Multidimentional ModDiv public key cryptosystem

Samir Bouftass

E-mail : crypticator@gmail.com

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Abstract

This paper presents Multidimentional Moddiv public key cryptosystem which is based on an instance of LWR problem consisting on finding a secret vector X in $\mathbb{Z}_r^n$ knowing vectors A and B respectively in $\mathbb{Z}_s^m$ and $\mathbb{Z}_t^l$, where elements of vector B are defined as follows :
$B(i) =(\ (\sum_{j=1}^{j=n} X(j) * A(j+i)) \ Mod(2^p)Div(2^q)$
Mod is integer modulo operation, Div is integer division operation, p and q are known integers satisfying $p > 2 \times q$. Size in bits of s equals p, size of bits of r equals q, and size in bits of t equals $p - q$, $m > 2 \times n$ and $l = m - n$.

Keywords : Diffie Hellman key exchange, Lattice based cryptography, Closest vector problem, Learn with rounding problem.

1 Introduction :

Since its invention by Withfield Diffie and Martin Hellman [1], public key cryptography has imposed itself as the necessary and indispensable building block of every IT Security architecture.
In the last decades, it has been proven that public key cryptosystems based on number theory problems are not immune againt quantum computing attacks [2], urging the necessity of inventing new algorithms not based on classical problems namely Factoring, Dicret log over multiplicative groups or elliptic curves.

In [3] is presented a one dimentional ModDiv public key cryptosystem which security have been shown in Barcau et all [4] to be based on CVP problem.
Y Zang [5] have proven that one dimentional ModDiv security problem can be reduced to a CVP problem in 2 dimentional lattice.

Present paper proposes Multidimentional ModDiv public key cryptosystem which security is based on an instance of learn with rounding problem, first proposed by Banerjee et all [6] .
In section 2 , we describe one dimentional ModDiv public key cryptosystem and provide a proof of corectness of its Key exchange protocol.
In section 3, we describe Multidimentional ModDiv public key cryptosystem.

2 One dimentional ModDiv public key protocol :

2.1 Notations:

$mdv2_{(p,q)}(A) = A\ Mod(2^p)Div(2^q)$. (A being an integer, Mod modulo operation , and Div integer division).

$\| A \|$: Size in bits of A .

2.1.1 Public parameters :

Integers A, p, q and S. while $p > 2 \times q$.

A is pseudorandom and $\| A \| = p$.

S is exchanged key size and equal to $p - (2 \times q)$.

2.1.2 Private Computations :

- Alice generates randomly a q bit number X, and calculates $U = mdv2_{(p,q)}(A \times X)$.

- Bob generates randomly a q bit number Y, and calculates $V = mdv2_{(p,q)}(A \times Y)$.

2.1.3 Publicly exchanged values :

- Alice sends U to Bob.

- Bob sends V to Alice.

2.1.4 Further Private Computations :

- Alice calculates $Wa = mdv2_{(p-q,q)}(X \times V)$.

- Bob calculates $Wb = mdv2_{(p-q,q)}(Y \times U)$.

Bob and Alice know that :

$Wa = Wb$ or $| Wa - Wb | = 1$.

2.2 Proof of correctness :

Lemma 1. *A, p and q are integers .*

if $p > q \Rightarrow mdv2_{(p,q)}(A \times 2^q) = A\ mod(2^{p-q})$.

Proof :

$$mdv2_{(p,q)}(A \times 2^q) = (A \times 2^q)mod(2^p)div(2^q).$$

Observe least significant q bits of $N = (A \times 2^q)\ mod(2^p)$ are zeros whereas its most significant $p-q$ bits are the least significant $p-q$ bits of A, dividing then N by (2^q) implies :

$$(A \times 2^q)mod(2^p)div(2^q) = A\ mod(2^{p-q}) = mdv2_{(p,q)}(A \times 2^q)\ .$$

Theoreme 1. *A, X, Y, p and q are integers where $p > q$, $\| A \| = p$, $\| X \| = \| Y \| = q$.*

$Wa = mdv2_{(p-q,q)}(X \times mdv_{(p,q)}(A \times Y))$.

$Wb = mdv2_{(p-q,q)}(Y \times mdv_{(p,q)}(A \times X))$.

There is two possibilities :

1 - $Wa = Wb$.

2 - $\mid Wa - Wb \mid = 1$.

Proof :

Let $H1$ and $H2$ be integers such as :

$$U_1 = mdv2_{(p,q)}(A \times X) \times 2^q = (A \times X - H1)mod(2^p).$$

$$V_1 = mdv2_{(p,q)}(A \times Y) \times 2^q = (A \times Y - H2)mod(2^p).$$

The fact that the least significant q bits of U_1 and V_1 are zeroes implies $\| H_1 \| = \| H_2 \| = q$.

Let's calculate :

$$Wa_1 = (X \times V_1)mod(2^p) = ((X \times Y \times A) - (X \times H_2))mod(2^p)\ (1)$$

$$Wb_1 = (Y \times U_1)mod(2^p) = ((Y \times X \times A) - (Y \times H_1))mod(2^p)\ (2)$$

$\| X \|=\| Y \|=\| H_1 \|=\| H_2 \|= q$ implies $\| X \times H_2 \|=\| Y \times H_1 \|= 2 \times q$, we have then :

$$Wa_1 div(2^{2\times q}) = (X \times Y \times A)mod(2^p)div(2^{2\times q}) - E_a$$

$$Wb_1 div(2^{2\times q}) = (Y \times X \times A)mod(2^p)div(2^{2\times q}) - E_b$$

where E_a and E_b are respectively the $2 \times q$'th borrows of binary substractions (1) and (2) .

E_a and E_b being bits, they can have then for values 0 or 1 implying :

if $E_a = E_b$ we have $Wa_1 div(2^{2\times q}) = Wb_1 div(2^{2\times q})$.

if $\mid E_a - E_b \mid= 1$ we have $\mid Wa_1 div(2^{2\times q}) - Wb_1 div(2^{2\times q}) \mid= 1$.

Now we'll show that :

$$Wa = Wa_1 div(2^{2\times q}) \text{ and } Wb = Wb_1 div(2^{2\times q})$$

ending thus theoreme's proof .

$$Wa_1 = (X \times V_1)mod(2^p) = (X \times mdv2_{(p,q)}(A \times Y) \times 2^q)mod(2^p)$$

$$Wa_1 div(2^q) = (X \times V_1)mod(2^p)div(2^q) = (X \times mdv2_{(p,q)}(A \times Y) \times 2^q)mod(2^p)div(2^q)$$

Applying Lemma 1, we get :

$$Wa_1 div(2^q) = (X \times V_1)mod(2^p)div(2^q) = (X \times mdv_{(p,q)}(A \times Y))mod(2^{p-q})$$

$$Wa_1 div(2^{2\times q}) = (X \times mdv2_{(p,q)}(A \times Y))mod(2^{p-q})div(2^q)$$

$$Wa_1 div(2^{2\times q}) = mdv2_{(p-q,q)}(X \times mdv_{(p,q)}(A \times Y))$$

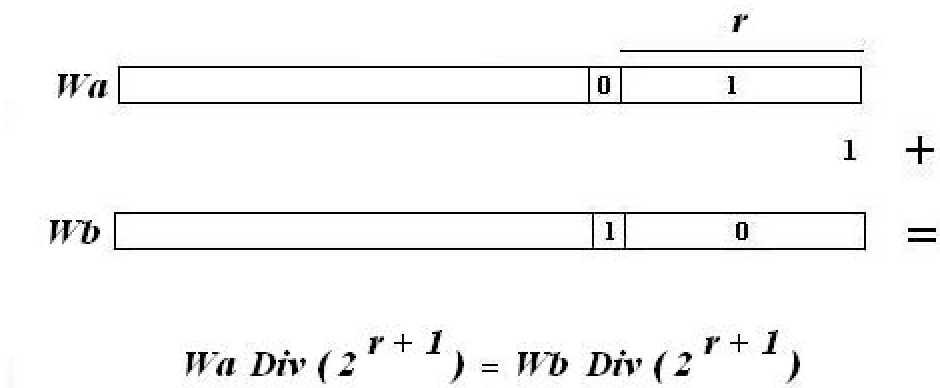
$$Wa_1 div(2^{2\times q}) = Wa$$

by the same way we can prove :

$$Wb_1 div(2^{2\times q}) = Wb$$

Observe, Max$(S) = \| Wa \| = \| Wb \| = p - (2 \times q)$.

Figure 1:



But there is one drawback, Alice or Bob don't get precisely the same value : they only know that $Wa = Wb$ or $|\ Wa - Wb\ | = 1$.

Meaning that if Alice encrypts a message M with key Wa and sends corresponding cipher text C to Bob. To get M, he should decrypts C with Wb, $Wb+1$ and $Wb-1$ and gets 3 plausible plain texts. To decide which one is correct, Alice should hash M and joins the computed digest as a header to M before encryption. Bob can then decide which plain text is correct by hashing obtained plaine text and compares it to joined hash value : Decryption can be then three times slower than encryption.

Now Lets suppose that computed values Wa and Wb are uniform so that the probability that their least r significant bits are ones is $(1/2)^r$.

To get M, Bob can perform only one decryption if he and Alice agreed on r. Alice should then encrypt with $Wa\ Div(2^r)$, Bob should decrypt with $Wb\ Div(2^r)$ (Figure 1).

We have experimentaly observed that $Pr[Wa = Wb] = 2/3$, implying :

$$Pr[\ Wa\ Div(2^r) = Wb\ Div(2^r)\ ] = 1 - ((1/3) \times (1/2)^r).$$

Bob can then decrypt C only one time, but the price going with it, is r bits security and a probability of $1 - ((1/3) \times (1/2)^r)$ to get the right plain text.

3 Multidimentional ModDiv public key protocol :

3.1 Notations:

polyDiv : Polynomial division operation , polyMod : Polynomial modulo operation.

$pmdv_{(m,n)}(\mathbf{A}) = (\mathbf{A})polyMod(x^m)polyDiv(x^n)$

3.1.1 Public parameters :

Integers m, n, p, q, r satisfying $m > 2 \times n$ and $p > (2 \times q) + r + log_2(n)$.

An m degree Polynomial **A**. which coefficients sizes in bits equals $\| 2^p \|$.

A coefficients are pseudorandomly generated.

3.1.2 Private Computations :

- Alice chooses an n degree Polynomial **X**, which coefficient sizes in bits equals $\| 2^q \|$, **X** coefficients are pseudorandomly generated.

- Alice calculates $\mathbf{U} = pmdv_{(m,n)}(\mathbf{A} \times \mathbf{X})$.

- For each **U** coefficient U(i), she calculates $U1(i) = mdv2_{(p,q)}U(i)$, getting thus :
a polynomial **U1**.

- Bob chooses an n degree Polynomial **Y**, which coefficient sizes in bits equal $\| 2^q \|$, **Y** coefficients are pseudorandomly generated.

- Bob calculates $\mathbf{V} = pmdv_{(m,n)}(\mathbf{A} \times \mathbf{Y})$.

- For each **V** coefficient V(i), she calculates $V1(i) = mdv2_{(p,q)}V(i)$, getting thus :
a polynomial **V1**.

3.1.3 Values exchanged publicly:

- Alice sends **U1** coefficients to Bob.

- Bob sends **V1** coefficients to Alice.

3.1.4 Further Private Computations :

- Alice calculates polynomial **Wa**$= pmdv_{(m-n,n)}(\mathbf{X}\times\mathbf{V1})$.

- foreach **Wa** coefficient $Wa(i)$, she computes $Wa1(i) = mdv2_{(p-q,q)}(Wa(i))Div(2^{r+log_2(n)})$

- Bob calculates polynomial $Wb = pmdv_{(m-n,n)}(\mathbf{X}\times\mathbf{U1})$.

- foreach **Wb** coefficient $Wb(i)$, he computes $Wb1(i) = mdv2_{(p-q,q)}(Wb(i))Div(2^{r+log_2(n)})$

Bob and Alice know that foreach coefficients $Wa^1(i)$ and $Wb^1(i)$:

$$Pr[Wa^1(i) = Wb^1(i)] = 1 - ((1/3) \times (1/2)^r).$$

Shared key sise in bits they could get equals : $(m - (2 \times n)) \times (p - ((2 \times q) + r + log_2(n)).$.

3.2 Proof of Correction :

Proof of correction straitly follows from :

- Polynomial multiplication comutativity.

- Theorem 1.

- Adding n, s bits numbers result's size is $s \times log_2(n)$.

3.3 Generalized Multidimentional Moddiv public key cryptosystem:

We have observed experimentally that generalized variant of proposed public key cryptosystem holds.

3.3.1 Public parameters :

Integers m, n, p, q, P, Q, r satisfying $m > 2 \times n$, $p > (2 \times q) + r + log_2(n)$, $\parallel P \parallel = \parallel 2^p \parallel$ and $\parallel Q \parallel = \parallel 2^q \parallel$.

An m degree Polynomial **A**. which coefficients sizes in bits equal $\parallel 2^p \parallel$.

P, Q and **A** coefficients are pseudorandomly generated.

3.3.2 Private Computations :

- Alice chooses an n degree Polynomial $\mathbf{X}$, which coefficient sizes in bits equal $\| 2^q \|$, $\mathbf{X}$ coefficients are pseudorandomly generated.

- Alice calculates $\mathbf{U} = pmdv_{(m,n)}(\mathbf{A} \times \mathbf{X})$.

- For each $\mathbf{U}$ coefficient $U(i)$, she calculates $U1(i) = U(i)Mod(P)Div(Q)$, getting thus :
a polynomial $\mathbf{U1}$.

- Bob chooses an n degree Polynomial $\mathbf{Y}$, which coefficient sizes in bits equal $\| 2^q \|$, $\mathbf{Y}$ coefficients are pseudorandomly generated.

- Bob calculates $\mathbf{V} = pmdv_{(m,n)}(\mathbf{A} \times \mathbf{Y})$.

- For each $\mathbf{V}$ coefficient $V(i)$, He calculates $V1(i) = V(i)Mod(P)Div(Q)$, getting thus :
a polynomial $\mathbf{V1}$.

3.3.3 Values exchanged publicly:

- Alice sends **U1** coefficients to Bob.

- Bob sends **V1** coefficients to Alice.

3.3.4 Further Private Computations :

- Alice calculates polynomial $\mathbf{Wa} = pmdv_{(m-n,n)}(\mathbf{X} \times \mathbf{V1})$.

- foreach $\mathbf{Wa}$ coefficient $Wa(i)$, she computes $Wa1(i) = Wa(i)Mod(PDiv(Q))Div(2^{q+r+log_2(n)})$.

- Bob calculates polynomial $\mathbf{Wb} = pmdv_{(m-n,n)}(\mathbf{X} \times \mathbf{U1})$.

- foreach $\mathbf{Wb}$ coefficient $Wb(i)$, he computes $Wb1(i) = Wb(i)Mod(PDiv(Q))Div(2^{q+r+log_2(n)})$.

Bob and Alice know that foreach coefficients $Wa1(i)$ and $Wb1(i)$:

$$Pr[Wa1(i) = Wb1(i)] = 1 - ((1/3) \times (1/2)^r).$$

The key sise in bits they can get equals : $(m - (2 \times n)) \times (p - ((2 \times q) + r + log_2(n))$.

3.4 Security :

To attack proposed public key cryptosystem, adversary Eve knows polynomials **A**, **U1**, **V1** and parameters p, q, m, n satistifying the following conditions :

- **A** is of degree m.
- **U1** and **V1** are of degree m-n.
- **A** coefficients sizes in bits are p.
- **U1** and **V1** coefficients sizes in bits are p-q.
- $m > 2\times$n and $p > 2 \times q$.

She equaly knows that there exists polynomials **U**, **V**, **X** and **Y** Satisfying :

- **X** and **Y** are of degree n.
- **U** and **V** are of degree m-n.
- **X** and **Y** coefficients sizes in bits are q.
- **U** and **V** coefficients sizes in bits are $p + q + log_2(n)$.

$\mathbf{U} = (\mathbf{X} \times \mathbf{A})polyMod(x^m)polyDiv(x^n)$.
$\mathbf{V} = (\mathbf{Y} \times \mathbf{A})polyMod(x^m)polyDiv(x^n)$.

Coefficients of **U** and **V** are respectively related to those of **U1** and **V1** by following equations :

$U1(i) = U(i)Mod(2^p)Div(2^q)$.
$V1(i) = V(i)Mod(2^p)Div(2^q)$.

Meaning to know secret polynomials **X** and **Y**, Adversary Eve have to solve the following equations set :

For n $\leq$ i $\leq$ m - n

$U1(i) =(\ (\sum_{j=1}^{j=n} X(j) * A(j+i)) \ Mod(2^p)Div(2^q) \ .$

$V1(i) =(\ (\sum_{j=1}^{j=n} Y(j) * A(j+i)) \ Mod(2^p)Div(2^q) \ .$

To Attack Generalized Multidimentional ModDiv public key, Eve had to solve the following equations set, knowing parameters P and Q satisfying $\parallel P \parallel = \parallel 2^p \parallel$ and $\parallel Q \parallel = \parallel 2^q \parallel$

For n $\leq$ i $\leq$ m - n

$U1(i) =(\ (\sum_{j=1}^{j=n} X(j) * A(j+i)) \ Mod(P)Div(Q) \ .$

$V1(i) =(\ (\sum_{j=1}^{j=n} Y(j) * A(j+i)) \ Mod(P)Div(Q) \ .$

3.4.1 ModDiv Learn with rounding problem:

Basically underlying problem of proposed public cryptosystem is an instance of Learn with rounding problem first proposed by Banerjee et all [6] , which we defined as ModDiv Learn with rounding problem.

Said instance is caracterized by the following features :

- Rounding is done from $\mathbb{Z}_{2^p}$ to $\mathbb{Z}_{2^{(p-q)}}$.

- In Generalized Multidimentional ModDiv, Rounding is done from $\mathbb{Z}_P$ to $\mathbb{Z}_{P/Q}$.

- Public vectors are not random, they reflect polynomial mutiplication algebraic structure :
  If vector **A1** [A(1), A(2) .... A(n)] is given as public, vector **A2** [R(1),A(2),.... A(n-1)].
  is also given as public. Element R(1) is pseudorandomly generated.

- Secret vector elements size is the same as derandomized Error vector elements induced by rounding, which is q bits.

Seemingly Multidimentional ModDiv learn with rounding problem is at most as hard as its generalized counterpart, but public key cryptosystem based on the first is easier to implement, the modulis are powers of two : bit shifting operations could be used instead of actual integer modulo and division operations.

Learn with rounding problem is considered to be at least as hard as Learn with error problem, and was used recently to construct public key cryptosystems like Saber [7] one of NIST third round finalists.

The open question is how hard ModDiv Learn with rounding problem, actually is ?.

4 Conclusion :

In this paper we have presented Multidimentional ModDiv public key cryptosystem, the multidimentional version of public key cryptosystem presented in [3].

We have shown that its underlying problem is an instance of Learn with rounding problem, we defined as ModDiv Learn with rounding problem.

One can construct public key encryption and digital signature ElGamal schemes based on presented Key exchange protocols, it is also quite possible to device symmetric key algorithms based on introduced problems namely hash functions and pseudo random numbers generators.

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