Public-key Cryptosystems and Signature Schemes from p-adic Lattices

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Abstract

In 2018, the longest vector problem and closest vector problem in local fields were introduced, as the p-adic analogues of the shortest vector problem and closest vector problem in lattices of Euclidean spaces. They are considered to be hard and useful in constructing cryptographic primitives, but no applications in cryptography were given. In this paper, we construct the first signature scheme and public-key encryption cryptosystem based on p-adic lattice by proposing a trapdoor function with the orthogonal basis of p-adic lattice. These cryptographic schemes have reasonable key size and efficiency, which shows that p-adic lattice can be a new alternative to construct cryptographic primitives and well worth studying.

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1. Introduction

Since Diffie and Hellman invented public-key cryptography in 1976 [4], quite a few public-key cryptosystems based on computationally hard mathematical problems have been proposed.

Two famous hard problems are integer factorization and discrete logarithm problem, based on which lots of cryptosystems have been constructed. For example, the first practical public-key cryptosystem RSA [18] is based on integer factorization. The ElGamal cryptosystem [6] is based on the discrete logarithm problem in finite fields. The elliptic curve cryptosystem is based on discrete logarithm of elliptic

curves over finite fields [9, 15], and hyperelliptic curve cryptosystem is based on discrete logarithm of Jacobian of hyperelliptic curves over finite fields [10]. The two problems have not been proven to be NP-hard yet, and Peter Shor [20] found quantum polynomial time algorithms in 1994 for them, which yields that the classical public-key cryptosystems such as RSA and ElGamal would be broken under future quantum computer.

However, there are also other computationally hard mathematical problems that can be employed to construct public key cryptosystems. For instance, multivariate cryptography [12] is based on solving system of nonlinear equations over finite fields. The McEliece system [13] is based on decoding a random linear code over finite fields. Lattice-based cryptography [14, 7] is based on the shortest vector problem and closest vector problem in lattices of Euclidean spaces. These computational problems have been shown to be NP-hard, and the corresponding cryptosystems are widely believed to be quantum-resistant, which are the main candidates in the standardization of post quantum cryptography initiated by NIST [22]. Specially, some new hard computational problems have been proposed in the standardization, such as computing isogeny between elliptic curves [5, 11]. But it is still unknown if computing isogeny between elliptic curves is NP-hard or not.

Motivated by lattice-based cryptosystems, one of the most promising post quantum cryptosystems, Deng *et al.* [2, 3] introduced some new computational problems in p-adic lattices of local fields, the longest vector problem and closest vector problem which are the p-adic analogues of the shortest vector problem and closest vector problem in lattices of Euclidean spaces. It was expected in [2, 3] that the new problems can be used to construct public key cryptosystems, which was left as an open problem.

In this paper, we try to solve the problem by constructing a signature scheme and a public-key encryption scheme. The basic idea is very similar to the code-based McEliece system [13] or the lattice-based GGH scheme [7], that is, we adopt a good basis as the private key and transform it into a bad basis as the public key. With the good basis, we can efficiently solve the hard problem in p-adic lattice, while the bad basis looks random that may not help solve the hard problem. We show that an orthogonal basis for a given p-adic lattice can be the good basis. More precisely, we show that if there is an orthogonal basis for a given p-adic lattice, then the longest vector problem and closest vector problem in local fields are easy to solve. Then the orthogonal bases can be used to construct trapdoor information for cryptographic schemes. Finally we propose a signature scheme and a public-key cryptosystem based on p-adic lattices.

We would like to point out that as main candidate of the post quantum cryptography, cryptography based on lattices in Euclidean spaces have obtained extensive study in recent years. However, p-adic lattices do not gain any attention. As the p-adic analogues of the lattices in Euclidean spaces, it is reasonable to expect that

the problem could be quantum-resistant. Our results shows that p-adic lattices may be useful in cryptography and it is worth for further study, which provides a new alternative candidate to construct cryptographic primitives.

Roadmap. The paper is organized as follows. We recall basic fact about local fields, the p-adic lattices, the longest vector problem (LVP) and closest vector problem (CVP) in Section 2 and present the fast algorithms to solve LVP and CVP in local fields with the help of an orthogonal basis in Section 3. We then construct a signature scheme in Section 4 and a public-key cryptosystem in Section 5. We give some possible attacks to our schemes in Section 6 and we report our experimental results in Section 7.

2. Local fields and p-adic lattices

In this section, we recall some basic facts about local fields, see [2, 3]. For detailed study of local fields, please see [8, 1, 19].

2.1. Basic facts about local fields

Let p be a prime number. For $x \in \mathbb{Q}$ with $x \neq 0$, write $x = p^t \frac{a}{b}$ with $t, a, b \in \mathbb{Z}$ and $p \nmid ab$. Define $\mid x \mid_p = p^{-t}$ and $\mid 0 \mid_p = 0$. Then $\mid \cdot \mid_p$ is a non-Archimedean absolute value on $\mathbb{Q}$. Namely, we have: (1) $\mid x \mid_p \geq 0$ and $\mid x \mid_p = 0$ if and only if x=0; (2) $\mid xy \mid_p = \mid x \mid_p \mid y \mid_p$; (3) $\mid x + y \mid_p \leq \max(\mid x \mid_p, \mid y \mid_p)$. If $\mid x \mid_p \neq \mid y \mid_p$, then $\mid x + y \mid_p = \max(\mid x \mid_p, \mid y \mid_p)$.

Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to $\mid \cdot \mid_p$. Denote $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid\mid x \mid_p \leq 1\}$. We have

$$\mathbb{Q}_p = \{\sum_{i=j}^{\infty} a_i p^i \mid a_i \in \{0, 1, 2, \ldots, p-1\}, i \geq j, j \in \mathbb{Z}\},$$

and

$$\mathbb{Z}_p = \{\sum_{i=0}^{\infty} a_i p^i \mid a_i \in \{0, 1, 2, \ldots, p-1\}, i \geq 0\}.$$

$\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is locally compact. $\mathbb{Z}_p$ is a discrete valuation ring, it has a unique nonzero principal maximal ideal $p\mathbb{Z}_p$ and p is called a uniformizer of $\mathbb{Q}_p$. The unit group of $\mathbb{Z}_p$ is $\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p \mid\mid x \mid_p = 1\}$. The residue class field $\mathbb{Z}_p/p\mathbb{Z}_p$ is a finite field with p elements.

Let n be a positive integer, and let K be an extension field of $\mathbb{Q}_p$ of degree n. We fix some algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and view K as a subfield of $\overline{\mathbb{Q}}_p$. Such K exists. For example, let $K = \mathbb{Q}_p(\alpha)$ with $\alpha^n = p$. Because $X^n - p$ is an Eisenstein polynomial over $\mathbb{Q}_p$, it is irreducible over $\mathbb{Q}_p$, then K has degree n over $\mathbb{Q}_p$. Further,

there are only finitely many extension fields of $\mathbb{Q}_p$ of degree n contained in $\overline{\mathbb{Q}}_p$, see [16]. The p-adic absolute value (or norm) $|\cdot|_p$ on $\mathbb{Q}_p$ can be extended uniquely to K, i.e., for $x \in K$, we have $| x |_p = | N_{K/\mathbb{Q}_p}(x) |_p^{\frac{1}{n}}$, where $N_{K/\mathbb{Q}_p}$ is the norm map from K to $\mathbb{Q}_p$. K is complete with respect to $|\cdot|_p$. See [1] for a proof.

Denote $\mathcal{O}_K = \{x \in K \mid\mid x |_p \leq 1\}$. $\mathcal{O}_K$ is also a discrete valuation ring, it has a unique nonzero principal maximal ideal $\pi\mathcal{O}_K$ and π is called a uniformizer of K. $\mathcal{O}_K$ is a free $\mathbb{Z}_p$-module of rank n. $\mathcal{O}_K$ is compact and K is locally compact. The unit group of $\mathcal{O}_K$ is $\mathcal{O}_K^\times = \{x \in K \mid\mid x |_p = 1\}$. The residue class field $\mathcal{O}_K/\pi\mathcal{O}_K$ is a finite extension of $\mathbb{Z}_p/p\mathbb{Z}_p$. Call the positive integer $f = [\mathcal{O}_K/\pi\mathcal{O}_K : \mathbb{Z}_p/p\mathbb{Z}_p]$ the residue field degree of $K/\mathbb{Q}_p$. As ideals in $\mathcal{O}_K$, we have $p\mathcal{O}_K = \pi^e\mathcal{O}_K$. Call the positive integer e the ramification index of $K/\mathbb{Q}_p$. We have $n = [K : \mathbb{Q}_p] = ef$. When $e = 1$, the extension $K/\mathbb{Q}_p$ is unramified, and when $e = n$, $K/\mathbb{Q}_p$ is totally ramified. Each element x of the multiplicative group $K^\times$ of nonzero elements of K can be written uniquely as $x = u\pi^t$ with $u \in \mathcal{O}_K^\times$ and $t \in \mathbb{Z}$. We have $p = u\pi^e$ with $u \in \mathcal{O}_K^\times$, so $| \pi |_p = p^{-\frac{1}{e}}$. The valuation group of K is

$$\{| x |_p \mid x \in K^\times\} = p^{\frac{\mathbb{Z}}{e}}.$$

2.2. Efficient computations in local fields

In this subsection, we describe how to do efficient computations in local fields.

We give a degree-n extension field K of $\mathbb{Q}_p$ by giving a monic degree-n irreducible polynomial $f(x) \in \mathbb{Z}_p[x]$. Let $\theta \in \overline{\mathbb{Q}_p}$ be a root of $f(x) = 0$, then let $K = \mathbb{Q}_p(\theta)$. If $f(x)$ is an Eisenstein polynomial, then K is totally ramified over $\mathbb{Q}_p$, see [1].

Let $\alpha \in K$, we express α as a polynomial of θ of degree $< n$ with coefficients in $\mathbb{Q}_p$. The map

$$\widehat{\alpha} : K \longrightarrow K$$

is defined as $\widehat{\alpha}(\beta) = \alpha \cdot \beta$, i.e., the map from K to K by multiplying α. This map is $\mathbb{Q}_p$-linear. The norm $N_{K/\mathbb{Q}_p}(\alpha)$ is the determinant of the map $\widehat{\alpha}$. We can take the basis $(1, \theta, \theta^2, \ldots, \theta^{n-1})$ of K over $\mathbb{Q}_p$, then representing the map $\widehat{\alpha}$ by an $n \times n$-matrix over $\mathbb{Q}_p$. The determinant of this matrix is the norm $N_{K/\mathbb{Q}_p}(\alpha)$. So we can efficiently calculate the p-adic absolute value $|\alpha|_p$.

Let $\alpha_1, \ldots, \alpha_n$ be a basis of K over $\mathbb{Q}_p$, and let $\beta \in K$. Using the following method, we can represent β as a $\mathbb{Q}_p$-linear combination of $\alpha_1, \ldots, \alpha_n$. Write

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = B \begin{pmatrix} 1 \\ \theta \\ \\ \theta^{n-1} \end{pmatrix}$$

with $B \in GL_n(\mathbb{Q}_p)$. It is clear that

$$\beta = (b_1, b_2, \ldots, b_n) \cdot \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{pmatrix} = (b_1, b_2, \ldots, b_n) \cdot B^{-1} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

2.3. p-adic lattices, LVP and CVP

In [2, 3], two new computational problems in p-adic lattices are introduced, they are the Longest Vector Problem and Closest Vector Problem. We first review it briefly.

Let p be a prime number, and let K be an extension field of $\mathbb{Q}_p$ of degree n, where n is a positive integer. Let m be a positive integer with $1 \leq m \leq n$. Let $\alpha_1, \ldots, \alpha_m \in K$ be m many $\mathbb{Q}_p$-linearly independent vectors. A lattice in K is the set

$$\mathcal{L}(\alpha_1, \ldots, \alpha_m) = \left\{ \sum_{i=1}^{m} a_i \alpha_i \mid a_i \in \mathbb{Z}_p, 1 \leq i \leq m \right\}$$

of all $\mathbb{Z}_p$-linear combinations of $\alpha_1, \ldots, \alpha_m$. The sequence of vectors $\alpha_1, \ldots, \alpha_m$ is called a basis of the lattice $\mathcal{L}(\alpha_1, \ldots, \alpha_m)$. The integers m and n are called the rank and dimension of the lattice, respectively. When $n = m$, we say that the lattice is of full rank.

2.3.1. Longest Vector Problem(LVP)

For any element $\alpha = \sum_{i=1}^{m} a_i \alpha_i \in \mathcal{L}$, since each $a_i \in \mathbb{Z}_p$, we have

$$\mid \alpha \mid_p = \mid \sum_{i=1}^{m} a_i \alpha_i \mid_p \leq \max_{1 \leq i \leq m} (\mid a_i \alpha_i \mid_p) \leq \max_{1 \leq i \leq m} (\mid \alpha_i \mid_p).$$

This indicates that the length $\mid \alpha \mid_p$ of any element of the p-adic lattice $\mathcal{L}$ is bounded above. Since the valuation group of K is discrete, as a subset of K, the set of lengths of elements of the lattice $\mathcal{L}$ is also discrete.

Definition 2.1. *[2, 3] Let $\mathcal{L} = \mathcal{L}(\alpha_1, \ldots, \alpha_m)$ be a lattice in K. We define recursively a sequence of positive real numbers: $\lambda_1, \lambda_2, \lambda_3, \ldots$ as follows.*

$$\lambda_1 = \max_{1 \leq i \leq m} (\mid \alpha_i \mid_p)$$

$$\lambda_{j+1} = \max\{\mid x \mid_p \mid x \in \mathcal{L}, \mid x \mid_p < \lambda_j\} \text{ for } j \geq 1.$$

We have $\lambda_1 > \lambda_2 > \lambda_3 > \ldots$ and $\lim_{j \to \infty} \lambda_j = 0$.

Definition 2.2. *[2, 3] Given a lattice $\mathcal{L} = \mathcal{L}(\alpha_1, \ldots, \alpha_m)$ in K, the longest vector problem (LVP) is to find a lattice vector $v \in \mathcal{L}$ such that $\mid v \mid_p = \lambda_2$.*

Theorem 2.3. *[2, 3] Given a lattice $\mathcal{L} = \mathcal{L}(\alpha_1, \ldots, \alpha_m)$ in K. Fix an integer $j \geq 2$. There exists an algorithm to find a lattice vector $v_j \in \mathcal{L}$ satisfying $\mid v_j \mid_p = \lambda_j$. The algorithm takes $O(p^{m(j-1)})$ many p-adic absolute value computations of elements of K.*

2.3.2. Closest Vector Problem(CVP)

Given a target vector $t \in K$. Since the function

$$\mathcal{L} \longrightarrow \mathbb{R}, v \longmapsto \mid t - v \mid_p$$

is continuous on the compact set $\mathcal{L}$, it can take the minimum and maximum on $\mathcal{L}$. Set

$$\mu_{\min} = \min_{v \in \mathcal{L}} \mid t - v \mid_p \text{ and } \mu_{\max} = \max_{v \in \mathcal{L}} \mid t - v \mid_p .$$

If $t \in \mathcal{L}$, it is obvious that we have $\mu_{\min} = 0$ and $\mu_{\max} = \lambda_1$. Here λ_1 is the same as in Definition 2.1. So we below assume $t \notin \mathcal{L}$. Hence $\mu_{\min} > 0$. Since the valuation group of K is discrete, the above distance function will take only finitely many values. So we have the following definition.

Definition 2.4. *[2, 3] Let $\mathcal{L} = \mathcal{L}(\alpha_1, \ldots, \alpha_m)$ be a lattice in K and let $t \in K - \mathcal{L}$ be a target vector. Define s positive real numbers $\mu_1 > \mu_2 > \mu_3 > \cdots > \mu_s$ as follows, where s is a positive integer.*

$$\{\mu_1, \mu_2, \mu_3, \ldots, \mu_s\} = \{\mid t - v \mid_p \mid v \in \mathcal{L}\}.$$

So $\mu_{\max} = \mu_1$ and $\mu_{\min} = \mu_s$.

If $\mid t \mid_p > \lambda_1$, since $\mid t - v \mid_p = \mid t \mid_p$, we have $\mu_{\min} = \mu_{\max} = \mid t \mid_p$. So we below assume $\mid t \mid_p \leq \lambda_1$.

Definition 2.5. *[2, 3] Let $\mathcal{L} = \mathcal{L}(\alpha_1, \ldots, \alpha_m)$ be a lattice in K and let $t \in K - \mathcal{L}$ be a target vector with $\mid t \mid_p \leq \lambda_1$. The closest vector problem (CVP) is to find a lattice vector $v \in \mathcal{L}$ such that $\mid t - v \mid_p = \mu_{\min}$.*

3. Solving LVP and CVP with orthogonal bases

3.1. Orthogonal bases

Let p be a prime. Let V be a left vector space over $\mathbb{Q}_p$. A norm on V is a function

$$\mid \cdot \mid : V \longrightarrow \mathbb{R}$$

such that:

$$(i) \quad |v| \geq 0, \forall v \in V, \text{ and } |v| = 0 \text{ if and only if } v = 0;$$

$$(ii) \quad |xv| = |x|_p \cdot |v|, \forall x \in \mathbb{Q}_p, v \in V;$$

$$(iii) \quad |v + w| \leq \max(|v|, |w|), \forall v, w \in V.$$

If $|\cdot|$ is a norm on V, and if $|v| \neq |w|$ for $v, w \in V$, then we have $|v + w| = \max(|v|, |w|)$. Weil [21] proved the following proposition:

Proposition 3.1. *[21, p.26] Let V be a left vector space over $\mathbb{Q}_p$ of finite dimension $n > 0$, and let $|\cdot|$ be a norm on V. Then there is a decomposition $V = V_1 + \cdots + V_n$ of V into a direct sum of subspaces V_i of dimension 1, such that*

$$|\sum_{i=1}^{n} v_i| = \max_{1 \leq i \leq n} |v_i|, \forall v_i \in V_i, i = 1, \ldots, n.$$

So we can define the orthogonal basis.

Definition 3.2 (Orthogonal basis)**.** *Let V be a left vector space over $\mathbb{Q}_p$ of finite dimension $n > 0$, and let $|\cdot|$ be a norm on V. We call $\alpha_1, \ldots, \alpha_n$ an orthogonal basis of V over $\mathbb{Q}_p$ if V can be decomposed into the direct sum of n 1-dimensional subspaces V_i's $(1 \leq i \leq n)$, such that*

$$|\sum_{i=1}^{n} v_i| = \max_{1 \leq i \leq n} |v_i|, \forall v_i \in V_i, i = 1, \ldots, n,$$

where V_i is spanned by α_i.

Weil's proof is not constructive, and he did not give a method to find out an orthogonal basis. The following is immediate.

Proposition 3.3. *Let V be a left vector space over $\mathbb{Q}_p$ of finite dimension $n > 0$, and let $|\cdot|$ be a norm on V. Let $\alpha_1, \ldots, \alpha_n$ be a basis of V over $\mathbb{Q}_p$. If*

$$\{|v_i| \mid v_i \in \mathbb{Q}_p \cdot \alpha_i\} \bigcap \{|v_j| \mid v_j \in \mathbb{Q}_p \cdot \alpha_j\} = \{0\}, \forall i, j = 1, \ldots, n, i \neq j,$$

then $\alpha_1, \ldots, \alpha_n$ is an orthogonal basis of V over $\mathbb{Q}_p$.

Proposition 3.4. *Let K be an extension field of degree n of $\mathbb{Q}_p$. Let π be a uniformizer of K. Set*

$$V = \sum_{i=0}^{e-1} \mathbb{Q}_p \cdot \pi^i$$

where e is the ramification index of $K/\mathbb{Q}_p$. Then V is an e-dimensional $\mathbb{Q}_p$-vector subspace of K, and $1, \pi, \ldots, \pi^{e-1}$ is an orthogonal basis of V.

Proof. We know $1, \pi, \ldots, \pi^{e-1}$ is $\mathbb{Q}_p$-linearly independent, see [1, p.125, Lemma 5.4]. Since $|\pi|_p = p^{-\frac{1}{e}}$,

$$\{|x|_p \mid x \in \mathbb{Q}_p \cdot \pi^i\} = \{0\} \bigcup p^{\mathbb{Z}-\frac{i}{e}}.$$

Now the result follows from Proposition 3.3. □

3.2. Solving LVP with orthogonal bases

We can prove the following theorem.

Theorem 3.5. *Given a lattice $\mathcal{L} = \mathcal{L}(\alpha_1, \ldots, \alpha_m)$ in K. Assume $\alpha_1, \ldots, \alpha_m$ is an orthogonal basis. Fix an integer $j \geq 2$. There exists an algorithm to find a lattice vector $v_j \in \mathcal{L}$ satisfying*

$$| v_j |_p = \lambda_j.$$

The algorithm takes $O(m)$ many p-adic absolute value computations of elements of K.

Proof. Without loss of generality, we can assume

$$\{|\alpha_i|_p \mid i = 1, \ldots, m\} = \{\nu_1, \ldots, \nu_s\} \text{ with } \nu_1 > \cdots > \nu_s,$$

and

$$|\alpha_i|_p = \nu_i, i = 1, \ldots, s.$$

Then, obviously,

$$\{\log_p |v|_p \mid v \in \mathcal{L}, v \neq 0\} = \{\log_p \nu_i - k \mid i = 1, \ldots, s, k = 0, 1, 2, \cdots\}.$$

Consider the set of valuations

$$S = \{\log_p \nu_i - k \mid 0 \leq k \leq j - i, 1 \leq i \leq \min(s, j)\}.$$

Obviously, in decreasing order, the first number of the set S is $\nu_1 = \lambda_1$, and the j-th number is λ_j. If $\lambda_j = \log_p \nu_i - k$, we can take the vector $v_j = p^k \alpha_i$.

The algorithm needs to compute the m many p-adic absolute values of the basis vectors. We ignore the time of comparing.

□

3.3. Solving CVP with orthogonal bases

We can prove the following theorem.

Theorem 3.6. *Let $\mathcal{L} = \mathcal{L}(\alpha_1, \ldots, \alpha_m)$ be a lattice in K. Let $t \in K - \mathcal{L}$ be a target vector with $\mid t \mid_p \leq \lambda_1$. Let $V(\supset \mathcal{L})$ be a k-dimensional $\mathbb{Q}_p$-vector subspace of the field K. Let $\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_k (k \geq m)$ be an orthogonal basis of V. Suppose the target vector $t \in V$. There is an algorithm to find a vector $v_i \in \mathcal{L}$ such that $|t - v_i|_p = \mu_i$, for each $i = 1, 2, \ldots, s$. The algorithm takes $O(n)$ many p-adic absolute value computations of elements of K, where n is the degree of K over $\mathbb{Q}_p$.*

Proof. Write

$$t = \sum_{i=1}^{k} b_i \alpha_i, \quad b_i \in \mathbb{Q}_p, i = 1, \ldots, k.$$

For any lattice vector

$$v = \sum_{i=1}^{m} a_i \alpha_i \in \mathcal{L}, a_i \in \mathbb{Z}_p, i = 1, \ldots, m,$$

we have

$$|t - v|_p = \max\{|b_i - a_i|_p \cdot |\alpha_i|_p (1 \leq i \leq m), |b_j \alpha_j|_p (m+1 \leq j \leq k)\}.$$

If $b_i \notin \mathbb{Z}_p$, then $|b_i - a_i|_p = |b_i|_p > 1$. If $b_i \in \mathbb{Z}_p$, then

$$\{|b_i - a_i|_p \mid a_i \in \mathbb{Z}_p\} = \{0, p^{-c} (c = 0, 1, 2, \ldots)\}.$$

Assume $b_i \notin \mathbb{Z}_p$ for $1 \leq i \leq l$ and $b_i \in \mathbb{Z}_p$ for $l+1 \leq i \leq m$, where l is an integer with $0 \leq l \leq m$. Thus,

$$\{|t - v|_p \mid v \in \mathcal{L}\} = \{\max\{|b_i|_p \cdot |\alpha_i|_p (1 \leq i \leq l); |b_j \alpha_j|_p (m+1 \leq j \leq k);$$

$$p^{-c_u} \cdot |\alpha_u|_p \mid c_u = 0, 1, 2, 3, \ldots, \infty (l+1 \leq u \leq m)\}\},$$

where we set $p^{-\infty} = 0$.

Set

$$N = \max\{|b_i|_p \cdot |\alpha_i|_p (1 \leq i \leq l); |b_j \alpha_j|_p (m+1 \leq j \leq k)\}.$$

Since $t \notin \mathcal{L}$, we have $N > 0$. Obviously, $\mu_s = \mu_{\min} = N$. To obtain the value

$$\max\{N, p^{-c_u} \cdot |\alpha_u|_p \mid c_u = 0, 1, 2, 3, \ldots, \infty (l+1 \leq u \leq m)\},$$

we can only consider those indices $l+1 \leq u \leq m$ for which $|\alpha_u|_p > N$. Denote d_u the largest non-negative integer with

$$p^{-d_u} \cdot |\alpha_u|_p > N.$$

Consider the set

$$T = \{N, p^{-c_u} \cdot |\alpha_u|_p \mid c_u = 0, 1, \ldots, d_u \text{ with } |\alpha_u|_p > N (l+1 \leq u \leq m)\}.$$

Listing the number of the set T, in decreasing order, we get the values of the distance function $|t-v|_p, v \in \mathcal{L}$, i.e. the s many positive real numbers $\mu_1 > \mu_2 > \mu_3 > \cdots > \mu_s$ such that

$$\{\mu_1, \mu_2, \mu_3, \ldots, \mu_s\} = \{|\ t - v\ |_p |\ v \in \mathcal{L}\}.$$

And it is easy to find all vectors $v_i \in \mathcal{L}$ such that $|t - v_i|_p = \mu_i, i = 1, 2, \ldots, s$.

□

Similar to Theorem 2.3, if there is no any orthogonal bases, exponential time algorithms are given in [2, 3].

4. A signature scheme

We present our signature scheme as follows.

Key Generation: We first choose a totally ramified K of degree n over $\mathbb{Q}_p$, i.e., choose an Eisenstein polynomial $f(x) = x^n + f_1 x^{n-1} + \ldots + f_{n-1}x + f_n \in \mathbb{Z}_p[x]$ satisfying $|f_n|_p = p^{-1}$ and $|f_i|_p < 1$ for $1 \leq i \leq n-1$. Let θ be a root of $f(x) = 0$. Choose another $\zeta \in \mathcal{O}_K = \mathbb{Z}_p[\theta]$ such that $\mathbb{Z}_p[\zeta] = \mathbb{Z}_p[\theta]$. Then $K = \mathbb{Q}_p(\zeta)$. Let $F(x) \in \mathbb{Z}_p[x]$ be the minimum polynomial of ζ over $\mathbb{Q}_p$ which is also monic and of degree n. Choose n non-negative integers $j_i \in \mathbb{Z}$ such that the $j_i \pmod n (1 \leq i \leq n)$ are distinct. Set $\alpha_i = \theta^{j_i} (1 \leq i \leq n)$. Then $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}_p$, thus $\alpha_1, \ldots, \alpha_n$ is an orthogonal basis. All elements of $\mathcal{O}_K$ should be expressed as polynomials in ζ of degree $< n$ with coefficients in $\mathbb{Z}_p$ and ζ is just a formal symbol.

Choose a matrix $A \in \mathrm{GL}_m(\mathbb{Z}_p)$. Put

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$$

such that the m vectors $\beta_1, \ldots, \beta_m$ have the same length or have almost same lengths. Set

$$\mathcal{L} = \mathbb{Z}_p \cdot \beta_1 + \ldots + \mathbb{Z}_p \cdot \beta_m = \mathbb{Z}_p \cdot \alpha_1 + \ldots + \mathbb{Z}_p \cdot \alpha_m.$$

We need a hash function

$$H : \{0,1\}^* \longrightarrow W := \{x \mid x \in K - \mathcal{L}, |x|_p = \lambda_1\},$$

where λ_1 is the maximum value of the lengths of all vectors in $\mathcal{L}$. This hash function can be implemented as follows. For the message $M \in \{0,1\}^*$, compute $seed = SHA-3(M)$, then using this seed to generate a random element in W.

Public key is set to be: $(F(x), H, (\beta_1, \ldots, \beta_m))$.

Private key is set to be: $(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n)$.
Signing algorithm: For any message $M \in \{0,1\}^*$, choose a random number r of fixed length, say, $r \in \{0,1\}^{256}$. Compute

$$t = H(M \parallel r).$$

Using the orthogonal basis $(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n)$, Bob computes a lattice vector $v \in \mathcal{L}$ which is the closest one to t. If the minimum value of the distance from t to $\mathcal{L}$ is strictly less than λ_1, then output the signature (r, v). If the minimum value of the distance from t to $\mathcal{L}$ is equal to λ_1, then choose a new $r \in \{0,1\}^{256}$ until the minimum value of the distance from t to $\mathcal{L}$ is strictly less than λ_1.
Verification algorithm: The signature is valid if and only if $t = H(M \parallel r), v \in \mathcal{L}$ and $|t - v|_p < \lambda_1$.

The correctness is obvious. For the efficiency, what needs to illustrate is how many random r's can yield a valid signature. We have not proven it in theory by now, but in our experiments, we always generated a valid signature with just one r. Hence, our signature scheme is very efficient.

Remark. Notice that, if (r, v) is a true signature, then we must have $|v|_p = \lambda_1$. Keeping the notation in the proof of Theorem 3.6, if there is an index u with $l + 1 \leq u \leq m$ such that $|b_u \alpha_u|_p > N$, then it holds that the minimum distance from t to $\mathcal{L}$ is strictly less than $|t|_p$.

5. A public-key cryptosystem

We first present an original public-key cryptosystem as follows.
Key Generation For the sake of clarity, we repeat the necessary notation. We first choose a totally ramified K of degree n over $\mathbb{Q}_p$, i.e., choose an Eisenstein polynomial $f(x) = x^n + f_1 x^{n-1} + \ldots + f_{n-1}x + f_n \in \mathbb{Z}_p[x]$ satisfying $|f_n|_p = p^{-1}$ and $|f_i|_p < 1$ for $1 \leq i \leq n-1$. Let θ be a root of $f(x) = 0$. Choose another $\zeta \in \mathcal{O}_K = \mathbb{Z}_p[\theta]$ such that $\mathbb{Z}_p[\zeta] = \mathbb{Z}_p[\theta]$. Then $K = \mathbb{Q}_p(\zeta)$. Let $F(x) \in \mathbb{Z}_p[x]$ be the minimum polynomial of ζ over $\mathbb{Q}_p$ which is also monic and of degree n. Choose n non-negative integers $j_i \in \mathbb{Z}$ such that the $j_i \pmod n (1 \leq i \leq n)$ are distinct. Set $\alpha_i = \theta^{j_i} (1 \leq i \leq n)$. Then $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}_p$, thus $\alpha_1, \ldots, \alpha_n$ is an orthogonal basis.

Choose a positive integer $m \leq n$. Choose a positive real number $0 < \delta < 1$. All elements of $\mathcal{O}_K$ should be expressed as polynomials in ζ of degree $< n$ with coefficients in $\mathbb{Z}_p$.

Choose a matrix $A \in \mathrm{GL}_m(\mathbb{Z}_p)$. Put

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$$

such that the m vectors $\beta_1, \ldots, \beta_m$ have the same length or have almost same lengths. Set

$$\mathcal{L} = \mathbb{Z}_p \cdot \beta_1 + \ldots + \mathbb{Z}_p \cdot \beta_m = \mathbb{Z}_p \cdot \alpha_1 + \ldots + \mathbb{Z}_p \cdot \alpha_m.$$

Public key is set to be: $(F(x), \delta, (\beta_1, \ldots, \beta_m))$.

Private key is set to be: $(A, (\alpha_1, \ldots, \alpha_n))$.

Encryption: For any plaintext $(a_1, \ldots, a_m) \in \{0, 1, \ldots, p-1\}^m - \{(0, \ldots, 0)\}$, Alice first chooses randomly $r \in K - \mathcal{L}$ with $p^{-1} < |r|_p < p^{-\delta}$, computes the ciphertext

$$C = a_1\beta_1 + \cdots + a_m\beta_m + r \in K$$

and sends C to Bob.

Decryption: When Bob receives the ciphertext C, using Theorem 3.6 with the orthogonal basis $(\alpha_1, \ldots, \alpha_n)$, he computes a lattice vector $v \in \mathcal{L}$ which is the closest one to C. Write

$$v = b_1\alpha_1 + \ldots + b_m\alpha_m, \quad b_i \in \mathbb{Z}_p,$$

then the plaintext is

$$(b_1, \ldots, b_m) \cdot A^{-1} \pmod{p}.$$

However, for the correctness, we can only prove that:

Theorem 5.1. *The decryption is correct for those indices $1 \le i \le m$ with $j_i \le \delta n$.*

Proof. Since there is a lattice vector $a_1\beta_1 + \cdots + a_m\beta_m$ such that $|C - (a_1\beta_1 + \cdots + a_m\beta_m)|_p < p^{-\delta}$, we have $|C - v|_p < p^{-\delta}$. Write $C = v + r'$ with $|r'|_p < p^{-\delta}$. We have

$$C = (a_1, \ldots, a_m) \cdot A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} + r = (b_1, \ldots, b_m) \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} + r'.$$

Set

$$(c_1, \ldots, c_m) = (a_1, \ldots, a_m) \cdot A, \quad c_i \in \mathbb{Z}_p.$$

We have

$$|\sum_{i=1}^{m}(c_i - b_i) \cdot \alpha_i|_p = |r' - r|_p \le \max\{|r|_p, |r'|_p\} < p^{-\delta}.$$

Since $\alpha_1, \ldots, \alpha_m$ is an orthogonal basis, we have

$$|\sum_{i=1}^{m}(c_i - b_i) \cdot \alpha_i|_p = \max_{1 \le i \le m}(|c_i - b_i|_p \cdot |\alpha_i|_p) < p^{-\delta}.$$

Since

$$|\alpha_i|_p = |\theta^{j_i}|_p = p^{-\frac{j_i}{n}},$$

if there is some index $1 \leq i \leq m$ with $c_i - b_i \in \mathbb{Z}_p^\times$, then we have

$$p^{-\frac{j_i}{n}} < p^{-\delta}$$

i.e., $\delta < \frac{j_i}{n}$. So $c_i \equiv b_i \pmod{p}$ for each $1 \leq i \leq m$ with $j_i \leq \delta n$. The decryption is correct for those indices $1 \leq i \leq m$ with $j_i \leq \delta n$. □

To make the decryption correct, we can revise the original cryptosystem a bit. In the key generation, after generating j_i's and δ, denote by I the set of indices j_i with $j_i \leq \delta n$. Alice additionally chooses a set $Z \subseteq I$ and sends the size $z = \#Z$ to Bob. Then Alice can permute the set of α_i's such that $\{\alpha_1, \cdots, \alpha_z\} = \{\theta^{j_i} | j_i \in Z\}$ and generates public key as before.

In the encryption algorithm, the plaintext becomes $(a_1, \ldots, a_z) \in \{0, 1, \ldots, p-1\}^z - \{(0, \ldots, 0)\}$. For any plaintext $(a_1, \ldots, a_z)$, Bob firstly extends it into $(a_1, \ldots, a_m) \in \{0, 1, \ldots, p-1\}^m$ with random $a_{z+1}, \ldots, a_m$, then encrypts it as before. In the decryption algorithm, Alice just accepts the first z components of the recovered vector.

The correctness is obvious. For the efficiency, what needs to illustrate is how efficient to generate a random $r \in K - \mathcal{L}$ with $|r|_p < p^{-\delta}$. As δ becomes bigger, it is harder to generate such r. In Section 7, we present some experiment results.

Remark. Our scheme is similar in its algorithmic nature to GGH scheme [7] based on lattices in Euclidean spaces and McEliece scheme [13], but the domains in which these operations take place are vastly different. On the other hand, the above theorem says that which positions of the plaintext can be correctly decrypted. This is a rare feature.

6. Security analysis

We do not present any security proof for our cryptosystems since the problems in local fields are relatively very new. Instead, we give some possible attacks to our schemes in this section.

6.1. Recovering a uniformizer

Given a local field K, if we could find out a uniformizer of K, then the above public-key cryptosystem and signature scheme would be broken completely. However, as mentioned in [3], uniformizers are just the second longest vectors in the p-adic lattice $\mathcal{O}_K$, so recovering a uniformizer is a LVP-instance in $\mathcal{O}_K$, which is assumed to be hard.

6.2. Finding an orthogonal basis

Now there are no any known algorithms to find out an orthogonal basis of a p-adic lattice if it has. Notice that, not all p-adic lattices necessarily have orthogonal bases. If we assume that an orthogonal basis of p-adic lattices is hard to compute, then it is difficult to recover the private key from the public key.

6.3. Solving CVP-instances

Obviously, if we could efficiently solve the CVP-instances, then the above public-key cryptosystem and signature scheme would be broken completely. If we assume the CVP is hard under a random basis of a lattice, then an illegal user is difficult to forge a true signature and recover the plaintext from the ciphertext.

One may argue that it may be not necessary to solve the CVP problem to break the encryption cryptosystem, since what we need to recover the plaintext is not $(d_1, \ldots, d_m) = (b_1, \ldots, b_m) \cdot A^{-1}$, but just $(d_1, \ldots, d_m) \mod p$. Notice that the ciphertext C is closest to $(d_1, \ldots, d_m) \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$ by the decryption process. Given an m-dimensional p-adic lattice basis B and a target t, we define $\mathrm{CVP}_p(B, t)$ as the problem to recover $(d_1, \ldots, d_m) \mod p$, where $(d_1, \ldots, d_m)$ is the coefficients of some lattice vector closest to t under the basis B. It is easy to show that given an oracle to solve $\mathrm{CVP}_p(B, t)$, we can find a very good approximation of the vector in $\mathcal{L}(B)$ closest to t, which means that even to find the coefficients modulo p is very difficult. Roughly speaking, we can run the oracle to solve $\mathrm{CVP}_p(B, t)$ and get a returned solution $(\bar{d}_1, \ldots, \bar{d}_m)$. Then we know there exists $(d_1, \ldots, d_m) \in \mathbb{Z}_p^m$ such that $(d_1, \ldots, d_m) \cdot B$ is a lattice vector closest to t, and $(d_1, \ldots, d_m) \mod p = (\bar{d}_1, \ldots, \bar{d}_m)$. We can continue to run the oracle to solve $\mathrm{CVP}_p(pB, t - (\bar{d}_1, \ldots, \bar{d}_m)B)$ and get a returned solution $(\tilde{d}_1, \ldots, \tilde{d}_m)$. Then we know there exists $(d_1, \ldots, d_m) \in \mathbb{Z}_p^m$ associated with a closest vector, such that $(d_1, \ldots, d_m) \mod p^2 = (\bar{d}_1 + p\tilde{d}_1, \ldots, \bar{d}_m + p\tilde{d}_1)$. Repeating the process several times, we can recover $(d_1, \ldots, d_m) \mod p^k$ for k polynomial in m, which is enough to yield a lattice vector that is very close to the target. Hence, to find the coefficients modulo p is still very difficult.

6.4. Modulo p attack

We look at a so-called Modulo p attack for the above public-key cryptosystem. It is similar to Nguyen's attack [17] for the GGH cryptosystem.

For the ciphertext

$$C = a_1\beta_1 + \cdots + a_m\beta_m + r$$

with $|r|_p < p^{-\delta}$. Denote the ring of integers of K by $\mathcal{O}_K$, i.e., those elements x of K with $|x|_p \leq 1$. We know $\mathcal{O}_K = \mathbb{Z}_p[\zeta]$. We express $C, \beta_1, \ldots, \beta_m, r$ as polynomials of ζ with coefficients in $\mathbb{Z}_p$, then equating the coefficients of $1, \zeta, \ldots, \zeta^{n-1}$ of the two sides of the above equation, we obtain a system of n linear equations with coefficients in $\mathbb{Z}_p$. Write

$$r = \sum_{i=0}^{n-1} r_i \cdot \zeta^i, \quad r_i \in \mathbb{Z}_p.$$

If ζ would be a uniformizer of K, then since $|\zeta^i|_p = p^{-\frac{i}{n}}$, we have

$$|r|_p = \max_{0 \leq i \leq n-1} (|r_i|_p \cdot p^{-\frac{i}{n}}).$$

Since $|r|_p < p^{-\delta}$, we have

$$|r_i|_p < p^{\frac{i-\delta n}{n}}.$$

If $i \leq \delta n$, then $p \mid r_i$. Since the $r_i's$ are unknown, reducing the above system of linear equations modulo p, we get a system of R linear equations over the finite field $\mathbb{F}_p$, and the unknowns are just the plaintext $(a_1, \ldots, a_m)$, where R is the number of indices $0 \leq i \leq n-1$ with $i \leq \delta n$. Assume the linear equations are linearly independent over $\mathbb{F}_p$, we can determine R unknowns as functions of other unknowns. We have proved the following.

Proposition 6.1. *If ζ would be a uniformizer of K, then the above Modulo p attack can at most reduce the search space of plaintexts from p^m to p^{m-R}, where R is the number of indices $0 \leq i \leq n-1$ with $i \leq \delta n$.*

However, in general, ζ is not a uniformizer of K, so $1, \zeta, \ldots, \zeta^{n-1}$ is not an orthogonal basis, so the above Modulo p attack fails. Further, we can let ζ be a unit of $\mathcal{O}_K$. Usually, the positive real number δ is < 1. Otherwise, if $\delta \geq 1$, it is easily seen that, when writing r as a polynomial of ζ, the coefficients are all divisible by p and the above Modulo p attack will apply.

7. Experimental results

To verify the efficiency of our cryptosystems, we did some experiments on a personal laptop with Windows 10 operation system, i5-10210U CPU and 8-GB memory. We report some experimental results in this section.

7.1. General strategies

Denote $\mathbb{Z}_{(p)} := \{x \in \mathbb{Q} \mid |x|_p \leq 1\}$, i.e., the localization of $\mathbb{Z}$ at p. We can choose an Eisenstein polynomial $f(x) = x^n + f_1 x^{n-1} + \ldots + f_{n-1}x + f_n \in \mathbb{Z}_{(p)}[x]$ satisfying

$|f_n|_p = p^{-1}$ and $|f_i|_p < 1$ for $1 \le i \le n-1$. Choose $\zeta \in \mathbb{Z}_{(p)}[\theta]$ such that $\theta \in \mathbb{Z}_{(p)}[\zeta]$. Then $F(x) \in \mathbb{Z}_{(p)}[x]$. Choose the matrix $A \in \mathrm{GL}_m(\mathbb{Z}_{(p)})$. Let

$$\begin{pmatrix} 1 \\ \zeta \\ \vdots \\ \zeta^{n-1} \end{pmatrix} = B \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{pmatrix}$$

such that the matrix B is in $\mathrm{GL}_m(\mathbb{Z}_{(p)})$, then we have $\theta \in \mathbb{Z}_{(p)}[\zeta]$. Then all computations can be done via polynomials in ζ of degree $< n$ with coefficients in $\mathbb{Z}_{(p)}$ and ζ is just a formal symbol. Note that $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{Z}_p$ and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, so $\mathbb{Z}_{(p)}$ is large enough to work with.

For a positive integer D, denote

$$C(D) = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, |a| \le D, 0 < b \le D, p \nmid b \right\} \subset \mathbb{Z}_{(p)}.$$

In our experiments, when randomly choosing elements of K, we choose a polynomial of ζ with coefficients in $C(D)$ for some $D > 0$.

7.2. Generating the keys

It is easy to generate the keys needed in our signature scheme and public-key cryptosystem. We provide a relatively small example as follows.

For $n = 100$ and $p = 2$, we construct polynomials $f(x) = x^{100} + \sum_{i=1}^{99} f_i 2x^i + 2$ with $f_i \in \{0, 1\}$. The calculations are all in GP/PARI Version 2.13.0. For instance,

$$\begin{aligned} f(x) = x^{100} &+ 2x^{98} + 2x^{95} + 2x^{93} + 2x^{90} + 2x^{88} + 2x^{87} + 2x^{85} + 2x^{83} + 2x^{82} \\ &+ 2x^{81} + 2x^{79} + 2x^{73} + 2x^{72} + 2x^{66} + 2x^{63} + 2x^{62} + 2x^{61} + 2x^{59} + 2x^{58} \\ &+ 2x^{57} + 2x^{55} + 2x^{53} + 2x^{52} + 2x^{51} + 2x^{49} + 2x^{48} + 2x^{44} + 2x^{42} + 2x^{38} \\ &+ 2x^{36} + 2x^{35} + 2x^{34} + 2x^{32} + 2x^{30} + 2x^{28} + 2x^{27} + 2x^{24} + 2x^{23} + 2x^{13} \\ &+ 2x^{12} + 2x^{11} + 2x^{10} + 2x^9 + 2x^8 + 2x^7 + 2x^6 + 2x^5 + 2x^3 + 2x + 2. \end{aligned}$$

Let $\zeta = 1 + \sum_{i=1}^{99} B_{2,i+1}\theta^i$ with $B_{2,i+1} \in \{0, 1\}$. We use the function "random()" to construct ζ in GP/PARI. Moreover, we get the matrix B and compute whether $B \in \mathrm{GL}_{100}(\mathbb{Z}_{(p)})$ or not. In this example, we construct 20 ζ's and there are 9 ζ's satisfying $\mathbb{Z}_{(2)}[\zeta] = \mathbb{Z}_{(2)}[\theta]$.

In our experimental results, for a fixed $f(x)$, the probability of $\mathbb{Z}_{(p)}[\zeta] = \mathbb{Z}_{(p)}[\theta]$ is about $1 - 1/p$. Note that we do not need to prove the probability. Our experimental results show that it is easy to construct B such that $\mathbb{Z}_{(p)}[\zeta] = \mathbb{Z}_{(p)}[\theta]$.

For $m = 50$, we construct a lattice $\mathcal{L} = \oplus_{i \in S} \mathbb{Z}_2 \theta^i$ with rank 50, where $S =$

$$\{0, 1, 2, 3, 4, 6, 12, 14, 15, 16, 18, 19, 21, 24, 28, 29, 30, 32, 33, 34, 35, 36, 38, 43, 44, 45,$$

$46, 47, 49, 52, 57, 59, 62, 65, 66, 68, 69, 71, 73, 74, 75, 81, 86, 89, 90, 91, 92, 95, 96, 98\}$.

Let

$$\zeta = 1 + \theta + \theta^3 + \theta^4 + \theta^6 + \theta^{14} + \theta^{16} + \theta^{19} + \theta^{21} + \theta^{29} + \theta^{32} + \theta^{33}$$
$$+\theta^{34} + \theta^{35} + \theta^{36} + \theta^{38} + \theta^{43} + \theta^{44} + \theta^{52} + \theta^{57} + \theta^{59} + \theta^{62} + \theta^{65}$$
$$+\theta^{66} + \theta^{68} + \theta^{69} + \theta^{74} + \theta^{75} + \theta^{81} + \theta^{90} + \theta^{91} + \theta^{95} + \theta^{96} + \theta^{98}.$$

We have that $\mathbb{Z}_{(2)}[\zeta] = \mathbb{Z}_{(2)}[\theta]$. Moreover, let $\beta_1 = 1$ and $\beta_2 = \zeta$, we can construct a matrix $A \in \mathrm{GL}_{50}(\mathbb{Z}_{(2)})$ and get a basis $\beta_1, \ldots, \beta_{50}$ of $\mathcal{L}$ and all $\beta_1, \ldots, \beta_{50}$ have length 1.

7.3. Generating a valid signature is easy

Using the example in the previous section, for any element in W (see Section 4), we find that we can always generate a valid signature with just one r. That is, for any generated $t \in W$ in our experiments, i.e., $t \in K - \mathcal{L}$ and $|t|_2 = \lambda_1 = 1$, we can always find a lattice vector $v \in \mathcal{L}$ such that $|t - v|_2 < 1$. So it is very easy to generate a valid signature for a legal user.

7.4. Generating an error vector in the public-key cryptosystem

In our experiments, we choose $r = \sum_{i=0}^{99} r_i \zeta^i$, where the r_i can be chosen randomly. For instance, we assume $r_i \in \{0, 1, 2, 3\}$. We choose randomly 200 such r's, the length distribution is as follows.

$\log_2 \lvert r\rvert_2$	0	$-\frac{1}{100}$	$-\frac{2}{100}$	$-\frac{3}{100}$	$-\frac{4}{100}$	$-\frac{6}{100}$	$-\frac{8}{100}$
# of r's	93	55	26	15	9	1	1

Thus, when we choose a suitable positive number δ, which depends on n, it is relatively easy to generate an error vector r needed in our public-key cryptosystem when δ is small. Notice that by modulo p attack, if δ is small, the schemes appear more secure. However, by Theorem 5.1, if δ is small, the size of plaintext should be also small to ensure the correct decryption. Our experiments show that the scheme can at least work well for short messages.

8. Conclusion

LVP and CVP in local fields may have further applications in cryptography and other areas. In this paper, we just mention one possibility. The signature scheme and the public-key cryptosystem constructed in this paper are just an illustration. LVP and CVP in local fields are new computationally difficult mathematical problems, it is worth for further study and there is much work to do.

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