A Survey on Perfectly-Secure Verifiable Secret-Sharing

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Abstract

Verifiable Secret-Sharing (VSS) is a fundamental primitive in secure distributed computing. It is used as a building block in several distributed computing tasks, such as Byzantine agreement and secure multi-party computation. In this article, we consider VSS schemes with *perfect* security, tolerating *computationally unbounded* adversaries. We comprehensively survey the existing perfectly-secure VSS schemes in three different communication settings, namely synchronous, asynchronous and hybrid setting and provide full details of the existing schemes in these settings. The aim of this survey is to provide a clear knowledge and foundation to researchers who are interested in knowing and extending the state-of-the-art perfectly-secure VSS schemes.

1 Introduction

A central concept in cryptographic protocols is that of *Secret Sharing* (SS) [55, 14]. Consider a set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of mutually distrusting parties, where the distrust is modeled by a centralized *adversary*, who can control up to t parties. Then a SS scheme allows a designated *dealer* $\mathsf{D} \in \mathcal{P}$ to share a secret s among $\mathcal{P}$, by providing each P_i a *share* of the secret s. The sharing is done in such a way that the adversary controlling any subset of at most t share-holders fails to learn anything about s, while any subset of at least $t + 1$ share-holders can jointly recover s. In a SS scheme, it is assumed that *all* the parties including the ones under the adversary's control follow the protocol instructions correctly (thus, the adversary is assumed to only *eavesdrop* the computation and communication of the parties under its control). VSS [18] extends the notion of SS to the more powerful *malicious/active* adversarial model, where the adversary can completely dictate the behaviour of the parties under its control during a protocol execution. Moreover, D is allowed to be potentially corrupted. A VSS scheme consists of a sharing phase and a reconstruction phase, each implemented by a publicly-known protocol. During the sharing phase, D shares its secret in a *verifiable* fashion, which is later reconstructed during the reconstruction phase. If D is *honest*, then the privacy of its secret is maintained during the sharing phase and the shared secret is later robustly reconstructed, irrespective of the behaviour of the corrupt parties. The interesting property of VSS is the *verifiability* property, which guarantees that even if D is *corrupt*, it has "consistently/correctly" shared some value among the parties and the same value is later reconstructed. One can interpret VSS

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as a distributed commitment, where, during the sharing phase, D publicly commits to a private input (known only to D) and later during the reconstruction phase, the committed value is reconstructed publicly (even if D does not cooperate). Due to its central importance in secure distributed-computing tasks, such as secure *multi-party computation* (MPC) [12, 54] and Byzantine agreement (BA) [30], VSS has been studied in various settings, based on the following categorizations.

- **Conditional vs Unconditional Security**: If the adversary is *computationally bounded* (where it is allowed to perform only polynomial amount of computations), then the notion of security achieved is conditional/cryptographic [53, 5, 6], whereas unconditionally-secure VSS provides security even against a *computationally unbounded* adversary. Unconditionally-secure VSS can be further categorized as *perfectly-secure* where all security guarantees are achieved in an error-free fashion [12], and *statistically-secure* where a negligible error is allowed [54, 25, 40].
- **Type of Communication**: Here we have three categories. The *synchronous* model assumes that the parties are synchronized through a global clock and there are strict (publicly-known) upper bounds on the message delays [12, 54, 33, 32, 39, 3]. The second category is the *asynchronous* model [11, 13, 7, 49, 50, 24], where the parties are not synchronized and where the messages can be arbitrarily (but finitely) delayed. A major challenge in the asynchronous model is that a *slow* sender party (whose messages are delayed) cannot be distinguished from a *corrupt* sender who does not send messages at all. Due to this inherent phenomenon, asynchronous VSS (AVSS) protocols are more complicated than their synchronous counter-parts. The third category is the *hybrid* communication setting [51], which is a mix of the synchronous and asynchronous models. Namely, the protocol starts with a few initial synchronous rounds, followed by a completely asynchronous execution. The main motivation for considering a hybrid setting is to "bridge" the feasibility and efficiency gaps between completely synchronous and completely asynchronous protocols.
- **Corruption Capacity**: Most of the works on VSS assume a *threshold* adversary which can corrupt any subset of t parties. A *non-threshold* adversary [26, 46, 22, 23] is a more generalized adversary, where the corruption capacity of adversary is specified by a publicly-known *adversary structure*, which is a collection of potentially corrupt subsets of parties. During the protocol execution, the adversary can choose any subset from the collection for corruption.

Our Contributions and Paper Organization We provide a comprehensive survey of all the existing *perfectly-secure* VSS schemes tolerating a *threshold* adversary. We cover three communication settings, namely synchronous, asynchronous and hybrid. These schemes are designed over a period of three decades. The nuances, subtleties and foundational ideas involved in these works need a holistic and unified treatment, which is the focus of this work. This survey is structured to provide an easy digest of the perfectly-secure VSS schemes. Through this survey, we hope to provide a clear knowledge and foundation to researchers who are interested in knowing and extending the state-of-the-art perfectly-secure VSS schemes. The survey is divided into three parts, each dealing with a separate communication model. We do *not* survey SS schemes and their share sizes, for which the interested readers are referred to the survey by Beimel [10].

Part I : Synchronous Communication Setting

2 Preliminaries and Definitions

Throughout part I, we consider a *synchronous* communication setting, where the parties in $\mathcal{P}$ are connected by pair-wise private and authentic channels. The distrust in the system is modelled by a *computationally unbounded* adversary Adv, who can corrupt at most t parties during the execution of a protocol in a malicious/Byzantine fashion and force them to behave in any arbitrary manner. The parties under the control of Adv are called *corrupt/malicious*, while the parties not under Adv's control are called *honest*. We assume a *static* adversary, who decides the set of corrupt parties at the beginning of a protocol. However, following [41] the protocols discussed can be proved to be secure even against an *adaptive* adversary, who can corrupt parties as the protocol proceeds.

We also assume the presence of a *broadcast* channel, which allows any designated party to send some message identically to all the parties. A protocol in the synchronous setting operates as a sequence of *rounds*. In each round, a party can (privately) send messages to other parties and broadcast a message. Moreover in a given round, each party can simultaneously use the broadcast channel. The messages sent or broadcast by a party is determined by its input, random coins and the messages received from other parties in previous rounds. The *view* of a party during a protocol execution consists of its inputs, random coins and all the messages received by the party throughout the protocol execution. The *view* of Adv is the collection of the views of the corrupt parties.

Structure of a VSS Scheme. Following [33], VSS schemes can be structured into two phases. A *sharing phase* executed by a protocol Sh, followed by a *reconstruction phase* executed by a protocol Rec. While the goal of Sh is to share a secret held by a designated *dealer* $\mathsf{D} \in \mathcal{P}$, the aim of Rec is to reconstruct back the shared secret. Specifically, during Sh, the input of D is some secret $s \in \mathcal{S}$, where $\mathcal{S}$ is some publicly-known *secret-space* which is the set of all possible D's secrets. Additionally, the parties may have random inputs for the protocol. Let view_i denote the view of P_i at the end of Sh. Based on view_i, each P_i outputs a *share* s_i, as determined by Sh.

During Rec, each P_i reveals a subset of view_i, as per Rec. The parties then apply a reconstruction function on the revealed views, as per Rec and reconstruct some output. Following [39], we say that *round-complexity* of Sh (resp. Rec) is (R, R'), if Sh (resp. Rec) involves total R rounds and among these R rounds, the broadcast channel is used for R' rounds. By *communication complexity* of a protocol, we mean the total number of bits communicated by the honest parties in the protocol.

2.1 Definitions

A *t-out-of-n secret-sharing* (SS) scheme is a pair of functions (G, R). While G is probabilistic, R is deterministic. Function G generates shares for the input secret, while R maps the shares back to the secret. The shares are generated in such a way that the probability distribution of any set of t shares is independent of the secret, while any set of $t+1$ shares uniquely determines the secret.

Definition 2.1 (t-out-of-n secret-sharing [34]). It is a pair of algorithms (G, R), such that:

- **Syntax**: The *share-generation function* G takes input a secret s and some randomness q and outputs a vector of n shares $(s_1, \ldots, s_n)$. The *recovery function* R takes input a set of $t+1$ shares corresponding to $t+1$ indices $\{i_1, \ldots, i_{t+1}\} \subset \{1, \ldots, n\}$ and outputs a value.
- **Correctness**: For any $s \in \mathcal{S}$ and any vector $(s_1, \ldots, s_n)$ where $(s_1, \ldots, s_n) = \mathsf{G}(s, q)$ for some randomness q, the condition $\mathsf{R}(s_{i_1}, \ldots, s_{i_{t+1}}) = s$ holds for any subset $\{i_1, \ldots, i_{t+1}\} \subset \{1, \ldots, n\}$.
- **Privacy**: For any subset of t indices, the probability distribution of the shares corresponding to these indices is independent of the underlying secret. That is, for any $I = \{i_1, \ldots, i_t\} \subset \{1, \ldots, n\}$, let $g_I(s) \overset{def}{=} (s_{i_1}, \ldots, s_{i_t})$, where $(s_1, \ldots, s_n) = \mathsf{G}(s, q)$ for some randomness q. Then we require that for every index-set I where $|I| = t$, the random variables $g_I(s)$ and $g_I(s')$ are identically distributed, for every $s, s' \in \mathcal{S}$, where $s \neq s'$.

Definition 2.2. Let $\Pi = (\Pi_{\mathsf{G}}, \Pi_{\mathsf{R}})$ be a t-out-of-n secret-sharing scheme. Then we say that *a value s is secret-shared among $\mathcal{P}$ as per* Π, if there exists some randomness q such that $(s_1, \ldots, s_n) = \Pi_{\mathsf{G}}(s, q)$ and each *honest* party $P_i \in \mathcal{P}$ has the share s_i.

In the literature, two types of VSS schemes have been considered. The type-I VSS schemes are "weaker" compared to the type-II VSS schemes. Namely, in type-II VSS, it is *guaranteed* that the dealer's secret is secret-shared as per the semantics of some *specified* secret-sharing scheme[1] (for instance, say Shamir's SS [55]). While type-I VSS is sufficient to study VSS as a stand-alone primitive (for instance, to study the round complexity of VSS [32] or to design a BA protocol [16]), type-II VSS schemes are desirable when VSS is used as a primitive in secure MPC protocols [39, 4, 3].

Definition 2.3 (Type-I VSS [33]). Let $(\mathsf{Sh}, \mathsf{Rec})$ be a pair of protocols for the parties in $\mathcal{P}$, where a designated *dealer* $\mathsf{D} \in \mathcal{P}$ has some private input $s \in \mathcal{S}$ for the protocol Sh. Then $(\mathsf{Sh}, \mathsf{Rec})$ is called a *Type-I perfectly-secure VSS scheme*, if the following requirements hold.

- **Privacy**: If D is *honest*, then the view of Adv during Sh is distributed independent of s.
- **Correctness**: If D is *honest*, then all honest parties output s at the end of Rec.
- **Strong Commitment**: Even if D is *corrupt*, in any execution of Sh, the joint view of the honest parties defines a unique value $s^\star \in \mathcal{S}$ (which could be different from s), such that all honest parties output $s^\star$ at the end of Rec, irrespective of the behaviour of Adv.

Definition 2.4 (Type-II VSS [34]). Let $\Pi = (\Pi_{\mathsf{G}}, \Pi_{\mathsf{R}})$ be a t-out-of-n SS scheme. Then $(\mathsf{Sh}, \mathsf{Rec})$ is called a *Type-II perfectly-secure VSS scheme with respect to* Π, if the following requirements hold.

- **Privacy**: If D is *honest*, then the view of Adv during Sh is distributed independent of s.
- **Correctness**: If D is *honest*, then at the end of Sh, the value s is secret-shared among $\mathcal{P}$ as per Π (see Definition 2.2). Moreover, all honest parties output s at the end of Rec.
- **Strong Commitment**: Even if D is *corrupt*, in any execution of Sh the joint view of the honest parties defines some value $s^\star \in \mathcal{S}$, such that $s^\star$ is secret-shared among $\mathcal{P}$ as per Π (see Definition 2.2). Moreover, all honest parties output $s^\star$ at the end of Rec.

Alternative Definition of VSS Definition 2.3-2.4 are called "property-based" definition, where we enumerate a list of desired security goals. One can instead follow other definitional frameworks such as the the *ideal-world/real-world* paradigm of Canetti [17] or the *constructive-cryptography* paradigm of Liu-Zhang and Maurer [42]. Proving the security of VSS schemes as per these paradigm brings in additional technicalities in the proofs. Since our main goal is to survey the existing VSS protocols, we will stick to the property-based definitions, which are easy to follow.

2.2 Properties of Polynomials Over a Finite Field

Let $\mathbb{F}$ be a finite field where $|\mathbb{F}| > n$ with $\alpha_1, \ldots, \alpha_n$ be distinct non-zero elements of $\mathbb{F}$. A degree-d *univariate polynomial* over $\mathbb{F}$ is of the form $f(x) = a_0 + \ldots + a_d x^d$, where $a_i \in \mathbb{F}$. A degree-(ℓ, m) *bivariate polynomial* over $\mathbb{F}$ is of the form $F(x, y) = \sum_{i,j=0}^{i=\ell, j=m} r_{ij} x^i y^j$, where $r_{ij} \in \mathbb{F}$. The polynomials $f_i(x) \stackrel{def}{=} F(x, \alpha_i)$ and $g_i(y) \stackrel{def}{=} F(\alpha_i, y)$ are called the i^{th} *row* and *column-polynomial* respectively of $F(x, y)$ as evaluating $f_i(x)$ and $g_i(y)$ at $x = \alpha_1, \ldots, \alpha_n$ and at $y = \alpha_1, \ldots, \alpha_n$ respectively results in an $n \times n$ matrix of points on $F(x, y)$ (see Fig 1). Note that $f_i(\alpha_j) = g_j(\alpha_i) = F(\alpha_j, \alpha_i)$ holds for all α_i, α_j. We say a degree-m polynomial $F_i(x)$ (resp. a degree-ℓ polynomial $G_i(y)$), where $i \in \{1, \ldots, n\}$, *lies* on a degree-(ℓ, m) bivariate polynomial $F(x, y)$, if $F(x, \alpha_i) = F_i(x)$ (resp. $F(\alpha_i, y) = G_i(y)$) holds. $F(x, y)$ is called *symmetric*, if $r_{ij} = r_{ji}$ holds, implying $F(\alpha_j, \alpha_i) = F(\alpha_i, \alpha_j)$ and $F(x, \alpha_i) = F(\alpha_i, x)$.

Definition 2.5 (d-sharing [28, 8]). A value $s \in \mathbb{F}$ is said to be d-shared, if there exists a degree-d polynomial, say $q(\cdot)$, with $q(0) = s$, such that each (honest) $P_i \in \mathcal{P}$ holds its *share* $s_i \stackrel{def}{=} q(\alpha_i)$

[1] In Type-I VSS, the underlying secret *need not* be secret-shared as per the semantics of any t-out-of-n SS scheme.

(we interchangeably use the term *shares of* s and *shares of the polynomial* $q(\cdot)$ to denote the values $q(\alpha_i)$). The vector of shares of s corresponding to the (honest) parties in $\mathcal{P}$ is denoted as $[s]_d$. A set of values $S = (s^{(1)}, \ldots, s^{(L)}) \in \mathbb{F}^L$ is said to be d-shared, if each $s^{(i)} \in S$ is d-shared.

2.2.1 Properties of Univariate Polynomials Over $\mathbb{F}$

Most of the type-II VSS schemes are with respect to the Shamir's t-out-of-n SS scheme $(\mathsf{Sha}_\mathsf{G}, \mathsf{Sha}_\mathsf{R})$ [55]. Algorithm Sha_G takes input a secret $s \in \mathbb{F}$. To compute the shares, it picks a *Shamir-sharing* polynomial $q(\cdot)$ randomly from the set $\mathcal{P}^{s,t}$ of all degree-t univariate polynomials over $\mathbb{F}$ whose constant term is s. The output of Sha_G is $(s_1, \ldots, s_n)$, where $s_i = q(\alpha_i)$. Since $q(\cdot)$ is chosen randomly, the probability distribution of the t shares learnt by Adv will be independent of the underlying secret. Formally:

Lemma 2.6 ([4])**.** *For any set of distinct non-zero elements $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, any pair of values $s, s' \in \mathbb{F}$, any subset $I \subset \{1, \ldots, n\}$ where $|I| = \ell \leq t$, and every $\vec{y} \in \mathbb{F}^\ell$, it holds that:*

$$\underset{f(x) \in_r \mathcal{P}^{s,t}}{Pr}\Big[\vec{y} = (\{f(\alpha_i)\}_{i \in I})\Big] = \underset{g(x) \in_r \mathcal{P}^{s',t}}{Pr}\Big[\vec{y} = (\{g(\alpha_i)\}_{I \in I})\Big],$$

where $f(x)$ and $g(x)$ are chosen randomly (denoted by the notation $\in_r$) from $\mathcal{P}^{s,t}$ and $\mathcal{P}^{s',t}$, respectively.

Let $(s_1, \ldots, s_n)$ be a vector of Shamir-shares for s, generated by Sha_G. Moreover, let $I \subset \{1, \ldots, n\}$, where $|I| = t + 1$. Then Sha_R takes input the shares $\{s_i\}_{i \in I}$ and outputs s by interpolating the unique degree-t Shamir-sharing polynomial passing through the points $\{(\alpha_i, s_i)\}_{i \in I}$.

Relationship Between d-sharing and Reed-Solomon (RS) Codes Let s be d-shared through a polynomial $q(\cdot)$ and let $(s_1, \ldots, s_n)$ be the vector of shares. Moreover, let W be a subset of these shares, such that it is ensured that at most r shares in W are incorrect (the exact identity of the incorrect shares are not known). The goal is to error-correct the incorrect shares in W and correctly reconstruct back the polynomial $q(\cdot)$. Coding-theory [47] says that this is possible if and only if $|W| \geq d + 2r + 1$ and the corresponding algorithm is denoted by $\mathsf{RS\text{-}Dec}(d, r, W)$. There are several well-known efficient instantiations of $\mathsf{RS\text{-}Dec}$, such as the Berlekamp-Welch algorithm [44].

2.2.2 Properties of Bivariate Polynomials Over $\mathbb{F}$

There always exists a unique degree-d univariate polynomial, passing through $d + 1$ distinct points. A generalization of this result for bivariate polynomials is that if there are "sufficiently many" univariate polynomials which are "pair-wise consistent", then together they lie on a unique bivariate polynomial. Formally:

Lemma 2.7 (**Pair-wise Consistency Lemma [16, 50, 4]**)**.** *Let $\{f_{i_1}(x), \ldots, f_{i_q}(x)\}$ and $\{g_{j_1}(y), \ldots, g_{j_r}(y)\}$ be degree-ℓ and degree-m polynomials respectively where $q \geq m + 1, r \geq \ell + 1$ and where $i_1, \ldots, i_q, j_1, \ldots, j_r \in \{1, \ldots, n\}$. Moreover, let for every $i \in \{i_1, \ldots, i_q\}$ and every $j \in \{j_1, \ldots, j_r\}$, the condition $f_i(\alpha_j) = g_j(\alpha_i)$ holds. Then there exists a unique degree-(ℓ, m) bivariate polynomial $F^\star(x, y)$, such that the polynomials $f_{i_1}(x), \ldots, f_{i_q}(x)$ and $g_{j_1}(y), \ldots, g_{j_r}(y)$ lie on $F^\star(x, y)$.*

In type-II VSS schemes based on Shamir's SS scheme, D on having input s first picks a random degree-t Shamir-sharing polynomial $q(\cdot) \in \mathcal{P}^{s,t}$ and then embeds $q(\cdot)$ into a random degree-(t, t) bivariate polynomial $F(x, y)$ at $x = 0$. Each P_i then receives $f_i(x) = F(x, \alpha_i)$ and $g_i(y) = F(\alpha_i, y)$ from D. Similar to Shamir SS, Adv by learning at most t row and column-polynomials, does not learn s. Intuitively, this is because $(t+1)^2$ distinct values are required to uniquely determine $F(x, y)$, but Adv learns at most $t^2 + 2t$ distinct values. In fact, it can be shown that for every two degree-t polynomials $q_1(\cdot), q_2(\cdot)$ such that $q_1(\alpha_i) = q_2(\alpha_i) = f_i(0)$ holds for every $P_i \in \mathcal{C}$ (where $\mathcal{C}$ is the set of corrupt parties), the distribution of the polynomials $\{f_i(x), g_i(y)\}_{P_i \in \mathcal{C}}$ when $F(x, y)$ is chosen based on $q_1(\cdot)$, is identical to the distribution when $F(x, y)$ is chosen based on $q_2(\cdot)$. Formally:

Lemma 2.8 ([4])**.** *Let $\mathcal{C} \subset \mathcal{P}$ where $|\mathcal{C}| \leq t$, and let $q_1(\cdot) \neq q_2(\cdot)$ be degree-t polynomials where $q_1(\alpha_i) = q_2(\alpha_i)$ holds for all $P_i \in \mathcal{C}$. Then the probability distributions $\Big\{\{F(x, \alpha_i), F(\alpha_i, y)\}_{P_i \in \mathcal{C}}\Big\}$ and $\Big\{\{F'(x, \alpha_i), F'(\alpha_i, y)\}_{P_i \in \mathcal{C}}\Big\}$ are identical, where $F(x, y) \neq F'(x, y)$ are different degree-(t, t) bivariate polynomials, chosen at random, under the constraints that $F(0, y) = q_1(\cdot)$ and $F'(0, y) = q_2(\cdot)$ holds.*

3 Lower Bounds

In *any* perfectly-secure VSS scheme, the joint view of the *honest* parties should uniquely determine the dealer's secret. Otherwise the *correctness* property will be violated if the corrupt parties produce incorrect view during the reconstruction phase. Since there can be only $n - t$ *honest* parties, to satisfy the *privacy* property, the condition $n - t > t$ should necessarily hold, as otherwise the view of the adversary will not be independent of the dealer's secret. We actually need a stricter necessary condition of $n > 3t$ to hold for *any* perfectly-secure VSS scheme, as stated in the following theorem.

Theorem 3.1 ([29])**.** *Let $\Pi = (\mathsf{Sh}, \mathsf{Rec})$ be a perfectly-secure VSS scheme. Then $n > 3t$ holds.*

Theorem 3.1 is first proved formally by Dolev, Dwork, Waarts and Yung in [29] by relating VSS with the problem of 1-*way perfectly-secure message transmission* (1-way PSMT) [29]. In the 1-way PSMT problem, there is a sender $\mathbf{S}$ and a receiver $\mathbf{R}$, such that there are n *disjoint uni-directional* communication channels $Ch_1, \ldots, Ch_n$ from $\mathbf{S}$ to $\mathbf{R}$ (i.e. only $\mathbf{S}$ can send messages to $\mathbf{R}$ along these channels). At most t out of these channels can be controlled by a *computationally unbounded* malicious/Byzantine adversary in any arbitrary fashion. The goal is to design a protocol, which allows $\mathbf{S}$ to send some input message m *reliably* (i.e. $\mathbf{R}$ should be able to receive m without any error) and *privately* (i.e. view of the adversary should be independent of m) to $\mathbf{R}$. In [29], it is shown that a 1-way PSMT protocol exists only if $n > 3t$. Moreover, if there exists a perfectly-secure VSS scheme $(\mathsf{Sh}, \mathsf{Rec})$ with $n \leq 3t$, then one can design a 1-way PSMT protocol with $n \leq 3t$, which is a contradiction. On a very high level, the reduction from 1-way PSMT to VSS can be shown as follows: $\mathbf{S}$ on having a message m, acts as a dealer and runs an instance of Sh with input m by playing the role of the parties $P_1, \ldots, P_n$ as per the protocol Sh. Let view_i be the view generated for P_i, which $\mathbf{S}$ communicates to $\mathbf{R}$ over Ch_i. Let $\mathbf{R}$ receives view'_i over Ch_i, where $\mathsf{view}'_i = \mathsf{view}_i$ if Ch_i is *not* under adversary's control. To recover m, $\mathbf{R}$ applies the reconstruction function as per Rec on $(\mathsf{view}'_1, \ldots, \mathsf{view}'_n)$. It is easy to see that the *correctness* of the VSS scheme implies that $\mathbf{R}$ correctly recovers m, while the *privacy* of the VSS scheme guarantees that the view of any adversary controlling at most t channels remains independent of m.

An alternative argument for the requirement of $n > 3t$ for perfect VSS follows from its reduction to a perfectly-secure *reliable-broadcast* (RB) protocol over the pair-wise channels, for which $n > 3t$ is required [52]. This is elaborated further later in the context of usage of a broadcast channel for designing VSS protocols (the paragraph entitled "On the Usage of Broadcast Channel" after Lemma 3.4 and Footnote 2).

The Round Complexity of VSS In Genarro, Ishai, Kushilevitz and Rabin [33], the round-complexity of a VSS scheme is defined to be the number of rounds in the sharing phase, as all (perfectly-secure) VSS schemes adhere to a single-round reconstruction. The interplay between the round-complexity of perfectly-secure VSS and resilience bounds is stated below.

Theorem 3.2 ([33]). *Let $R \geq 1$ be a positive integer and let $R' \leq R$. Then:*

- *If $R = 1$, then there exists no perfectly-secure VSS scheme with (R, R') rounds in the sharing phase if either $t > 1$ (irrespective of the value of n) or if ($t = 1$ and $n \leq 4$).*
- *If $R = 2$, then perfectly-secure VSS with (R, R') rounds in sharing phase is possible only if $n > 4t$.*
- *If $R \geq 3$, then perfectly-secure VSS with (R, R') rounds in sharing phase is possible only if $n > 3t$.*

We give a very high level overview of the proof of Theorem 3.2. We focus only on 2 and R-round sharing phase VSS schemes where $R \geq 3$, as one can use standard hybrid arguments to derive the bounds related to VSS schemes with 1-round sharing phase (see for instance [48]). The lower bound for 2-round VSS schemes (namely $n > 4t$) is derived by relating VSS to the *secure multi-cast* (SM) problem. In the SM problem, there exists a designated *sender* $\mathsf{S} \in \mathcal{P}$ with some private message m and a designated *receiving set* $\mathcal{R} \subseteq \mathcal{P}$, where $\mathsf{S} \in \mathcal{R}$ and where $|\mathcal{R}| > 2$. The goal is to design a protocol which allows S to send its message identically to all the parties *only* in $\mathcal{R}$, even in the presence of an adversary who can control any t parties, possibly including S. Moreover, if all the parties in $\mathcal{R}$ are *honest*, then the view of the adversary should be independent of m. Genarro et al. [33] establishes the following relationship between VSS and SM.

Lemma 3.3 ([33]). *Let $(\mathsf{Sh}, \mathsf{Rec})$ be a VSS scheme with a k-round sharing phase where $k \geq 2$. Then there exists a k-round SM protocol.*

The proof of Lemma 3.3 proceeds in two steps.

- It is first shown that for any R-round protocol Π, there exists an R-round protocol Π' with the same security guarantees as Π, such that all the messages in rounds $2, \ldots, R$ of Π' are broadcast messages. The idea is to let each P_i exchange a "sufficiently-large" random pad with every P_j apriori during the first round. Then in any subsequent round of Π, if P_i is supposed to *privately* send v_{ij} to P_j, then in Π', party P_i instead broadcasts v_{ij} being masked with appropriate pad, exchanged with P_j. Since P_j is supposed to hold the same pad, it can unmask the broadcasted value and recover v_{ij} and process it as per Π.
- Let $(\mathsf{Sh}, \mathsf{Rec})$ be a VSS scheme where Sh requires k rounds such that $k \geq 2$. Using the previous implication, we can assume that all the messages during the *final* round of Sh are broadcast messages. Using $(\mathsf{Sh}, \mathsf{Rec})$, one can get a k-round SM protocol as follows: the parties invoke an instance of Sh, with S playing the role of D with input m. In the final round, apart from the messages broadcasted by the parties as part of Sh, every party P_i *privately* sends its entire view of the first $k-1$ rounds during Sh to every party in $\mathcal{R}$. Based on this, every party in $\mathcal{R}$ will get the view of all the parties in $\mathcal{P}$ for all the k rounds of Sh. Intuitively, this also gives every (honest) party in $\mathcal{R}$ the share of every (honest) party from Sh. Every party in $\mathcal{R}$ then applies the reconstruction function on the n extrapolated views as per Rec and outputs the result. It is easy to see that the *correctness* of the VSS implies that if S is *honest*, then all the honest parties in $\mathcal{R}$ obtains m. Moreover, the *privacy* of the VSS scheme implies that if $\mathcal{R}$ contains only honest parties, then the view of the adversary remains independent of m. Finally, the *strong commitment* of the VSS guarantees that even if S is *corrupt*, all *honest* parties in $\mathcal{R}$ obtain the same output.

Next, Genarro et al. [33] shows the impossibility of any 2-round SM protocol with $n \leq 4t$.

Lemma 3.4. *There exists no 2-round SM protocol where $n \leq 4t$.*

By a standard *player-partitioning* argument [43], Lemma 3.4 reduces to showing the impossibility of any 2-round SM protocol with $n = 4$ and $t = 1$. At a high-level, the player-partitioning argument goes as follows: if there exists a 2-round SM protocol Π with $n = 4t$, then one can also design a 2-round SM protocol Π' for 4 parties, where each of the 4 parties in Π' plays the role of a disjoint set of t parties as per Π. However, one can show that there does not exist any 2-round SM protocol with $n = 4$ and $t = 1$, thus ruling out the existence of any 2-round SM protocol with $n \leq 4t$. We refer the interested readers to [33] for the proof of *non-existence* of any 2-round SM protocol with $n = 4$ and $t = 1$. Now combining Lemma 3.3 with Lemma 3.4, we get that there exists no VSS scheme with $n \leq 4t$ and a 2-round sharing phase. Since $n > 3t$ is necessary for *any* VSS scheme, this automatically implies that a VSS scheme with 3 or more rounds of sharing phase will necessarily require $n > 3t$.

On the Usage of Broadcast Channel All *synchronous* VSS schemes assume the existence of a broadcast channel. However, this is just a *simplifying abstraction*, as the parties can "emulate" the effect of a broadcast channel by executing a perfectly-secure *reliable-broadcast* (RB) protocol over the pair-wise channels, provided $n > 3t$ holds [52]. RB protocols with *guaranteed termination* in the presence of malicious/Byzantine adversaries require $\Omega(t)$ rounds of communication [31], while the RB protocols with *probabilistic termination* guarantees require $\mathcal{O}(1)$ expected number of rounds [30, 38] where the constants are high. Given the fact that the usage of broadcast channel is an "expensive resource", a natural question is whether one can design a VSS scheme with a *constant* number of rounds in the sharing phase and which *does not* require the usage of broadcast channel in any of these rounds. Unfortunately, the answer is no. This is because such a VSS scheme will imply the existence of a strict constant round RB protocol with *guaranteed termination* in the presence of a *malicious* adversary (the message to be broadcast by the sender can be shared using the VSS scheme with sender playing the role of the dealer, followed by reconstructing the shared message), which is *impossible* as per the result of[2] [31]. Hence the best that one can hope for is to design VSS schemes which invoke the broadcast channel only in a fewer rounds.

4 Upper Bounds

We now discuss the optimality of the bounds in Theorem 3.2 by presenting VSS schemes with various round-complexities. The sharing phase of these schemes are summarized in Table 1 and their reconstruction phase require one round. The prefix in the names of the schemes denotes the number of rounds in the sharing phase. In the table, $\mathbb{G}$ denotes a finite group and RSS stands for replicated secret-sharing [37] (see Section 4.1.4). The 3AKP-VSS scheme has some special properties, compared to 3FGGRS-VSS, 3KKK-VSS schemes, which are useful for designing round-optimal perfectly-secure MPC protocols (see Section 4.1.7).

[2] This reduction from RB to VSS is another way to argue the necessity of the $n > 3t$ condition for VSS, since the necessary condition for RB is $n > 3t$ [52].

Scheme	n	Round Complexity	Type	Sharing Semantic	Algebraic Structure	Communication Complexity
7BGW-VSS [12]	$n > 3t$	$(7, 5)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log \lvert\mathbb{F}\rvert))$
5BGW-VSS [33]	$n > 3t$	$(5, 3)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log \lvert\mathbb{F}\rvert))$
4GIKR-VSS [33]	$n > 3t$	$(4, 3)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log \lvert\mathbb{F}\rvert))$
3GIKR-VSS [33]	$n > 3t$	$(3, 2)$	Type-II	RSS	$\mathbb{G}$	$\mathcal{O}(n \cdot \binom{n}{t} \log \lvert\mathbb{G}\rvert + \mathcal{BC}(n \cdot \binom{n}{t} \log \lvert\mathbb{G}\rvert))$
3FGGRS-VSS [32]	$n > 3t$	$(3, 2)$	Type-I	Not Applicable	$\mathbb{F}$	$\mathcal{O}(n^3 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^3 \log \lvert\mathbb{F}\rvert))$
3KKK-VSS [39]	$n > 3t$	$(3, 1)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^3 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^3 \log \lvert\mathbb{F}\rvert))$
3AKP-VSS [3]	$n > 3t$	$(3, 2)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^3 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^3 \log \lvert\mathbb{F}\rvert))$
2GIKR-VSS [33]	$n > 4t$	$(2, 1)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log \lvert\mathbb{F}\rvert))$
1GIKR-VSS [33]	$n = 5, t = 1$	$(1, 0)$	Type-I	Not Applicable	$\mathbb{F}$	$\mathcal{O}(n \log \lvert\mathbb{F}\rvert)$

Table 1: Summary of the sharing phase of the perfectly-secure VSS schemes, with $\mathcal{BC}$ denoting communication over the broadcast channel.

4.1 VSS Schemes with $n > 3t$

We start with perfectly-secure VSS schemes with $n > 3t$. While presenting these schemes, we use the following simplifying conventions. If in a protocol a party is expecting some message from a sender party and if it either receives no message or semantically/syntactically incorrect message, then the receiving party substitutes some default value and proceeds with the steps of the protocol. Similarly, if the dealer is *publicly* identified to be cheating then the parties discard the dealer and terminate the protocol execution with a default sharing of 0.

4.1.1 The 7BGW-VSS Scheme

The scheme (Fig 2) consists of the protocols 7BGW-VSS-Sh and 7BGW-VSS-Rec. Protocol 7BGW-VSS-Sh is actually a simplified version of the original protocol, taken from [36]. To share s, the dealer D picks a random degree-t *Shamir-sharing polynomial* $q(\cdot)$ and the goal is to ensure that D *verifiably* distributes the Shamir-shares of s as per $q(\cdot)$. Later, during 7BGW-VSS-Rec, the parties exchange these Shamir-shares and reconstruct $q(\cdot)$ by error-correcting up to t incorrect shares using the algorithm RS-Dec. The verifiability in 7BGW-VSS-Sh ensures that even if D is *corrupt*, the shares distributed by D to the (honest) parties are as per some degree-t Shamir-sharing polynomial, say $q^\star(\cdot)$, thus ensuring that $s^\star \overset{def}{=} q^\star(0)$ is t-shared. Moreover, the same $s^\star$ is reconstructed during 7BGW-VSS-Rec. This ensures that the scheme is of type-II.

To prove that D is sharing its secret using a degree-t polynomial $q(\cdot)$, the dealer D embeds $q(\cdot)$ in a random degree-(t, t) bivariate polynomial $F(x, y)$. As shown in Fig 1, there are two approaches to do this embedding. The polynomial $q(\cdot)$ could be either embedded at $x = 0$ (the approach shown in part (a)) or it could be embedded at $y = 0$ (the approach shown in part (b)). For our description, we follow the first approach. The dealer then distributes distinct row and column-polynomials to respective parties. If D is *honest*, then this distribution of information maintains the privacy of dealer's secret (follows from Lemma 2.8). Also, if D is *honest*, then constant term of the individual row-polynomials actually constitute the Shamir-shares of s, as they constitute distinct points on $q(\cdot)$. However, a potentially *corrupt* D may distribute polynomials, which *may not* be derived from a single degree-(t, t) bivariate polynomial. Hence, the parties interact to verify if D has distributed "consistent" row and column-polynomials, *without* revealing any additional information about s.

Every pair of parties P_i, P_j upon receiving the polynomials $f_i(x), g_i(y)$ and $f_j(x), g_j(y)$ respectively, interact and check if $f_i(\alpha_j) = g_j(\alpha_i)$ and $f_j(\alpha_i) = g_i(\alpha_j)$ holds. If the checks pass for all the pairs of (honest) parties, then from Lemma 2.7, it follows that D has distributed consistent row (and column-polynomials) to the parties. However, if the checks do not pass, then either D has distributed inconsistent polynomials or the parties have not exchanged the correct common values. In this case, the parties interact publicly with D to resolve these inconsistencies. The details follow.

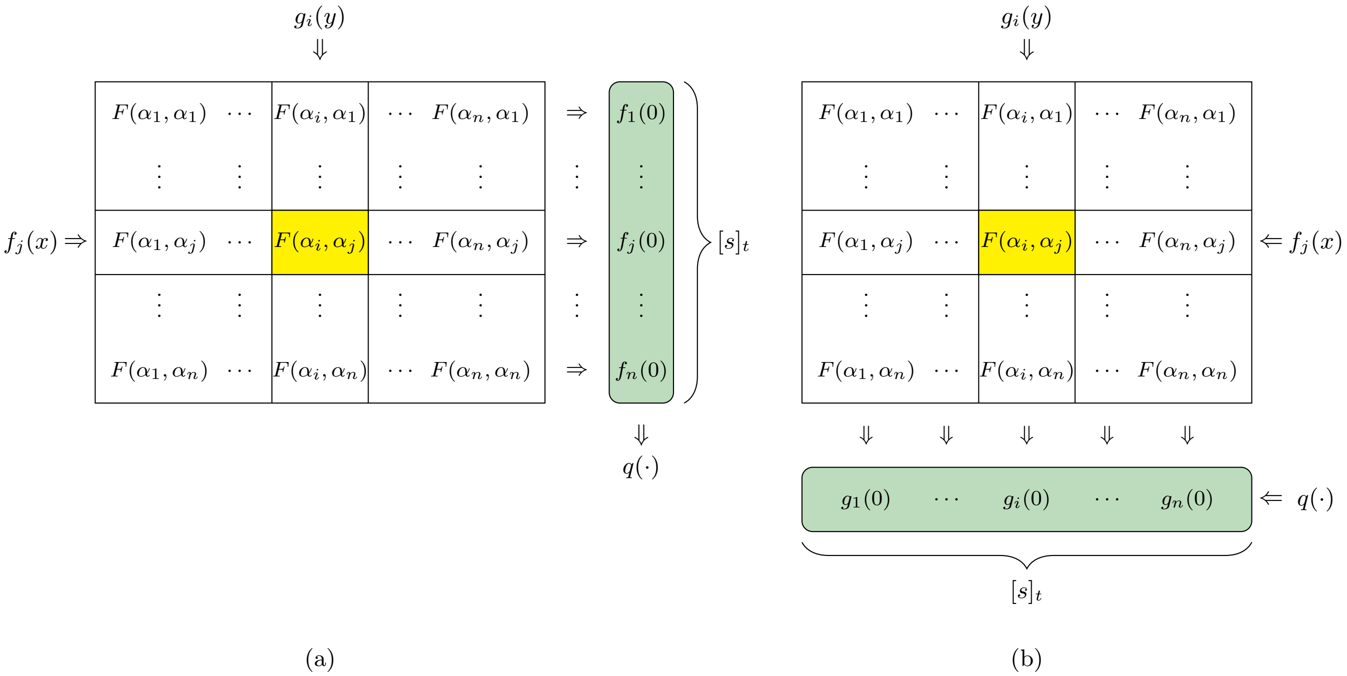

$g_i(y)$
$F(\alpha_1,\alpha_1)$ $\cdots$ $F(\alpha_i,\alpha_1)$ $\cdots$ $F(\alpha_n,\alpha_1)$
$f_j(x) \Rightarrow$
$F(\alpha_1,\alpha_j)$ $\cdots$ $F(\alpha_i,\alpha_j)$ $\cdots$ $F(\alpha_n,\alpha_j)$
$F(\alpha_1,\alpha_n)$ $\cdots$ $F(\alpha_i,\alpha_n)$ $\cdots$ $F(\alpha_n,\alpha_n)$
$f_1(0)$
$f_j(0)$
$f_n(0)$
$[s]_t$
$q(\cdot)$
(a)
$g_i(y)$
$F(\alpha_1,\alpha_1)$ $\cdots$ $F(\alpha_i,\alpha_1)$ $\cdots$ $F(\alpha_n,\alpha_1)$
$F(\alpha_1,\alpha_j)$ $\cdots$ $F(\alpha_i,\alpha_j)$ $\cdots$ $F(\alpha_n,\alpha_j)$
$\Leftarrow f_j(x)$
$F(\alpha_1,\alpha_n)$ $\cdots$ $F(\alpha_i,\alpha_n)$ $\cdots$ $F(\alpha_n,\alpha_n)$
$g_1(0)$ $\cdots$ $g_i(0)$ $\cdots$ $g_n(0)$
$\Leftarrow q(\cdot)$
$[s]_t$
(b)

Figure 1: Pictorial depiction of the values on the degree-(t,t) polynomial $F(x,y)$ distributed by D and how they constitute $[s]_t$. The value highlighted in the yellow color denotes a common value held by every pair of parties (P_i,P_j). In the first approach, the Shamir-sharing polynomial $q(\cdot)$ is set as $F(0,y)$ and the Shamir-shares are the constant terms of the individual row-polynomials. In the second approach, $q(\cdot)$ is set as $F(x,0)$ and the Shamir-shares are the constant terms of the individual column-polynomials.

Every P_i upon receiving the supposedly common values on its column polynomial prepares a *complaint-list* L_i, which includes all the parties P_j whose received value is inconsistent with P_i's column-polynomial (this is interpreted as if there is a dispute between P_i and P_j). If there is a dispute between P_i and P_j, then at least one of the three parties D, P_i, P_j is *corrupt*. Each party then broadcasts its complaint-list. In response, for every dispute reported by a party, the dealer D makes public its version of the disputed value, namely the corresponding value on its bivariate polynomial. This is followed by the first-stage accusations against the dealer. Namely, a party publicly "accuses" D, if the party is in dispute with more than t parties or if it finds D making public a value, which is not consistent with its column-polynomial. In response, D makes public the row and column-polynomials of such accusing parties. However, care has to taken to ensure that these broadcasted polynomials are consistent with the polynomials of the parties who have not yet accused D. This is done through the second-stage of public accusations against the dealer, where a party (who has not yet accused D) publicly accuses D, if it finds any inconsistency between the row and column-polynomials held by the party and the polynomials which were made public by D.

An *honest* D always respond correctly against any accusation or dispute. Moreover, there will be at most t accusations against D. Consequently, if the parties find D not responding to any accusation/dispute or if more than t parties accuse D, then D is *corrupt* and hence the parties discard D. If D is *honest*, then all the values which are made public correspond to corrupt parties (already known to Adv) and hence *does not* violate *privacy*. On the other hand, if D is *corrupt* but not discarded, then it ensures that the polynomials of all honest parties are consistent.

Scheme $\mathsf{7BGW\text{-}VSS}$

Sharing Phase: Protocol $\mathsf{7BGW\text{-}VSS\text{-}Sh}$

- **Round I (sending polynomials) — the dealer does the following**:
  - On having the input $s \in \mathbb{F}$, pick a random degree-t *Shamir-sharing polynomial* $q(\cdot)$, such that $q(0) = s$ holds. Then pick a random degree-(t,t) bivariate polynomial $F(x,y)$, such that $F(0,y) = q(\cdot)$ holds.
  - For $i = 1, \ldots, n$, send the polynomials $f_i(x) \stackrel{def}{=} F(x, \alpha_i)$ and $g_i(y) \stackrel{def}{=} F(\alpha_i, y)$ to party P_i.
- **Round II (pair-wise consistency checks) — each party P_i does the following**:
  - On receiving degree-t polynomials $f_i(x), g_i(y)$ from D, send $f_{ij} \stackrel{def}{=} f_i(\alpha_j)$ to P_j, for $j = 1, \ldots, n$.
- **Round III (broadcast complaints) — each party P_i does the following**:
  - Initialize a *complaint-list* L_i to $\emptyset$. For $j = 1, \ldots, n$, include P_j to L_i, if $f_{ji} \neq g_i(\alpha_j)$. Broadcast L_i.
- **Round IV (resolving complaints) — the dealer does the following**:
  - For $i = 1, \ldots, n$, if P_i has broadcast $L_i \neq \emptyset$, then for every $P_j \in L_i$, broadcast the value $F(\alpha_i, \alpha_j)$.
- **Round V (first-stage accusations) — each party P_i does the following**:
  - Broadcast the message $(P_i, \mathtt{accuse}, \mathsf{D})$, if any of the following conditions hold.
    1. $|L_i| > t$ or if $P_i \in L_i$;
    2. If $\exists k \in \{1, \ldots, n\}$, such that $P_i \in L_k$ and $F(\alpha_k, \alpha_i) \neq f_i(\alpha_k)$;
    3. If for any $P_j \in L_i$, the condition $F(\alpha_i, \alpha_j) \neq g_i(\alpha_j)$ holds.
- **Round VI (resolving first-stage accusations) — the dealer does the following**:
  - For every P_i who has broadcast $(P_i, \mathtt{accuse}, \mathsf{D})$, broadcast the degree-$t$ polynomials $f_i(x)$ and $g_i(y)$.
- **Round VII (second-stage accusations) — each party P_i does the following**:
  - Broadcast the message $(P_i, \mathtt{accuse}, \mathsf{D})$ if there exists any $j \in \{1, \ldots, n\}$ such that D has broadcast degree-t polynomials $f_j(x), g_j(y)$ and either $f_j(\alpha_i) \neq g_i(\alpha_j)$ or $g_j(\alpha_i) \neq f_i(\alpha_j)$ holds.
- **Output decision — each party P_i does the following**:
  - If more than t parties P_j broadcast $(P_j, \mathtt{accuse}, \mathsf{D})$ throughout the protocol, then discard D.
  - Else output the *share*[a] $s_i = f_i(0)$.

Reconstruction Phase: Protocol $\mathsf{7BGW\text{-}VSS\text{-}Rec}$

Each party P_i sends the share s_i to every party $P_j \in \mathcal{P}$. Let W_i be the set of shares received by P_i from the parties. Party P_i executes $\mathsf{RS\text{-}Dec}(t, t, W_i)$ to reconstruct s.

[a] If D has broadcast new polynomials $f_i(x), g_i(y)$ for P_i during Round VI, then consider these new polynomials.

Figure 2: The perfectly-secure VSS scheme of Ben-Or, Goldwasser and Wigderson [12].

Since the idea of bivariate polynomials has been used in all the followup works on perfectly-secure VSS, we give a very high level overview of the proof of the properties of $\mathsf{7BGW\text{-}VSS}$ scheme.

Theorem 4.1. *Protocols* $(\mathsf{7BGW\text{-}VSS\text{-}Sh}, \mathsf{7BGW\text{-}VSS\text{-}Rec})$ *constitute a Type-II perfectly-secure VSS scheme with respect to Shamir's t-out-of-n secret-sharing scheme. The protocol incurs a communication of $\mathcal{O}(n^2 \log |\mathbb{F}|)$ bits over the point-to-point channels and broadcast of $\mathcal{O}(n^2 \log |\mathbb{F}|)$ bits.*

Proof. If D is *honest*, then $f_i(\alpha_j) = g_j(\alpha_i)$ and $f_j(\alpha_i) = g_i(\alpha_j)$ holds for every pair of parties (P_i, P_j). Consequently, no honest P_j will be present in the list L_i of any honest P_i. Moreover, D honestly resolves the first-stage accusations as well as second-stage accusations and consequently, no honest party accuses and discards D. Hence s will be t-shared through the degree-t polynomial $q(\cdot)$. Moreover, during $\mathsf{7BGW\text{-}VSS\text{-}Rec}$, the honest parties correctly reconstruct $q(\cdot)$ and hence s. This follows from the properties of $\mathsf{RS\text{-}Dec}$ and the fact that $q(\cdot)$ is a degree-t polynomial and at most t corrupt parties can send incorrect shares. Hence, the *correctness* property is guaranteed.

Let $\mathcal{C}$ be the set of corrupt parties. If D is *honest*, then throughout 7BGW-VSS-Sh, the view of Adv consists of $\{f_i(x), g_i(y)\}_{P_i \in \mathcal{C}}$. Moreover, no honest party accuses D and hence all the information which D makes public can be derived from $\{f_i(x), g_i(y)\}_{P_i \in \mathcal{C}}$. Now since these polynomials are derived from $F(x, y)$, which is a randomly chosen polynomial embedding $q(\cdot)$, it follows from Lemma 2.8 that the view of Adv is distributed independently of s, thus guaranteeing *privacy*.

For *strong commitment* we have to consider a *corrupt* D. If the honest parties discard D, then clearly the value $s^\star \overset{def}{=} 0$ will be t-shared and the same value 0 gets reconstructed during 7BGW-VSS-Rec. On the other hand, consider the case when the honest parties do not discard D during 7BGW-VSS-Sh. In this case we claim that the row and column-polynomials of all the *honest* parties are derived from a single degree-(t, t) bivariate polynomial, say $F^\star(x, y)$, which we call as D's committed bivariate polynomial. Consequently, $s^\star \overset{def}{=} F^\star(0, 0)$ will be t-shared through the Shamir-sharing polynomial $q^\star(\cdot) \overset{def}{=} F^\star(0, y)$ and $s^\star$ gets reconstructed during 7BGW-VSS-Rec.

To prove the claim, we first note that there are at least $n - t \geq 2t + 1$ parties who do not broadcast an accuse message against D (as otherwise D is discarded). Let $\mathcal{H}$ be the set of *honest* parties among these $n - t$ parties. It is easy to see that $|\mathcal{H}| \geq n - 2t \geq t + 1$. The parties in $\mathcal{H}$ receive degree-t row and column-polynomials from D. Moreover, for every $P_i, P_j \in \mathcal{H}$, their row and column-polynomials are pair-wise consistent (as otherwise either P_i or P_j would broadcast an accuse message against D). It then follows from Lemma 2.7 that the row and column-polynomials of all the parties in $\mathcal{H}$ lie on a single degree-(t, t) bivariate polynomial, say $F^\star(x, y)$. Next consider any *honest* party $P_j \notin \mathcal{H}$, who broadcasts an accuse message against D and corresponding to which D makes public the row and column-polynomials of P_j. To complete the proof of the claim, we need to show that these polynomials also lie on $F^\star(x, y)$. We show it for the row-polynomial of P_j and a similar argument can be used for the column-polynomial as well. So let $f_j(x)$ be the degree-t row-polynomial broadcast by D for P_j. It follows that $f_j(\alpha_i) = g_i(\alpha_j)$ holds for *every* $P_i \in \mathcal{H}$, where $g_i(y)$ is the degree-t column polynomial held by P_i (otherwise P_i would have broadcast an accuse message against D). Now $g_i(y) = F^\star(\alpha_i, y)$ holds. Moreover, since $|\mathcal{H}| \geq t + 1$, the distinct points $\{(\alpha_j, g_i(\alpha_j))\}_{P_i \in \mathcal{H}}$ uniquely determine the degree-t polynomial $F^\star(x, \alpha_j)$. This implies that $f_j(x) = F^\star(x, \alpha_j)$, as two different degree-t polynomials can have at most t common points.

In the protocol, D sends two degree-t polynomials to each party and every pair of parties exchange 2 common values, which requires a communication of $\mathcal{O}(n^2)$ field elements. There could be at most t parties corresponding to which D makes public their polynomials and this requires a broadcast of $\mathcal{O}(n^2)$ field elements. □

4.1.2 A 5-round Version of 7BGW-VSS-Sh

Protocol 7BGW-VSS-Sh follows the "share-complaint-resolve" paradigm, where D first distributes the information on its bivariate polynomial, followed by parties complaining about any "inconsistency", which is followed by D resolving these inconsistencies. At the end, either all (honest) parties held consistent polynomials derived from a single degree-(t, t) bivariate polynomial or D is discarded. The "complaint" and "resolve" phases of 7BGW-VSS-Sh occupied *five* rounds. In Gennaro, Ishai, Kushilevitz and Rabin [33], the authors proposed a *round-reducing* technique, which collapses these phases to *three* rounds, thus reducing the overall number of rounds to five. The modified protocol 5BGW-VSS-Sh is presented in Fig 3.

The high level idea of 5BGW-VSS-Sh is as follows. In 7BGW-VSS-Sh, during the third round, P_i broadcasts *only* the identity of the parties with which it has a dispute (through L_i), followed by D making the corresponding disputed values public during Round IV, which is further followed by P_i accusing D during Round V, if P_i finds D's version to mis-match with P_i's version. Let us call P_i to be *unhappy*, if it accuses D during Round V. The round-reducing technique of [33] enables to identify the set of unhappy parties $\mathcal{UH}$ by the end of Round IV as follows. During Round III, apart from broadcasting the list of disputed parties, party P_i also makes public its version of the corresponding disputed values. In response, both D and the corresponding complainee party makes public their respective version of the disputed value. Now based on whether D's version matches the complainant's version or complainee's version, the parties can identify the set $\mathcal{UH}$.

In 7BGW-VSS-Sh, once $\mathcal{UH}$ is identified, D makes public the polynomials of the parties in $\mathcal{UH}$ during Round VI. And to verify if D made public the correct polynomials, during Round VII, the parties *not in* $\mathcal{UH}$ raise accusations against D, if they find any inconsistency between the polynomials held by them and the polynomials made public by D. The round-reducing technique of [33] collapses these two rounds into a single round. Namely, once $\mathcal{UH}$ is decided, D makes public the row-polynomials of these parties. In *parallel*, the parties not in $\mathcal{UH}$ make public the corresponding supposedly common values on these row-polynomials. Now to check if D broadcasted correct row-polynomials, one just has to verify whether each broadcasted row-polynomial is pair-wise consistent with at least $2t + 1$ corresponding values, broadcasted by the parties *not* in $\mathcal{UH}$.

Protocol 5BGW-VSS-Sh

- **Round I** — D picks $F(x, y)$ as in 7BGW-VSS-Sh and distributes $f_i(x), g_i(y)$ lying on $F(x, y)$ to each P_i.
- **Round II** — As in 7BGW-VSS-Sh, each P_i upon receiving $f_i(x), g_i(y)$ from D, sends $f_{ij} = f_i(\alpha_j)$ to P_j.
- **Round III — each party P_i does the following**:
  - For every $P_j \in \mathcal{P}$ where $f_{ji} \neq g_i(\alpha_j)$, broadcast $(\texttt{complaint}, i, j, g_i(\alpha_j))$.
- **Round IV (making disputed values public) — the dealer and each party P_j does the following**:
  - If P_i broadcasts $(\texttt{complaint}, i, j, g_i(\alpha_j))$, then D and P_j broadcasts $F(\alpha_i, \alpha_j)$ and $f_j(\alpha_i)$ respectively.
- **Local computation (deciding unhappy parties) — each party P_k does the following**:
  - Initialize a set of *unhappy parties* $\mathcal{UH}$ to $\emptyset$.
  - For every pair of parties (P_i, P_j), such that P_i has broadcast $(\texttt{complaint}, i, j, g_i(\alpha_j))$, D has broadcast $F(\alpha_i, \alpha_j)$ and P_j has broadcast $f_j(\alpha_i)$, do the following.
    - If $g_i(\alpha_j) \neq F(\alpha_i, \alpha_j)$, then include P_i to $\mathcal{UH}$.
    - If $f_j(\alpha_i) \neq F(\alpha_i, \alpha_j)$, then include P_j to $\mathcal{UH}$.
  - If $|\mathcal{UH}| > t$, then discard D.
- **Round V (resolving unhappy parties) — the dealer and each $P_j \notin \mathcal{UH}$ does the following**:
  - For every $P_i \in \mathcal{UH}$, the dealer D broadcasts degree-t polynomial $f_i(x)$.
  - For every $P_i \in \mathcal{UH}$, party P_j broadcasts $g_j(\alpha_i)$.
- **Output decision — each party P_k does the following**:
  - If there exists any $P_i \in \mathcal{UH}$ for which D broadcasts $f_i(x)$ and at most $2t$ parties $P_j \notin \mathcal{UH}$ broadcast $g_j(\alpha_i)$ values where $f_i(\alpha_j) = g_j(\alpha_i)$ holds, then discard D.
  - Else output the *share* $f_k(0)$.

Figure 3: A simplified 5-round version of 7BGW-VSS-Sh due to Genarro, Ishai, Kushilevitz and Rabin [33].

4.1.3 The 4-round 4GIKR-VSS Scheme

Genarro et al [33] proposed another round-reducing technique to reduce the number of rounds of 5BGW-VSS-Sh by one. The modified protocol 4GIKR-VSS-Sh is presented in Fig 4. The idea is to ensure that the set $\mathcal{UH}$ is decided by the end of Round III, even though this might look like an impossible task. This is because $\mathcal{UH}$ can be decided during Round IV, only after the results of pair-wise consistency checks are available during Round III. The key-observation of [33] is that the parties can "initiate" the pair-wise consistency checks from Round I itself. More specifically, every

P_i, P_j exchange random pads *privately* during the Round I, independently of D's distribution of the polynomials. During the second round, P_i, P_j can then broadcast a masked version of the supposedly common values on their polynomials, using the exchanged pads as the masks. If D, P_i and P_j are *honest*, then the masked version of the common values will be the same and nothing about the common values will be learnt, as the corresponding masks will be private. By comparing the masked versions of the common values, the parties publicly learn about the results of pair-wise consistency checks by the end of the Round II.

Protocol 4GIKR-VSS-Sh

- **Round I (sending polynomials and exchanging random pads)**
  - D picks $F(x, y)$ as in 7BGW-VSS-Sh and distributes $f_i(x), g_i(y)$ lying on $F(x, y)$ to each P_i.
  - Each $P_i \in \mathcal{P}$ picks a random pad $r_{ij} \in \mathbb{F}$ corresponding to every $P_j \in \mathcal{P}$ and sends r_{ij} to P_j.
- **Round II (broadcasting common values in a masked fashion) — each P_i does the following**:
  - Broadcast $a_{ij} \stackrel{def}{=} f_i(\alpha_j) + r_{ij}$ and $b_{ij} \stackrel{def}{=} g_i(\alpha_j) + r'_{ji}$, where r'_{ji} is the pad, received from P_j.
- **Round III (making disputed values public) — for all P_i, P_j where $a_{ij} \neq b_{ji}$, party P_i, P_j and D does the following**:
  - D broadcasts $F(\alpha_j, \alpha_i)$; P_i broadcasts $f_i(\alpha_j)$ and P_j broadcasts $g_j(\alpha_i)$.
- **Local computation (deciding unhappy parties) — each party P_k does the following**:
  - Initialize a set of *unhappy parties* $\mathcal{UH}$ to $\emptyset$. For every (P_i, P_j) where $a_{ij} \neq b_{ji}$ and where P_i has broadcast $f_i(\alpha_j)$, D has broadcast $F(\alpha_j, \alpha_i)$ and P_j has broadcast $g_j(\alpha_i)$, do the following.
    - If $f_i(\alpha_j) \neq F(\alpha_j, \alpha_i)$, then include P_i to $\mathcal{UH}$.
    - If $g_j(\alpha_i) \neq F(\alpha_j, \alpha_i)$, then include P_j to $\mathcal{UH}$.
  - If $|\mathcal{UH}| > t$, then discard D.
- **Round IV (resolving unhappy parties) — the dealer and each $P_j \notin \mathcal{UH}$ does the following**:
  - For every $P_i \in \mathcal{UH}$, D broadcasts degree-t polynomial $f_i(x)$ and P_j broadcasts $g_j(\alpha_i)$.
- **Output decision — each party P_k does the following**:
  - If there exists any $P_i \in \mathcal{UH}$ for which D broadcasts $f_i(x)$ and at most $2t$ parties $P_j \notin \mathcal{UH}$ broadcast $g_j(\alpha_i)$ values where $f_i(\alpha_j) = g_j(\alpha_i)$ holds, then discard D.
  - Else output the *share* $f_k(0)$.

Figure 4: A 4-round sharing phase protocol due to Genarro, Ishai, Kushilevitz and Rabin [33].

4.1.4 The 3-round 3GIKR-VSS Scheme

We now present the 3GIKR-VSS scheme from [33], which has a *round-optimal* sharing phase, namely a 3-round sharing phase. However, the protocol is *inefficient*, as it requires an exponential (in n and t) amount of computation and communication. The computations in 3GIKR-VSS scheme are done over a finite group $(\mathbb{G}, +)$. We first explain t-out-of-n *replicated secret-sharing* (RSS) [37]. Let $K \stackrel{def}{=} \binom{n}{t}$ and $A_1, \ldots, A_K$ denote the set of all possible subsets of $\mathcal{P}$ of size t. For $k = 1, \ldots, K$, let $\mathcal{G}_k = \mathcal{P} \setminus A_k$. It is easy to see that in each $\mathcal{G}_k$, the majority of the parties are *honest*. To share $s \in \mathbb{G}$, the share-generation algorithm of RSS outputs $(v^{(1)}, \ldots, v^{(K)}) \in \mathbb{G}^K$, where $v^{(1)}, \ldots, v^{(K)}$ are random elements, such that $v^{(1)} + \ldots + v^{(K)} = s$ holds. The *share* s_i for P_i is defined as $s_i \stackrel{def}{=} \{v^{(j)}\}_{P_i \in \mathcal{G}_j}$. Any t-sized subset of $(s_1, \ldots, s_n)$ will have at least one "missing" element from $v^{(1)}, \ldots, v^{(K)}$, say $v^{(l)}$, whose probability distribution will be independent of s, thus ensuring privacy. On the other hand, any $(t+1)$-sized subset of $(s_1, \ldots, s_n)$ will have all values $v^{(1)}, \ldots, v^{(K)}$ which can be added to reconstruct back s, thus ensuring correctness. We say that $s \in \mathbb{G}$ *is RSS-shared*, if it is secret-shared as per t-out-of-n RSS. That is, if there exist $v^{(1)}, \ldots, v^{(K)}$ where $s = v^{(1)} + \ldots + v^{(K)}$, with all the parties in $\mathcal{G}_k$ holding $v^{(k)}$.

Scheme 3GIKR-VSS is presented in Fig 5. The sharing protocol 3GIKR-VSS-Sh verifiably generates a replicated secret-sharing of dealer's secret, maintaining its privacy if D is *honest*. The verifiability ensures that even if D is *corrupt*, there exists some value which has been shared as per RSS by D. The reconstruction protocol 3GIKR-VSS-Rec allows the parties to reconstruct the RSS-shared value of D. During 3GIKR-VSS-Sh, D generates a vector of values $(v^{(1)}, \ldots, v^{(K)})$ as per the share-generation algorithm of RSS and sends $v^{(k)}$ to all the parties in $\mathcal{G}_k$. If D is *honest*, then this ensures the privacy of s. This is because if Adv corrupts the parties in A_k, then it will not know $v^{(k)}$. To ensure that a potentially *corrupt* D has distributed the same v_k to all the (honest) parties in $\mathcal{G}_k$, each pair of parties in $\mathcal{G}_k$ privately exchange their respective copies of $v^{(k)}$ and publicly raise a complaint if they find any inconsistency. To resolve any complaint raised for $\mathcal{G}_k$, D makes public the value $v^{(k)}$ for $\mathcal{G}_k$, thus ensuring that all the parties in $\mathcal{G}_k$ have the same $v^{(k)}$. Notice that this does not violate privacy, since if any inconsistency is reported for $\mathcal{G}_k$, then either D is *corrupt* or $\mathcal{G}_k$ consists of at least one corrupt party and so adversary already knows $v^{(k)}$.

The above process will require *four* rounds, which can be collapsed to three, based on the idea of pre-exchanging pair-wise random pads. During 3GIKR-VSS-Rec, the goal of each P_i is to correctly obtain $v^{(1)}, \ldots, v^{(K)}$. If $P_i \in \mathcal{G}_k$, then it already has $v^{(k)}$. However, if $P_i \notin \mathcal{G}_k$, then every party in $\mathcal{G}_k$ sends $v^{(k)}$ to P_i, who applies the majority rule to filter out the correct $v^{(k)}$.

Scheme 3GIKR-VSS

Sharing Phase: Protocol 3GIKR-VSS-Sh

- **Round I (distributing shares and exchanging random pads)**: Let $K = \binom{n}{t}$ and $A_1, \ldots, A_K$ denote the set of all possible subsets of $\mathcal{P}$ of size t and let $\mathcal{G}_k = \mathcal{P} \setminus A_k$.
  - D on having the input $s \in \mathbb{G}$, randomly selects $v^{(1)}, \ldots, v^{(K)} \in \mathbb{G}$ such that $s = v^{(1)} + \ldots + v^{(K)}$ holds. It then sends $v^{(k)}$ to every party $P_i \in \mathcal{G}_k$, for $k = 1, \ldots, K$.
  - For $k = 1, \ldots, K$, each $P_i \in \mathcal{G}_k$ sends a randomly chosen pad $r_{ij}^{(k)} \in \mathbb{G}$ to every $P_j \in \mathcal{G}_k$ where $i < j$.
- **Round II (pair-wise consistency check within each group)**
  - For $k = 1, \ldots, K$, each pair of parties $P_i, P_j \in \mathcal{G}_k$ with $i < j$ do the following:
    - P_i broadcasts $a_{ij}^{(k)} = v_i^{(k)} + r_{ij}^{(k)}$, where $v_i^{(k)}$ denotes the version of $v^{(k)}$ received by P_i from D.
    - P_j broadcasts $a_{ji}^{(k)} = v_j^{(k)} + r_{ij}'^{(k)}$, where $v_j^{(k)}$ denotes the version of $v^{(k)}$ received by P_j from D and $r_{ij}'^{(k)}$ denotes the pad received by P_j from P_i.
- **Round III (resolving conflicts)**
  - For $k = 1, \ldots, K$, if there exists $P_i, P_j \in \mathcal{G}_k$ such that $a_{ij}^{(k)} \neq a_{ji}^{(k)}$, then D broadcasts the value $v^{(k)}$.
- **Output determination — each party P_i does the following**:
  - If $\exists k \in \{1, \ldots, K\}$ where $P_i \in \mathcal{G}_k$ such that D broadcast $v^{(k)}$, then set $v_i^{(k)}$ to $v^{(k)}$.
  - Output the *share* $s_i \overset{def}{=} \{v_i^{(k)}\}_{P_i \in \mathcal{G}_k}$.

Reconstruction Phase: Protocol 3GIKR-VSS-Rec

Each party $P_i \in \mathcal{P}$ does the following:

- $\forall k \in \{1, \ldots, K\}$, such that $P_i \in \mathcal{G}_k$, send $v_i^{(k)}$ to every party in $\mathcal{P} \setminus \mathcal{G}_k$.
- $\forall k \in \{1, \ldots, K\}$ where $P_i \notin \mathcal{G}_k$, set $v^{(k)}$ to be the value $v_j^{(k)}$ received from at least $t+1$ parties $P_j \in \mathcal{G}_k$.
- Output $s = \sum_{\mathcal{G}_k : P_i \in \mathcal{G}_k} v_i^{(k)} + \sum_{\mathcal{G}_k : P_i \notin \mathcal{G}_k} v^{(k)}$.

Figure 5: The 3-round 3GIKR-VSS scheme due to Genarro, Ishai, Kushilevitz and Rabin [33].

4.1.5 The 3-round 3FGGRS-VSS Scheme

The 4GIKR-VSS protocol comes closest in terms of the number of rounds to obtain a round optimal and *efficient* VSS scheme. Round IV of 4GIKR-VSS-Sh consists of D making public the polynomials of the *unhappy* parties. Fitzi et al. [32] observed that the elimination of Round IV results in a primitive that satisfies a *weaker* commitment property, where the reconstructed value may be some predefined default value, when the dealer is *corrupt*. This primitive is called *weak* verifiable secret sharing (WSS) [54] and is used as a building block to construct a VSS scheme. We next present the required background and the WSS scheme of [32].

Definition 4.2 ((n, t)-WSS [54])**.** Let (Sh, Rec) be a pair of protocols where $\mathsf{D} \in \mathcal{P}$ has a private input $s \in \mathcal{S}$ for Sh. Then (Sh, Rec) is a *perfectly-secure* (n, t)*-WSS scheme*, if the following hold.

- **Privacy and Correctness**: Same as in VSS.
- **Weak Commitment**: Even if D is *corrupt*, in any execution of Sh the joint view of the honest parties defines a unique value $s^\star \in \mathcal{S}$ (which could be different from s), such that each honest party outputs either $s^\star$ or some default value $\perp$ at the end of Rec, irrespective of Adv.

The WSS scheme 3FGGRS-WSS of Fitzi et al. [32] is given in Fig. 6. Protocol 3FGGRS-WSS-Sh is the same as 4GIKR-VSS-Sh, except that the parties do not execute Round IV. Consequently, the parties in $\mathcal{UH}$ will not posses their shares. Hence, during 3FGGRS-WSS-Rec, only the *happy* parties (who are *not* in $\mathcal{UH}$) participate by broadcasting their respective polynomials. To verify that correct polynomials are broadcasted, the pair-wise consistency of these polynomials is checked. The polynomials which are *not* found to be pair-wise consistent with "sufficiently many" polynomials are not considered. If the parties are left with at least $n - t$ polynomials, then they are used to reconstruct back D's committed Shamir-sharing polynomial, else the parties output $\perp$. The idea is that if the parties are left with $n - t$ polynomials, then they lie on the same degree-(t, t) bivariate polynomial as committed by D to *honest* happy parties during the sharing phase, as among these polynomials, at least $t + 1$ belong to the honest parties who are happy.

For an *honest* D, *no* honest party will be in $\mathcal{UH}$ and hence all honest parties will have their respective shares of D's Shamir-sharing polynomial. Moreover, if any *corrupt* party produces an incorrect polynomial during the reconstruction phase, then it will be ignored due to the pair-wise consistency checks. Thus 3FGGRS-WSS achieves the properties of a type-II VSS, for an *honest* D. However if D is *corrupt*, then up to t *honest* parties may belong to $\mathcal{UH}$. Moreover, during the reconstruction phase, even if a single corrupt party produces incorrect polynomials, then the parties reconstruct $\perp$. And this prevents 3FGGRS-WSS from being a VSS scheme.

Scheme 3FGGRS-WSS

Sharing Phase: Protocol 3FGGRS-WSS-Sh

- The parties execute the first 3 rounds of 4GIKR-VSS-Sh. Let $\mathcal{UH}$ be the set of *unhappy* parties and let $\mathcal{W} \stackrel{def}{=} \mathcal{P} \setminus \mathcal{UH}$ be the set of *happy* parties. If $|\mathcal{UH}| > t$, then discard D.

Reconstruction Phase: Protocol 3FGGRS-WSS-Rec

- **Revealing private information**: each $P_i \in \mathcal{W}$ broadcasts $f_i(x)$ and $g_i(y)$.
- **Consistency check and output decision — each party P_i does the following**:
  - Construct a *consistency graph* G over the set of parties $\mathcal{W}$ with an edge between P_j and P_k if and only if $f_j(\alpha_k) = g_k(\alpha_j)$ and $g_j(\alpha_k) = f_k(\alpha_j)$.
  - Remove P_j from G, if it has degree less than $n-t$ in G. Repeat till no more nodes can be removed from G. Redefine $\mathcal{W}$ to be the set of parties, whose corresponding nodes remain in G.
  - If $|\mathcal{W}| < n-t$ then output a default value $\perp$. Else interpolate a degree-t polynomial $q(\cdot)$ through

the points $\{(\alpha_j, f_j(0))\}_{P_j \in \mathcal{W}}$ and output $q(\cdot), s = q(0)$.

Figure 6: The 3-round 3FGGRS-WSS scheme due to Fitzi, Garay, Gollakota, Rangan and Srinathan [32].

Sharing and Reconstructing Polynomial Using 3FGGRS-WSS One can interpret D's computation in 3FGGRS-WSS-Sh as if D wants to share the degree-t Shamir-sharing polynomial $F^{\star}(0, y)$. If D is not discarded, then each (honest) $P_j \in \mathcal{W}$ receives the share $F^{\star}(0, \alpha_j)$ from D through its degree-t row-polynomial $F^{\star}(x, \alpha_j)$. Here $F^{\star}(x, y)$ is the degree-(t, t) bivariate polynomial committed by D to the honest parties in $\mathcal{W}$, which is the same as $F(x, y)$ for an *honest* D. If D is *honest*, then adversary learns at most t shares lying on $F(0, y)$ and hence its view will be independent of $F(0, 0)$. Similarly, the computations during 3FGGRS-WSS-Rec can be interpreted as if the parties publicly try to reconstruct a degree-t Shamir-sharing polynomial $F^{\star}(0, \text{y})$, which has been shared by D during 3FGGRS-WSS-Sh. If D is *honest*, then the parties robustly reconstruct the shared polynomial. Else, the parties either reconstruct the shared polynomial or output $\perp$. Hence we propose the following notations for 3FGGRS-WSS, which later simplifies the presentation of 3FGGRS-VSS.

Notation 4.3 (Notations for using 3FGGRS-WSS). We use the following notations.

- We say that party $P_j \in \mathcal{P}$ *shares a degree-t polynomial* $r(\cdot)$ held by P_j, to denote that P_j plays the role of D and invokes an instance of 3FGGRS-WSS-Sh by selecting $r(\cdot)$ as its Shamir-sharing polynomial and all the parties participate in this instance.
- We say that P_i *receives a wss-share* r_{ji} *from* P_j, to denote that in Round I of the 3FGGRS-WSS-Sh instance invoked by P_j, P_i receives a degree-t row-polynomial from P_j, whose constant term is r_{ji}. If P_j is not discarded during the 3FGGRS-WSS-Sh instance, then the wss-shares r_{ji} of all the *honest* parties in $\mathcal{W}$ lie on a unique degree-t Shamir-sharing polynomial held by P_j.
- Let $r(\cdot)$ be a degree-t polynomial shared by P_j through an instance of 3FGGRS-WSS-Sh. We say that the *parties try to reconstruct P_j's shared polynomial*, to denote that the parties execute the corresponding instance of 3FGGRS-WSS-Rec, which either outputs $r(\cdot)$ or $\perp$.

From WSS to VSS The sharing-phase protocol of 3FGGRS-VSS (see Fig. 7) is the same as the first three rounds of 4GIKR-VSS-Sh with the following twist. The random pads r_{ji} used by P_j to verify the pair-wise consistency of its row-polynomial with the other parties' column-polynomials are "tied together" by letting these pads lie on a random degree-t *blinding-polynomial* $r_j(\cdot)$, which is shared by P_j (see Notation 4.3). For pair-wise consistency, P_j makes public the polynomial $A_j(\cdot)$, which is a masked version of its row-polynomial and its blinding-polynomial, while every P_i makes public the supposedly common value on its column-polynomial, blinded with the wss-share of P_j's blinding-polynomial. This new way of performing pair-wise consistency checks achieves the same "goals" as earlier. Moreover, privacy is maintained for an *honest* D, as for every *honest* P_j, the adversary obtains at most t wss-shares of P_j's blinding-polynomial, which are randomly distributed. The set of happy parties for the VSS is identified based on the results of pair-wise consistency and conflict-resolutions, as done in 4GIKR-VSS-Sh. Moreover, the parties also ensure that for every happy party P_j for the VSS, there is an overlap of at least $n - t$ between the set of happy parties for the VSS and the set of happy parties for P_j's WSS-sharing instance. This is crucial for ensuring the strong commitment property during the reconstruction phase.

Let $F^{\star}(x, y)$ be the degree-(t, t) bivariate polynomial, committed by D during the sharing phase. During the reconstruction phase, instead of asking the *happy* parties to make their row-polynomials public, the parties reconstruct their blinding-polynomials (by executing instances of 3FGGRS-WSS-Rec), which are then unmasked from the corresponding $A_j(\cdot)$ polynomials to get back the row-polynomials of the happy parties. For the *honest* happy parties P_j, robust reconstruction of their blinding-polynomials is always guaranteed, thus ensuring that their row-polynomials $F^{\star}(x, \alpha_j)$ are robustly reconstructed. If the reconstruction of P_j's blinding-polynomial fails, then P_j is *corrupt* and hence can be safely discarded from consideration. However, if P_j is *corrupt* and the parties reconstruct a degree-t polynomial during the corresponding 3FGGRS-WSS-Rec instance, then the *weak commitment* property of the WSS ensures that reconstructed polynomial is the correct blinding-polynomial. Hence unmasking it from $A_j(\cdot)$ will return back the row-polynomial $F^{\star}(x, \alpha_j)$. This is because there are at least $n-t$ parties, which belong to the happy set of *both* the VSS instance, as well as P_j's WSS instance. Among these $n-t$ common happy-parties, at least $t+1$ are *honest.* And the wss-shares received by these honest parties P_k from P_j uniquely define P_j's blinding-polynomial $r_j(\cdot)$, while evaluations of the column-polynomials of the same honest-parties P_k at $y = \alpha_j$ uniquely determine the degree-t row-polynomial $F^{\star}(x, \alpha_j)$. Moreover, during the sharing phase these honest parties P_k collectively ensured that $A_j(\cdot) = F^{\star}(x, \alpha_j) + r_j(\cdot)$ holds, as otherwise they do not belong to the happy set of P_j's WSS instance.

Scheme 3FGGRS-VSS

Sharing Phase: Protocol 3FGGRS-VSS$_{\mathsf{Sh}}$

- **Round I (sending polynomials and exchanging random pads)**:
  - D picks $F(x, y)$ and distributes $f_i(x), g_i(y)$ lying on $F(x, y)$ to each P_i.
  - Each party $P_i \in \mathcal{P}$ (including D) picks a random degree-t *blinding-polynomial* $r_i(\cdot)$ and shares it through an instance WSS−Sh$_i$ of 3FGGRS-WSS-Sh.
- **Round II (broadcasting common values in a masked fashion) — each P_i does the following**:
  - Broadcast the degree-t polynomial $A_i(\cdot) \overset{def}{=} f_i(x) + r_i(\cdot)$.
  - Broadcast $b_{ij} \overset{def}{=} g_i(\alpha_j) + r'_{ji}$, where r'_{ji} denotes the wss-share received from P_j during WSS−Sh$_j$.
  - For every $k \in \{1, \ldots, n\}$, concurrently execute Round II of the instance WSS−Sh$_k$.
- **Round III**:
  - **(making disputed values public)** — for all P_i, P_j where $A_i(\alpha_j) \neq b_{ji}$, dealer D broadcasts $F(\alpha_j, \alpha_i)$, party P_i broadcasts $f_i(\alpha_j)$ and party P_j broadcasts $g_j(\alpha_i)$.
  - For every $k \in \{1, \ldots, n\}$, the parties concurrently execute Round III of the instance WSS−Sh$_k$.
- **Local computation at the end of Round III — each party P_k does the following**:
  - Initialize a set $\mathcal{UH}$ of *unhappy* parties. For every P_i, P_j where $A_i(\alpha_j) \neq b_{ji}$, do the following.
    - Include $P_i \in \mathcal{UH}$ (resp. $P_j \in \mathcal{UH}$), if $F(\alpha_j, \alpha_i) \neq f_i(\alpha_j)$ (resp. $F(\alpha_j, \alpha_i) \neq g_j(\alpha_i)$) holds.

    Let $\mathcal{V} \overset{def}{=} \mathcal{P} \setminus \mathcal{UH}$ be the set of *happy* parties.
  - For $j \in \{1, \ldots, n\}$, let $\mathcal{W}_j$ denote the set of *happy* parties during the instance WSS−Sh$_j$. Remove P_i from $\mathcal{W}_j$, if during Round II, $A_j(\alpha_i) \neq b_{ij}$ holds.
  - For every $P_j \in \mathcal{V}$, if $|\mathcal{V} \cap \mathcal{W}_j| < n - t$, then remove P_j from $\mathcal{V}$. Repeat this step till no more parties can be removed from $\mathcal{V}$. If $|\mathcal{V}| < n - t$, then discard D.

Reconstruction Phase: Protocol 3FGGRS-VSS$_{\mathsf{Rec}}$

- **Reconstructing the blinding-polynomials**:
  - $\forall P_j \in \mathcal{V}$, the parties try to reconstruct P_j's blinding-polynomial by participating in an instance WSS−Rec$_j$ of 3FGGRS-WSS-Rec. P_j is removed from $\mathcal{V}$ if $\perp$ is the output during WSS−Rec$_j$.
- **Output decision — each party P_i does the following**:
  - For each $P_j \in \mathcal{V}$, compute $f_j(x) = A_j(x) - r_j(\cdot)$, where $r_j(\cdot)$ is reconstructed during WSS−Rec$_j$. Interpolate a degree-t polynomial $q(\cdot)$ through $\{(\alpha_j, f_j(0))\}_{P_j \in \mathcal{V}}$. Output $s = q(0)$.

Figure 7: The 3-round 3FGGRS-VSS scheme due to Fitzi, Garay, Gollakota, Rangan and Srinathan [32].

4.1.6 The 3-round 3KKK-VSS Scheme

The 3FGGRS-VSS scheme is a Type-I VSS, because if D is *corrupt*, then *only* the honest parties in $\mathcal{V}$ get their shares. Moreover, it makes use of the broadcast channel during two of the rounds of the sharing phase, which is *not* optimal (the optimal is one round). The 3KKK-VSS scheme due to Katz et al. [39] rectifies both these problems.

The optimal broadcast-channel usage must first be rectified for 3FGGRS-WSS-Sh. The modified construction 3KKK-WSS-Sh (see Fig 8) is based on the observation that during Round II of 3FGGRS-WSS-Sh (which is the same as Round II of 4GIKR-VSS-Sh), there is no need to *publicly* perform the pair-wise consistency checks over *masked* values. Instead, parties can first *privately* perform the pair-wise consistency checks over *unmasked* values and later *publicly* announce the results during the third round. However, we also need to add a provision for the dealer to resolve any potential conflicts during the third round itself. For this, during the second round, every P_i, P_j privately exchange their supposedly common values and also the random pads. Additionally, the pads are also "registered" with the dealer. Later during the third round, upon a disagreement between P_i and P_j, they broadcast their respective common values and their respective random pads, else they broadcast their appropriately masked common values. In parallel, dealer either broadcasts the common value in a masked fashion if the pads it received from P_i, P_j are same, else it just broadcasts the common value. The secrecy of the common values is maintained if P_i, P_j and the dealer are honest. On the other hand, if dealer is corrupt and if there is a disagreement between honest P_i, P_j , then the dealer can "take side" with at most one of them during the third round.

Protocol 3KKK-WSS-Sh

- **Round I (sending polynomials and exchanging random pads)**:
  - D distributes $f_i(x), g_i(y)$ lying on $F(x, y)$ to each P_i. In parallel, each P_i sends a random pad r_{ij} to P_j and additionally the pad-list $\{r_{ij}\}_{P_j \in \mathcal{P}}$ to D.
- **Round II (exchanging common values and confirming pad) — each P_i does the following**:
  - Send $a_{ij} = f_i(\alpha_j)$ and $b_{ij} = g_i(\alpha_j)$ to P_j.
  - Send $\{r'_{ji}\}_{P_j \in \mathcal{P}}$ to D, where r'_{ji} denotes the pad received from P_j during Round I.
- **Round III (complaint and resolution) — each party P_i does the following**:
  - For $j \in \{1, \ldots, n\}$, let a'_{ji} and b'_{ji} be the values received from P_j during Round II.
    - If $b'_{ji} \neq f_i(\alpha_j)$, broadcast $(j, \texttt{disagree-row}, f_i(\alpha_j), r_{ij})$, else broadcast $(j, \texttt{agree-row}, f_i(\alpha_j) + r_{ij})$.
    - If $a'_{ji} \neq g_i(\alpha_j)$, broadcast $(j, \texttt{disagree-column}, g_i(\alpha_j), r'_{ji})$, else broadcast $(j, \texttt{agree-column}, g_i(\alpha_j) + r'_{ji})$.
  - If $P_i =$ D, then for every ordered pair of parties (P_j, P_k), *additionally* do the following.
    - Let $r^{(1)}_{jk}$ and $r^{(2)}_{jk}$ be the pads received from P_j and P_k respectively during Round I and Round II. If $r^{(1)}_{jk} \neq r^{(2)}_{jk}$, then broadcast $((j, k), \texttt{NEQ}, F(\alpha_k, \alpha_j))$, else broadcast $((j, k), \texttt{EQ}, F(\alpha_k, \alpha_j) + r^{(1)}_{jk})$.
- **Local computation (identifying unhappy parties) — each party P_k does the following**:
  - Initialize a set of *unhappy* parties $\mathcal{UH}$ to $\emptyset$. For every P_i, P_j such that P_i broadcasts $(j, \texttt{disagree-row}, f_i(\alpha_j), r_{ij})$ and P_j broadcasts $(i, \texttt{disagree-column}, g_j(\alpha_j), r'_{ij})$ where $r_{ij} = r'_{ij}$, do the following.
    - Include P_i to $\mathcal{UH}$, if one of the following holds.
      - During Round III, D broadcasts $((i, j), \texttt{NEQ}, d_{ij})$, such that $d_{ij} \neq f_i(\alpha_j)$.
      - During Round III, D broadcasts $((i, j), \texttt{EQ}, d_{ij})$, such that $d_{ij} \neq f_i(\alpha_j) + r_{ij}$.
    - Include P_j to $\mathcal{UH}$, if one of the following holds.
      - During Round III, D broadcasts $((i, j), \texttt{NEQ}, d_{ij})$, such that $d_{ij} \neq g_j(\alpha_i)$.
      - During Round III, D broadcasts $((i, j), \texttt{EQ}, d_{ij})$, such that $d_{ij} \neq g_j(\alpha_i) + r'_{ij}$.
  - If $|\mathcal{UH}| > t$, then discard D. Else let $\mathcal{W} = \mathcal{P} \setminus \mathcal{UH}$ be the set of *happy* parties.

Figure 8: The 3-round 3KKK-WSS-Sh protocol due to Katz, Koo and Kumaresan [39].

From 3KKK-WSS to 3KKK-VSS The notion of sharing and reconstructing degree-t polynomials (as per Notation 4.3) is applicable even for 3KKK-WSS. Replacing 3FGGRS-WSS with 3KKK-WSS in the 3FGGRS-VSS readily provides a VSS scheme with the optimal usage of the broadcast channel. However, the resultant VSS *need not* be of Type-II if D is *corrupt*, as the *unhappy honest* parties may not get their shares. Hence 3KKK-VSS deploys an additional trick to get rid of this problem.

Let P_i be an *unhappy* party during 3FGGRS-VSS-Sh and let $F^\star(x, y)$ be D's committed bivariate polynomial. Note that each happy P_j broadcasts a masking $A_j(\cdot)$ of $F^\star(x, \alpha_j)$. If P_i is *happy* in $\mathsf{WSS}{-}\mathsf{Sh}_j$, then P_i can compute the point $F^\star(\alpha_i, \alpha_j)$ on its supposedly column-polynomial $F^\star(\alpha_i, y)$ by unmasking the wss-share r'_{ji} computed during $\mathsf{WSS}{-}\mathsf{Sh}_j$ from $A_j(\alpha_i)$. Hence if there is a set $\mathcal{SS}_i$ of at least $t + 1$ happy parties P_j, who keep P_i happy during $\mathsf{WSS}{-}\mathsf{Sh}_j$ instances, then P_i can compute $t + 1$ distinct points on $F^\star(\alpha_i, y)$ and hence get $F^\star(\alpha_i, y)$. However, it is not clear how to extend the above approach to enable P_i obtain its row-polynomial $F^\star(x, \alpha_i)$, which is required for P_i to obtain its share of D's Shamir-sharing polynomial $F^\star(0, y)$. The way-out is to let D use a *symmetric bivariate polynomial*, which ensures that $F^\star(x, \alpha_i) = F^\star(\alpha_i, y)$ holds.

The above idea requires broadcast during the second and third round. To ensure the optimal usage of the broadcast channel, the technique used in 3KKK-WSS-Sh is deployed, along with postponing the broadcast of masked row-polynomials to the third round. However, this brings additional challenges to filter out the parties from $\mathcal{W}_j$ sets for the individual WSS instances. For example, a *corrupt* D can distribute pair-wise *inconsistent* polynomials in such a way that the masked polynomial $A_j(\cdot)$ broadcast by an *honest happy* party P_j is inconsistent with the corresponding masked value broadcast by an *honest unhappy* party P_i, even though P_i belongs to $\mathcal{W}_j$ during $\mathsf{WSS}{-}\mathsf{Sh}_j$. Simply removing P_i from $\mathcal{W}_j$ in this case (as done in 3FGGRS-VSS-Sh) might end up resulting in P_i being removed from the $\mathcal{W}_j$ sets of every honest happy party. And this may lead to $\mathcal{SS}_i$ set of size less than $t + 1$. To prevent this, apart from pair-wise consistency checks of masked row-polynomials, the parties also carefully consider the results of *private* pair-wise consistency checks performed during the second round, whose results are *public* during the third round (see Fig. 9). We stress that even though each party obtains a single polynomial from D, it is treated both as row as well as column-polynomial to perform the pair-wise consistency checks. Accordingly, if P_i finds a "negative" result for the *private* pair-wise consistency check with P_j, then it broadcasts both `disagree-row` and `disagree-column` messages against P_j. Else it broadcasts just an `agree-column` message for P_j; the `agree-row` message for P_j is assumed to be *implicitly* present in the latter case. The reconstruction phase of 3KKK-VSS scheme is the same as 7BGW-VSS.

Protocol 3KKK-VSS

Sharing Phase: Protocol 3KKK-VSS-Sh

- **Round I (sending polynomials and exchanging random pads)**:
  - D embeds its *Shamir-sharing polynomial* $q(\cdot)$ in a *random* degree-(t, t) *symmetric* bivariate polynomial $F(x, y)$ at $x = 0$ and sends only the row-polynomial $f_i(x) = F(x, \alpha_i)$ to party P_i.
  - Each party $P_i \in \mathcal{P}$ (including D) picks a random degree-t *blinding polynomial* $r_i(\cdot)$ and shares it through an instance $\mathsf{WSS}{-}\mathsf{Sh}_i$ of 3KKK-WSS-Sh. In addition, P_i sends the polynomial $r_i(\cdot)$ to D.
- **Round II (exchanging common values and confirming pad) — each P_i does the following**:
  - For $j = 1, \ldots, n$, send $a_{ij} = f_i(\alpha_j)$ to P_j. Send $\{r'_{ji}\}_{j=1,\ldots,n}$ to D, where r'_{ji} is the wss-share received from P_j during Round I of $\mathsf{WSS}{-}\mathsf{Sh}_j$.
  - For $j = 1, \ldots, n$ execute Round II of the instance $\mathsf{WSS}{-}\mathsf{Sh}_j$.
- **Round III (complaint and resolution) — each party P_i does the following**:
  - Broadcast the degree-t polynomial $A_i(\cdot) = f_i(x) + r_i(\cdot)$.
  - For $j \in \{1, \ldots, n\}$, let a'_{ji} be the value received from P_j during Round II.
    - If $a'_{ji} \neq f_i(\alpha_j)$, then broadcast $(j, \texttt{disagree-row}, f_i(\alpha_j), r_{ij})$ and $(j, \texttt{disagree-column}, f_i(\alpha_j), r_{ji})$.

- Else broadcast $(j, \texttt{agree-column}, f_i(\alpha_j) + r'_{ji})$.
- If $P_i = \mathsf{D}$, then for every ordered pair of parties (P_j, P_k), *additionally* do the following.
  - Let $r_{jk}^{(1)}$ and $r_{jk}^{(2)}$ be the pads received from P_j and P_k respectively during Round I and Round II. If $r_{jk}^{(1)} \neq r_{jk}^{(2)}$, then broadcast $((j,k), \texttt{NEQ}, F(\alpha_k, \alpha_j))$, else broadcast $((j,k), \texttt{EQ}, F(\alpha_k, \alpha_j) + r_{jk}^{(1)})$.
- For every $j \in \{1, \ldots, n\}$, concurrently execute Round III of $\mathsf{WSS{-}Sh}_j$.

- **Local computation at the end of Round III — each party P_k does the following**:
  - Initialize a set of *unhappy* parties $\mathcal{UH}$ to $\emptyset$. For every P_i, P_j such that P_i broadcasts $(j, \texttt{disagree-row}, f_i(\alpha_j), r_{ij})$ and P_j broadcasts $(i, \texttt{disagree-column}, f_j(\alpha_i), r'_{ij})$ where $r_{ij} = r'_{ij}$, do the following.
    - Include P_i to $\mathcal{UH}$, if one of the following holds.
      - During Round III, D broadcasts $((i,j), \texttt{NEQ}, d_{ij})$, such that $d_{ij} \neq f_i(\alpha_j)$.
      - During Round III, D broadcasts $((i,j), \texttt{EQ}, d_{ij})$, such that $d_{ij} \neq f_i(\alpha_j) + r_{ij}$.
    - Include P_j to $\mathcal{UH}$, if one of the following holds.
      - During Round III, D broadcasts $((i,j), \texttt{NEQ}, d_{ij})$, such that $d_{ij} \neq f_j(\alpha_i)$.
      - During Round III, D broadcasts $((i,j), \texttt{EQ}, d_{ij})$, such that $d_{ij} \neq f_j(\alpha_i) + r'_{ij}$.
  - Let $\mathcal{V} = \mathcal{P} \setminus \mathcal{UH}$ be the set of *happy* parties and for every $P_j \in \mathcal{V}$, let $\mathcal{W}_j$ be the set of happy parties during $\mathsf{WSS{-}Sh}_j$. Remove P_j from $\mathcal{V}$, if any of the following holds:
    - $|\mathcal{W}_j| < n - t$.
    - $\exists i \in \{1, \ldots, n\} : P_j$ broadcasts $(i, \texttt{disagree-row}, f_j(\alpha_i), r_{ji})$ where $A_j(\alpha_i) \neq f_j(\alpha_i) + r_{ji}$.
  - For every $P_j \in \mathcal{V}$, remove P_i from $\mathcal{W}_j$, if any of the following holds.
    - P_i broadcasts $(j, \texttt{agree-column}, y)$ such that $A_j(\alpha_i) \neq y$.
    - P_j broadcasts $(i, \texttt{disagree-row}, f_j(\alpha_i), r_{ji})$ and P_i broadcasts either $(j, \texttt{agree-column}, \star)$ or $(j, \texttt{disagree-column}, \star, r'_{ji})$, where $r'_{ji} \neq r_{ji}$.
  - Remove P_j from $\mathcal{V}$ if $|\mathcal{V} \cap \mathcal{W}_j| < n - t$. Repeat, till no more parties can be removed from $\mathcal{V}$.
  - If $|\mathcal{V}| < n - t$, then discard D.
- **Computing shares — each party $P_i \in \mathcal{P}$ does the following**:
  - If $P_i \in \mathcal{V}$, then output the *share* $f_i(0)$. Else recompute $f_i(x)$ as follows and output the *share* $f_i(0)$.
    - Add P_j to $\mathcal{SS}_i$, if $P_j \in \mathcal{V}$ and $P_i \in \mathcal{W}_j$. Interpolate $\{(\alpha_j, A_j(\alpha_i) - r'_{ji})\}_{P_j \in \mathcal{SS}_i}$ to compute $f_i(x)$.

Figure 9: The 3-round sharing-phase protocol due to Katz, Koo and Kumaresan [39].

4.1.7 The 3-round 3AKP-VSS Scheme

In Applebaum, Kachlon and Patra [3], it is shown that 4 rounds are necessary and sufficient for securely computing any n-party degree-2 functionality with perfect security and optimal resilience of $t < n/3$. To design their 4-round MPC protocol, they rely on a 3-round Type-II perfectly-secure VSS which should ensure that if D is not discarded at the end of sharing phase, then one of the following holds for every *honest* party P_i at the end of Round II.

- P_i holds its *tentative Shamir-share* of the underlying secret;
- P_i holds at least $t + 1$ *tentative shares* of its Shamir-share, which we call as *sub-shares*.

Moreover, it is also required that at the end of Round III, either the tentative share or the tentative sub-shares should turn out to be correct (the exact case need not be known to P_i at the end of Round II). The 3FGGRS-VSS *does not* satisfy the above requirements, as it is not a Type-II VSS. The 3KKK-VSS scheme also fails because if $P_i \notin \mathcal{V}$, then it obtains its sub-shares through $\mathcal{SS}_i$ only at the end of Round III. Hence in Applebaum et al. [3], a new 3-round VSS scheme (see Fig 10) is presented, satisfying the above requirements. The scheme is obtained by tweaking the 3FGGRS-VSS scheme and by borrowing the idea of symmetric bivariate polynomial from the 3KKK-VSS scheme. The reconstruction phase of the scheme is same as 7BGW-VSS-Rec.

Protocol 3AKP-VSS

Sharing Phase: Protocol 3AKP-VSS-Sh

- **Round I**: Same as Round I of 3FGGRS-VSS-Sh, except that D uses a random degree-(t,t) *symmetric* bivariate polynomial $F(x,y)$ and distributes only the row-polynomial $f_i(x) = F(x,\alpha_i)$ to every P_i.
- **Round II**: Same as Round II of 3FGGRS-VSS-Sh, except that $b_{ij} \overset{def}{=} f_i(\alpha_j) + r'_{ji}$. Moreover, every party P_i sets $s_i \overset{def}{=} f_i(0)$ as its *tentative Shamir-share* and $\{A_j(\alpha_i) - r'_{ji}\}_{P_j \in \mathcal{P}}$ as its *tentative sub-shares.*
- **Round III**: Same as Round III of 3FGGRS-VSS-Sh, except that for every P_i, P_j where $A_i(\alpha_j) \neq b_{ji}$, party P_i broadcasts $(f_i(\alpha_j), r_{ij})$, party P_j broadcasts $(f_j(\alpha_i), r'_{ij})$ and D broadcasts $F(\alpha_j, \alpha_i)$.
- **Local computation at the end of Round III — each party P_k does the following**:
  - Compute the sets $\mathcal{UH}$ and $\mathcal{V}$ as in 3FGGRS-VSS-Sh, based on every P_i, P_j for which $A_i(\alpha_j) \neq b_{ji}$.
  - Remove P_i from $\mathcal{W}_j$, if $A_j(\alpha_i) \neq b_{ij}$ during Round II and $r_{ji} \neq r'_{ji}$ during Round III.
  - Remove P_j from $\mathcal{V}$ if there exists some $i \in \{1,\ldots,n\}$, such that P_j broadcasts $(f_j(\alpha_i), r_{ji})$ during Round III and $A_j(\alpha_i) \neq f_j(\alpha_i) + r_j(\alpha_i)$.
  - Remove P_j from $\mathcal{V}$, if $|\mathcal{V} \cap \mathcal{W}_j| < n - t$. Repeat, till no more parties can be removed from $\mathcal{V}$. If $|\mathcal{V}| < n - t$, then discard D.
- **Computing shares — each party $P_i \in \mathcal{P}$**: compute the shares as in the protocol 3KKK-VSS$_{\mathsf{Sh}}$.

Figure 10: The 3-round sharing-phase protocol due to Applebaum, Kachlon and Patra [3].

4.2 VSS Scheme with $n > 4t$

We present a perfectly-secure VSS scheme with $n > 4t$ and a 2-round sharing phase due to Genarro et al. [33]. We first present a data structure called (n,t)-star (which we often call as just star).

Definition 4.4 ((n,t)-star [11])**.** Let G be an undirected graph over $\mathcal{P}$. Then a pair of subsets of nodes $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{P}$ is called an (n,t)-star, if all the following hold.
– $|\mathcal{C}| \geq n - 2t$ and $|\mathcal{D}| \geq n - t$.
– For every $P_i \in \mathcal{C}$ and every $P_j \in \mathcal{D}$, the edge (P_i, P_j) is present in G.

Ben-Or et al. [11] presents an efficient algorithm for checking the presence of a star. Whenever the input graph contains a clique of size at least $n - t$, then the algorithm outputs a star.

Protocol 2GIKR-VSS-Sh is based on a simplification of the *asynchronous* VSS scheme of Ben-Or et al. [11], adapted to the synchronous setting (see Section 6.1). As done in the earlier protocols, D distributes row and column-polynomials to the respective parties. The goal is then to verify if the row and column-polynomials of all (honest) parties are derived from a single degree-(t,t) bivariate polynomial. However, since $n > 4t$ (compared to $n > 3t$ in the earlier protocols), the above verification task is significantly simplified. For simplicity, we first explain an *inefficient* verification, followed by the actual *efficient* method used in the protocol.

Once the parties receive their respective polynomials, they perform the pair-wise consistency checks (based on the idea of using random pads). The parties next construct a *consistency-graph* G over $\mathcal{P}$, where there exists an edge between a pair of parties if no dispute is reported between them. The parties next check for the presence of a *clique* of size $n - t$ in G, which is bound to exist if D is *honest.* If no clique is obtained then clearly D is *corrupt* and hence discarded. If a clique $\mathcal{C}$ of size $n - t$ is obtained, then the polynomials of the *honest* parties in $\mathcal{C}$ are pair-wise consistent and lie on a single degree-(t,t) bivariate polynomial, say $F^\star(x,y)$. However, there could be up to t (honest) parties *outside* $\mathcal{C}$ and the goal is to let each such "outsider" $P_i \notin \mathcal{C}$ obtain its degree-t row-polynomial $F^\star(x,\alpha_i)$. The crucial observation here is that since $n > 4t$, achieving this goal *does not* require any additional interaction. That is, each $P_i \notin \mathcal{C}$ considers the set of $n - t \geq 3t + 1$ g_{ji} values, received from the parties $P_j \in \mathcal{C}$ as part of the pair-wise consistency test. Among these g_{ji} values, at least

$2t + 1$ are sent by the *honest* parties $P_j \in \mathcal{C}$, which uniquely define $F^\star(x, \alpha_i)$. Since $F^\star(x, \alpha_i)$ is a degree-t polynomial and there can be at most t *corrupt* parties $P_j \in \mathcal{C}$ who may provide incorrect values of g_{ji}, party P_i can error-correct these values and obtain $F^\star(x, \alpha_i)$.

The above method is *inefficient*, as finding a maximum-sized clique is an NP-complete problem. Instead, the parties check for the presence of a star. If D is *honest* then the set of honest parties constitute a potential clique of size $n - t$ in G and so a star is always obtained. If a star $(\mathcal{C}, \mathcal{D})$ is obtained, then there are at least $|\mathcal{C}| - t = t + 1$ *honest* parties in $\mathcal{C}$ holding degree-t row-polynomials and at least $|\mathcal{D}| - t = 2t + 1$ *honest* parties in $\mathcal{D}$ holding degree-t column-polynomials, which are pair-wise consistent and hence lie on a single degree-(t, t) bivariate polynomial $F^\star(x, y)$. Any $P_i \notin \mathcal{C}$ obtains its corresponding row-polynomial $F^\star(x, \alpha_i)$ by applying the error-correction procedure as discussed above on the g_{ji} values received from $P_j \in \mathcal{D}$. Protocol 2GIKR-VSS-Sh is presented in Fig 11; the reconstruction protocol 2GIKR-VSS-Rec is the same as 7BGW-VSS-Rec.

Protocol 2GIKR-VSS-Sh

- **Round I**: D picks $F(x, y)$ and distributes $f_i(x), g_i(y)$ lying on $F(x, y)$ to each P_i. In parallel, each P_i sends a random mask r_{ij} to each P_j.
- **Round II**: Each P_i broadcasts $a_{ij} = f_i(\alpha_j) + r_{ij}$ and $b_{ij} = g_i(\alpha_j) + r'_{ji}$, where r'_{ji} is the pad, received from P_j.
- **Local computation at the end of round II — each party P_i does the following**:
  - Construct an undirected graph G over $\mathcal{P}$, where the edge (P_l, P_m) is present if $a_{lm} = b_{ml}$ and $a_{ml} = b_{lm}$ holds. Run the star-finding algorithm over G.
  - If no star is obtained, then discard D. Else let $(\mathcal{C}, \mathcal{D})$ be the star obtained in G.
    - If $P_i \in \mathcal{C}$, then output the *share* $f_i(0)$.
    - Else recompute the row-polynomial $f_i(x)$ as follows and output the *share* $f_i(0)$.
      - $\forall P_j \in \mathcal{D}$, compute $g_{ji} = b_{ji} - r_{ij}$. Execute RS-Dec(t, t, S_i) to get $f_i(x)$, where $S_i \overset{def}{=} \{g_{ji}\}_{P_j \in \mathcal{D}}$.

Figure 11: The 2-round 2GIKR-VSS-Sh protocol with $n > 4t$ due to Genarro, Ishai, Kushilevitz and Rabin [33].

4.3 VSS Scheme with a Single Round

Genarro et al. [33] presented a VSS scheme (Fig 12) with 1-round sharing-phase, where $n = 5$ and $t = 1$. For simplicity, let P_1 be the dealer. During the sharing phase, P_1 distributes Shamir-shares of its secret. Since no additional rounds are available, the parties cannot verify whether D has distributed consistent shares. In the reconstruction phase, the dealer is *not allowed* to participate. The remaining parties exchange their respective shares and try to error-correct one potential incorrect share. If the error-correction is successful then the parties output the constant term of the reconstructed degree-1 polynomial, else they output $\perp$. Since there can be one corrupt party, there are two possible cases. If P_1 is *honest*, then *privacy* is ensured and the shares of P_2, P_3, P_4 and P_5 lie on a degree-1 polynomial. Hence during the reconstruction phase, even if a potentially corrupt party provides incorrect share, it can be error-corrected, thus guaranteeing the *correctness* property.

The case when P_1 is *corrupt* can be divided into *three* sub-cases. If P_1 has distributed valid Shamir-shares (all lying on degree-1 polynomial), then the underlying Shamir-shared value will be reconstructed correctly. The second sub-case is when P_1 has distributed valid Shamir-shares to *exactly three* parties, say P_2, P_3 and P_4, and let their shares lie on a degree-1 polynomial $q^\star(\cdot)$. Hence during the reconstruction phase, all honest parties reconstruct $s^\star = q^\star(0)$ by error-correcting the share provided by P_5. The remaining sub-case is when the shares of no three parties among $\{P_2, P_3, P_4, P_5\}$ lie on a degree-1 polynomial. In this case we define $s^\star \overset{def}{=} \perp$, where $\perp \notin \mathbb{F}$, indicating that the sharing dealt by P_1 is "invalid". During the reconstruction phase, the error-correction will fail (since the shares of no three parties lie on a degree-1 polynomial) and hence the honest parties output $\bot$. Thus in all the 3 sub-cases, *strong commitment* property is achieved.

Scheme 1GIKR-VSS

Sharing Phase: Protocol 1GIKR-VSS-Sh

The dealer P_1 on having input $s \in \mathbb{F}$, picks a random degree-1 polynomial $q(\cdot)$ over $\mathbb{F}$ such that $q(0) = s$. For $i = 2, \ldots, 5$, it sends the *share* $s_i = q(\alpha_i)$ to P_i.

Reconstruction Phase: Protocol 1GIKR-VSS-Rec

Each party $P_i \in \{P_2, P_3, P_4, P_5\}$ does the following:

- Send s_i to every $P_j \in \mathcal{P} \setminus \{P_1\}$. On receiving s_j from P_j, include s_j in a list W_i. Execute $\mathsf{RS\text{-}Dec}(1, 1, W_i)$.
  - If RS-Dec outputs a degree-1 polynomial $q(\cdot)$, then output $q(0)$. Else output $\bot$.

Figure 12: The one round perfectly-secure VSS scheme of Genarro, Ishai, Kushilevitz and Rabin [33].

Part III : Asynchronous Communication Setting

5 Preliminaries and Definitions

In the *synchronous* communication setting, each party knows in advance how long it has to wait for an expected message, and if the message does not arrive within that time-bound, then the sender party is *corrupt*. Unfortunately, it is impossible to ensure such strict time-outs in real-world networks like the Internet. Motivated by this, [11, 16] introduced the *asynchronous* communication model with *eventual message delivery*. Apart from a better modelling of real-world networks, asynchronous protocols have the advantage of running at the *actual* speed of the underlying network. More specifically, for a synchronous protocol, the participants have to pessimistically set the global delay Δ to a large value to ensure that the messages sent by every party at the beginning of a round reach their destination within time Δ. But if the *actual* delay δ is such that $\delta << \Delta$, then the protocol fails to take advantage of the faster network and its running time will be proportional to Δ.

In the asynchronous model, the messages can be *arbitrarily, but finitely* delayed. The only guarantee is that any sent message is *eventually* delivered, but probably in a different order. The sequence of message delivery is controlled by a *scheduler* and to model the worst case scenario, we assume that the scheduler is under the control of Adv. Due to the lack of any upper bound on the message delays, no party can wait to receive communication from *all* its neighbours to avoid an endless wait (as a corrupt neighbour may not send any message). As a result, a party can afford to wait for messages from *at most* $n - t$ parties (including itself), thus ignoring communication from t potentially *honest* neighbours. Consequently, all synchronous VSS protocols become insecure when executed in an asynchronous environment, as they depend upon the fact that the messages of *all* the honest parties are considered. Due to the absence of any global clock, the execution of any asynchronous protocol is event-based, rather than round-based.

Informally, an AVSS scheme consists of an *asynchronous* sharing and an *asynchronous* reconstruction phase, providing privacy, correctness and strong commitment guarantees. However, we need these properties to hold *eventually*. Additionally, we need termination guarantees. Namely, if D is *honest*, then we require that these phases eventually terminate. On the other hand, if D is *corrupt*, then the termination demands are "weaker". Namely, we require the honest parties to terminate the sharing and reconstruction phase *only if* some honest party has terminated the sharing phase. This models the fact that a potentially corrupt D may not invoke the sharing phase in the first place (this is unlike synchronous VSS, where protocols always terminate after a "finite" number of communication rounds).

Definition 5.1 (Perfectly-Secure AVSS [16]). Let $(\mathsf{Sh}, \mathsf{Rec})$ be a pair of asynchronous protocols for the n parties, where a designated *dealer* $\mathsf{D} \in \mathcal{P}$ has a private input s for Sh and where each (honest) party who completes Sh, subsequently invokes Rec, with its local output of Sh. Then $(\mathsf{Sh}, \mathsf{Rec})$ constitute a *perfectly-secure AVSS scheme*, if all the following holds for every possible Adv.

- **Termination**:
  - If D is *honest*, then every honest party will eventually complete Sh.
  - If some honest party has completed Sh, then all honest parties will eventually complete Sh.
  - If some honest party has completed Sh, then it will eventually complete Rec.
- **Privacy and Correctness**: Same as for synchronous VSS.
- **Strong Commitment**: If D is *corrupt* and some honest party completes Sh, then the joint view of the honest parties at the end of Sh defines a value $s^\star$ (possibly different from s), such that all honest parties eventually output $s^\star$ at the end of Rec.

We can have Type-I and Type-II AVSS schemes. All the existing perfectly-secure AVSS schemes are of Type-II, where the secret is *always* shared as per the Shamir's secret-sharing scheme.

5.1 Asynchronous Tools

Asynchronous Reliable-Broadcast (ACast) An ACast protocol allows a designated *sender* $\mathsf{S} \in \mathcal{P}$ to identically send a message m to all the parties. If S is *honest*, then all honest parties eventually terminate with output m. While, if S is *corrupt and* some honest party terminates with output $m^\star$, then eventually every other honest party should terminate with output $m^\star$. Hence the termination guarantees are "weaker" than *synchronous* reliable-broadcast (RB), where the protocol *always* terminates, irrespective of S. Bracha [15] presented a very elegant instantiation of ACast for any $n > 3t$. We use the term P_i *broadcasts* m to mean that P_i acts as S and invokes an instance of ACast protocol to broadcast m. Similarly, the term P_j *receives* m *from the broadcast of* P_i means that P_j completes the instance of ACast protocol where P_i is S, with m as output.

Online Error-Correction (OEC) Let s be d-shared among $\mathcal{P}' \subseteq \mathcal{P}$ such that $d < (|\mathcal{P}'| - 2t)$ holds. That is, there exists some degree-d polynomial $q(\cdot)$ with $q(0) = s$ and each $P_i \in \mathcal{P}'$ has a share $q(\alpha_i)$. The goal is to make some *designated* party, say P_R, reconstruct s (actually OEC allows P_R to reconstruct $q(\cdot)$). In the *synchronous* setting, this is achieved by letting *every* party in $\mathcal{P}'$ to send its share to P_R, who can apply the algorithm $\mathsf{RS\text{-}Dec}$ and error-correct up to t potentially incorrect shares. Given that $d < (|\mathcal{P}'| - 2t)$, the reconstruction will be robust. In the *asynchronous* setting, achieving the same goal requires a bit of trick. The intuition behind OEC is that P_R keeps waiting till it receives $d + t + 1$ shares, all of which lie on a *unique* degree-d polynomial, which eventually happens for P_R (even if the *corrupt* parties in $\mathcal{P}'$ do not send their shares to P_R) as there are at least $|\mathcal{P}'| - t \geq d + t + 1$ honest parties in $\mathcal{P}'$. This step requires P_R to repeatedly apply $\mathsf{RS\text{-}Dec}$ and try recover $q(\cdot)$, upon asynchronously receiving every new share from the parties in $\mathcal{P}'$. Once P_R receives $d + 1 + 1$ shares lying on a degree-d polynomial, say $q'(\cdot)$, then $q'(\cdot) = q(\cdot)$. This is because among the $d + t + 1$ shares, at least $d + 1$ are from the *honest* parties, which uniquely determine $q(\cdot)$. We denote the OEC procedure by $\mathsf{OEC}(\mathcal{P}', d)$.

6 Perfectly-Secure AVSS Schemes

We discuss the various perfectly-secure AVSS schemes [11, 50, 21]. While the sharing phase requires $n > 4t$, with the exception of [50] the reconstruction phase requires $n > 3t$. The necessity of the condition $n > 4t$ follows from [13, 1], where it is shown that in any AVSS scheme designed with $n \leq 4t$, there is a *non-zero* probability in the termination property. We summarize the AVSS schemes in Table 2. While the degree of sharing d is t for [11, 21], the degree d could be more than t for [50].

	Sharing Phase					Reconstruction Phase	
Scheme	n	d	L	$\lvert\mathbb{F}\rvert$	Communication Complexity (CC)	n	CC
BCG-AVSS [11]	$n > 4t$	t	1	$\lvert\mathbb{F}\rvert > n$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log n))$	$n > 3t$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert)$
PCR-AVSS [50]	$n > 4t$	$t < d < n - 2t$	1	$\lvert\mathbb{F}\rvert > n$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log n))$	$n > 4t$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert)$
CHP-AVSS [21]	$n > 4t$	t	$\geq n - 3t$	$\lvert\mathbb{F}\rvert > 2n - 3t$	$\mathcal{O}(L \cdot n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log n))$	$n > 3t$	$\mathcal{O}(L \cdot n^2 \log \lvert\mathbb{F}\rvert)$

Table 2: Summary of the perfectly-secure AVSS schemes. Here L denotes the number of values shared through a single AVSS instance and $\mathcal{BC}$ denotes the communication happening through ACast.

6.1 BCG-AVSS Scheme

The BCG-AVSS scheme is presented in Fig 13. The sharing phase protocol BCG-AVSS-Sh is a slightly modified and simplified version of the original protocol [11], based on the simplifications suggested in [7, 50]. Protocol BCG-AVSS-Sh is similar to 2GIKR-VSS-Sh (see Fig 11), executed in the asynchronous setting. The protocol has four stages, each of which is executed asynchronously.

During the first stage, D distributes the row and column-polynomials to the respective parties. During the second stage, the parties perform perform pair-wise consistency checks and publicly announce the results, based on which parties build a consistency-graph. Since the results of the consistency checks are broadcasted *asynchronously*, the consistency-graph might be different for different parties (however, the edges which are present in the graph of one honest party will be *eventually* included in the graph of every other honest party). During the third stage, D checks for a star $(\mathcal{C}, \mathcal{D})$ in its consistency-graph, which it then broadcasts as a "proof" that the row-polynomials of the (honest) parties in $\mathcal{C}$ and the column-polynomials of the (honest) parties in $\mathcal{D}$ lie a single degree-(t, t) bivariate polynomial, which is considered as D's "committed" bivariate polynomial. A party upon receiving $(\mathcal{C}, \mathcal{D})$ from D *accepts* it, when $(\mathcal{C}, \mathcal{D})$ constitutes a star in its own consistency-graph. Fo an *honest* D, the set of honest parties eventually constitute a clique and hence D eventually finds a star, which will be eventually accepted by every honest party.

Once a star is accepted by P_i then in the last stage, its goal is to compute its share, for which P_i should hold its degree-t row-polynomial, lying on D's committed bivariate polynomial. If $P_i \in \mathcal{C}$, then it already has this polynomial. Else, P_i waits for the common values on the required row-polynomial from the parties in $\mathcal{D}$ and error-corrects the incorrectly received values using OEC. Since P_i's desired row-polynomial has degree-t and since each P_j in $\mathcal{D}$ holds a share of this polynomial in the form of a common value on its column-polynomial, OEC eventually outputs the desired row-polynomial for P_i, as $|\mathcal{D}| \geq 3t + 1$ and $\mathcal{D}$ contains at most t corrupt parties[3].

During reconstruction phase, every party sends its share to every other party. The parties then reconstruct the secret by using OEC on the received shares (this step will work even if $n > 3t$).

Scheme BCG-AVSS

Sharing Phase: Protocol BCG-AVSS-Sh

- **Stage I : Distributing Polynomials — the dealer D does the following**

[3] For this step, it is necessary that $n > 4t$. Else $|\mathcal{D}| \leq 3t$ and OEC will fail to let P_i obtain its desired row-polynomial.

  - On having the input $s \in \mathbb{F}$, pick a random degree-t *Shamir-sharing polynomial* $q(\cdot)$, such that $q(0) = s$ holds. Then pick a random degree-(t, t) bivariate polynomial $F(x, y)$, such that $F(0, y) = q(\cdot)$ holds.
  - For $i = 1, \ldots, n$, send the polynomials $f_i(x) = F(x, \alpha_i)$ and $g_i(y) = F(\alpha_i, y)$. to P_i.
- **Stage II : Pair-wise consistency checks and building consistency-graph — each party P_i**
  - Upon receiving $f_i(x), g_i(y)$ from D, send $f_{ij} = f_i(\alpha_j)$ and $g_{ij} = g_i(\alpha_j)$ to P_j, for $j = 1, \ldots, n$.
  - Upon receiving f_{ji}, g_{ji} from P_j, broadcast $(\mathtt{OK}, i, j)$ if $f_{ji} = g_i(\alpha_j)$ and $g_{ji} = f_i(\alpha_j)$ hold.
  - Construct a graph G_i over $\mathcal{P}$. Add the edge (P_j, P_k) in G_i, if $(\mathtt{OK}, j, k)$ and $(\mathtt{OK}, k, j)$ are received from the broadcast of P_j and P_k respectively. Keep updating G_i, upon receiving new $(\mathtt{OK}, \star, \star)$ messages.
- **Stage III : Finding star in the consistency-graph — the dealer D does the following**
  - Let G_D be the consistency-graph built by D. After every update in G_D, run the star-finding algorithm to check for the presence of a star in G_D. If a star $(\mathcal{C}, \mathcal{D})$ is found in G_D, then broadcast $(\mathcal{C}, \mathcal{D})$.
- **Stage IV : share computation — each party P_i does the following**
  - If $(\mathcal{C}, \mathcal{D})$ is received from the broadcast of D, then *accept* it if $(\mathcal{C}, \mathcal{D})$ is a star in the graph G_i.
  - If $(\mathcal{C}, \mathcal{D})$ is accepted, then compute the *share* s_i as follows and terminate.
    - If $P_i \in \mathcal{C}$, then set $s_i = f_i(0)$, where $f_i(x)$ is the degree-t row-polynomial received from D.
    - Else initialize W_i to $\emptyset$. Upon receiving g_{ji} from $P_j \in \mathcal{D}$, include g_{ji} to W_i. Keep updating W_i and keep executing OEC(W_i, t) till a degree-t polynomial $f_i(x)$ is obtained. Then set $s_i = f_i(0)$.

Reconstruction Phase: Protocol BCG-AVSS-Rec

Each party $P_i \in \mathcal{P}$ does the following.

- Send the share s_i to every party $P_j \in \mathcal{P}$.
- Initialize a set R_i to $\emptyset$. Upon receiving s_j from P_j, include s_j to R_i. Keep updating R_i and executing OEC(R_i, t) till a degree-t polynomial $q(\cdot)$ is obtained. Then output $s = q(0)$ and terminate.

Figure 13: The perfectly-secure AVSS scheme of Ben-Or, Canetti and Goldreich [11].

For the sake of completeness, we prove the properties of the scheme BCG-AVSS, as stated in Theorem 6.1. The proof for the follow-up AVSS schemes also use similar arguments.

Theorem 6.1. (BCG-AVSS-Sh, BCG-AVSS-Rec) *constitute a Type-II perfectly-secure AVSS scheme with respect to Shamir's t-out-of-n secret-sharing scheme.*

Proof. Let us first consider an *honest* D. The *privacy* simply follows from the fact that during BCG-AVSS-Sh, the adversary learns at most t rows and column-polynomials, lying on $F(x, y)$. If D is *honest*, then every pair of *honest* parties P_i, P_j *eventually* receive their respective polynomials and exchange the common points on their polynomials. Since the pair-wise consistency test will pass, P_i and P_j *eventually* broadcasts $(\mathtt{OK}, i, j)$ and $(\mathtt{OK}, j, i)$ messages respectively and from the properties of ACast, these messages are *eventually* delivered to every honest party. Since there are at least $n - t$ honest parties, the honest parties will *eventually* constitute a clique in every honest party's consistency graph. Consequently, D *eventually* finds a star $(\mathcal{C}, \mathcal{D})$ in its consistency graph and broadcasts it, which is *eventually* delivered to every honest party. Moreover, $(\mathcal{C}, \mathcal{D})$ is *eventually* accepted by every honest party, as $(\mathcal{C}, \mathcal{D})$ *eventually* forms a star in every honest party's consistency graph. Now consider an arbitrary *honest* P_i. If $P_i \in \mathcal{C}$, then it already has the polynomial $f_i(x)$ and hence it outputs $f_i(0)$ as its share and terminates BCG-AVSS-Sh. On the other hand, even if $P_i \notin \mathcal{C}$, it *eventually* receives the common point on its row polynomial from the honest parties in $\mathcal{D}$ as part of the pair-wise consistency checks and hence by applying OEC, it *eventually* computes $f_i(0)$ as its share and terminates BCG-AVSS-Sh. During BCG-AVSS-Rec, the share of every honest party is *eventually* delivered to every honest party. Consequently, by applying OEC, every honest party *eventually* reconstructs s. This proves the *correctness* property.

Next consider a *corrupt* D and let P_h be the *first honest* party to terminate Sh. This implies that P_h has received $(\mathcal{C}, \mathcal{D})$ from the broadcast of D, which constitutes a star in P_h's consistency graph. From the properties of ACast, every honest party will eventually receive and accept $(\mathcal{C}, \mathcal{D})$. This implies that the row-polynomials of the *honest* parties in $\mathcal{C}$ and the column-polynomials of the *honest* parties in $\mathcal{D}$ are pair-wise consistent and lie on a single degree-(t, t) bivariate polynomial, say $F^\star(x, y)$. Let $q^\star(\cdot) = F^\star(0, y)$ and $s^\star = q^\star(0)$. Consider an arbitrary *honest* P_i. If $P_i \in \mathcal{C}$, then it already has the row-polynomial $F^\star(x, \alpha_i)$ and hence it outputs $F^\star(0, \alpha_i) = q^\star(\alpha_i)$ as its share of $s^\star$. On the other hand, even if $P_i \notin \mathcal{C}$, it *eventually* receives the common point on its row polynomial from the honest parties in $\mathcal{D}$ as part of the pair-wise consistency checks and hence by applying OEC, it *eventually* computes $F^\star(0, \alpha_i)$ as its share. Moreover, it is easy to see that during $\mathsf{BCG\text{-}AVSS\text{-}Rec}$, every honest party eventually outputs $s^\star$. This proves the *strong commitment* property. □

6.2 PCR-AVSS Scheme

Patra, Choudhury and Rangan [50] observed that the $\mathsf{BCG\text{-}AVSS}$ scheme can be modified in a non-trivial way to generate a d-sharing of D's input in a verifiable fashion, for *any* given d in the range $t \leq d < n - 2t$. The resultant scheme is presented in Fig 14. The main motivation for an AVSS with the degree of sharing *greater* than t is to get efficient MPC protocols (see [28, 7, 8, 50]).

To d-share s, D picks a random degree-d *Shamir-sharing polynomial* $q(\cdot)$ with $q(0) = s$ and embeds it in a random degree-(d, t) bivariate polynomial $F(x, y)$ at $y = 0$. Thus the row and column-polynomials have *different* degrees. This is in *contrast* to the earlier schemes where $q(\cdot)$ has degree-t and where $q(\cdot)$ is embedded at $x = 0$ (instead of $y = 0$) in a random degree-(t, t) bivariate polynomial. D then distributes the row and column-polynomials and the parties publicly announce the results of pair-wise consistency checks and build consistency-graphs. D then proves that it has distributed consistent polynomials to "sufficiently many" parties, derived from a single degree-(d, t) bivariate polynomial, say $F^\star(x, y)$ (where $F^\star(x, y) = F(x, y)$ for an *honest* D). This stage is *different* from $\mathsf{BCG\text{-}AVSS\text{-}Sh}$ and constitutes the core of $\mathsf{PCR\text{-}AVSS\text{-}Sh}$. In a more detail, D publicly proves that it has delivered degree-d row-polynomials lying on $F^\star(x, y)$, to at least $n - t = 3t + 1$ parties $\mathcal{E}$ and degree-t column-polynomials lying on $F^\star(x, y)$, to at least $n - t = 3t + 1$ parties $\mathcal{F}$ (the sets $\mathcal{E}$ and $\mathcal{F}$ need not be the same). Once the existence of $(\mathcal{E}, \mathcal{F})$ is confirmed (we will discuss in the sequel how such sets are identified), D's sharing is completed by ensuring that every P_i gets its degree-d column-polynomial $g_i(y)$. Party P_i can then output $g_i(0)$ (which is the same as $F^\star(\alpha_i, 0)$) as its share and the value $s^\star = F^\star(0, 0)$ will be d-shared through $F^\star(x, 0)$. If $P_i \in \mathcal{F}$ then it will already have its $g_i(y)$ polynomial. For $P_i \notin \mathcal{F}$, we observe that *every* $P_j \in \mathcal{E}$ possesses a share on $g_i(y)$ in the form of a point on P_j's row-polynomial and which P_j would have sent to P_i as part of pair-wise consistency test. Since there are at least $3t + 1$ such parties P_j in $\mathcal{E}$ and since $g_i(y)$ has degree-t, party P_i can reconstruct $g_i(y)$ by using the OEC mechanism.

If D is *honest*, then Adv learns at most t rows and column-polynomials lying on $F(x, y)$. Now similar to Lemma 2.8, one can show that the probability distribution of these polynomials will be independent of $q(\cdot) = F(0, y)$. That is, for every candidate degree-d polynomial $q(\cdot)$, there exists some degree-(d, t) bivariate polynomial, consistent with the row and column-polynomials learnt by Adv. Intuitively this is because $(d + 1)(t + 1)$ distinct points are required to uniquely determine $F(x, y)$, but Adv learns at most $t(d + 1) + t$ distinct points on $F(x, y)$ through the polynomials of corrupt parties, leaving $d + 1 - t$ "degree of freedom" from the view-point of Adv. This ensures the privacy of D's input. We next discuss how the $(\mathcal{E}, \mathcal{F})$ sets are identified.

Dealer first finds a star $(\mathcal{C}, \mathcal{D})$ in its consistency-graph, proving that the row and column-polynomials of the honest parties in $\mathcal{C}$ and $\mathcal{D}$ respectively lie on $F^\star(x, y)$. This follows from Lemma 2.7 and the fact that the degree-d row-polynomials and the degree-t column-polynomials of *honest* parties in $\mathcal{C}$ and $\mathcal{D}$ respectively are pair-wise consistent, where $\mathcal{C}$ and $\mathcal{D}$ has $t+1$ and $n-2t = d+1$ *honest* parties respectively. To find $(\mathcal{E}, \mathcal{F})$, the dealer finds *additional* "supportive parties" whose row and column-polynomials also lie on $F^\star(x, y)$. The idea is that for an *honest* D, the row and column-polynomials of *all* honest parties lie on $F(x, y)$ and there are $n-t$ honest parties. To hunt for these additional supportive parties, D follows the following two-stage non-intuitive approach.

- It first tries to "expand" $\mathcal{D}$ by identifying additional parties whose degree-t column-polynomials also lie on $F^\star(x, y)$. The expanded set $\mathcal{F}$, includes all the parties having edges with at least $2t+1$ parties from $\mathcal{C}$. Thus $\mathcal{D} \subseteq \mathcal{F}$. It is easy to see that the column-polynomial of every $P_j \in \mathcal{F}$ lies on $F^\star(x, y)$, as it will be pair-wise consistent with the row-polynomials of at least $t+1$ *honest* parties from $\mathcal{C}$, all of which lie on $F^\star(x, y)$.
- D then tries to "expand" $\mathcal{C}$ by searching for the parties P_j, who have an edge with at least $d+t+1$ parties from $\mathcal{F}$. This will guarantee that P_j's degree-d row-polynomial lies on $F^\star(x, y)$, as it will be pair-wise consistent with the column-polynomials of at least $d+1$ *honest* parties from $\mathcal{F}$, all of which lie on $F^\star(x, y)$. Such parties P_j are included by D in a set $\mathcal{E}$. Notice that the parties in $\mathcal{C}$ will satisfy the above condition and so $\mathcal{C} \subseteq \mathcal{E}$.

Once D finds $(\mathcal{E}, \mathcal{F})$, it broadcasts them and then the parties verify whether indeed they satisfy the above conditions. However there is a subtle issue, as an *honest* D may have to wait *indefinitely* for the "expansion" of $\mathcal{D}$ and $\mathcal{C}$ sets, beyond their initial cardinalities. For instance, let $n = 4t+1$, $d = 2t$ and let $\mathcal{C}$ and $\mathcal{D}$ be of size $2t+1$ and $3t+1$ respectively, containing t corrupt parties. If the *corrupt* parties P_i in $\mathcal{C}$ choose to be *inconsistent* with the parties P_j *outside* $\mathcal{D}$ (by *not* broadcasting the $(\mathtt{OK}, i, j)$ messages), then the *honest* parties P_j *outside* $\mathcal{D}$ will have edges with only $t+1$ parties from $\mathcal{C}$ and will *not* be included in the set $\mathcal{F}$. So $\mathcal{F}$ will remain the same as $\mathcal{D}$. Similarly, the *corrupt* parties in $\mathcal{F}$ may choose to be inconsistent with the parties *outside* $\mathcal{C}$, due to which $\mathcal{E}$ will remain the same as $\mathcal{C}$. To deal with the above, Patra et al. [50] observed that if D is *honest* then eventually the *honest* parties form a clique in the consistency-graph. Moreover, if the star-finding algorithm is executed on "this" instance of the consistency graph, then the $\mathcal{C}$ component of the obtained star will have at least $2t+1$ *honest* parties. Now if $\mathcal{C}$ contains at least $2t+1$ honest parties, then eventually $\mathcal{D}$ will expand to $\mathcal{F}$, which will contain all $n-t$ honest parties and eventually $\mathcal{C}$ will expand to $\mathcal{E}$ containing $n-t = 3t+1$ parties. This crucial observation is at the heart of PCR-AVSS-Sh. However, it is difficult for D to identify an instance of its dynamic consistency-graph that contains a clique involving at least $n-t$ honest parties. The way-out is to repeatedly run the star-finding algorithm and try the expansion of every instance of star $(\mathcal{C}, \mathcal{D})$ obtained in the consistency-graph.

Scheme PCR-AVSS

Sharing Phase: Protocol PCR-AVSS-Sh

- **Stage I : Distributing Polynomials** — same steps as BCG-AVSS-Sh except that the *Shamir-sharing polynomial* $q(\cdot)$ is of degree-d, the bivariate polynomial $F(x, y)$ is of degree-(d, t) and $F(x, 0) = q(\cdot)$.
- **Stage II : Pair-wise consistency checks and building consistency-graph** — same as BCG-AVSS-Sh.
- **Stage III : Finding $(\mathcal{E}, \mathcal{F})$ in the consistency-graph — the dealer D does the following**
  - After every update in G_D, run the star-finding algorithm to check for the presence of a star. Let there are α number of distinct stars that are found till now in G_D, where $\alpha \geq 0$.
    - If a new star $(\mathcal{C}_{\alpha+1}, \mathcal{D}_{\alpha+1})$ is found in G_D, then do the following:
      1. Add P_j to a set $\mathcal{F}_{\alpha+1}$ if P_j has an edge with at least $2t+1$ parties from $\mathcal{C}_{\alpha+1}$ in G_D.
      2. Add P_j to a set $\mathcal{E}_{\alpha+1}$ if P_j has an edge with at least $d+t+1$ parties from $\mathcal{F}_{\alpha+1}$ in G_D.
      3. For $\beta = 1, \ldots, \alpha$, update the existing $\mathcal{F}_\beta$ and $\mathcal{E}_\beta$ sets as follows:
         - Add P_j to $\mathcal{F}_\beta$, if $P_j \notin \mathcal{F}_\beta$ and P_j has an edge with at least $2t+1$ parties from $\mathcal{C}_\beta$ in

G_D.

- Add P_j to $\mathcal{E}_\beta$, if $P_j \notin \mathcal{E}_\beta$ and P_j has an edge with at least $d+t+1$ parties from $\mathcal{F}_\beta$ in G_D.

  – If no new star is obtained, then update the existing sets $\mathcal{F}_\beta, \mathcal{E}_\beta$ by executing the step 3 as above.

– Let $(\mathcal{E}_\gamma, \mathcal{F}_\gamma)$ be the first pair among the generated pairs $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ such that $|\mathcal{E}_\gamma| \geq 3t+1$ and $|\mathcal{F}_\gamma| \geq 3t+1$. Then broadcast $((\mathcal{C}_\gamma, \mathcal{D}_\gamma), (\mathcal{E}_\gamma, \mathcal{F}_\gamma))$.

- **Stage IV : share computation — each party P_i does the following**
  - If $((\mathcal{C}_\gamma, \mathcal{D}_\gamma), (\mathcal{E}_\gamma, \mathcal{F}_\gamma))$ is received from the broadcast of D, *accept* it if all the following hold.
    - $|\mathcal{E}_\gamma| \geq 3t+1$ and $|\mathcal{F}_\gamma| \geq 3t+1$ and $(\mathcal{C}_\gamma, \mathcal{D}_\gamma)$ is a star in the consistency-graph G_i.
    - Every party $P_j \in \mathcal{F}_\gamma$ has an edge with at least $2t+1$ parties from $\mathcal{C}_\gamma$ in G_i.
    - Every party $P_j \in \mathcal{E}_\gamma$ has an edge with at least $d+t+1$ parties from $\mathcal{F}_\gamma$ in G_i.
  - If $((\mathcal{C}_\gamma, \mathcal{D}_\gamma), (\mathcal{E}_\gamma, \mathcal{F}_\gamma))$ is accepted, then compute the *share* s_i as follows and terminate.
    - If $P_i \in \mathcal{F}_\gamma$, then set $s_i = g_i(0)$, where $g_i(y)$ is the degree-t column-polynomial received from D.
    - Else initialize W_i to $\emptyset$. Upon receiving f_{ji} from $P_j \in \mathcal{E}$, include f_{ji} to W_i. Keep updating W_i and keep executing $\mathsf{OEC}(W_i, t)$ till a degree-t polynomial $g_i(y)$ is obtained. Then set $s_i = g_i(0)$.

Reconstruction Phase: Protocol PCR-AVSS-Rec

Same steps as BCG-AVSS-Rec, except that the parties now run $\mathsf{OEC}(\star, d)$ to recover a degree-d polynomial.

Figure 14: The perfectly-secure AVSS scheme of Patra, Choudhury and Rangan [50].

6.3 CHP-AVSS Scheme

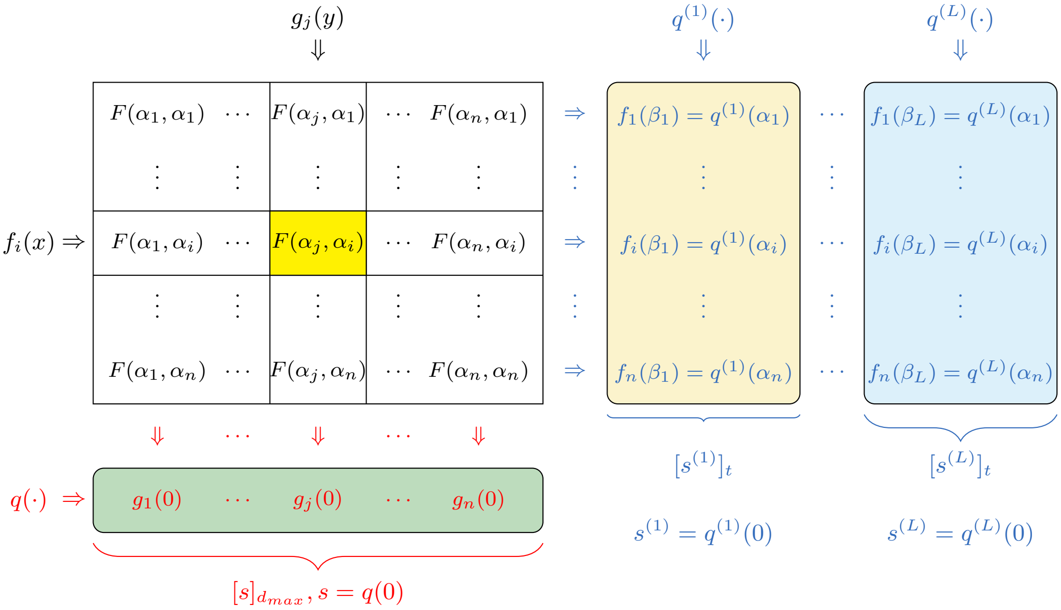
Protocol BCG-AVSS-Sh requires a communication of $\mathcal{O}(n^2 \log |\mathbb{F}|)$ bits over the pair-wise channels, apart from the broadcast of $\Theta(n^2)$ OK messages and the broadcast of star. The protocol generates t-sharing of a *single* secret. If D wants to t-share L secrets, then it can invoke L instances of BCG-AVSS-Sh. This makes the *broadcast-complexity* (namely the number of bits to be broadcast) proportional to L. Instead Choudhury, Hirt and Patra [21] proposed a modification of [4] PCR-AVSS-Sh, which allows D to t-share L secrets for *any* given $L \geq n-3t$, *without* incurring any additional communication complexity. The broadcast-complexity of CHP-AVSS-Sh will be *independent* of L, which is a significant saving. This is because each instance of the broadcast in the asynchronous setting needs to be emulated by running the costly Bracha's ACast protocol.

We explain the idea of CHP-AVSS-Sh assuming $L = n-3t$. If $L > n-3t$, then D can divide its inputs into multiple batches of $n-3t$ and invoke an instance of CHP-AVSS-Sh for each batch. Recall that in PCR-AVSS-Sh, if D is *honest*, then the adversary's view leaves $d+1-t$ "degree of freedom" in the degree-(d,t) bivariate polynomial $F(x,y)$, where $t < d < n-2t$. If we consider the *maximum* value d_{max} of d which is $n-2t-1$, this implies $n-3t$ degree of freedom. While PCR-AVSS-Sh uses this degree of freedom for generating a d_{max}-sharing of a *single* secret by embedding a *single* degree-d_{max} sharing-polynomial in $F(x,y)$, CHP-AVSS-Sh uses it for t-sharing of $n-3t$ values by embedding $n-3t$ degree-t Shamir-sharing polynomials in $F(x,y)$.

In a more detail, given $s^{(1)}, \ldots, s^{(n-3t)}$ for t-sharing, D picks $n-3t$ random degree-t Shamir-sharing polynomials $q^{(1)}(\cdot), \ldots, q^{(n-3t)}(\cdot)$, where $q^{(k)}(0) = s^{(k)}$. These polynomials are embedded in a degree-(d_{max}, t) bivariate polynomial, which is otherwise a random polynomial, except that $F(\beta_k, y) = q^{(k)}(\cdot)$ holds. Here $\beta_1, \ldots, \beta_{n-3t}$ are distinct, publicly-known non-zero elements from $\mathbb{F}$,

[4] It is easy to see that the communication complexity of PCR-AVSS-Sh is the same as BCG-AVSS-Sh.

different from the evaluation-points $\alpha_1, \ldots, \alpha_n$ (this requires $|\mathbb{F}| > 2n - 3t$). Notice that the embedding and the degree of the sharing-polynomials are different in PCR-AVSS-Sh and CHP-AVSS-Sh. Accordingly, the shares of the parties are different (see Fig 15). The *shares* of P_i in CHP-AVSS-Sh will be $\{F(\beta_k, \alpha_i)\}_{k \in \{1,\ldots,n-3t\}}$. And to compute them, P_i should get its row-polynomial $f_i(x) = F(x, \alpha_i)$, as P_i can then compute its shares by evaluating $f_i(x)$ at $x = \beta_1, \ldots, \beta_{n-3t}$.



$g_j(y)$
$q^{(1)}(\cdot)$
$q^{(L)}(\cdot)$
$F(\alpha_1,\alpha_1)$ $\cdots$ $F(\alpha_j,\alpha_1)$ $\cdots$ $F(\alpha_n,\alpha_1)$
$f_i(x) \Rightarrow$
$F(\alpha_1,\alpha_i)$ $\cdots$ $F(\alpha_j,\alpha_i)$ $\cdots$ $F(\alpha_n,\alpha_i)$
$F(\alpha_1,\alpha_n)$ $\cdots$ $F(\alpha_j,\alpha_n)$ $\cdots$ $F(\alpha_n,\alpha_n)$
$f_1(\beta_1) = q^{(1)}(\alpha_1)$
$f_i(\beta_1) = q^{(1)}(\alpha_i)$
$f_n(\beta_1) = q^{(1)}(\alpha_n)$
$f_1(\beta_L) = q^{(L)}(\alpha_1)$
$f_i(\beta_L) = q^{(L)}(\alpha_i)$
$f_n(\beta_L) = q^{(L)}(\alpha_n)$
$q(\cdot) \Rightarrow$
$g_1(0)$ $\cdots$ $g_j(0)$ $\cdots$ $g_n(0)$
$[s]_{d_{max}}, s = q(0)$
$[s^{(1)}]_t$
$s^{(1)} = q^{(1)}(0)$
$[s^{(L)}]_t$
$s^{(L)} = q^{(L)}(0)$

Figure 15: Values distributed by D in PCR-AVSS-Sh and CHP-AVSS-Sh on degree-(d_{max}, t) polynomial $F(x, y)$, where $d_{max} = n - 2t - 1$. In PCR-AVSS-Sh, s is d_{max}-shared through $F(x, 0)$ (shown in red color), while in CHP-AVSS-Sh, $s^{(1)}, \ldots, s^{(L)}$ are t-shared through $F(\beta_1, y), \ldots, F(\beta_L, y)$ (shown in blue color), where $L = n - 3t$.

To achieve the above goal, we observe that if D invokes PCR-AVSS-Sh (with the above modifications) and if the protocol terminates, then it ensures that D has "committed" a degree-(d_{max}, t) bivariate polynomial $F^\star(x, y)$, such that each (honest) party P_j possesses its column-polynomial $g_j(y) = F^\star(\alpha_j, y)$. We also observe that for each row-polynomial $f_i(x) = F^\star(x, \alpha_i)$, every P_j holds a share $F^\star(\alpha_j, \alpha_i)$ in the form of $g_j(\alpha_i)$. Moreover, the degree of $f_i(x)$ is $d_{max} = n - 2t - 1$. Hence, if every party P_j sends its share $g_j(\alpha_i)$ of $f_i(x)$ to P_i, then P_i can reconstruct its desired row-polynomial $f_i(x)$ by applying OEC on the received values. Hence the values $\vec{S'} = (F^\star(\beta_1, 0), \ldots, F^\star(\beta_{n-3t}, 0))$ will be t-shared, where $\vec{S'} = (s^{(1)}, \ldots, s^{(n-3t)})$ for an *honest* D.

Part III : Hybrid Communication Setting

7 Preliminaries and Definitions for Hybrid Communication Setting

Even though the asynchronous model is practically more relevant compared to the synchronous setting, there are some inherent downsides in general with asynchronous protocols. Synchronous protocols offer better *resilience*, compared to asynchronous protocols (for instance $t < n/4$ for AVSS compared to $t < n/3$ for VSS). Additionally, asynchronous protocols are more complex. An inherent drawback of *asynchronous* MPC (AMPC) protocols is the lack of *input-provision* from *all honest* parties; i.e. inputs of up to t *honest* parties *may not* be considered for the computation [7]. To get rid of these drawbacks, several forms of "synchronization" have been considered in the literature ([27, 9, 35, 45, 2, 51]). One of these is the *hybrid* communication setting [7, 9, 50, 51, 20, 19], which is a "mix" of synchronous and asynchronous setting. Namely, the first R rounds are assumed to be synchronous, after which the network behaves *asynchronously.* There are several practical advantages of the hybrid setting compared to a completely asynchronous setting. For instance, one can guarantee *input-provision* in MPC protocols. Moreover, one can design protocols with the *same* resilience, as synchronous protocols, *without* letting the network to be synchronous for the *entire* duration of the protocol. Also, it is theoretically interesting to study the trade-off between network synchrony and resilience, round and communication-complexity of various distributed-computing tasks. In the context of VSS, Patra and Ravi [51] have shown that one can design a Type-II perfectly-secure AVSS satisfying Definition 5.1 with $t < n/3$ in the *hybrid* setting (which otherwise would require $t < n/4$ in a *completely asynchronous* setting), where $R = 1$. Notice that the protocol has optimal resilience as well as it requires the optimal number of synchronous rounds.

We next define *weak polynomial sharing* (WPS), which is used in [51]. The primitive allows a dealer D to distribute shares of a degree-t polynomial held by D. If D is *honest*, then every honest party should eventually terminate with its share. Moreover, even if D is *corrupt* and some honest party terminates Sh, then it is ensured that D has distributed shares of some degree-t polynomial to at least $t + 1$ honest parties. Property-wise, WPS is a *weaker* primitive than WSS, as it has only a sharing phase and *may not* allow even a "weak reconstruction" of D's shared polynomial.

Definition 7.1 (Weak Polynomial Sharing (WPS) [51]). Let Sh be protocol for the n parties in the hybrid setting, where a designated *dealer* $\mathsf{D} \in \mathcal{P}$ has a degree-t polynomial $f(x)$ over $\mathbb{F}$ as input for Sh. Then Sh constitutes a *perfectly-secure WPS*, if all the following hold.

- **Termination and Privacy**: Same as AVSS.
- **Correctness**: If some honest party terminates Sh, then there exists a degree-t *weakly committed polynomial* $f^\star(x)$ over $\mathbb{F}$ such that.
  - If D is *honest*, then $f^\star(x) = f(x)$ and each honest P_i outputs $f(\alpha_i)$ at the end of Sh.
  - If D is *corrupt*, then every honest P_i outputs either $f^\star(\alpha_i)$ or some default value $\perp$, with at least $t + 1$ honest parties outputting $f^\star(\alpha_i)$.

8 Hybrid AVSS Protocol with $t < n/3$

In the AVSS schemes, the parties not receiving their shares from D, deploy OEC to recompute their shares from the sub-shares received from the parties, who have received their shares from D. This inherently requires $n > 4t$. On contrary, the hybrid AVSS of Patra et al. [51] is designed with $n > 3t$. The presence of a synchronous round at the beginning simplifies certain aspects of verifiability and completely avoids the need for OEC. We first start with the WPS construction of [51], which is similar to the $\mathsf{3KKK\text{-}WSS\text{-}Sh}$ protocol. The dealer embeds its degree-t polynomial in a random *symmetric* degree-(t, t) bivariate polynomial and distributes its row-polynomials. In parallel, the parties pair-wise exchange random pads. Since the pads are exchanged during the *synchronous* round, each P_i receives the pad selected for it by every other party, at the end of the synchronous round. The sent and received pads are also "registered" with D for the comparison purpose. Based on this, D sends to P_i its list of conflicting-parties $\mathcal{C}_i$, who did not concur on the pads. Based on $\mathcal{C}_i$, party P_i broadcasts its common values in a masked fashion for the parties who are not in $\mathcal{C}_i$ and in an unmasked fashion for the parties who are in $\mathcal{C}_i$. The dealer then checks whether P_i's public values are consistent with the bivariate polynomial and the pads registered with D and accordingly includes P_i to a set $\mathcal{W}$. Once $\mathcal{W}$ achieves the size of $2t + 1$ (which *eventually* happens for an *honest* D), D broadcasts $\mathcal{W}$. It is then publicly verified that no pair of parties in $\mathcal{W}$ publicly conflicts over their supposedly common values that are either in padded or in clear form.

If $\mathcal{W}$ is successfully verified, then the row-polynomials of the (honest) parties in $\mathcal{W}$ lie on a single degree-(t,t) symmetric bivariate polynomial $F^{\star}(x,y)$. While every $P_i \in \mathcal{W}$ can output the constant term of its row-polynomial as its share, any $P_i \notin \mathcal{W}$ tries to compute its row-polynomial by interpolating the common values with the parties in $\mathcal{W}$. If P_i has a conflict with any party in $\mathcal{W}$, then the common value is publicly available. Else, it subtracts the pad it sent to that party in the synchronous round from the padded value available publicly. If the interpolation does not give a degree-t polynomial (which can happen only for a *corrupt* D), then P_i outputs $\perp$.

Protocol WPS

Synchronous Phase

- **Sending polynomials and exchanging random pads**:
  - D with input $f(\cdot)$ chooses a random symmetric degree-(t,t) bivariate $F(x,y)$ such that $F(0,y) = f(\cdot)$ and sends $f_i(x) = F(x,\alpha_i)$ to each party $P_i \in \mathcal{P}$
  - Each party $P_i \in \mathcal{P}$ picks a random pad r_{ij} for every $P_j \in \mathcal{P}$ and sends r_{ij} to P_j.
  - Each P_i sends $\{r_{ij}\}_{P_j \in \mathcal{P}}$ to D. Let $\{r_{ij}^{(1)}\}_{P_j \in \mathcal{P}}$ be the list of *sent-pads* received by D from P_i.

Asynchronous Phase

- **Verifying masks**:
  - Each P_i sends the pads $\{r'_{ji}\}_{P_j \in \mathcal{P}}$ received from various parties to D. Let $\{r_{ji}^{(2)}\}_{P_j \in \mathcal{P}}$ be the list of *received-pads* which D receives from P_i.
  - Upon receiving $\{r_{ji}^{(2)}\}_{P_j \in \mathcal{P}}$ from P_i, dealer D sends to P_i a set $\mathcal{C}_i = \{P_j : r_{ji}^{(2)} \neq r_{ji}^{(1)}\}$.
- **Broadcasting masked/unmasked common values — each party** P_i: Broadcasts $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i)$, where:
  - $\mathcal{A}_i = \{a_{ij} = f_i(\alpha_j) + r_{ij}\}_{P_j \in \mathcal{P}}$.
  - $\mathcal{B}_i = \{b_{ij}\}_{P_j \in \mathcal{P}}$, where $b_{ij} = f_i(\alpha_j)$ if $P_j \in \mathcal{C}_i$ and $b_{ij} = f_i(\alpha_j) + r'_{ji}$ otherwise.
- **Constructing and broadcasting $\mathcal{W}$ — Dealer D does the following**:
  - Upon receiving $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i)$ from P_i, mark P_i as *correct* and include in $\mathcal{W}$, if all the following holds.
    - $a_{ij} - r_{ij}^{(1)} = F(\alpha_j, \alpha_i)$
    - $b_{ij} = F(\alpha_j, \alpha_i)$ for all $P_j \in \mathcal{C}_i$ and $b_{ij} - r_{ji}^{(2)} = F(\alpha_j, \alpha_i)$ otherwise
    - $\mathcal{C}_i$ is the same set sent by D to P_i.
  - Wait until $|\mathcal{W}| \geq 2t+1$ and then broadcast $\mathcal{W}$.
- **Verifying $\mathcal{W}$ — each party** P_i: *Accept* $\mathcal{W}$ received from the broadcast of D if all the following hold.
  - $|\mathcal{W}| \geq 2t+1$ and $(\mathcal{A}_j, \mathcal{B}_j, \mathcal{C}_j)$ is received from the broadcast of each $P_j \in \mathcal{W}$.
  - Every $P_j, P_k \in \mathcal{W}$ are *pair-wise* consistent, as per the following conditions.
    - if $P_j \in \mathcal{C}_k$ and $P_k \in \mathcal{C}_j$ then $b_{jk} = b_{kj}$
    - if $P_j \in \mathcal{C}_k$ and $P_k \notin \mathcal{C}_j$ then $a_{kj} = b_{jk}$
    - if $P_j \notin \mathcal{C}_k$ and $P_k \in \mathcal{C}_j$ then $a_{jk} = b_{kj}$
    - Else $a_{jk} = b_{kj}$ and $a_{kj} = b_{jk}$
- **Output stage — each party** P_i: If $\mathcal{W}$ is accepted, then terminate with output s_i, computed as follows.
  - If $P_i \in \mathcal{W}$ then set $s_i = f_i(0)$.
  - Else interpolate the points $\{(\alpha_j, s_{ij})\}_{P_j \in \mathcal{W}}$ where $s_{ij} = b_{ji}$ if $P_i \in \mathcal{C}_j$ and $s_{ij} = b_{ji} - r_{ij}$ otherwise. If the interpolation outputs a degree-t polynomial $f_i(x)$ then set $s_i = f_i(0)$, else set $s_i = \perp$.

Figure 16: WPS protocol of Patra and Ravi [51].

From WPS to VSS WPS fails to serve as a VSS because if D is *corrupt*, then the parties outside $\mathcal{W}$ may output $\perp$. Protocol PR-Sh fixes this shortcoming. The protocol has two "layers" of communication. The first layer is similar to WPS and identifies a set $\mathcal{V}$ of $2t+1$ parties whose row-polynomials lie on a single degree-(t, t) symmetric bivariate polynomial $F^{\star}(x, y)$. The second layer enables the parties *outside* $\mathcal{V}$ to obtain their row-polynomials lying on $F^{\star}(x, y)$. In a more detail, every P_j picks a random *blinding-polynomial* $r_j(\cdot)$ and shares it by invoking an instance WPS_j of WPS. Additionally, it makes public the polynomial $r_j(\cdot) + f_j(x)$. The idea is that if later P_j is a part of $\mathcal{V}$, then any $P_i \notin \mathcal{V}$ can compute the point $f_j(\alpha_i)$ (which is the same as $f_i(\alpha_j)$) on P_i's row-polynomial, if P_i obtains the output $r_j(\alpha_i)$ during WPS_j. While an *honest* $P_j \in \mathcal{V}$ makes public the correct $r_j(\cdot) + f_j(x)$, care has to be taken to ensure that even a *corrupt* $P_j \in \mathcal{V}$ has made public the correct polynomial. This is done as follows. First, each P_k participates *conditionally* during WPS_j only if the blinded polynomial of P_k is consistent with respect to its received point on $r_k(\cdot)$ during WPS_k and the supposedly common value $f_k(j)$. Second, P_j is included in $\mathcal{V}$ only when during WPS_j the generated $\mathcal{W}$ set $\mathcal{W}_j$ is *accepted* and which has an overlap of $2t + 1$ with $\mathcal{V}$.

For a *corrupt* $P_j \in \mathcal{V}$, an *honest* $P_i \notin \mathcal{V}$ may end up obtaining $\perp$ during WPS_j. However there will be at least $t + 1$ *honest* $P_j \in \mathcal{V}$, corresponding to whom P_i eventually obtains $r_j(\alpha_i)$ during WPS_j, using which P_i obtains $t + 1$ points on $f_i(x)$, which are sufficient to compute $f_i(x)$.

Protocol PR-Sh

Synchronous Phase

D and parties execute the same steps as in the synchronous phase of WPS. Additionally, each P_i picks a random degree-t *blinding-polynomial* $r_i(\cdot)$ and as a dealer invokes an instance WPS_i of WPS to share $r_i(\cdot)$. Moreover, P_i also participates in the synchronous phase of the instance WPS_j for every $P_j \in \mathcal{P}$.

Asynchronous Phase

- **Verifying masks**: Parties and D execute the same steps as in WPS.
- **Broadcasting values — each party** P_i: Broadcast $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, d_i(x))$, where $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$ are same as in WPS and $d_i(x) = r_i(\cdot) + f_i(x)$.
- **Participating in WPS instances — each party** P_i: For $j = 1, \ldots, n$, participate in WPS_j if a degree-t polynomial $d_j(x)$ is received from the broadcast of P_j and if $d_j(\alpha_i) = r_j(\alpha_i) + f_i(\alpha_j)$ holds.
- **Computing and Broadcasting** $\mathcal{V}$ **— the** D: compute and broadcast $(\mathcal{V}, \{\mathcal{W}_i\}_{P_i \in \mathcal{V}})$, such that:
  - Every $P_i \in \mathcal{V}$ is marked as *correct* (by satisfying the same conditions as in WPS).
  - For every $P_i \in \mathcal{V}$, the set $\mathcal{W}_i$ is *accepted* during the instance WPS_i.
  - $|\mathcal{V}| \geq 2t + 1$ and $|\mathcal{V} \cap \mathcal{W}_i| \geq 2t + 1$ for every $P_i \in \mathcal{V}$.
- **Verifying** $\mathcal{V}$ **— each party** P_i: *Accept* $(\mathcal{V}, \{\mathcal{W}_i\}_{P_i \in \mathcal{V}})$ received from the broadcast of D, if:
  - Every $P_j, P_k \in \mathcal{V}$ are *pair-wise consistent* (using the same criteria as in WPS).
  - For all $P_j \in \mathcal{V}$, the set $\mathcal{W}_j$ is *accepted* during the instance WPS_j.
  - $|\mathcal{V}| \geq 2t + 1$ and for every $P_j \in \mathcal{V}$, the condition $|\mathcal{V} \cap \mathcal{W}_j| \geq 2t + 1$ holds.
- **Output stage — each party** P_i: If $(\mathcal{V}, \{\mathcal{W}_i\}_{P_i \in \mathcal{V}})$ is accepted then terminate with output s_i, where:
  - If $P_i \in \mathcal{V}$ then set $s_i = f_i(0)$.
  - Else compute the output r_{ji} in WPS_j for every $P_j \in \mathcal{V}$, interpolate degree-t polynomial $f_i(x)$ through the points $\{(\alpha_i, s_{ij} = d_j(\alpha_i) - r_{ji})\}_{P_j \in \mathcal{V} \wedge r_{ji} \neq \perp}$ and set $s_i = f_i(0)$.

Figure 17: The hybrid AVSS protocol of Patra and Ravi [51].

9 Open Problems

We identify the following open problems in the domain of *perfectly-secure* VSS.

- The communication complexity of existing *efficient* VSS schemes with a 3-round sharing phase is n times more compared to the VSS schemes which allow four or more rounds in the sharing phase (see Table 1). It is interesting to see if one can close this gap.
- We are unaware of any non-trivial lower bound on the communication complexity of perfectly-secure VSS schemes. One could explore deriving any non-trivial lower bound.

- The *broadcast complexity* (namely the number of bits broadcasted) of all VSS schemes with $n > 3t$ is proportional to the number of values L shared by the scheme (namely $\mathcal{O}(L \cdot n^3 \log |\mathbb{F}|)$ bits). This is unlike the AVSS scheme of Patra et al. [21], where the broadcast complexity is $\mathcal{O}(n^2 \log n)$ for sharing L values. Given that each instance of broadcast needs to be emulated by running a costly *reliable broadcast* (RB) protocol, it is interesting to see if one can design a VSS scheme with $n > 3t$ where the broadcast complexity is *independent* of L.
- The 3AKP-VSS VSS scheme [3] uses the broadcast channel during two rounds, while the optimal usage is *one* round. One could explore to achieve the same properties as the 3AKP-VSS scheme, with the broadcast channel being used only during one round.

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