(a,b)$, where $\mathcal{N}(a,b)$ is defined as in (6). Then the Hamming weights can be obtained from Lemma 9. If $a \in \mathbb{F}_q \setminus \mathbb{F}_p^\star$ or if $a^{-1}b \notin \mathcal{S}_g$ where $a \in \mathbb{F}_p^\star$, then we have $W_H(\mathbf{c}_{(a,b)}) = (p-1)p^{2n-2}$, and its weight distribution is $p^{2n} - (p-1)p^{n-s_g} - 1$ by Lemma 2. Additionally, for every $a^{-1}b \in \mathcal{S}_g$ where $a \in \mathbb{F}_p^\star$, we have

$$W_H(\mathbf{c}_{(a,b)}) = \begin{cases} (p-1)p^{2n-2} - \epsilon_g(p-1)p^{n-2}\sqrt{p^*}^{\,n+s_g}, & \text{if } g^\star(a^{-1}b) = 0, \\ (p-1)p^{2n-2} + \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g}, & \text{if not,} \end{cases}$$

when $n + s_g$ is even; otherwise,

$$W_H(\mathbf{c}_{(a,b)}) = \begin{cases} (p-1)p^{2n-2}, & \text{if } g^\star(a^{-1}b) = 0, \\ (p-1)p^{2n-2} - \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g+1}, & \text{if } g^\star(a^{-1}b) \in SQ, \\ (p-1)p^{2n-2} + \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g+1}, & \text{if } g^\star(a^{-1}b) \in NSQ. \end{cases}$$

In this case, the weight distribution of each weight is derived from Lemma 5. All Hamming weights with their weight distributions are given in Tables 1 and 2, completing the proof. □

We below give an example of $\mathcal{C}_\mathcal{D}$ constructed in Theorem 1, which is verified by MAGMA in [2].

Example 1. The function $g : \mathbb{F}_{3^4} \to \mathbb{F}_3$ defined as $g(x) = \mathrm{Tr}^4(2x^{92})$ is weakly regular 2-plateaued function from *WRP*, and $\mathcal{W}_g(\beta) \in \{0, \epsilon_g \eta_0^3(-1)3^3\xi_3^{g^\star(\beta)}\}$, where $\epsilon_g = 1$ and $g^\star$ is a ternary function with $g^\star(0) = 0$. Then, $\mathcal{C}_\mathcal{D}$ is a 3-weight minimal ternary $[2186, 8, 1215]$ code with the weight enumerator $1 + 16y^{1215} + 6542y^{1458} + 2y^{1944}$.

Remark 1. When g is a weakly regular 0-plateaued (bent) function in Theorem 1, one can easily obtain the linear code given in [26, Theorem 3].

3.3 Three-weight and four-weight linear codes derived from $f(x), g(y) \in WRP$

In this subsection, we construct the linear code $\mathcal{C}_{\mathcal{D}}$ of the form (2) based on the set

$$\mathcal{D} = \{(x, y) \in \mathbb{F}_q^2 \setminus \{(0,0)\}\colon f(x) + g(y) = 0\}, \tag{8}$$

where $f, g \in WRP$ are s_f-plateaued and s_g-plateaued functions from $\mathbb{F}_q$ to $\mathbb{F}_p$, respectively, for $0 \le s_f, s_g < n$. For $(a, b) \in (\mathbb{F}_q^2)^\star$, we define

$$\mathcal{N}(a, b) = \#\{(x, y) \in \mathcal{D}\colon \mathrm{Tr}^n(ax + by) = 0\}. \tag{9}$$

We first introduce three lemmas by using the exponential sums and Walsh spectrum of a weakly regular plateaued function.

The size of the set $\mathcal{D}$ can be calculated by using the Walsh transform values of the employed functions at the zero points.

Lemma 10. *Let $f, g \in WRP$ and let $\mathcal{D}$ be defined as in (8). Then*

$$\#\mathcal{D} = \begin{cases} p^{2n-1} - 1, & \text{if } 2n + s_f + s_g \text{ is odd,} \\ p^{2n-1} - 1 + \epsilon_f \epsilon_g \frac{p-1}{p} \sqrt{p^*}^{2n+s_f+s_g}, & \text{if not.} \end{cases}$$

Proof. We can write $\mathcal{W}_f(0) = \epsilon_f \sqrt{p^*}^{n+s_f}$ and $\mathcal{W}_g(0) = \epsilon_g \sqrt{p^*}^{n+s_g}$ from Lemma 3, where $\epsilon_f, \epsilon_g \in \{\pm 1\}$, since we know $f^\star(0) = g^\star(0) = 0$ from Proposition 1. Hence, from the orthogonality of exponential sums, we have

$$\begin{aligned} \#\mathcal{D} + 1 &= \tfrac{1}{p} \sum_{x,y \in \mathbb{F}_q} \sum_{z \in \mathbb{F}_p} \xi_p^{z(f(x)+g(y))} \\ &= \tfrac{1}{p} \left(p^{2n} + \sum_{z \in \mathbb{F}_p^\star} \sigma_z \left(\sum_{x \in \mathbb{F}_q} \xi_p^{f(x)} \sum_{y \in \mathbb{F}_q} \xi_p^{g(y)} \right) \right) \\ &= \tfrac{1}{p} \left(p^{2n} + \sum_{z \in \mathbb{F}_p^\star} \sigma_z (\epsilon_f \epsilon_g \sqrt{p^*}^{2n+s_f+s_g}) \right) \\ &= \tfrac{1}{p} \left(p^{2n} + \epsilon_f \epsilon_g \sqrt{p^*}^{2n+s_f+s_g} \sum_{z \in \mathbb{F}_p^\star} \eta_0^{2n+s_f+s_g}(z) \right). \end{aligned}$$

The proof is completed from Lemma 1. □

Lemma 11. *Let $f, g \in WRP$ and $\mathcal{S}_f, \mathcal{S}_g$ be their Walsh supports. Let l_f, l_g be defined as in Proposition 1. For $(a, b) \in (\mathbb{F}_q^2)^\star$, define*

$$B(a, b) = \sum_{z_1, z_2 \in \mathbb{F}_p^\star} \sum_{x,y \in \mathbb{F}_q} \xi_p^{z_1(f(x)+g(y)) - z_2 \mathrm{Tr}^n(ax+by)}.$$

Then, for $(a, b) \notin \mathcal{S}_f \times \mathcal{S}_g$ we have $B(a, b) = 0$, and for $(a, b) \in \mathcal{S}_f \times \mathcal{S}_g$, we have the following cases.

– *When $2n+s_f+s_g$ is odd and $l_f=l_g$, we have*

$$B(a,b)=\begin{cases}0, & \text{if } A_1,\\ \epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g+1}, & \text{if } A_2,\\ -\epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g+1}, & \text{if } A_3,\end{cases}$$

where

A_1 *denotes* $f^\star(a)=b=0$ *or* $a=g^\star(b)=0$ *or* $f^\star(a)+g^\star(b)=0$ *for* $ab\neq 0$,
A_2 *denotes* $f^\star(a)\in SQ$ *for* $b=0$ *or* $g^\star(b)\in SQ$ *for* $a=0$ *or* $f^\star(a)+g^\star(b)\in SQ$ *for* $ab\neq 0$,
A_3 *denotes* $f^\star(a)\in NSQ$ *for* $b=0$ *or* $g^\star(b)\in NSQ$ *for* $a=0$ *or* $f^\star(a)+g^\star(b)\in NSQ$ *for* $ab\neq 0$.

– *When $2n+s_f+s_g$ is even, we have for $l_f=l_g$*

$$B(a,b)=\begin{cases}\epsilon_f\epsilon_g(p-1)^2\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } C_1,\\ -\epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{otherwise},\end{cases}$$

and for $l_f\neq l_g$

$$B(a,b)=\begin{cases}\epsilon_f\epsilon_g(p-1)^2\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } C_2,\\ \epsilon_f\epsilon_g(p+1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } -\frac{f^\star(a)}{g^\star(b)}\in SQ,\\ -\epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{otherwise},\end{cases}$$

where

C_1 *denotes* $a=g^\star(b)=0$ *or* $f^\star(a)=b=0$ *or* $f^\star(a)+g^\star(b)=0$ *for* $ab\neq 0$,
C_2 *denotes* $a=g^\star(b)=0$ *or* $f^\star(a)=b=0$ *or* $f^\star(a)=g^\star(b)=0$ *for* $ab\neq 0$.

Proof. From the definition of $B(a,b)$, we have

$$\begin{aligned}B(a,b)&=\sum_{z_1,z_2\in\mathbb{F}_p^\star}\sum_{x\in\mathbb{F}_q}\xi_p^{z_1(f(x)-\mathrm{Tr}^n(z_2ax))}\sum_{y\in\mathbb{F}_q}\xi_p^{z_1(g(y)-\mathrm{Tr}^n(z_2by))}\\ &=\sum_{z_1\in\mathbb{F}_p^\star}\sigma_{z_1}\Big(\sum_{z_2\in\mathbb{F}_p^\star}\mathcal{W}_f(z_2a)\mathcal{W}_g(z_2b)\Big),\end{aligned}$$

where we use the fact that $\frac{z_2}{z_1}$ passes all over $\mathbb{F}_p^\star$ for a fixed z_1 while z_2 passes through $\mathbb{F}_p^\star$ in the first equality.

– If $(a,b)\notin\mathcal{S}_f\times\mathcal{S}_g$, equivalently, $(z_2a,z_2b)\notin\mathcal{S}_f\times\mathcal{S}_g$ for every $z_2\in\mathbb{F}_p^\star$ (see Lemma 6), then we clearly say that $B(a,b)=0$.
– If $(a,b)\in\mathcal{S}_f\times\mathcal{S}_g$, equivalently, $(z_2a,z_2b)\in\mathcal{S}_f\times\mathcal{S}_g$ for every $z_2\in\mathbb{F}_p^\star$, then there are two cases: $ab=0$ and $ab\neq 0$.
  - In the case of $ab=0$, suppose $a=0$ and $b\neq 0$, without loss of generality. We then have

$$\begin{aligned}B(a,b)&=\sum_{z_1\in\mathbb{F}_p^\star}\sigma_{z_1}\left(\sum_{z_2\in\mathbb{F}_p^\star}\epsilon_f\sqrt{p^*}^{\,n+s_f}\epsilon_g\sqrt{p^*}^{\,n+s_g}\xi_p^{g^\star(z_2b)}\right)\\ &=\sum_{z_1\in\mathbb{F}_p^\star}\sigma_{z_1}\left(\sum_{z_2\in\mathbb{F}_p^\star}\epsilon_f\epsilon_g\sqrt{p^*}^{2n+s_f+s_g}\xi_p^{z_2^{l_g}g^\star(b)}\right)\\ &=\epsilon_f\epsilon_g\sqrt{p^*}^{2n+s_f+s_g}\sum_{z_1\in\mathbb{F}_p^\star}\eta_0^{2n+s_f+s_g}(z_1)\sum_{z_2\in\mathbb{F}_p^\star}\xi_p^{z_1z_2^{l_g}g^\star(b)},\end{aligned}$$

where we use Lemmas 3, 6 and Proposition 1 in the first and second equality, respectively. With the help of Lemma 1, we get

$$B(a,b)=\begin{cases}0, & \text{if } g^\star(b)=0,\\ \epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g+1}, & \text{if } g^\star(b)\in SQ,\\ -\epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g+1}, & \text{if } g^\star(b)\in NSQ,\end{cases}$$

when $2n+s_f+s_g$ is odd; otherwise,

$$B(a,b)=\begin{cases}\epsilon_f\epsilon_g(p-1)^2\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } g^\star(b)=0,\\ -\epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{if not.}\end{cases}$$

Similarly, for $b=0$ and $a\neq 0$, the analogous computations yield the same results above with respect to the parameter a.

- In the case of $ab\neq 0$, we get

$$\begin{aligned}B(a,b)&=\sum_{z_1\in\mathbb{F}_p^\star}\sigma_{z_1}\left(\sum_{z_2\in\mathbb{F}_p^\star}\epsilon_f\sqrt{p^*}^{\,n+s_f}\xi_p^{f^\star(z_2a)}\epsilon_g\sqrt{p^*}^{\,n+s_g}\xi_p^{g^\star(z_2b)}\right)\\ &=\sum_{z_1\in\mathbb{F}_p^\star}\sigma_{z_1}\left(\sum_{z_2\in\mathbb{F}_p^\star}\epsilon_f\epsilon_g\sqrt{p^*}^{2n+s_f+s_g}\xi_p^{z_2^{l_f}f^\star(a)+z_2^{l_g}g^\star(b)}\right)\\ &=\epsilon_f\epsilon_g\sqrt{p^*}^{2n+s_f+s_g}\sum_{z_1\in\mathbb{F}_p^\star}\eta_0^{2n+s_f+s_g}(z_1)\sum_{z_2\in\mathbb{F}_p^\star}\xi_p^{z_1(z_2^{l_f}f^\star(a)+z_2^{l_g}g^\star(b))},\end{aligned}$$

where we use Lemmas 3, 6 and Proposition 1 in the first and second equality, respectively. We hence compute the value $B(a,b)$ by using Lemma 1 and some properties of the cyclotomic field. When $2n+s_f+s_g$ is odd, we get for $l_f=l_g$

$$B(a,b)=\begin{cases}0, & \text{if } f^\star(a)+g^\star(b)=0,\\ \epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g+1}, & \text{if } f^\star(a)+g^\star(b)\in SQ,\\ -\epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g+1}, & \text{if } f^\star(a)+g^\star(b)\in NSQ.\end{cases}$$

When $2n+s_f+s_g$ is even, we get for $l_f=l_g$

$$B(a,b)=\begin{cases}\epsilon_f\epsilon_g(p-1)^2\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } f^\star(a)+g^\star(b)=0,\\ -\epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{otherwise,}\end{cases}$$

and for $l_f\neq l_g$

$$B(a,b)=\begin{cases}\epsilon_f\epsilon_g(p-1)^2\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } f^\star(a)=g^\star(b)=0,\\ \epsilon_f\epsilon_g(p+1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } -\frac{f^\star(a)}{g^\star(b)}\in SQ,\\ -\epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{otherwise.}\end{cases}$$

□

The following lemma helps to compute the weights and their weight distributions.

Lemma 12. *Let $f, g \in WRP$ and $\mathcal{S}_f, \mathcal{S}_g$ be their Walsh supports. Let l_f, l_g be defined as in Proposition 1. Let $\mathcal{N}(a,b)$ be defined as in (9) for $(a,b) \in (\mathbb{F}_q^2)^\star$.*

- *Suppose that $2n + s_f + s_g$ is odd and $l_f = l_g$. For every $(a,b) \notin \mathcal{S}_f \times \mathcal{S}_g$, we have $\mathcal{N}(a,b) = p^{2n-2} - 1$, and for every $(a,b) \in \mathcal{S}_f \times \mathcal{S}_g$, we have*

$$\mathcal{N}(a,b) = \begin{cases} p^{2n-2} - 1, & \text{if } A_1, \\ p^{2n-2} - 1 + \epsilon_f \epsilon_g \frac{1}{p^2}(p-1)\sqrt{p^*}^{2n+s_f+s_g+1}, & \text{if } A_2, \\ p^{2n-2} - 1 - \epsilon_f \epsilon_g \frac{1}{p^2}(p-1)\sqrt{p^*}^{2n+s_f+s_g+1}, & \text{if } A_3, \end{cases}$$

where

A_1 *denotes $f^\star(a) = b = 0$ or $a = g^\star(b) = 0$ or $f^\star(a) + g^\star(b) = 0$ for $ab \neq 0$,*
A_2 *denotes $f^\star(a) \in SQ$ for $b = 0$ or $g^\star(b) \in SQ$ for $a = 0$ or $f^\star(a) + g^\star(b) \in SQ$ for $ab \neq 0$,*
A_3 *denotes $f^\star(a) \in NSQ$ for $b = 0$ or $g^\star(b) \in NSQ$ for $a = 0$ or $f^\star(a) + g^\star(b) \in NSQ$ for $ab \neq 0$.*

- *Suppose that $2n + s_f + s_g$ is even. For every $(a,b) \notin \mathcal{S}_f \times \mathcal{S}_g$, we have $\mathcal{N}(a,b) = p^{2n-2} - 1 + \epsilon_f \epsilon_g \frac{1}{p^2}(p-1)\sqrt{p^*}^{2n+s_f+s_g}$. For every $(a,b) \in \mathcal{S}_f \times \mathcal{S}_g$, we have for $l_f = l_g$*

$$\mathcal{N}(a,b) = \begin{cases} p^{2n-2} - 1 + \epsilon_f \epsilon_g \frac{1}{p}(p-1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } C_1, \\ p^{2n-2} - 1, & \text{otherwise}, \end{cases}$$

and for $l_f \neq l_g$

$$\mathcal{N}(a,b) = \begin{cases} p^{2n-2} - 1 + \epsilon_f \epsilon_g \frac{1}{p}(p-1)\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } C_2, \\ p^{2n-2} - 1 + \epsilon_f \epsilon_g \frac{2}{p}\sqrt{p^*}^{2n+s_f+s_g}, & \text{if } -\frac{f^\star(a)}{g^\star(b)} \in SQ, \\ p^{2n-2} - 1, & \text{otherwise}, \end{cases}$$

where

C_1 *denotes $a = g^\star(b) = 0$ or $f^\star(a) = b = 0$ or $f^\star(a) + g^\star(b) = 0$ for $ab \neq 0$,*
C_2 *denotes $a = g^\star(b) = 0$ or $f^\star(a) = b = 0$ or $f^\star(a) = g^\star(b) = 0$ for $ab \neq 0$.*

Proof. By the definition of $\mathcal{N}(a,b)$ and using the orthogonality of exponential sums, we get

$$\mathcal{N}(a,b) + 1 = p^{-2} \sum_{x,y \in \mathbb{F}_q} \sum_{z_1 \in \mathbb{F}_p} \xi_p^{z_1(f(x)+g(y))} \sum_{z_2 \in \mathbb{F}_p} \xi_p^{-z_2 \mathrm{Tr}^n(ax+by)} = p^{2n-2} + \frac{1}{p^2}(A + B(a,b)),$$

where

$$A = \sum_{z_1 \in \mathbb{F}_p^\star} \sum_{x,y \in \mathbb{F}_q} \xi_p^{z_1(f(x)+g(y))} \quad \text{and} \quad B(a,b) = \sum_{z_1, z_2 \in \mathbb{F}_p^\star} \sum_{x,y \in \mathbb{F}_q} \xi_p^{z_1(f(x)+g(y)) - z_2 \mathrm{Tr}^n(ax+by)}.$$

We clearly have $A = p(\#\mathcal{D} + 1) - p^{2n}$ in the light of Lemma 10. The proof is then complete from Lemmas 10 and 11. $\square$

The following lemma is needed to determine the weight distribution.

Lemma 13. *Let $f, g \in$ WRP with $0 \leq s_f, s_g < n$ and $\mathcal{S}_f, \mathcal{S}_g$ be their Walsh supports. For $a, b \in \mathbb{F}_q$ and $t \in \mathbb{F}_p$, define $S_{f,g}(t) = \#\{(a,b) \in \mathcal{S}_f \times \mathcal{S}_g \colon f^\star(a) + g^\star(b) = t\}$. Then, we have*

$$S_{f,g}(t) = \begin{cases} p^{2n-s_f-s_g-1}, & \text{if } t = 0, \\ p^{2n-s_f-s_g-1} + \epsilon_f \epsilon_g \eta_0(-t) \frac{1}{p} \sqrt{p^*}^{\,2n-s_f-s_g+1}, & \text{if } t \neq 0 \end{cases}$$

when $2n - s_f - s_g$ is odd; otherwise,

$$S_{f,g}(t) = \begin{cases} p^{2n-s_f-s_g-1} + \epsilon_f \epsilon_g \frac{p-1}{p} \sqrt{p^*}^{2n-s_f-s_g}, & \text{if } t = 0, \\ p^{2n-s_f-s_g-1} - \epsilon_f \epsilon_g \frac{1}{p} \sqrt{p^*}^{\,2n-s_f-s_g}, & \text{if } t \neq 0. \end{cases}$$

Proof. From the orthogonality of exponential sums, we have

$$\begin{aligned} S_{f,g}(t) &= \tfrac{1}{p} \sum_{a \in \mathcal{S}_f} \sum_{b \in \mathcal{S}_g} \sum_{z \in \mathbb{F}_p} \xi_p^{z(f^\star(a)+g^\star(b))-zt} \\ &= \tfrac{1}{p} (p^{2n-s_f-s_g} + \sum_{z \in \mathbb{F}_p^\star} \xi_p^{-zt} \sigma_z (\sum_{a \in \mathcal{S}_f} \xi_p^{f^\star(a)} \sum_{b \in \mathcal{S}_g} \xi_p^{g^\star(b)})) \\ &= p^{2n-s_f-s_g-1} + \epsilon_f \epsilon_g \tfrac{1}{p} \sqrt{p^*}^{\,2n-s_f-s_g} \sum_{z \in \mathbb{F}_p^\star} \eta_0^{2n-s_f-s_g}(z) \xi_p^{-zt}, \end{aligned}$$

where Lemma 4 is used in the last equality. The proof is hence complete by Lemma 1. □

We now construct the code $\mathcal{C}_\mathcal{D}$ of the form (2) when $2n + s_f + s_g$ is odd.

Theorem 2. *Let $f, g \in$ WRP with their Walsh supports $\mathcal{S}_f, \mathcal{S}_g$ and with $l_f = l_g$, where l_f, l_g are defined as in Proposition 1. Let $\mathcal{D}$ be defined as in (8). Suppose that $n + s_f$ is odd and $n + s_g$ is even with $0 \leq s_f, s_g < n$. Then, the code $\mathcal{C}_\mathcal{D}$ of the from (2) is a 3-weight linear $[p^{2n-1} - 1, 2n]$ code with parameters listed in Table 3.*

Proof. From the definition of $\mathcal{C}_\mathcal{D}$, its length equals the size of $\mathcal{D}$, and the weight of each codeword is $W_H(\mathbf{c}_{(a,b)}) = \#\mathcal{D} - \mathcal{N}(a,b)$ for every $(a,b) \in (\mathbb{F}_q^2)^\star$, where $\mathcal{N}(a,b)$ is defined as in (9). By Lemma 10, we have $\#\mathcal{D} = p^{2n-1} - 1$, and the Hamming weights can be derived from Lemma 12. To put it more explicitly, for every $(a,b) \notin \mathcal{S}_f \times \mathcal{S}_g$, we have $W_H(\mathbf{c}_{(a,b)}) = (p-1)p^{2n-2}$, and the number of such codewords equals $p^{2n} - p^{2n-s_f-s_g}$ by Lemma 2. Additionally, for every $(a,b) \in \mathcal{S}_f \times \mathcal{S}_g$, we get

$$W_H(\mathbf{c}_{(a,b)}) = \begin{cases} (p-1)p^{2n-2}, & A_1 \text{ times}, \\ (p-1)p^{2n-2} - \epsilon_f \epsilon_g \frac{1}{p^2} (p-1) \sqrt{p^*}^{2n+s_f+s_g+1}, & A_2 \text{ times}, \\ (p-1)p^{2n-2} + \epsilon_f \epsilon_g \frac{1}{p^2} (p-1) \sqrt{p^*}^{2n+s_f+s_g+1}, & A_3 \text{ times}, \end{cases}$$

whose weight distribution is determined by Lemmas 12 and 13. Firstly, to compute A_1, we define the following three sets and it can be expressed as the sum of their sizes:

$$\begin{aligned} A_1 &= \#\{(a,b) \in \mathcal{S}_f^\star \times \mathcal{S}_g^\star \colon f^\star(a) + g^\star(b) = 0\} \\ &\quad + \#\{a \in \mathcal{S}_f^\star \colon f^\star(a) = 0\} + \#\{b \in \mathcal{S}_g^\star \colon g^\star(b) = 0\} \\ &= S_{f,g}(0) - 1 = p^{2n-s_f-s_g-1} - 1 \end{aligned}$$

where $S_{f,g}(0)$ is defined in Lemma 13. Similarly, A_2 and A_3 can be expressed as

$$\begin{aligned} A_2 &= \#\{(a,b) \in \mathcal{S}_f^\star \times \mathcal{S}_g^\star\colon f^\star(a) + g^\star(b) \in SQ\} \\ &\quad + \#\{a \in \mathcal{S}_f^\star\colon f^\star(a) \in SQ\} + \#\{b \in \mathcal{S}_g^\star\colon g^\star(b) \in SQ\} \\ &= (\tfrac{p-1}{2})S_{f,g}(i), \\ A_3 &= \#\{(a,b) \in \mathcal{S}_f^\star \times \mathcal{S}_g^\star\colon f^\star(a) + g^\star(b) \in NSQ\} \\ &\quad + \#\{a \in \mathcal{S}_f^\star\colon f^\star(a) \in NSQ\} + \#\{b \in \mathcal{S}_g^\star\colon g^\star(b) \in NSQ\} \\ &= (\tfrac{p-1}{2})S_{f,g}(j) \end{aligned}$$

where $S_{f,g}(i)$ and $S_{f,g}(j)$ are given in Lemma 13 for $i \in SQ$ and $j \in NSQ$. This completes the proof. □

The following examples for the code $\mathcal{C}_\mathcal{D}$ given in Theorem 2 are verified by MAGMA in [2].

Example 2. Let $f, g : \mathbb{F}_{3^2} \to \mathbb{F}_3$ be defined as $f(x) = \mathrm{Tr}^2(\zeta x^4 + \zeta^8 x^2)$ and $g(x) = \mathrm{Tr}^2(x^{10})$, for a primitive element ζ of $\mathbb{F}_{3^2}$. Then $f, g \in \mathit{WRP}$ with $s_f = 1, s_g = 0$ and $\epsilon_f = \epsilon_g = 1$, and hence $\mathcal{C}_\mathcal{D}$ is a 3-weight ternary $[26, 4, 12]$ code with $1 + 12y^{12} + 62y^{18} + 6y^{24}$.

Example 3. Let $f, g : \mathbb{F}_{3^3} \to \mathbb{F}_3$ be defined as $f(x) = \mathrm{Tr}^3(x^{10})$ and $g(x) = \mathrm{Tr}^3(\zeta x^4 + \zeta^8 x^2)$, for a primitive element ζ of $\mathbb{F}_{3^3}$. Then $f, g \in \mathit{WRP}$ with $s_f = 0, s_g = 1$ and $\epsilon_f = \epsilon_g = 1$, and hence $\mathcal{C}_\mathcal{D}$ is a 3-weight minimal ternary $[242, 6, 144]$ code with $1 + 90y^{144} + 566y^{162} + 72y^{180}$. It is worth noting that this code is better than the code $[242, 6, 135]_3$, which is obtained in [26, Example 6] only from quadratic weakly regular bent $f(x) = \mathrm{Tr}^3(x^{10})$.

We below construct the code $\mathcal{C}_\mathcal{D}$ of the form (2) when $2n + s_f + s_g$ is even.

Theorem 3. *Let $f, g \in \mathit{WRP}$ with their Walsh supports $\mathcal{S}_f, \mathcal{S}_g$ and let l_f, l_g be defined as in Proposition 1. Let $\mathcal{D}$ be defined as in (8). Suppose that $2n + s_f + s_g$ is even with $0 \le s_f, s_g < n$. Then, the code $\mathcal{C}_\mathcal{D}$ of the form (2) with parameters $[p^{2n-1} - 1 + \epsilon_f\epsilon_g \frac{1}{p}(p-1)\sqrt{p^*}^{2n+s_f+s_g}, 2n]$*

- *is a 3-weight linear p-ary code over $\mathbb{F}_p$ when $l_f = l_g$,*
- *is a 4-weight linear p-ary code over $\mathbb{F}_p$ when $l_f \neq l_g$ and $p > 3$.*

The Hamming weights with their weight distributions are given in Tables 4 and 5 when $l_f = l_g$ and $l_f \neq l_g$, respectively.

Proof. The length of the code $\mathcal{C}_\mathcal{D}$ follows from Lemma 10, and for every $(a, b) \in (\mathbb{F}_q^2)^\star$, the weight $W_H(\mathbf{c}_{(a,b)}) = \#\mathcal{D} - \mathcal{N}(a, b)$ can be obtained from Lemmas 10 and 12. To be more precise, when $(a, b) \notin \mathcal{S}_f \times \mathcal{S}_g$, we have

$$W_H(\mathbf{c}_{(a,b)}) = (p-1)(p^{2n-2} + \epsilon_f\epsilon_g \frac{1}{p^2}(p-1)\sqrt{p^*}^{2n+s_f+s_g}),$$

whose weight distribution is $p^{2n} - p^{2n-s_f-s_g}$ from Lemma 2. In addition, when $(a, b) \in \mathcal{S}_f \times \mathcal{S}_g$, there are two distinct cases.

- When $l_f = l_g$,

$$W_H(\mathbf{c}_{(a,b)}) = \begin{cases} (p-1)p^{2n-2}, & A_{\beta_1} \text{ times}, \\ (p-1)(p^{2n-2} + \epsilon_f\epsilon_g \frac{1}{p}\sqrt{p^*}^{2n+s_f+s_g}), & A_{\beta_2} \text{ times}. \end{cases}$$

To determine A_{β_1}, we define the following three sets by using the condition C_1 given in Lemma 12, and so

$$\begin{aligned}A_{\beta_1} &= \#\{(a,b) \in \mathcal{S}_f^\star \times \mathcal{S}_g^\star \colon f^\star(a) + g^\star(b) = 0\} \\ &\quad + \#\{a \in \mathcal{S}_f^\star \colon f^\star(a) = 0\} + \#\{b \in \mathcal{S}_g^\star \colon g^\star(b) = 0\}.\end{aligned}$$

We hence conclude that $A_{\beta_1} = S_{f,g}(0) - 1$. Similarly, we can see that $A_{\beta_2} = (p-1)S_{f,g}(t)$, where $t \in \mathbb{F}_p^\star$. Hence, the weight distributions follow from Lemma 13. The Hamming weights with their weight distributions are given in Table 4.

– When $l_f \neq l_g$,

$$W_H(\mathbf{c}_{(a,b)}) = \begin{cases} (p-1)p^{2n-2}, & A_{\omega_1} \text{ times}, \\ (p-1)p^{2n-2} + \epsilon_f\epsilon_g \frac{(p-3)}{p}\sqrt{p^*}^{2n+s_f+s_g}, & A_{\omega_2} \text{ times}, \\ (p-1)(p^{2n-2} + \epsilon_f\epsilon_g \frac{1}{p}\sqrt{p^*}^{2n+s_f+s_g}), & A_{\omega_3} \text{ times}. \end{cases}$$

In this case, to determine the weight distribution, we define the following four sets by using the condition C_2 given in Lemma 12. A_{ω_1} and A_{ω_2} can be written as

$$\begin{aligned}A_{\omega_1} &= \#\{(a,b) \in \mathcal{S}_f^\star \times \mathcal{S}_g^\star \colon f^\star(a) = g^\star(b) = 0\} + \#\{a \in \mathcal{S}_f^\star \colon f^\star(a) = 0\} \\ &\quad + \#\{b \in \mathcal{S}_g^\star \colon g^\star(b) = 0\} = \#\{(a,b) \in \mathcal{S}_f \times \mathcal{S}_g \colon f^\star(a) = g^\star(b) = 0\} - 1 \\ &= \mathcal{N}_{f^\star}(0) * \mathcal{N}_{g^\star}(0) - 1, \\ A_{\omega_2} &= \#\{(a,b) \in \mathcal{S}_f^\star \times \mathcal{S}_g^\star \colon -\tfrac{f^\star(a)}{g^\star(b)} \in SQ\} = \tfrac{(p-1)^2}{2}\mathcal{N}_{f^\star}(i) * \mathcal{N}_{g^\star}(j),\end{aligned}$$

where $i, j \in SQ$. Here, the numbers $\mathcal{N}_{f^\star}(i)$ and $\mathcal{N}_{g^\star}(j)$ depend on the parity of s_f and s_g, and they are given in Lemma 5. Additionally, we have $A_{\omega_3} = p^{2n-s_f-s_g} - 1 - A_{\omega_1} - A_{\omega_2}$ due to the fact that the dimension is $2n$. Hence, the weight distributions follow from Lemma 5, and the Hamming weights are given in Table 5.

The proof of this theorem is complete. □

We end this subsection by giving an example for the code $\mathcal{C}_\mathcal{D}$ constructed in Theorem 3, verified by MAGMA in [2].

Example 4. Let $f, g : \mathbb{F}_{3^5} \to \mathbb{F}_3$ be defined as $f(x) = \mathrm{Tr}^5(\zeta x^{10} + \zeta^{20}x^4)$ and $g(x) = \mathrm{Tr}^5(\zeta x^{10} + 2x^4 + x^2)$, for a primitive element ζ of $\mathbb{F}_{3^5}$. Then $f, g \in \mathit{WRP}$ with $s_f = s_g = 1$, $l_f = l_g = 2$, $\epsilon_f = 1$ and $\epsilon_g = -1$. Hence, $\mathcal{C}_\mathcal{D}$ is a 3-weight minimal ternary $[19196, 10, 12636]$ code with $1 + 4428y^{12636} + 52488y^{12798} + 2132y^{13122}$.

Remark 2. If f and g are two weakly regular 0-plateaued (bent) functions in Theorem 3, then we get 2-weight and 3-weight linear codes when $l_f = l_g$ and $l_f \neq l_g$, respectively. They were presented in [26, Theorem 4].

3.4 Three-weight punctured codes

In this subsection, we derive shorter linear codes from the constructed codes by using a special subset of the defining set $\mathcal{D}$. Such a code is said to be a *punctured code* of the original code. It is known that the minimum distance and length of a punctured code are rather smaller than the original code while its dimension is the same as the original code.

We deal with the code $\mathcal{C}_{\mathcal{D}}$ of the form (2) for the defining set $\mathcal{D}$ of the form (8). In Theorems 2 and 3, the length and Hamming weights of $\mathcal{C}_{\mathcal{D}}$ have a common factor $(p-1)$, which suggests that $\mathcal{C}_{\mathcal{D}}$ can be punctured into a shorter linear code over $\mathbb{F}_p$. Let $f, g \in WRP$ with $k_f = k_g$. For every $x, y \in \mathbb{F}_q$, $f(cx) + g(cy) = 0$ iff $f(x) + g(y) = 0$ for every $c \in \mathbb{F}_p^{\star}$ because $f(cx) + g(cy) = c^{k_f}(f(x) + g(y))$. We can then choose a subset $\overline{\mathcal{D}}$ of the set $\mathcal{D}$ such that $\bigcup_{c\in\mathbb{F}_p^{\star}} c\overline{\mathcal{D}}$ is a partition of $\mathcal{D}$:

$$\mathcal{D} = \mathbb{F}_p^{\star}\overline{\mathcal{D}} = \{c\overline{(x,y)} \colon c \in \mathbb{F}_p^{\star} \text{ and } \overline{(x,y)} \in \overline{\mathcal{D}}\}.$$

Thus, $\mathcal{C}_{\mathcal{D}}$ can be punctured into a shorter one $\mathcal{C}_{\overline{\mathcal{D}}}$ based on the defining set $\overline{\mathcal{D}}$. Since $\#\mathcal{D} = (p-1)\#\overline{\mathcal{D}}$, the length and Hamming weights of the punctured code $\mathcal{C}_{\overline{\mathcal{D}}}$ can be derived from that of $\mathcal{C}_{\mathcal{D}}$ by dividing by $(p-1)$.

We introduce the parameters of the punctured codes in the following corollaries.

Corollary 1. *Let $f, g \in WRP$ with $l_f = l_g$ and $k_f = k_g$. Let $\mathcal{D}$ be defined as in (8). Let $\mathcal{C}_{\mathcal{D}}$ be the 3-weight code proposed in Theorem 2. Then, its punctured code $\mathcal{C}_{\overline{\mathcal{D}}}$ is a 3-weight $[(p^{2n-1}-1)/(p-1), 2n]$ linear code with parameters documented in Table 6.*

As examples, we give the following punctured codes, which are almost optimal.

Example 5. The punctured code $\mathcal{C}_{\overline{\mathcal{D}}}$ of the code given in Example 2 is a 3-weight ternary $[13, 4, 6]$ code with $1 + 12y^6 + 62y^9 + 6y^{12}$. This punctured code is almost optimal ternary code because the best ternary code with length 13 and dimension 4 has $d = 7$ in [12].

Example 6. The punctured code $\mathcal{C}_{\overline{\mathcal{D}}}$ of the code given in Example 3 is a 3-weight ternary $[121, 6, 72]$ minimal code with $1 + 90y^{72} + 566y^{81} + 72y^{90}$. Note that $d = 78$ for the best ternary code with length 121 and dimension 6 in [12].

Corollary 2. *Let $f, g \in WRP$ with $k_f = k_g$ and $l_f = l_g$. Let $\mathcal{D}$ be defined as in (8). Let $\mathcal{C}_{\mathcal{D}}$ be the 3-weight code proposed in Theorem 3. Then, its punctured code $\mathcal{C}_{\overline{\mathcal{D}}}$ is a 3-weight $[(p^{2n-1}-1)/(p-1) + \epsilon_f\epsilon_g\frac{1}{p}\sqrt{p^*}^{2n+s_f+s_g}, 2n]$ code whose parameters are listed in Table 7.*

3.5 Minimality of the constructed codes

In this subsection, we show that the constructed codes are minimal and investigate the minimum Hamming distances of their dual codes.

A linear code $\mathcal{C}$ is *minimal* if every nonzero codeword $\mathbf{v}$ in $\mathcal{C}$ covers only the codewords $j\mathbf{v}$ for all $j \in \mathbb{F}_p$. The following lemma introduces the well-known sufficient condition on the minimal codes.

Lemma 14. *(Ashikhmin-Barg,1998) [1] Let $\mathcal{C}$ be a linear code over $\mathbb{F}_p$, and let $w_{\min}$ and $w_{\max}$ represent, respectively, the minimum and maximum Hamming weights of $\mathcal{C}$. Then, $\mathcal{C}$ is minimal if*

$$\frac{p-1}{p} < \frac{w_{\min}}{w_{\max}}. \tag{10}$$

By (10), our linear codes are minimal codes for almost all integers s_f and s_g with $0 \le s_f, s_g < n$. The following proposition finds the bounds on the integers s_f and s_g that make the associated codes are minimal.

Proposition 2. *Let $f, g \in WRP$, and let s_f and s_g be two integers with $0 \leq s_f, s_g < n$. We have the following bounds on the parameters.*

i.) *The code $\mathcal{C}_{\mathcal{D}}$ in Theorem 1 is minimal for $0 \leq s_g \leq n-3$ if $n+s_g$ is even; otherwise, it is minimal for $0 \leq s_g \leq n-2$ and $4 \leq n$.*

ii.) *The code $\mathcal{C}_{\mathcal{D}}$ in Theorem 2 is minimal when $0 \leq s_f + s_g \leq 2n-4$ and $3 \leq n$.*

iii.) *The code $\mathcal{C}_{\mathcal{D}}$ in Theorem 3 is minimal for $0 \leq s_f+s_g \leq 2n-4$ if $\epsilon_f \epsilon_g \eta_0^{(2n+s_f+s_g+1)/2}(-1) = 1$; otherwise, it is minimal for $0 \leq s_f + s_g \leq 2n-6$ and $3 \leq n$.*

Remark 3. Our punctured codes are minimal for almost all cases.

Since our codes are minimal, we can describe the access structures of the secret sharing schemes based on their dual codes as described in [4, Theorem 17]. We first consider the minimum distances $d^{\perp}$ of the dual codes of our minimal codes.

For the codes $\mathcal{C}_{\mathcal{D}}$ constructed in Theorems 1, 2 and 3, their dual codes $\mathcal{C}_{\mathcal{D}}^{\perp}$ have $d^{\perp} = 2$ due to the fact that two entries of each codeword in $\mathcal{C}_{\mathcal{D}}$ are linearly dependent iff the minimum distance $d^{\perp}$ of $\mathcal{C}_{\mathcal{D}}^{\perp}$ is equal to 2. This suggests that these minimal codes can be used to design high democratic secret sharing schemes with good access structures as introduced in [4, Theorem 17] (and developed in [9, Proposition 2]).

On the other hand, for the punctured codes $\mathcal{C}_{\overline{\mathcal{D}}}$ given in Corollaries 1 and 2, the minimum distances of their dual codes are at least 3 since no two of the vectors are dependent. As a consequence, the punctured codes are projective minimal codes. The projective 3-weight codes given in Corollaries 1 and 2 can be employed to design association schemes introduced in [3]. Additionally, they can be employed to design democratic secret sharing schemes as introduced in [4, Theorem 17].

4 Conclusion

In this paper, motivated by the work of [13,15,26], to construct minimal codes, we consider weakly regular plateaued unbalanced functions in the recent construction method of linear codes. As far as we search, our minimal codes have new parameters since we for the first time use a new class *WRP* of functions in the recent construction method proposed in [13,15,26]. In conclusion, the main results of the paper are given as follows.

- We construct new infinite classes of 3-weight and 4-weight linear codes from the class *WRP* of plateaued functions over $\mathbb{F}_p$. To find the Hamming weights, we benefit from the exponential sums and Walsh spectrum of the employed functions $f, g \in WRP$. To determine the weight distributions, we use the exponential sums and Walsh distributions of $f, g \in WRP$ as well as the numbers of the pre-images of the associated functions $f^\star$ and $g^\star$ on the Walsh supports $\mathcal{S}_f$ and $\mathcal{S}_g$.
- We derive 3-weight punctured codes from the constructed codes, by deleting some special coordinates in the defining set. Note that they contain almost optimal codes due to the Griesmer bound.
- We show that our obtained codes are minimal, which says that they can be used to design high democratic secret sharing schemes with new parameters under the framework introduced in [9, Proposition 2].
- We lastly consider the minimum distances of the dual codes of our minimal codes. We conclude that the proposed 3-weight punctured codes are projective, which approves that they can be used to design association schemes in [3].

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5 Appendix

The appendix lists the Hamming weights with their weight distributions for the proposed minimal codes in this paper. For simplicity, we denote $m = 2n + s_f + s_g$ and $r = 2n - s_f - s_g$, where $0 \leq s_f, s_g < n$, in the following tables.

Table 1. The code $\mathcal{C}_\mathcal{D}$ in Theorem 1 when $n + s_g$ is even

Hamming weight ω	Multiplicity A_ω
0	1
$(p-1)p^{2n-2}$	$p^{2n} - (p-1)p^{n-s_g} - 1$
$(p-1)(p^{2n-2} - \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g})$	$(p-1)(p^{n-s_g-1} + \epsilon_g \eta_0^{n+1}(-1)(p-1)\sqrt{p^*}^{\,n-s_g-2})$
$(p-1)p^{2n-2} + \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g}$	$(p-1)^2(p^{n-s_g-1} - \epsilon_g \eta_0^{n+1}(-1)\sqrt{p^*}^{\,n-s_g-2})$

Table 2. The code $\mathcal{C}_\mathcal{D}$ in Theorem 1 when $n + s_g$ is odd

Hamming weight ω	Multiplicity A_ω
0	1
$(p-1)p^{2n-2}$	$p^{2n} - (p-1)^2 p^{n-s_g-1} - 1$
$(p-1)p^{2n-2} - \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g+1}$	$\frac{(p-1)^2}{2}(p^{n-s_g-1} + \epsilon_g \eta_0^{n}(-1)\sqrt{p^*}^{\,n-s_g-1})$
$(p-1)p^{2n-2} + \epsilon_g p^{n-2}\sqrt{p^*}^{\,n+s_g+1}$	$\frac{(p-1)^2}{2}(p^{n-s_g-1} - \epsilon_g \eta_0^{n}(-1)\sqrt{p^*}^{\,n-s_g-1})$

Table 3. The code $\mathcal{C}_\mathcal{D}$ in Theorem 2 when $n + s_f$ is odd and $n + s_g$ is even

Hamming weight ω	Multiplicity A_ω
0	1
$(p-1)p^{2n-2}$	$p^{2n} - p^r + p^{r-1} - 1$
$(p-1)(p^{2n-2} - \epsilon_f\epsilon_g\sqrt{p^*}^{m-3})$	$\frac{(p-1)}{2}(p^{r-1} + \epsilon_f\epsilon_g\eta_0(-1)\frac{1}{p}\sqrt{p^*}^{\,r+1})$
$(p-1)(p^{2n-2} + \epsilon_f\epsilon_g\sqrt{p^*}^{m-3})$	$\frac{(p-1)}{2}(p^{r-1} - \epsilon_f\epsilon_g\eta_0(-1)\frac{1}{p}\sqrt{p^*}^{\,r+1})$

Table 4. The code $\mathcal{C}_\mathcal{D}$ in Theorem 3 when m is even and $l_f = l_g$

Hamming weight ω	Multiplicity A_ω
0	1
$(p-1)(p^{2n-2} + \epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{m-4})$	$p^{2n} - p^r$
$(p-1)p^{2n-2}$	$p^{r-1} + \epsilon_f\epsilon_g\frac{1}{p}(p-1)\sqrt{p^*}^{\,r} - 1$
$(p-1)(p^{2n-2} + \epsilon_f\epsilon_g\frac{1}{p}\sqrt{p^*}^{m})$	$(p-1)(p^{r-1} - \epsilon_f\epsilon_g\frac{1}{p}\sqrt{p^*}^{\,r})$

Table 5. The code $\mathcal{C}_{\mathcal{D}}$ in Theorem 3 when $p > 3$, m is even and $l_f \neq l_g$

Hamming weight ω	Multiplicity A_ω
0	1
$(p-1)(p^{2n-2} + \epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{m-4})$	$p^{2n} - p^r$
$(p-1)p^{2n-2}$	A_{ω_1}
$(p-1)p^{2n-2} + \epsilon_f\epsilon_g\frac{(p-3)}{p}\sqrt{p^*}^m$	A_{ω_2}
$(p-1)(p^{2n-2} + \epsilon_f\epsilon_g\frac{1}{p}\sqrt{p^*}^m)$	$p^r - 1 - A_{\omega_1} - A_{\omega_2}$

Table 6. The code $\mathcal{C}_{\overline{\mathcal{D}}}$ in Corollary 1 when m is odd and $k_f = k_g$

Hamming weight ω	Multiplicity A_ω
0	1
p^{2n-2}	$p^{2n} - p^r + p^{r-1} - 1$
$p^{2n-2} - \epsilon_f\epsilon_g\sqrt{p^*}^{m-3}$	$\frac{(p-1)}{2}(p^{r-1} + \epsilon_f\epsilon_g\eta_0(-1)\frac{1}{p}\sqrt{p^*}^{r+1})$
$p^{2n-2} + \epsilon_f\epsilon_g\sqrt{p^*}^{m-3}$	$\frac{(p-1)}{2}(p^{r-1} - \epsilon_f\epsilon_g\eta_0(-1)\frac{1}{p}\sqrt{p^*}^{r+1})$

Table 7. The code $\mathcal{C}_{\overline{\mathcal{D}}}$ in Corollary 2 when m is even, $k_f = k_g$ and $l_f = l_g$

Hamming weight ω	Multiplicity A_ω
0	1
$p^{2n-2} + \epsilon_f\epsilon_g(p-1)\sqrt{p^*}^{m-4}$	$p^{2n} - p^r$
p^{2n-2}	$p^{r-1} + \epsilon_f\epsilon_g\frac{1}{p}(p-1)\sqrt{p^*}^r - 1$
$p^{2n-2} + \epsilon_f\epsilon_g\frac{1}{p}\sqrt{p^*}^m$	$(p-1)(p^{r-1} - \epsilon_f\epsilon_g\frac{1}{p}\sqrt{p^*}^r)$