Resolvable Block Designs in Construction of Approximate Real MUBs that are Sparse

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Abstract Several constructions of Mutually Unbiased Bases (MUBs) borrow tools from combinatorial objects. In this paper we focus how one can construct Approximate Real MUBs (ARMUBs) with improved parameters using results from the domain of Resolvable Block Designs (RBDs). We first explain the generic idea of our strategy in relating the RBDs with MUBs/ARMUBs, which are sparse (the basis vectors have small number of non-zero co-ordinates). Then specific parameters are presented, for which we can obtain new classes and improve the existing results. To be specific, we present an infinite family of $\lceil \sqrt{d} \rceil$ many ARMUBs for dimension $d = q(q+1)$, where $q \equiv 3 \bmod 4$ and it is a prime power, such that for any two vectors v_1, v_2 belonging to different bases, $|\langle v_1 | v_2 \rangle| < \frac{2}{\sqrt{d}}$. We also demonstrate certain cases, such as $d = sq^2$, where q is a prime power and $sq \equiv 0 \bmod 4$. These findings subsume and improve our earlier results in [Cryptogr. Commun. 13, 321-329, January 2021]. This present construction idea provides several infinite families of such objects, not known in the literature, which can find efficient applications in quantum information processing for the sparsity, apart from suggesting that parallel classes of RBDs are intimately linked with MUBs/ARMUBs.

Keywords (Approximate Real) Mutually Unbiased Bases, Combinatorial Design, Cryptology, Hadamard Matrices, Quantum Information Theory, Resolvable Block Design.

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1 Introduction

Mutually Unbiased Bases (MUBs) are unique mathematical structures on a complex vector space linked with quantum mechanics and having fundamental applications in quantum information processing. The usefulness of the MUBs are evident in different aspects of quantum cryptology and communications (one may

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see [1] and the references therein). In a d-dimensional complex vector space ($\mathbb{C}^d$), it is known that maximum number of MUBs can be $d+1$. The known methods to construct MUBs are based on Galois Field [23, 7] or through construction of maximal commuting unitary operators using generalized Pauli matrices [2]. Both of them provide the complete set of MUBs only when dimension is in the form of some power of a prime. The lower bound for number of MUBs in $\mathbb{C}^d$, when the dimension is a composite integer (say $d = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$), is given by $p_r^{n_r} + 1$ where $p_r^{n_r}$ is equal to $\min\{p_1^{n_1}, p_2^{n_2}, \dots, p_s^{n_s}\}$. There are known cases [21] where the lower bound is better than this. In case of $\mathbb{R}^d$, except for $d = 4^i s^2$, $s, i \in \mathbb{N}$, in all other dimensions, mostly a pair of Real MUB exist. Related results in this direction are available in [3].

Inspired by the fact that known methods to construct MUBs provide complete set only when dimension is certain power of a prime, there are strong conjectures relating existence of the complete set of MUBs and objects from combinatorial design. Though it is to be noted that the MUBs are constructed on Hilbert spaces which are continuum, whereas structures of combinatorial design like affine planes are built on finite number of points and lines for any order. Hence, the conjectures connecting existence of complete set of MUBs and certain combinatorial designs are intriguing. For example, one can refer to the conjecture [16] that states "non existence of a projective plane of the given order d implies that there are less than $d+1$ MUBs in $\mathbb{C}^d$."

Zauner studied quantum designs [24], which are orthogonal projection matrices on finite dimensional Hilbert space ($\mathbb{C}^d$) with certain features, and emphasized its parallel with combinatorial design theory. Noteworthy is the analogy with regular affine quantum design, which are equivalent to MUBs for rank one projection matrices, with combinatorial affine designs that consist of resolvable parallel classes. In the thesis [24], he also provided the solution of maximal regular affine quantum design, drawing parallels from combinatorial affine 2-design. The solution was shown to exist for prime power dimensions as was the case for combinatorial affine 2-design. However, for composite dimensions, the method did not offer solution.

Wooters [22] drew parallel between the known numbers of Mutually Orthogonal Latin Square (MOLS) of order q with the number of known MUBs in $\mathbb{C}^d$, where $d = q^2$. Based on this parallel, the analogy between lines in finite geometry and pure state in quantum mechanics can be understood. The study further argues that the complete set of MUBs in d-dimensional Hilbert space are analogous to combinatorial structure of affine plane of order d. In order to prove or disprove the conjecture, attempts had been made to construct MUBs from MOLS(q) and vice versa. One interesting work in this direction was by Wocjan [21] who used MOLS(q) to construct MUBs in $\mathbb{C}^d$, when $d = q^2$. This also improves the lower bound of MUBs for many different dimensions. Further, Paterek [13] devised a method to generate complete set of MUBs in prime power dimension using augmented set of MOLS(q) and Weyl-Schwinger unitary operators. However, in [14], the authors analysed the idea deeply and concluded that the method cannot relate the MUBs to MOLSs completely. They further concluded that the "problem of MUBs might not be equivalent to the mathematical problem of MOLS".

Our construction for ARMUBs (see Construction 1 later) is an independent and generalized approach based on RBDs, but it should be noted that for special cases related to exact MUBs, the MOLS based approach of [21] uses similar kind of combinatorial objects. The main difference is corresponding to each block. The construction idea of [21, Theorem 3, Example 4] considered different components of a vector in dimension d and a single Hadamard matrix of a specific order has been used. In our case, same sub-components of a vector are used corresponding to the elements of a block and those are disjoint for different blocks inside the same parallel class. In a special case, while generating exact real MUBs, we obtain similar results as in [21], but have the flexibility of exploiting different non-equivalent Hadamard matrices of the same order. Further, we have the advantage of using unitary matrices of the different orders to provide approximate MUBs, in case of different block sizes in designs which are not regular or balanced. These are not achievable for a large range of parameters by tweaking the construction idea in [21].

It is now well known for decades that obtaining new classes of MUBs and reaching the upper bounds are quite challenging problems. Some relaxation is thus considered in literature and there are efforts towards the concept of Approximate Mutually Unbiased Bases (AMUBs), where the inner product of two vectors drawn from two different bases is upper bounded by some value, rather than the exact $\frac{1}{\sqrt{d}}$ for dimension d. In this direction the works of [10,17] are pioneering, particularly, the result [17, Theorem 2] remains the best known construction of Approximate Mutually Unbiased Bases in $\mathbb{C}^d$. The vectors in the bases due to this construction are inherently complex in nature. Thus it is interesting to explore some novel construction method when one considers only the real components. In this direction we have studied certain results in [11] very recently.

However, further examination pointed out that the work [11] considered a very restricted class and further generalization beyond that is possible given richer combinatorial structures in literature. In this direction, we propose a generic method to construct Approximate MUBs (AMUBs) using Resolvable Block Designs (RBDs). RBDs consist of parallel classes. We provide a method to convert each parallel class into an orthonormal basis and show that these bases are intimately linked with AMUBs. Certain kinds of parallel classes in RBDs, meeting appropriate exact conditions can generate exact MUBs too. When these conditions are not met with, the parallel classes will generate approximate ones. The number of such MUBs or AMUBs depends on the number of parallel classes in RBDs. To convert parallel classes of RBDs into orthonormal bases, our construction strategy exploits unitary matrices, mostly in smaller dimension, depending on the parameters of the resolvable design.

In this article our main focus is to construct RBDs with suitable parameters where real Hadamard matrix (a subset of unitary matrices) can be used. The technique described here provides novel results in obtaining very sparse Approximate Real MUBs (ARMUBs), that can find application in quantum information processing. It is well known that sparsity can be exploited for efficient computations. With this backdrop, let us present the organization and contribution of this paper.

1.1 Organization & Contribution

In Section 2 we begin with various terms and notations formally. We define parameters to characterize Approximate MUBs and its sparsity. Then, in Section 2.1, we briefly explain the basics of Resolvable Block Design (RBD), Balanced Incomplete Block Design (BIBD) and Affine Resolvable BIBD. We provide examples of such block designs and clarify the symbols used for different parameters. Further, we highlight important relationships between the parameters of block designs and the necessary conditions for their existence. Thereafter, we present the novel results of this paper.

- In Section 3, we present the generic method to construct an orthonormal basis using a parallel class of RBD. This is explained in Construction 1. We also prove important bound on inner product between basis vectors from different orthonormal bases constructed from different parallel classes in the RBD. These are presented in Lemma 1 and Theorem 1.
- Then, in Section 4, we use different Resolvable Balanced Incomplete Block Designs (RBIBDs) to construct ARMUBs. The parameters of the BIBDs and the existence of certain matrices, particularly Hadamard, are identified from literature and then we plug those into our construction. Our main result is presented in Theorem 2. Several novel structures with new parameters are identified in this process in Section 4.1 through Affine Resolvable BIBDs (ARBIBDs).
  - Finally in Remark 1, we explain the construction of exact real MUBs as a special case. We can attain the results of similar quality as it is mentioned in [21]. However, the focus of this paper is on ARMUBs, and it will be evident that our proposal is much generalized and tuned towards the approximate results, that cannot be achieved through [21] or any other existing methods.
- In Section 5 we exploit the RBDs which are not balanced. We construct the unbalanced designs mostly by assimilating or modifying the Affine Resolvable $(q^2, q, 1)$-BIBDs, whose construction are known to exist for whenever q is some power of prime. Clear improvements over presently known parameters are described here. The treatment here provides significant generalization and improvement over our earlier result in [11] in different aspects. In [11], it was shown that $\frac{\sqrt{d}}{4} + 1$ ARMUBs with maximum value of inner product as $\frac{4}{\sqrt{d}}$ could be achieved.
  - To show the breadth of this new approach, one special case under Theorem 3 provides ARMUBs with the same quality as in [11]. This happens when $d = sq^2$, where q is a prime power and $sq \equiv 0 \bmod 4$. The special case, $s = 16$ as well as q a prime itself, takes care of the result in [11].
  - The parameters are improved too in some other classes. Theorem 4 shows that it is possible to construct $\lceil \sqrt{d} \rceil$ many ARMUBs with maximum value of inner product, between the vectors of two different bases, upper bounded by $\frac{2}{\sqrt{d}}$. That is,we have more number of classes with improved counts of MUBs and a better upper bound on the absolute values of the inner products. This happens when $d = q(q+1)$, where q is a prime power and $q \equiv 3 \bmod 4$.

We conclude the paper in Section 6 with directions to future research. While constructing the approximate MUBs, sometimes we also refer how exact MUBs can be obtained from our strategy. Indeed, this is not the main focus of this paper and those results are not better than the state-of-the-art ones, in terms of number of MUBs constructed. However, large sparsity is a novel feature of our construction, which is almost absent in the existing methods. However, we expect to obtain certain improvements if these techniques can be explored further.

Before proceeding, let us now define various notations and parameters characterizing the MUBs and the AMUBs, and highlight the relevant combinatorial objects and block designs. We will present certain examples to explain the ideas whenever required.

2 Background and Preliminaries

As stated previously, $\mathbb{C}^d$ (respectively $\mathbb{R}^d$) denotes the Complex (Real) vector space over dimension d. Throughout this article, depending on the context, the underlying vector space would be assumed to be $\mathbb{C}^d$ or $\mathbb{R}^d$, without mentioning explicitly.

In quantum information theory, two orthonormal bases in the d-dimensional complex vector space $\mathbb{C}^d$, given by $\{|e_1\rangle, \ldots, |e_d\rangle\}$ and $\{|f_1\rangle, \ldots, |f_d\rangle\}$ are called Mutually Unbiased, if the inner product, between the vectors from different bases satisfy

$$|\langle e_i|f_j\rangle| = \frac{1}{\sqrt{d}}, \forall i,j \in \{1,2,\ldots,d\}.$$

Similarly, a set of orthonormal bases are called Mutually Unbiased Bases (MUBs) if every pair of bases in the set is Mutually Unbiased. A set of MUBs are called real, if the imaginary components of the vectors in all of the bases are zero. The well known and age old problem is to maximize the number of MUBs for a dimension d and it is still open in composite dimensions. Further, the situation is more complicated in $\mathbb{R}^d$, where very few MUBs are known in general. Knowledge of relatively large number of Approximate MUBs can be helpful in practical situations. To characterize the Approximate ones and quantitatively compare it with the MUBs, let us formally define a few quantities and notations which we will be using throughout this article.

By $\mathbb{M} = \{M_1, M_2, \ldots, M_r\}$, we denote the set of r orthonormal bases in $\mathbb{C}^d$ or $\mathbb{R}^d$. We will denote the vectors of a basis M_l by $\{|\psi_i^l\rangle\}$, where $1 \le i \le d$. As explained, two orthonormal bases $M_l, M_m \in \mathbb{M}$, $l \neq m$, such that $M_l = \left\{|\psi_1^l\rangle, |\psi_2^l\rangle, \ldots, |\psi_d^l\rangle\right\}$ and $M_m = \{|\psi_1^m\rangle, |\psi_2^m\rangle, \ldots, |\psi_d^m\rangle\}$ will be called Mutually Unbiased if and only if $\left|\langle\psi_j^l|\psi_i^m\rangle\right| = \frac{1}{\sqrt{d}}$, $\forall i,j \in \{1,2,\ldots,d\}$. The set $\mathbb{M} = \{M_1, M_2, \ldots, M_r\}$ consisting of such orthonormal bases will form an MUB of size r provided $M_l, M_m \in \mathbb{M}$ are mutually unbiased for $\forall l \neq m$.

Given a set of orthonormal bases $\mathbb{M} = \{M_1, M_2, \ldots, M_r\}$ (may not be MUBs) of dimension d, we define Δ to be the set of inner products between the vectors from different orthonormal bases. That is, Δ contains the distinct values of $\left|\langle\psi_i^l|\psi_j^m\rangle\right|$ for all $i,j \in \{1,2,\ldots,d\}$ and $l \neq m \in \{1,\ldots,r\}$. In case $\mathbb{M}$ is an MUB, Δ is a singleton set with the only element $\frac{1}{\sqrt{d}}$. However, for the AMUBs, there will be more than one values in the set and we will try to minimize the maximum absolute value. In this regard, we like to define β-AMUB or β-ARMUB, for which the maximum value in Δ is bounded by $\frac{\beta}{\sqrt{d}}$.

To characterize the closeness of orthonormal bases M_l and M_m to MUBs, we define the variance of the inner products between the vectors of M_l and M_m from $\frac{1}{\sqrt{d}}$. For this we define $\sigma^{l,m} = \frac{1}{d}\sqrt{\sum_{i,j}\left(\frac{1}{\sqrt{d}} - \left|\langle\psi_j^l|\psi_i^m\rangle\right|\right)^2}$, as there are d^2 different elements in the calculation. For the set $\mathbb{M}$ of orthonormal bases, σ is accordingly defined as $\sigma = \max_{l\neq m}\{\sigma^{l,m}\}$.

Another way to characterize the closeness of a pair of orthonormal bases to MUBs is by the value of maximum difference of the inner product between any pair of vectors, say from M_l and M_m, with the value of $\frac{1}{\sqrt{d}}$. For this we define, $\tau^{l,m} = \max\left\{\left|\frac{1}{\sqrt{d}} - |\langle\psi_j^l|\psi_i^m\rangle|\right|\right\} \forall i,j$. For a set $\mathbb{M}$ of orthonormal bases, τ is accordingly defined as maximum of $\tau^{l,m}$, i.e., $\tau = \max_{l\neq m}\{\tau^{l,m}\}$.

Note that if M_l and M_m constitute a pair of MUBs, then $\beta = 1$, $\Delta = \left\{\frac{1}{\sqrt{d}}\right\}$, $\sigma^{l,m} = 0$ and $\tau^{l,m} = 0$. Similarly, if $\mathbb{M} = \{M_1, M_2, \ldots, M_r\}$ is a set of MUBs, then $\beta = 1, \Delta = \left\{\frac{1}{\sqrt{d}}\right\}$, $\sigma = 0$ and $\tau = 0$. In a certain sense, the vectors in two different bases in an MUB set should make maximum and same angles with others. Thus, the projective measurements associated with them are maximally uncorrelated. This will be deviated for the approximate MUBs.

It is clear that a particular basis of MUBs or AMUBs in $\mathbb{C}^d$ (resp. ARMUBs in $\mathbb{R}^d$) can be thought of as a $d \times d$ unitary matrix (resp. orthogonal matrix in real case) with their columns as orthonormal basis vectors. To characterize the sparsity of such matrices, we define ϵ as the ratio of the number of zero elements in the matrix to the total number of elements, i.e., d^2. It is clear to see that, $0 \leq \epsilon \leq 1$. Closer the value of ϵ to 1, more the number of zeros in the matrix and therefore larger the sparsity. MUBs, which have been constructed for prime or prime power dimensions using finite fields [23] or those constructed using maximal class of commuting operators [2], are invariably having almost all nonzero entries in the MUBs except for the standard basis. Thus, ϵ is close to 0 in these constructions. Regarding sparsity, the situation is similar with real MUBs constructed in [4]. The construction provided mutually unbiased Hadamard matrices, which by nature has all the entries $\{1, -1\}$, thereby ϵ is 0, i.e., not sparse at all. The MUBs constructed using MOLS [21,3] show relatively better sparsity. This is because the MOLS related constructions are equivalent to the RBDs in certain cases [5, Part III.3]. We like to reiterate that this is the first time the sparsity of the (approximate) MUBs are being quantified in literature. In case of actual implementation or computation, the sparsity might provide efficiency in practice.

2.1 Basics of Resolvable Block Design

Let us now explain the combinatorial object that we relate to construct (approximate) MUBs. The notations for combinatorial designs are borrowed from [19, Chapter 1].

Definition 1 A design can be expressed as a pair (X, A) such that the following properties are satisfied.

1. X is a set of elements, called points, and
2. A is a collection of non-empty subsets of X, called blocks.

A design is called simple, if there is no repeated block in A. In this paper, we will restrict our analysis to simple designs only.

Definition 2 A parallel class in design (X, A) is a subset of disjoint blocks in A whose union is X. For a design (X, A), if A can be partitioned into $r \geq 1$ parallel classes, called resolution, then the design (X, A) is called Resolvable Block Design (RBD).

For example, consider the combinatorial design (X, A_1) and (X, A_2) with $X = \{1,2,3,4,5,6,7,8\}$, $A_1 = \{(1,2),(2,3,4),(5,6,7),(1,8,6),(2,5),(6,7),(2,6,8)\}$ and $A_2 = \{(1,2,3),(2,4,6),(3,5,8),(6,8),(1,7),(4,5,7)\}$. Then (X, A_2) is a resolvable design since $A_2 = P_1 \cup P_2$ where $P_1 = \{(1,2,3),(6,8),(4,5,7)\}$ and $P_2 = \{(1,7),(2,4,6),(3,5,8)\}$ form two parallel classes consisting of disjoint sets whose union is set X. We say P_1 and P_2 form resolutions of A_2. On the other hand, the design (X, A_1) is not resolvable as such resolutions are not possible in this case.

Definition 3 A Balanced Incomplete Block Design (BIBD) is a design (X, A), with parameters $\{v, k, \lambda\} \in \mathbb{N}$ and $v > k \geq 2$ and $\lambda \geq 1$ such that the following properties are satisfied:

1. $|X| = v$,
2. each block contains exactly k points, and
3. every pair of distinct points is contained in exactly λ blocks.

The third property relates to balancedness. It can be shown that every point occurs in exactly $r = \frac{\lambda(v-1)}{k-1}$ blocks and a BIBD has exactly $b = \frac{vr}{k} = \frac{\lambda(v^2-v)}{k^2-k}$ blocks. A (v, k, λ)-BIBD (X, A) is resolvable if A has at least one resolution. Note that, the design (X, A_2) has resolution and hence it is a RBD. However, it is not a BIBD as properties 2, 3 are not satisfied.

The necessary condition for (v, k, λ)-BIBD to be resolvable is $b \geq v + r - 1$ or equivalently $r \geq k + \lambda$. A Resolvable (v, k, λ)-BIBD is called Affine Resolvable (ARBIBD) if $b = v + r - 1$ or equivalently $r = k + \lambda$. Further, any two blocks from different parallel classes of ARBIBD have exactly $\frac{k^2}{v}$ points in common.

An Affine Plane of order q is an example of $(q^2, q, 1)$-ARBIBD. The construction of such Affine Planes are known only when q is some power of a prime. A finite projective plane of order q is an example of $(q^2 + q + 1, q + 1, 1)$-BIBD. Finite projective planes are equivalent to finite affine planes and vice versa. Detailed understanding on these structures are presented in [19, Chapters 2, 5].

3 Our Generic idea of Construction

Here we connect how one can design MUBs or approximate MUBs from the above mentioned combinatorial objects, namely RBDs. We provide a generic construction of an orthonormal basis from a parallel class of any Resolvable Block Design (RBD). If the parallel class contain s blocks then the construction would also require s many unitary matrices each of the order which would be equal to the the size of blocks in the parallel class under consideration. If there are r many parallel classes in (X, A), then each one of them can be used to construct an orthonormal basis in $\mathbb{C}^d$ or $\mathbb{R}^d$. Next we show that the inner product between two vectors, each from different orthonormal basis, constructed using parallel classes from design (X, A), are bounded if the Hadamard matrices are exploited as unitary matrices. The set of Orthonormal Basis so constructed are β-AMUBs (see Theorem 1 later). This β will depend on the parameters of the RBD and if the parameters are such that $\beta = 1$ then the set of orthonormal bases, constructed using parallel classes, will be MUBs.

Let us now describe the steps for construction of an orthonormal basis using a parallel class from an RBD (X, A). Then we present a simple example to explain the technique.

Construction 1

1. *In a design (X, A), choose the elements of X as some orthonormal basis vectors of $\mathbb{C}^d$. That is, if $|X| = d$ then $X = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_d\rangle\}$, such that $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. Hence A, which contains blocks consisting of elements from X, would consist of blocks consisting the elements from the set of chosen orthonormal basis vectors.*
2. *Let $B = \{b_1, b_2, \ldots, b_s\}$ be one of the parallel class of the design (X, A), where b_i's are disjoint blocks containing elements from X. Since B is a parallel class, this implies $X = b_1 \cup b_2 \cup \ldots \cup b_s$.*
3. *Consider one of the blocks $b_r = \{|\psi_{r_1}\rangle, |\psi_{r_2}\rangle, \ldots, |\psi_{r_{n_r}}\rangle\} \in B$ and let $|b_r| = n_r$. Corresponding to this block, choose any $n_r \times n_r$ unitary matrix whose elements are say $u^r_{ij},\ i, j = 1, 2, \ldots, n_r$.*
4. *Next construct n_r many vectors in the following manner, using b_r and u^r_{ij}.*

$$|\phi^r_i\rangle = u^r_{i1}|\psi_{r_1}\rangle + u^r_{i2}|\psi_{r_2}\rangle + \ldots + u^r_{in_r}|\psi_{r_{n_r}}\rangle = \sum_{k=1}^{n_r} u^r_{ik}|\psi_{r_k}\rangle : i = 1, 2, \ldots, n_r.$$

5. *In a similar fashion, corresponding to each block $b_j \in B$, construct n_j many vectors where $|b_j| = n_j$, using any $n_j \times n_j$ unitary matrix. Since $\sum_{j=1}^{s} n_j = d$, we will get exactly d many vectors.*

Note that if all the blocks in a parallel class used in the above construction consist of only single element, then it will result into vectors which will be some permutation of X. Similarly if identity matrices are chosen corresponding to all blocks b_j of the parallel class, again the above construction will result into vectors which will be some permutation of X. Hence, in order to get vectors different from the initial chosen orthonormal vectors X, at least one of the blocks of the parallel class should have more than one elements and at least one of the unitary matrices, chosen corresponding to some block of the parallel class, should be different from the identity matrix.

Lemma 1 *Refer to Construction 1. The vectors, $|\phi^r_i\rangle$ for $i = 1, 2, \ldots, n_r$ and $r = 1, 2, \ldots, s$, such that $\sum_{j=1}^{s} n_j = d$, form an orthonormal basis.*

Proof Consider n_r many vectors constructed from the block b_r of a parallel class B. The inner product of any two vectors constructed from b_r would give

$$\langle\phi^r_j|\phi^r_i\rangle = \sum_{k,l=1}^{n_r} \overline{u^r_{jl}}\, u^r_{ik}\, \langle\psi_{r_l}|\psi_{r_k}\rangle = \sum_{k,l=1}^{n_r} \overline{u^r_{jl}}\, u^r_{ik}\, \delta_{kl} = \sum_{k=1}^{n_r} \overline{u^r_{jk}}\, u^r_{ik} = \delta_{ij}.$$

Hence n_r many vectors constructed from the block b_r are orthogonal. Note that, the vectors constructed from a block are linear combinations of vectors $\{|\psi_i\rangle\} \in X$ in the corresponding block. Since different blocks of the parallel class are disjoint subsets of X, the vectors constructed from different blocks of the parallel class would lie on the orthogonal subspace of $\mathbb{C}^d$ and hence will be orthogonal. Since $\sum_{j=1}^{s} n_j = d$, the construction will generate an orthonormal basis in $\mathbb{C}^d$. □

Note that in Construction 1, if X is chosen from some orthonormal basis vectors of $\mathbb{R}^d$ along with the orthogonal matrix (i.e., all the entries real) corresponding to each b_r in step 1 and 3 respectively, then the Construction 1 will result into real orthonormal basis vectors in $\mathbb{R}^d$ corresponding to the parallel class under consideration. Similarly, as noted above, the construction will provide vectors different from X in $\mathbb{R}^d$, if at least one block of the parallel class consist of more than one elements or at least one orthogonal matrix corresponding to some block is chosen different from the identity matrix.

Let us illustrate above construction method by applying it on Resolvable Block Design (X, A_2) mentioned in Section 2.1. The two resolutions of A_2 are P_1 and P_2, where $P_1 = \{(1,2,3),(6,8),(4,5,7)\}$ and $P_2 = \{(1,7),(2,4,6),(3,5,8)\}$. We will show how to convert P_1 into one orthonormal basis and on a similar manner P_2 can be converted to another orthonormal basis. Let $X = \{|1\rangle, |2\rangle, \ldots, |8\rangle\}$ be the computational basis in $\mathbb{C}^8$. Using above notations, consider the parallel class $P_1 = \{b_1, b_2, b_3\}$, where $b_1 = (1,2,3)$, $b_2 = (6,8)$ and $b_3 = (4,5,7)$. Thus we see that, $|b_1| = |b_3| = 3$ and $|b_2| = 2$. Hence, we require at least two unitary matrices, one of order 2 and another of order 3. We will choose the Hadamard matrices in this direction. Let us choose $U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ and $U_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}$. For simplicity, we will use the same U_3 for both the blocks, b_1 and b_3. Following the notations and methods given in Construction 1, we obtain total eight orthogonal vectors of $\mathbb{C}^8$ from the parallel class P_1. Two orthogonal vectors are constructed from b_2 and three each from b_1 and b_3 in the following manner.

$$\begin{aligned}
|\phi_1^1\rangle &= \frac{1}{\sqrt{3}}\left(|1\rangle + |2\rangle + |3\rangle\right) = \frac{1}{\sqrt{3}}(1\ 1\ 1\ 0\ 0\ 0\ 0\ 0)^T\\
|\phi_2^1\rangle &= \frac{1}{\sqrt{3}}\Big(|1\rangle + \omega\,|2\rangle + \omega^2\,|3\rangle\Big) = \frac{1}{\sqrt{3}}(1\ \omega\ \omega^2\ 0\ 0\ 0\ 0\ 0)^T\\
|\phi_3^1\rangle &= \frac{1}{\sqrt{3}}\Big(|1\rangle + \omega^2\,|2\rangle + \omega\,|3\rangle\Big) = \frac{1}{\sqrt{3}}(1\ \omega^2\ \omega\ 0\ 0\ 0\ 0\ 0)^T\\
|\phi_1^2\rangle &= \frac{1}{\sqrt{2}}\left(|6\rangle + |8\rangle\right) = \frac{1}{\sqrt{2}}(0\ 0\ 0\ 0\ 0\ 1\ 0\ 1)^T\\
|\phi_1^2\rangle &= \frac{1}{\sqrt{2}}\left(|6\rangle - |8\rangle\right) = \frac{1}{\sqrt{2}}(0\ 0\ 0\ 0\ 0\ 1\ 0\ \ -1)^T\\
|\phi_1^3\rangle &= \frac{1}{\sqrt{3}}\left(|4\rangle + |5\rangle + |7\rangle\right) = \frac{1}{\sqrt{3}}(0\ 0\ 0\ 1\ 1\ 0\ 1\ 0)^T\\
|\phi_2^3\rangle &= \frac{1}{\sqrt{3}}\Big(|4\rangle + \omega\,|5\rangle + \omega^2\,|7\rangle\Big) = \frac{1}{\sqrt{3}}(0\ 0\ 0\ 1\ \omega\ 0\ \omega^2\ 0)^T\\
|\phi_3^3\rangle &= \frac{1}{\sqrt{3}}\Big(|4\rangle + \omega^2\,|5\rangle + \omega\,|7\rangle\Big) = \frac{1}{\sqrt{3}}(0\ 0\ 0\ 1\ \omega^2\ 0\ \omega\ 0)^T.
\end{aligned}$$

Note that the first three vectors $|\phi_1^1\rangle, |\phi_2^1\rangle, |\phi_3^1\rangle$ corresponding to one block in a parallel class works with $|1\rangle, |2\rangle, |3\rangle$ only, and the orthogonality among themselves is achieved by using U_3. This is different from [21, Theorem 3, Example 4] as there the vectors corresponding to each block may have other components of the vector. The kind of separate grouping that we use here and use the unitary matrices for orthogonality between the vectors is different from that of [21]. In our case the between block orthogonality in the same parallel class is achieved as the components of the vectors are different. This helps us to exactly calculate the different inner product values (as we are considering approximate MUBs rather than exact MUBs) when blocks (and the vectors corresponding to that) from two different parallel classes (different orthogonal bases) interact.

Now arranging the above 8 Orthogonal vectors as columns of 8×8 unitary matrix we have have the following.

$$M_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2}\omega & \sqrt{2}\omega^2 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2}\omega^2 & \sqrt{2}\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2}\omega & \sqrt{2}\omega^2 \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2}\omega^2 & \sqrt{2}\omega \\ 0 & 0 & 0 & \sqrt{3} & -\sqrt{3} & 0 & 0 & 0 \end{pmatrix}.$$

In a similar manner, the parallel class P_2 can be converted into another orthonormal basis of $\mathbb{C}^8$. Following the Construction 1 in a similar manner, and using the unitary matrix U_2 for block $(1,7)$ and U_3 for both the blocks $(2,4,6)$ and $(3,5,8)$ we obtain the following.

$$M_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} & \sqrt{2}\omega & \sqrt{2}\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2}\omega & \sqrt{2}\omega^2 \\ 0 & 0 & \sqrt{2} & \sqrt{2}\omega^2 & \sqrt{2}\omega & 0 & 0 & 0 \\ \sqrt{3} & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2}\omega^2 & \sqrt{2}\omega \end{pmatrix}.$$

Let us now denote $\{|\psi_i^1\rangle\}, 1 \leq i \leq 8$ for column vectors of M_1 and $\{|\psi_j^2\rangle\}, 1 \leq j \leq 8$ for column vectors of M_2. Through explicit calculations we get, $\Delta = \left\{ |\langle \psi_i^1 | \psi_j^2 \rangle| \text{ where } i,j = 1,\ldots,8 \right\} = \left\{ \frac{1}{2}, \frac{1}{\sqrt{6}}, \frac{1}{3} \right\}$. In order to calculate $\sigma^{1,2}$, note that out of 64 many inner products formed between the vectors of M_1 and M_2, 36 of them have the value $\frac{1}{3}$, 24 of them have the value of $\frac{1}{\sqrt{6}}$ and remaining 4 has the value $\frac{1}{2}$, whereas MUBs in $\mathbb{C}^8$ would have inner product value of $\frac{1}{\sqrt{8}}$ for all the cases. Hence,

$$\left(\sigma^{1,2}\right)^2 = \frac{1}{64}\left(36\left(\frac{1}{3} - \frac{1}{\sqrt{8}}\right)^2 + 24\left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{8}}\right)^2 + 4\left(\frac{1}{2} - \frac{1}{\sqrt{8}}\right)^2\right),$$

which evaluates to $\sigma^{1,2} \approx 0.052$. Note that $\max_{i,j} \left|\langle \psi_i^1 | \psi_j^2 \rangle\right| = \frac{1}{2}$. Hence $\tau^{1,2} = \left|\frac{1}{2} - \frac{1}{\sqrt{8}}\right| \approx 0.12$. We also obtain $\beta_{1,2} = \frac{\sqrt{8}}{2} = \sqrt{2}$ and the sparsity $\epsilon = \frac{42}{64} \approx 0.66$ for both M_1 and M_2. From calculations it is evident that the $\max_{i,j} \left|\langle \psi_i^1 | \psi_j^2 \rangle\right|$ is dependent on the block sizes and number of points common between the blocks from which $|\psi_i^1\rangle, |\psi_j^2\rangle$ are constructed. The following proposition examines the same, when Hadamard matrices are used as unitary matrices, and presents an upper bound on this value. The Hadamard matrices are subset of unitary matrices, and atleast one such (Fourier) matrix exists for every dimension. In the following proposition, and the subsequent constructions, we will use Hadamard matrices of order dependent on the block size of parallel class under consideration. That is, in this paper, we will use the real Hadamard matrices for unitarity/orthogonality.

Theorem 1 *Let P_1 and P_2 be two parallel classes of Resolvable Block Design (X, A) having constant block sizes k_1 and k_2 respectively, such that the maximum intersection points between the blocks of parallel classes is μ. Then corresponding to the parallel classes P_1 and P_2, orthonormal bases in $\mathbb{C}^d$ can be constructed which is β-AMUB with $\beta = \mu\sqrt{\frac{d}{k_1 k_2}}$ where $|X| = d$.*

Proof The proof follows from Construction 1, where the Hadamard matrices are chosen as the unitary matrices in step 3. We show this as follows.

Let $X = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_d\rangle\}$ be some orthonormal basis vectors in $\mathbb{C}^d$. Let the blocks in the parallel classes are $P_1 = \{b_1^1, b_2^1, \ldots, b_p^1\}$ and $P_2 = \{b_1^2, b_2^2, \ldots, b_q^2\}$. We have $|b_1^1| = |b_2^1| = \ldots = |b_p^1| = k_1$ and $|b_1^2| = |b_1^2| = \ldots = |b_q^2| = k_2$ and $X = b_1^1 \cup b_2^1 \cup \ldots \cup b_p^1 = b_1^2 \cup b_2^2 \cup \ldots \cup b_q^2$.

Following the steps of the Construction 1, let $M_1 = \{|\zeta_1\rangle, |\zeta_2\rangle, \ldots, |\zeta_d\rangle\}$ and $M_2 = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_d\rangle\}$ be the orthonormal matrices constructed from parallel classes P_1 and P_2 respectively of the RBD (X, A). Consider the vectors constructed from r^{th} $(r \leq p)$ block of P_1, say $|\zeta_{r_i}\rangle$, and from s^{th} $(s \leq q)$ block of P_2, say $|\phi_{s_j}\rangle$. Let H_{k_1} be the Hadamard matrix of order k_1 used for constructing $|\zeta_{r_i}\rangle$ and H_{k_2} be the Hadamard matrix of order k_2 used for constructing $|\phi_{s_j}\rangle$. Then we have

$$|\zeta_{r_i}\rangle = h_{i1}^1 |\psi_{r_1}\rangle + h_{i2}^1 |\psi_{r_2}\rangle + \ldots + h_{ik_1}^1 |\psi_{r_{k_1}}\rangle = \sum_{u=1}^{k_1} h_{iu}^1 |\psi_{r_u}\rangle : b_r^1 = \{|\psi_{r_u}\rangle\} \subset X,$$

$$|\phi_{s_j}\rangle = h_{j1}^2 |\psi_{s_1}\rangle + h_{j2}^2 |\psi_{s_2}\rangle + \ldots + h_{jk_2}^2 |\psi_{s_{k_2}}\rangle = \sum_{v=1}^{k_2} h_{jv}^2 |\psi_{s_v}\rangle : b_s^2 = \{|\psi_{s_v}\rangle\} \subset X,$$

where $(H_{k_1})_{i,j} = h_{i,j}^1$ and $(H_{k_2})_{i,j} = h_{i,j}^2$. Hence,

$$\langle\zeta_{r_i}|\phi_{s_j}\rangle = \sum_{u,v=1}^{k_1,k_2} \overline{h_{iu}^1}\ h_{jv}^2\ \langle\psi_{r_u}|\psi_{s_v}\rangle = \sum_{u,v=1}^{k_1,k_2} \overline{h_{iu}^1}\ h_{jv}^2\ \delta_{r_u,s_v}.$$

Since $\{|\psi_{r_u}\rangle\}$ and $\{|\psi_{s_v}\rangle\}$ are subsets of X, which consist of orthonormal basis vectors, therefore $\langle\psi_{r_u}|\psi_{s_v}\rangle = \delta_{r_u,s_v}$. Let $b_r^1 \cap b_s^2$ be the set of points common in the two blocks, which has been used in the construction for $|\zeta_{r_i}\rangle$ and $|\phi_{s_j}\rangle$. It is given that $\max_{i,j}\left\{\left|b_i^1 \cap b_j^2\right|\right\} = \mu$, where $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$. Hence $\left|b_r^1 \cap b_s^2\right| \leq \mu$. Further, note that $\left|h_{r_u}^1\right| = \frac{1}{\sqrt{k_1}}$ and $\left|h_{s_v}^2\right| = \frac{1}{\sqrt{k_2}}$. Hence

$$\langle\zeta_{r_i}|\phi_{s_j}\rangle = \sum_{b_r^1 \cap b_s^2} \overline{h_{iu}^1}\ h_{jv}^2 \leq \sum_{b_r^1 \cap b_s^2} |\overline{h_{iu}^1}|\ |h_{jv}^2| = \sum_{b_r^1 \cap b_s^2} \frac{1}{\sqrt{k_1 k_2}} \leq \frac{\mu}{\sqrt{k_1 k_2}} = \frac{\mu\sqrt{\frac{d}{k_1 k_2}}}{\sqrt{d}},$$

where $\beta = \mu\sqrt{\frac{d}{k_1 k_2}}$. The vectors $|\zeta_{r_i}\rangle$ and $|\phi_{s_j}\rangle$ are constructed from any r^{th} and s^{th} block of P_1 and P_2 respectively. Hence the above relationship will hold for any two vectors constructed from different parallel classes. Hence $|\langle\zeta_i|\phi_j\rangle| \leq \frac{\mu}{\sqrt{k_1 k_2}}$ for any $1 \leq i, j \leq d$. Thereby, the orthonormal bases constructed corresponding to the parallel classes P_1 and P_2 are β-AMUB. $\square$

Suitable choices of Hadamard matrices, for specific situations may improve the inequality. Particularly, whenever parametric form of Hadamard matrices are available, they may be used and parameters may be optimized, which can result into orthonormal bases closer to MUBs. Another method to improve this inequality would be through reducing the μ, which is dependent on parameters of the Resolvable Block Design. In fact, if $\mu = 1$ and $d = k_1 \cdot k_2$ then $\beta = 1$, and the above constructions will present MUBs. In our present work we will focus on making β close to 1 (from the higher side) by altering the parameters of RBD.

In a similar manner, we can convert Resolvable Block Design consisting of r resolutions into set of r orthonormal bases for $\mathbb{R}^d$ using real Hadamard matrices in set 3 of Construction 1. A real Hadamard matrix exists for $d = 2^s, s \in \mathbb{N}$ (Sylvester Construction [9]) and for $d = 2^s(q+1)$, where q is some power of odd prime (Paley Construction [12]), apart from other known constructions [9]. In fact, the Hadamard Conjecture [6] says that the real Hadamard matrix exists for all dimensions $d > 2$ such that $4|d$. This is longstanding unproven conjecture which has been found to be true for all $d < 668$ [6].

The order of Hadamard matrix to be exploited in the step 3 of construction 1, is decided by the block size of the corresponding parallel class. Hence to obtain real MUBs, we will ensure that the block size (denoted by k) is either 2 or divisible by 4. Though our focus is on ARMUBs, the results hold equally well for complex AMUBs. In fact to obtain complex AMUBs, there would be no restriction on the parameters of the Resolvable Block Design (X, A), as there are Hadamard matrices available for every order, namely the Fourier matrices.

For all our examples and constructions in following sections, the points (or elements) in X would consist of computational basis vectors and would be simply denoted as $\{1, 2, \ldots, d\}$. For example, $|X| = 4$ implies $X = \{1, 2, 3, 4\}$ where 1 represents $(1, 0, 0, 0)^T$, 2 represents $(0, 1, 0, 0)^T$, 3 represent $(0, 0, 1, 0)^T$ and 4 represent $(0, 0, 0, 1)^T$. Since a real Hadamard matrix consists of only $\{-1, +1\}$ entries, our construction for ARMUBs will have vectors whose entries will consist of $\{-1, 0, +1\}$ with some normalization factor for the corresponding vectors.

4 ARMUBs using Resolvable BIBDs

In this section, we will explore the designs which are resolvable and also $(v, k, 1)$-BIBDs. Necessary condition for a $(v, k, 1)$-BIBD to be resolvable can be derived by fact that $k|v$ and $(k-1)|(v-1)$ for b and r to be integers. It turns out that the necessary condition for resolvable $(v, k, 1)$-BIBD is $v = k(k-1)t + k$ for $t \in \mathbb{N}$ [8]. It has been shown that with finitely many exceptions, resolvable $(v, k, 1)$-BIBDs exist whenever necessary condition is satisfied. More specifically, given $k \geq 2$, there exists a constant $C(k)$ such that if $v \geq C(k)$ and $v \equiv k \mod [k(k-1)]$, then $(v, k, 1)$-resolvable BIBDs exist [15]. This implies that there exist infinite families of resolvable $(v, k, 1)$-BIBDs for every $k \in \mathbb{N}$. In all the following theorems and constructions, the dimension d of the underlying vector space will be equal to v i.e., $d = v$.

Theorem 2 *Suppose, there exists a resolvable $(v, k, 1)$-BIBD. Let $d = v = k(k-1)t + k$, where $t \in \mathbb{N}$. If $t > 1$, then one can construct $(kt+1)$ many Approximate MUBs in $\mathbb{C}^d$ with $\Delta = \left\{0, \frac{1}{k}\right\}$, $\beta = \sqrt{\frac{(k-1)t+1}{k}}$, $\sigma^2 = \frac{2}{d}\left[1 - \frac{k}{\sqrt{d}}\right]$, $\tau < \frac{1}{k}$ and the* *sparsity* $\epsilon = 1 - \frac{1}{(k-1)t+1}$. *If* $t = 1$, *then one can construct* $(k+1)$ *many MUBs in* $\mathbb{C}^d$ *with* $\Delta = \left\{\frac{1}{k}\right\}$, $\beta = 1$, $\sigma = \tau = 0$ *and the sparsity* $\epsilon = \left(1 - \frac{1}{k}\right)$. *Further, if a real Hadamard matrix of order* k *exists, then one can construct ARMUBs in* $\mathbb{R}^d$ *with the same parameters.*

Proof The necessary condition for the existence of a Resolvable $(v, k, 1)$-BIBD is $v = k(k-1)t + k$ for some $t \in \mathbb{N}$ [8]. Let us consider $t > 1$. Since $\lambda = 1$ in this BIBD, every pair of points will occur in a single block. Thus, any two blocks will have maximum one point in common. This implies that the blocks from different parallel classes will have at most one point in common. Now we can use any Hadamard matrix of order k to convert each parallel class having block size k into orthonormal basis as per Construction 1. The Δ would consist of $\left\{\frac{1}{k}, 0\right\}$ corresponding to whether there is one point in common or there is no point is common between the blocks used to generate corresponding vectors of the orthonormal basis. Hence $\beta = \frac{\sqrt{d}}{k} = \frac{\sqrt{k(k-1)t+k}}{k} = \sqrt{\frac{(k-1)t+1}{k}}$. Now to compute σ, note that each vector in an orthonormal basis will have inner product value equal to $\frac{1}{k}$ with k^2 vectors of any other orthonormal basis and will have inner product equal to 0 with remaining $(d - k^2)$ basis vectors. Hence,

$$\sigma^2 = \frac{1}{d}\left[k^2\left(\frac{1}{\sqrt{d}} - \frac{1}{k}\right)^2 + \left(d - k^2\right)\left(\frac{1}{\sqrt{d}} - 0\right)^2\right] = \frac{2}{d}\left[1 - \frac{k}{\sqrt{d}}\right].$$

In order to calculate τ, note that $\left|\frac{1}{k} - \frac{1}{\sqrt{d}}\right| \geq \frac{1}{\sqrt{d}}$ for $d \geq 4k^2$ which implies $t \geq \frac{4k-1}{k-1} \approx 4$ for sufficiently large k. Hence,

$$\tau = \begin{cases} \frac{1}{\sqrt{d}}, \text{ for } 1 < t \leq \frac{4k-1}{k-1} \ \left(\text{i.e., } k^2 < d \leq 4k^2\right) \\ \frac{1}{k} - \frac{1}{\sqrt{d}}, \text{ for } t > \frac{4k-1}{k-1} \ \left(\text{i.e., } d > 4k^2\right) \end{cases}.$$

Hence $\tau \leq \frac{1}{k}$ for any d. Note that for for a fixed k, σ decreases as the dimension d increases, whereas τ goes towards $\frac{1}{k}$. In this construction Δ is independent of dimension but σ and τ are not.

To calculate sparsity, note that each vector in $\mathbb{C}^d$ (or $\mathbb{R}^d$), constructed from block of size k, will have exactly k many non-zero and $d - k$ many zero entries. Since the construction provides d orthonormal basis vectors, we get

$$\epsilon = \frac{d^2 - dk}{d^2} = 1 - \frac{k}{d} = 1 - \frac{1}{(k-1)t+1}.$$

Now consider the case $t = 1$. Here $d = v = k^2$ which implies combinatorial design is $(k^2, k, 1)$- ARBIBD. Hence blocks from different parallel classes have exactly one point in common. Hence $\Delta = \left\{\frac{1}{k}\right\}$. But in this case $\frac{1}{k} = \frac{1}{\sqrt{d}}$. Hence $\sigma = \tau = 0$. The expression for sparsity will remain unchanged i.e. $\epsilon = 1 - \frac{k}{d} = 1 - \frac{1}{k}$. This completes the proof. □

Hence the constructed orthonormal bases are very sparse even for moderate size k and t. Let us consider two simple cases, for $k = 2, 4$. This will enable us to choose real Hadamard matrices for converting the parallel classes of resolvable $(v, k, 1)$-BIBD into orthonormal base. It has been shown in [15,8] that for these values of k, necessary condition is also a sufficient condition for the existence of resolvable $(v, k, 1)$-BIBD without any exception. Hence we obtain the following result.

Corollary 1 *For any even dimension $d > 4$, there exist $d-1$ ARMUBs in $\mathbb{R}^d$ such that $\Delta = \left\{\frac{1}{2}, 0\right\}$, $\beta = \sqrt{\frac{d}{4}}$, $\sigma^2 = \frac{2}{d}\left[1 - \frac{2}{\sqrt{d}}\right]$, $\tau < \frac{1}{2}$ and sparsity $\epsilon = 1 - \frac{2}{d}$. Further, for $d = 4$ we can construct 3 Real MUBs with sparsity $\epsilon = \frac{1}{2}$*

Proof This directly follows from Theorem 2. Taking $k = 2$ gives $d = 2t + 2$. Hence for $t > 1$, we can construct $2t + 1 = d - 1$ many Approximate MUBs in $\mathbb{R}^d$ with $\Delta = \left\{0, \frac{1}{2}\right\}$, $\beta = \sqrt{\frac{t+1}{2}} = \sqrt{\frac{d}{4}}$, $\sigma^2 = \frac{2}{d}\left[1 - \frac{2}{\sqrt{d}}\right]$, $\tau < \frac{1}{2}$ and the sparsity $\epsilon = \left(1 - \frac{2}{d}\right)$. If $t = 1$, then $d = 2 + 2 = 4$ and we can construct $(2 + 1) = 3$ many MUBs in $\mathbb{R}^d$ with $\Delta = \left\{\frac{1}{2}\right\}$, $\beta = 1$, $\sigma = \tau = 0$ and the sparsity $\epsilon = 1 - \frac{1}{2} = \frac{1}{2}$. This completes the proof. □

To explicitly demonstrate the construction of real MUBs in $\mathbb{R}^4$, using Resolvable $(4, 2, 1)$-BIBD, let $X = \{1, 2, 3, 4\}$ be the four standard basis vectors in $\mathbb{R}^4$ and let $A = \{P_1, P_2, P_3\}$ be the three parallel classes of the resolvable design. Explicitly, one such design would be:

$$P_1 = \{(1,2),(3,4)\} \quad P_2 = \{(1,3),(2,4)\} \quad P_3 = \{(1,4),(2,3)\}.$$

Now using $H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for each block of parallel class, and exploiting Construction 1, we obtain three set of orthonormal basis vectors corresponding to each parallel class as follows.

$$M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, M_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, M_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

The columns of $\{M_1, M_2, M_3\}$ form the orthonormal basis vectors, and these orthonormal bases are MUBs in $\mathbb{R}^4$. Note that maximum number of real MUBs in $d = 4^s, s \in \mathbb{N}$ is equal to $\frac{d}{2} + 1$ [3]. Hence for $d = 4$, the three MUBs constructed are also maximal for $\mathbb{R}^4$.

To illustrate construction of ARMUBs with a specific example, let us consider $d = 6$. Let $X = \{1, 2, 3, 4, 5, 6\}$ be the six standard basis vectors in $\mathbb{R}^6$ and let $A = \{P_1, P_2, P_3, P_4, P_5\}$ be the five parallel classes of the design. Explicitly, one such design would be

$$\begin{aligned} P_1 &= \{(1,4),(2,3),(5,6)\} \\ P_2 &= \{(2,6),(3,4),(1,5)\} \\ P_3 &= \{(5,2),(3,1),(4,6)\} \\ P_4 &= \{(5,3),(4,2),(6,1)\} \\ P_5 &= \{(5,4),(6,3),(1,2)\}. \end{aligned}$$

Now again using $H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ for each block of parallel class, and following Construction 1, we obtain five set of orthonormal basis vectors as follows.

$$M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&1&-1&0&0\\1&-1&0&0&0&0\\0&0&0&0&1&1\\0&0&0&0&1&-1\end{pmatrix}, M_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0&1&1\\1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&1&-1&0&0\\0&0&0&0&1&-1\\1&-1&0&0&0&0\end{pmatrix},$$

$$M_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1&1&0&0\\1&1&0&0&0&0\\0&0&1&-1&0&0\\0&0&0&0&1&1\\1&-1&0&0&0&0\\0&0&0&0&1&-1\end{pmatrix}, M_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0&1&1\\0&0&1&1&0&0\\1&1&0&0&0&0\\0&0&1&-1&0&0\\1&-1&0&0&0&0\\0&0&0&0&1&-1\end{pmatrix},$$

$$M_5 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0&1&1\\0&0&0&0&1&-1\\0&0&1&1&0&0\\1&1&0&0&0&0\\1&-1&0&0&0&0\\0&0&1&-1&0&0\end{pmatrix}.$$

Here again, the columns form the orthonormal basis vectors. Note that, for any basis vector say from M_1, there are four vectors in M_2 which has inner product value of $\frac{1}{2}$ and the remaining two have the values 0 i.e., two of them are orthogonal. Here $\Delta = \left\{0, \frac{1}{2}\right\}$, $\beta = \frac{\sqrt{6}}{2} \approx 1.22$. Further, for this construction $\sigma = \sqrt{\frac{2}{6}\left(1 - \frac{2}{\sqrt{6}}\right)} \approx 0.247$, $\tau = \max\left\{\frac{1}{\sqrt{6}}, |\frac{1}{2} - \frac{1}{\sqrt{6}}|\right\} = \frac{1}{\sqrt{6}} \approx 0.408$ and sparsity $\epsilon = 1 - \frac{2}{6} = \frac{2}{3}$. Now let us consider the case for $k = 4$.

Corollary 2 *For any dimension $d > 16$ where $d = 4 \bmod (12)$, there exist $\frac{d-1}{3}$ many ARMUBs in $\mathbb{R}^d$ such that $\Delta = \left\{0, \frac{1}{4}\right\}$, $\beta = \frac{\sqrt{d}}{4}$, $\sigma^2 = \frac{2}{d}\left(1 - \frac{4}{\sqrt{d}}\right)$, $\tau < \frac{1}{4}$ and the sparsity $\epsilon = 1 - \frac{4}{d}$. For $d = 16$, we can construct five real MUBs with sparsity $\epsilon = \frac{3}{4}$.*

Proof The result follows directly from Theorem 2 by taking $k = 4$. This gives $d = 4 \times 3t + 4 = 4 \bmod (12)$. Hence for $t > 1$, using Theorem 2, we get $4t + 1 = \frac{d-1}{3}$ many ARMUBs with $\Delta = \left\{0, \frac{1}{4}\right\}$, $\beta = \sqrt{\frac{3t+1}{4}} = \frac{\sqrt{d}}{4}$, $\sigma^2 = \frac{2}{d}\left(1 - \frac{4}{\sqrt{d}}\right)$, $\tau < \frac{1}{4}$ and the sparsity $\epsilon = 1 - \frac{4}{d}$.

If $t = 1$, then in $d = 4 \times 3 + 4 = 16$ and we can construct $(4 + 1) = 5$ many MUBs in $\mathbb{R}^d$ with $\Delta = \left\{\frac{1}{4}\right\}$, $\beta = 1$, $\sigma = \tau = 0$ and the sparsity $\epsilon = 1 - \frac{1}{4} = \frac{3}{4}$. This completes the proof. □

Note that in $d = 4^2$, the maximum number of Real MUBs are $\frac{4^2}{2} + 1 = 9$ [3]. However, from our construction, we only obtain five MUBs which is not maximal. However, we like to point out here that these 5 MUBs are very sparse ($\epsilon = 0.75$), whereas 9 Real MUBs constructed using [4] would give MUBs in the form of mutually unbiased Hadamard matrices, where all the entries from $\{+1, -1\}$ hence no sparsity, except the standard basis.

Note that the construction using Theorem 2 gives very good Approximate MUBs for a fixed k ($d \geq k^2$) in lower dimensions, with β close to 1. However, as dimension increases β increases as $\sqrt{d}$ hence approximation deteriorates. On the other hand, as d increases the sparsity also increases. In this regard, let us present two instances.

Example 1 The resolvable $(28, 4, 1)$-BIBD will provide 9 ARMUBs in $\mathbb{R}^{28}$ with $\beta = \sqrt{\frac{7}{4}} \approx 1.3$ and $\epsilon = \frac{6}{7}$. Similarly, the resolvable $(40, 4, 1)$-BIBD will generate 13 ARMUBs in $\mathbb{R}^{40}$ with $\beta = \sqrt{\frac{5}{2}} \approx 1.6$ and $\epsilon = \frac{9}{10}$. Note that for both of the dimension only a pair of Real MUBs can exist.

4.1 Constructions following ARBIBD

Among all the resolvable BIBDs, affine resolvable BIBDs ($r = k + \lambda$) provide very interesting class of MUBs and AMUBs because of the fact that any two blocks from different parallel classes intersect at exactly $\frac{k^2}{v}$ points. By using the Hadamard matrices in Construction 1 at Step 3 provides approximate MUBs with small values of σ and τ, as well as β close to 1.

The affine resolvable BIBD (v, k, λ) can be parameterised in terms of two positive integer variables n and μ. The other parameters of the design in terms of n and μ are given as $v = n^2\mu$, $k = n\mu$, $\lambda = \frac{n\mu-1}{n-1}$, $b = \frac{n(n^2\mu-1)}{n-1}$, $r = \frac{n^2\mu-1}{n-1}$, and $\frac{k^2}{v} = \mu$. Hence one may consider it as $(n^2\mu, n\mu, \frac{n\mu-1}{n-1})$-ARBIBD. Conversely, any resolvable BIBD having parameters of this form is affine resolvable. We will denote such BIBD as an (n, μ)-ARBIBD [19, Chapter 5].

Lemma 2 *If there exists an affine resolvable BIBD of the form $(n^2\mu, n\mu, \frac{n\mu-1}{n-1})$, then for $d = n^2\mu$, we can construct $\frac{n^2\mu-1}{n-1}$ many approximate MUBs with $\beta = \sqrt{\mu}$, $\sigma \leq \frac{1}{n}$, $\tau \leq \frac{1}{n}$ and sparsity $\epsilon = 1 - \frac{1}{n}$. If real Hadamard matrix of order $n\mu$ exist, then we can construct Approximate Real MUBs with the same parameters.*

Proof Using Affine resolvable $(n^2\mu, n\mu, \frac{n\mu-1}{n-1})$-BIBD, we can convert $r = \frac{n^2\mu-1}{n-1}$ number of parallel classes using Hadamard matrix of order $n\mu$ into r many orthonormal Basis. Since exactly μ points are common between any two blocks from different parallel classes, maximum inner product between vectors from different orthonormal bases should be less than or equal to $\mu \times \frac{1}{n\mu} = \frac{1}{n}$.

Now, $\max\left\{\frac{1}{n\sqrt{\mu}}, \left|\frac{1}{n} - \frac{1}{n\sqrt{\mu}}\right|\right\}$ is equal to $\frac{\sqrt{\mu}-1}{\sqrt{d}}$ for $\mu \geq 4$ and it is equal to $\frac{1}{\sqrt{d}}$ for $1 < \mu < 4$. Therefore $\sigma^2 \leq \left(\frac{1}{n} - \frac{1}{\sqrt{n^2\mu}}\right)^2 = \frac{(\sqrt{\mu}-1)^2}{d}$ for $\mu \geq 4$ and $\sigma^2 \leq \left(\frac{1}{n\sqrt{\mu}} - 0\right)^2 = \frac{1}{d}$ for $1 < \mu < 4$. Hence $\sigma^2 \leq \frac{\mu}{d} = \frac{1}{n^2} \forall \mu > 1$. That is, $\sigma \leq \frac{1}{n}$ for $\mu > 1$. Similarly $\tau = \max\left\{\frac{1}{n\sqrt{\mu}}, \left|\frac{1}{n} - \frac{1}{n\sqrt{\mu}}\right|\right\}$, which implies $\tau \leq \sqrt{\frac{\mu}{d}} = \frac{1}{n}\ \forall \mu > 1$.

Note that when $\mu = 1$, both σ and $\tau = 0$. This happens because, $\mu = 1$ means, between a pair of blocks from different parallel classes has exactly one point in common, implying $\beta = 1$, and we will get exact MUBs.

The sparsity can be calculated as $\epsilon = 1 - \frac{k}{d} = 1 - \frac{1}{n}$. Moreover, if real Hadamard matrix of order $n\mu$ exists, then we can choose them as the unitary matrix in step 3 of Construction 1 to construct ARMUBs, where the parameters will remain unchanged. As noted, if $\mu = 1$ then both σ and τ will be zero, hence we will get exactly $n + 1$ many MUBs. $\square$

However, there are not many known families of affine resolvable $(n^2\mu, n\mu, \frac{n\mu-1}{n-1})$-BIBDs [18,19]. One well known family of ARBIBD can be constructed from affine geometry of order d. However, this construction is known only if d is some power of prime. In particular when $d = q^2$, where q is some power of prime, affine resolvable $(q^2, q, 1)$-BIBDs can be constructed. This immediately gives the following corollary.

Corollary 3 *When $d = q^2$ where q is some power of prime, then we can construct $q + 1$ MUBs in $\mathbb{C}^d$ with $\epsilon = 1 - \frac{1}{q}$.*

Proof For any prime power, an affine resolvable $(q^2, q, 1)$-BIBD exists, such that between any two blocks from different parallel classes there is only one point in common. Thus, using any Hadamard matrix of order q in Step 3 of Construction 1, we we obtain $q+1$ many MUBs in $\mathbb{C}^d$. If Fourier matrix of order q is used, then resulting entries of MUBs will consist of only q^{th} roots of unity and zeros. These $q + 1$ MUBs so constructed have sparsity $\epsilon = 1 - \frac{1}{q}$. $\square$

In such constructions of MUBs, any kind of Hadamard matrix can be used in step 3 of Construction 1. This immediately suggests the ways to generate MUBs in a dimension q^2, where q is some power of a prime. Such constructions are not possible using Galois Field [23,7] or through construction of maximal commuting unitary operators using generalized Pauli matrices [2]. The caveat here is, using Construction 1, the number of MUBs would be $q+1$, which is considerably less than the upper bound of $q^2 + 1$. For example, we can construct five MUBs in $d = 4^2$ using affine resolvable $(4^2, 4, 1)$-BIBD and using parametric form of Hadamard matrix $F_4^{(1)(a)}$ [20, Example 1.2.1]. Similarly, using Buston Hadamard matrices like $BH(n^{2k}, 6)$ [20, Corollary 1.4.42], which exist for every $n, k \in \mathbb{N}$, we can construct $q^2 + 1$ many MUBs in $\mathbb{C}^{q^4}$ where q is some power of prime. Here we need to use affine resolvable $(q^4, q^2, 1)$-BIBD, whose non zero entries would consist of only sixth roots of unity. Further, using Petrescu's construction for parametric form of Hadamard matrices, for primes $p = 7, 13, 19, 31$ [20, Theorem 3.1.2], one can construct corresponding parametric MUBs in $d = 7^2, 13^2, 19^2, 31^2$, which would not be equivalent to the MUBs from known methods, based on Galois Field [23,7] or through construction of maximal commuting unitary operators using generalized Pauli matrices [2]. Further knowledge of Hadamard matrices, whose orders are some powers of prime, can be exploited to construct interesting sparse MUBs using this method.

Since there always exist real Hadamard matrices of order $2^s, s \in \mathbb{N}$ [9], we have the following corollary.

Corollary 4 *For $d = 4^s, s \in \mathbb{N}$, there exist $2^s + 1$ many real MUBs with sparsity $\epsilon = 1 - 2^{-s}$.*

Note that these are very sparse real MUBs and hence can be used for efficient computations. However, these do not improve the existing parameters in literature. However, we present these for exposure as we expect that further analysis of our techniques may improve the parameters.

It should also be noted that the existence of real Hadamard matrix of order $4m$ implies the existence for Affine Resolvable $(4m, 2m, 2m-1)$-BIBD [19,18]. Thus we have the following result.

Proposition 1 *Consider that a real Hadamard matrix of order $2m$ $(m > 1)$ exists. Then for $d = 4m$, we can construct $4m-1$ many β-ARMUBs where $\beta \leq \sqrt{m}$. Further, $\Delta = \frac{1}{\sqrt{d}} \times \left\{0, \frac{2}{\sqrt{m}}, \frac{4}{\sqrt{m}}, \ldots, \frac{m}{\sqrt{m}}\right\}$, $\sigma \leq \frac{1}{2}$, $\tau \leq \frac{1}{2}$ and the sparsity $\epsilon = \frac{1}{2}$.*

Proof The case for $m = 1$, i.e., $d = 4$, is covered in Corollary 1. Hence we assume $m > 1$ here. If there exists a Hadamard matrix of order $2m$, then for $m > 1$, m must be even. We can use this Hadamard matrix of order $2m$ to construct Hadamard matrix of order $4m$ by taking its tensor product with the Hadamard matrix of order 2 [9]. Then using Hadamard matrix of order $4m$, we obtain Affine Resolvable $(4m, 2m, 2m-1)$-BIBD [19,18]. Now following the Construction 1 and choosing the given real Hadamard matrix of order $2m$ in step 3, we obtain the desired ARMUB.

Since the design is affine resolvable, there are exactly same number of points are common between blocks from different resolution, which is $\frac{k^2}{v} = \frac{(2m)^2}{4m} = m$ implying $\beta = \sqrt{m}$. This also implies that the inner product between vectors from different basis would be of the form $\frac{1}{2m} \times w$, where w will be the sum of m many 1's putting $\pm$ before each 1. Since m is even and $d = 4m$, this implies $\Delta = \left\{0, \frac{2}{2m}, \frac{4}{2m}, \ldots, \frac{m}{2m}\right\} = \frac{1}{\sqrt{d}} \times \left\{0, \frac{2}{\sqrt{m}}, \frac{4}{\sqrt{m}}, \ldots, \frac{m}{\sqrt{m}}\right\}$.

In order to estimate σ, note that the maximum inner product between the vectors from different bases is $\frac{m}{2m} = \frac{1}{2}$ which implies $\sigma^2 \leq \left(\frac{1}{\sqrt{4m}} - \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4m} - \frac{1}{\sqrt{4m}} \leq \frac{1}{4}$. Similarly $\tau \leq \left|\frac{1}{\sqrt{4m}} - \frac{1}{2}\right| \leq \frac{1}{2}$ and the sparsity $\epsilon = 1 - \frac{k}{d} = \frac{1}{2}$. Refer to the definition that this k is the block size. □

In a first look, it appears that ARMUBs with $\beta = \sqrt{m}$ might not be very interesting. However, using this construction we get $d-1$ many ARMUBs and β is not very large for certain moderate values of d. Let us take the example for $d = 64 = 4 \cdot 16$ and thus, $\beta \leq 4$. Here we obtain 63 ARMUBs. However, for the same dimension, our construction in the earlier work [11] provided only three β-ARMUBs respectively with $\beta \leq 4$. This is a significant improvement for this specific $d = 64$. For larger dimensions, this construction will provide significantly more number of ARMUBs than [11] but the value of β will be greater than 4.

Now we present the case for $d = 2^s$, where the existence of Hadamard matrix is guaranteed.

Corollary 5 *For $d = 2^s, s \geq 2$ there exist $2^s - 1$ many β-ARMUBs where $\beta \leq \sqrt{2^{s-2}}$, with $\sigma \leq \frac{1}{2}$, $\tau \leq \frac{1}{2}$ and with sparsity $\epsilon = \frac{1}{2}$.*

Proof There always exists a Hadamard matrix of the order 2^n where $n \in \mathbb{N}$ (Sylvester construction [9]). Then in the affine resolvable $(4m, 2m, 2m-1)$-BIBD, one can substitute $m = 2^{s-2}$ with $s > 2$, and the result follows immediately. □

We conclude this section with the following remark that compares our construction idea for exact real MUBs with [21].

Remark 1 Consider that w many MOLS(q) are available. Such a structure can be used to construct an RBD (X, A), such that $|X| = q^2$ having $w + 2$ parallel classes, each having q blocks of constant size q and any two blocks from different parallel classes will have exactly one point in common. This idea follows from [19, Section 6.4.1, Theorem 6.32] in relating MOLS and Affine Plane. One may note that such an RBD will provide $w + 2$ MUBs in $\mathbb{C}^d$ following our Construction 1. Further, if real Hadamard matrix of order q exists, then the construction will provide $w + 2$ Real MUBs in $\mathbb{R}^d$. Our numerical results related to exact MUBs in this direction will be the same as [21], but our construction is different and we have more flexibility of using different suitable unitary matrices. Further, our main focus in this paper is the relaxed model of approximate MUBs, rather that exact ones, and their we have the opportunity of different avenues to explore through Construction 1, which we could not see immediately through the work of [21].

In the next section we explore the designs which are not balanced, and that provide us further results in this direction.

5 ARMUBs using Resolvable Block Designs that are not Balanced

Now we will focus on resolvable block designs which are not balanced. This implies that either one or both the conditions given in 2 or 3 of BIBD (Definition 3) are not satisfied. However, these kinds of customized designs, for the purpose of obtaining ARMUBs provide generic and improved results. In the first construction, we use multiple affine resolvable BIBDs which are identical, and in the next one we add new elements in the design.

Theorem 3 *Consider $d = sq^2$, where q is a prime power and $sq \equiv 0 \bmod 4$. Assuming a real Hadamard matrix of order sq exists, we can construct $q + 1$ many β-ARMUB, where $\beta \leq \sqrt{s}$. Further, $\sigma \leq \sqrt{\frac{s}{d}}$, $\tau = \frac{1}{\sqrt{d}}$ for $1 \leq s \leq 4$ and $\tau = \frac{\sqrt{s}-1}{\sqrt{d}}$ for $s > 4$ and the sparsity $\epsilon = 1 - \frac{1}{q}$.*

Proof We split $d = sq^2$ orthonormal vectors in s sets of q^2 vectors. Now for each set of q^2 vectors, one can construct affine resolvable $(q^2, q, 1)$-BIBD, where each one of them will have all blocks of size q and total $q + 1$ many parallel classes, such that blocks from different parallel classes will have only one point in common. Now, consider the union of s such ARBIBDs, each having an identical structure, but different points. It will give resolvable design of sq^2 points, with each block of size of sq, consisting of $q + 1$ many parallel classes, such that blocks from two different parallel classes will have exactly s points in common. If we assume that Hadamard matrix of order sq exits, that can be used to convert each parallel classes into orthonormal bases as in Construction 1. Thus we obtain $q + 1$ many β-ARMUBs.

To explain the values in Δ, note that inner products between the vectors from different parallel classes would be of the form $\frac{1}{sq} \times w$, where w will be the sum of s many 1's putting $\pm$ before each 1. This implies $\Delta = \left\{0, \frac{2}{sq}, \frac{4}{sq}, \ldots, \frac{s}{sq}\right\}$ if s is even and $\Delta = \left\{\frac{1}{sq}, \frac{3}{sq}, \frac{5}{sq}, \ldots, \frac{s}{sq}\right\}$ if s is odd. Hence $\beta = \frac{\sqrt{d}}{q} = \sqrt{s}$.

The largest inner product value between the vectors from different parallel classes is equal to $\frac{1}{q}$. Further, we have $\max\left\{\frac{1}{\sqrt{d}}, \left|\frac{1}{\sqrt{d}} - \frac{1}{q}\right|\right\}$ is equal to $\frac{1}{\sqrt{d}}$ for $1 \le s \le 4$ and is equal to $\frac{1}{q} - \frac{1}{\sqrt{d}}$ for $s \ge 4$. Hence $\sigma^2 \le \left(\frac{1}{\sqrt{sq}} - 0\right)^2$ for $1 < s \le 4$, and $\sigma^2 \le \left(\frac{1}{\sqrt{sq}} - \frac{1}{q}\right)^2$, for $s \ge 4$ which we can conveniently state $\sigma \le \sqrt{\frac{s}{d}}$. In order to ascertain τ, we have $\max\left\{\frac{1}{\sqrt{d}}, \left|\frac{1}{\sqrt{d}} - \frac{1}{q}\right|\right\}$ is equal to $\frac{1}{\sqrt{d}}$ for $1 \le s \le 4$ and is equal to $\frac{1}{q} - \frac{1}{\sqrt{d}}$ for $s \ge 4$. Hence $\tau = \frac{\sqrt{s}-1}{\sqrt{d}}$ for $s > 4$, else $\tau = \frac{1}{\sqrt{d}}$ for $s \le 4$. The sparsity $\epsilon = 1 - \frac{k}{d} = 1 - \frac{1}{q}$, where k is the block size. □

This case subsumes the result in [11] for $s = 16$ and q prime, i.e., $d = (4q)^2$. That is, with the method of [11], one can obtain $q+1 = \frac{\sqrt{d}}{4}+1$ ARMUBs such that for two vectors from different orthogonal bases, the inner product will be upper bounded by $\frac{4}{\sqrt{d}}$. This is the same quality result presented in [11, Corollary 1]. The clear extension in our case is that, here u can be any power of prime, whereas the construction given [11] was applicable only to $d = (4q)^2$ for a prime q. For example, using above corollary, we can construct β-ARMUBs with $\beta \le 4$, even for dimensions 4×9, 4×25 etc., whereas one can not construct β-ARMUBs for these dimensions using the construction given in [11]. Thus this result subsumes the result of our previous work [11].

Now we present a result, where we can improve the number of MUBs as well as upper bound the inner product value. This we explore for a case where real Hadamard matrices exist. For this we have the following result.

Theorem 4 *Consider $d = q(q+1)$ such that q is a prime power and $q \equiv 3 \bmod 4$. Then we can construct $(q+1)$ many ARMUBs with $\Delta = \left\{0, \frac{1}{q+1}, \frac{2}{q+1}\right\}$, $\beta = 2\sqrt{\frac{q}{q+1}}$ and $\sigma_o^2\left(1 - \frac{1}{\sqrt{d}}\right) \le \sigma^2 \le \sigma_o^2\left(1 + \frac{1}{\sqrt{d}}\right)$, where $\sigma_o^2 = \frac{2}{d}\left(1 - \sqrt{\frac{q}{q+1}}\right)$. Further, $\tau = \frac{1}{\sqrt{d}}$ and the sparsity is given by $\epsilon = 1 - \frac{1}{q}$.*

Proof Consider an affine resolvable $(q^2, q, 1)$-BIBD. There will be $r = q+1$ parallel classes, consisting of q blocks each having q elements. Any two blocks from different parallel classes will have only one point in common. Add q more elements in the set X, which implies $|X| = q^2 + q$. Add these q elements, one in each block of every parallel class. Now all the parallel classes will have blocks of size $q+1$ and the number of blocks will remain unchanged, which is q. In this situation, any block in a parallel class will have one element in common with $q-1$ blocks and two elements in common with the remaining block of any other parallel class. Hence we obtain a set of $q+1$ parallel classes each having q blocks and each block consisting of $q+1$ elements. This is the desired resolvable design.

Since $q \equiv 3 \bmod 4$, the Paley construction [12] will always provide real Hadamard matrix of order $q+1$. Hence we use this for constructing ARMUBs following Construction 1. Note that the blocks from different parallel classes have maximum two points in common, implying $\Delta = \left\{0, \frac{1}{q+1}, \frac{2}{q+1}\right\}$. In order to calculate σ, note that every block has only one point in common with $q-1$ blocks of any other parallel classes and two points in common with the remaining blocks of that parallel class. Thus, any vector from one basis will have the inner product value of $\frac{1}{q+1}$ with

$(q-1)(q+1)$ vectors and will have inner product either 0 or $\frac{2}{q+1}$ with $(q+1)$ vectors of any other orthogonal basis. Thus, we have

$$\left(\frac{1}{\sqrt{q(q+1)}}-\frac{1}{q+1}\right)^2(q-1)(q+1)+\left(\frac{1}{\sqrt{q(q+1)}}-\frac{2}{q+1}\right)^2(q+1)\leq d\times\sigma^2$$
$$\leq\left(\frac{1}{\sqrt{q(q+1)}}-\frac{1}{q+1}\right)^2(q-1)(q+1)+\left(\frac{1}{\sqrt{q(q+1)}}-0\right)^2(q+1).$$

This simplifies to $\sigma_o^2\left(1-\frac{1}{\sqrt{d}}\right)\leq\sigma^2\leq\sigma_o^2\left(1+\frac{1}{\sqrt{d}}\right)$ where $\sigma_o^2=\frac{2}{d}\left(1-\sqrt{\frac{q}{q+1}}\right)$.

On the other hand, there will be vectors between two different orthonormal bases which will also be orthogonal, corresponding to blocks having two points in common such that one gives $+1$ and another provides -1 in the inner product or vice versa, thereby, making the inner product between the vectors 0. Hence $\tau=\frac{1}{\sqrt{q(q+1)}}=\frac{1}{\sqrt{d}}$ and the sparsity would be given by $\epsilon=1-\frac{1}{q}$. □

This result clearly shows that there are $\lceil\sqrt{d}\rceil$ real MUBs with $\beta<2$ and it substantially improves the result of [11] from both in number of MUBs as well as in terms of upper bound of the inner products. As numerical examples, for $d=12,56$, there would be respectively $4,8$ ARMUBs of above type.

For clarity, let us present the case for $q=3$, i.e., $d=3(3+1)=12$. To begin with, consider the design of Affine Resolvable $(3^2,3,1)$-BIBD. Below we represent each parallel class as 3×3 matrix, where the each row represent one block of the parallel class. Hence there would be 4 such matrices. Writing them explicitly

$$P_1=\begin{pmatrix}1&5&9\\2&6&7\\3&4&8\end{pmatrix},P_2=\begin{pmatrix}1&6&8\\2&4&9\\3&5&7\end{pmatrix},P_3=\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix},P_4=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}.$$

Now add three more point in the design, namely $\{10,11,12\}$, and as stated, one point is added in each block of every parallel class. The resulting parallel classes would be

$$P_1=\begin{pmatrix}1&5&9&10\\2&6&7&11\\3&4&8&12\end{pmatrix},P_2=\begin{pmatrix}1&6&8&10\\2&4&9&11\\3&5&7&12\end{pmatrix},P_3=\begin{pmatrix}1&4&7&10\\2&5&8&11\\3&6&9&12\end{pmatrix},P_4=\begin{pmatrix}1&2&3&10\\4&5&6&11\\7&8&9&12\end{pmatrix}.$$

Above is the desired RBD, consisting of 4 many parallel class, each having 3 books of constant size 4, such that between any two blocks from different parallel classes, either one or two points will be in common. The existence of real Hadamard matrix of order 4 will produce four ARMUBs here, with the inner product value bounded by $\frac{2}{q+1}=\frac{2}{4}=\frac{1}{2}<\frac{2}{\sqrt{d}}=\frac{2}{\sqrt{12}}=\frac{1}{\sqrt{3}}$.

As a passing remark, in the above construction of RBD, we can add q elements as one block in say, $(q+1)$-th parallel class, and then one elements of this block, into each block of all other parallel classes. This will make all the parallel classes to have q blocks of size $q+1$ except the $(q+1)$-th parallel class which will have $q+1$ blocks each having size q. Now the $(q+1)$-th parallel class has $(q+1)$ blocks, each having q elements. Each block of this parallel class will have only one element in common with any block of other parallel classes. Since q is a prime power, and if it is odd, we need a complex Hadamard matrix to convert $(q+1)$-th parallel class into orthonormal basis. This complex orthonormal basis would be mutually unbiased with all the other q sets of real orthonormal bases so constructed.

6 Conclusion

In this paper we have described a generic approach that connects an object of combinatorial design, namely Resolvable Block Design (RBD) with Mutually Unbiased Bases (MUBs) which are structures on Hilbert spaces. We have presented a method which takes an RBD as input and use this to construct the orthonormal bases. The parallel classes of RBD play the most important role here. Each orthonormal basis is constructed out of a parallel class, and the parameters of the approximate MUBs are dependent on that of the parallel classes. Our construction method also exploits unitary matrices, dependent on the block sizes of a parallel class to generate the Approximate Real MUBs (in some cases MUBs too, but those are not main focus of this work). Throughout the paper, we mostly concentrate on Hadamard construction while using the unitary matrices. To characterize the approximate nature of the MUBs, we define certain parameters namely β, Δ, σ and τ. It has been shown that in most of the cases variance goes to zero as dimension increases, hence making the approximation quite close to actual MUBs. In certain cases, where the variance is zero, exact MUBs are obtained. The sparsity ϵ has been characterized by simple ratio of the number of zero elements divided by the total elements in the matrix that corresponds to a basis. In general, our construction provides very high sparsity and we obtain $\epsilon = 1 - \frac{1}{\sqrt{d}}$, in most of the cases. In summary, we provide a generic approach for the first time to obtain ARMUBs for a large class of parameters that were not known earlier. The kinds of constructions we studied are different from the existing efforts in this domain of research. Thus, it will be interesting if these ideas can be extended further to obtain ARMUBs with improved inner product values or exact MUBs with more numbers than what is available in the state of the art literature. We are working in this direction as the combinatorial designs offer rich structures in such analysis.

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