Semilinear Transformations in Coding Theory: A New Technique in Code-Based Cryptography

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Abstract

This paper presents a new technique for disturbing the algebraic structure of linear codes in code-based cryptography. Specifically, we introduce the so-called semilinear transformations in coding theory and then creatively apply them to the construction of code-based cryptosystems. Note that $\mathbb{F}_{q^m}$ can be viewed as an $\mathbb{F}_q$-linear space of dimension m, a semilinear transformation φ is therefore defined as an $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$. Then we impose this transformation to a linear code $\mathcal{C}$ over $\mathbb{F}_{q^m}$. It is clear that $\varphi(\mathcal{C})$ forms an $\mathbb{F}_q$-linear space, but generally does not preserve the $\mathbb{F}_{q^m}$-linearity any longer. Inspired by this observation, a new technique for masking the structure of linear codes is developed in this paper. Meanwhile, we endow the underlying Gabidulin code with the so-called partial cyclic structure to reduce the public-key size. Compared to some other code-based cryptosystems, our proposal admits a much more compact representation of public keys. For instance, 2592 bytes are enough to achieve the security of 256 bits, almost 403 times smaller than that of Classic McEliece entering the third round of the NIST PQC project.

Keywords Post-quantum cryptography · Code-based cryptography · Rank metric codes · Gabidulin codes · Partial cyclic codes · Semilinear transformations

1 Introduction

Over the past decades, post-quantum cryptosystems (PQCs) have been drawing more and more attention from the cryptographic community. The most important advantage of PQCs is their potential resistance against attacks from quantum computers. In post-quantum cryptography, cryptosystems based on coding theory are one of the most promising candidates. In addition to security in the future quantum era, these cryptosystems generally have fast encryption and decryption procedures. Code-based cryptography has quite a long history, nearly as old as RSA–one of the best known public-key cryptosystems. However, this family of cryptosystems has never been used in practical situations for the reason that it requires large memory for public keys. For instance, Classic McEliece [1] submitted to the NIST PQC project has a public-key size of 255 kilobytes for the

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128-bit security. To overcome this drawback, a variety of improvements for McEliece's original scheme [16] were proposed one after another. Generally these improvements can be divided into two categories: one is to substitute Goppa codes used in the McEliece system with other families of codes endowed with special structures, the other is to use codes endowed with the rank metric. However, most of these variants have been shown to be insecure against structural attacks.

The first cryptosystem based on rank metric codes, known as the GPT cryptosystem, was proposed by Gabidulin et al. in [2]. The main advantage of rank-based cryptosystems consists in their compact representation of public keys. For instance, 600 bytes are enough to reach the 100-bit security for the original GPT cryptosystem. After that, applying rank metric codes to the construction of cryptosystems became an important topic in code-based cryptography. Some of the representative variants based on Gabidulin codes can be found in [3–7]. Unfortunately, most of these variants, including the original GPT cryptosystem, have been completely broken because of Gabidulin codes being highly structrued. Concretely, Gabidulin codes contain a large subspace invariant under the Frobenius transformation, which provides the feasibility for us to distinguish Gabidulin codes from general ones. Based on this observation, various structural attacks [25–29] on the GPT cryptosystem and some of their variants were designed. Apart from Gabidulin codes, another family of rank metric codes, known as the Low Rank Parity Check (LRPC) codes, and a probabilistic encryption scheme based on these codes were proposed in [8, 9]. Compared to Gabidulin codes, LRPC codes admit a weak algebraic structure. Encryption schemes based on these codes can therefore resist structural attacks designed for Gabidulin codes based cryptosystems. However, this type of cryptosystems generally has a decrypting failure rate, which can be used to devise a reaction attack [10] to recover the private key.

Our contribution. In the paper [11], Guo and Fu mentioned a fact that $\varphi(\mathcal{C})$ generally does not preserve the $\mathbb{F}_{q^m}$-linearity according to their experiments on Magma, where $\mathcal{C}$ is a linear code over $\mathbb{F}_{q^m}$ and φ is an $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$. This inspires us to apply this family of transformations to the construction of code-based cryptosystems. According to our analysis in the present paper, a transformation that preserves the $\mathbb{F}_{q^m}$-linearity of all linear codes over $\mathbb{F}_{q^m}$ is indeed a composition of the Frobenius transformation and stretching transformation. An $\mathbb{F}_q$-linear automorphism φ of $\mathbb{F}_{q^m}$ admitting this property is called a fully linear transformation over $\mathbb{F}_{q^m}$, otherwise we call it a semilinear transformation. Furthermore, we find that an $\mathbb{F}_q$-linear transformation preserving the $\mathbb{F}_{q^m}$-linearity of a linear code is fully linear in general cases. Note that the total number of $\mathbb{F}_q$-linear automorphisms of $\mathbb{F}_{q^m}$ is $\prod_{i=0}^{m-1}(q^m - q^i)$, while the total number of fully linear transformations is $m(q^m - 1)$ according to our analysis. This implies that a fully linear transformation occurs with an extremely low probability when q^m is large enough, which provides the feasibility for us to exploit this kind of transformation in code-based cryptography. As an application of semilinear transformations, we propose an encryption scheme based on the so-called partial cyclic Gabidulin code.

The rest of this paper is organized as follows. Section 2 introduces some basic notations used throughout this paper, as well as the definitions of Moore matrices and partial cyclic codes. Section 3 presents two hard problems in coding theory and two types of attacks on them that will be useful to estimate the security level of our proposal. In Section 4, we will introduce the concept of semilinear transformations, and investigate their algebraic properties when acting on linear codes. Section 5 is devoted to the description of our new proposal and some notes on the choice of the secret keys, then we present the security analysis of our proposal in Section 6. After that, we give some suggested parameters for different security levels and make a comparison on public-key size with some other code-based cryptosystems in Section 7. A few concluding remarks will be presented in Section 8.

2 Preliminaries

In this section, we first introduce some notations in finite field and coding theory used throughout this paper. After that, we will recall some basic concepts about Gabidulin codes and the so-called partial cyclic codes.

2.1 Notations and basic concepts

Let $\mathbb{F}_q$ be a finite field, and $\mathbb{F}_{q^m}$ be an extension field of $\mathbb{F}_q$ of degree m. A vector $\boldsymbol{a} \in \mathbb{F}_{q^m}^m$ is called a basis vector of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ if components of $\boldsymbol{a}$ are linearly independent over $\mathbb{F}_q$. Particularly, we call α a polynomial element if $\boldsymbol{a} = (1, \alpha, \cdots, \alpha^{m-1})$ forms a basis vector of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, and call α a normal element if $\boldsymbol{a} = (\alpha, \alpha^q, \cdots, \alpha^{q^{m-1}})$ forms a basis vector of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. For two positive integers k and n, denote by $\mathcal{M}_{k,n}(\mathbb{F}_q)$ the space of all $k \times n$ matrices over $\mathbb{F}_q$, and by $GL_n(\mathbb{F}_q)$ the set of all invertible matrices in $\mathcal{M}_{n,n}(\mathbb{F}_q)$. For a matrix $M \in \mathcal{M}_{k,n}(\mathbb{F}_q)$, let $\langle M \rangle_q$ be the vector space spanned by the rows of M over $\mathbb{F}_q$.

An $[n, k]$ linear code $\mathcal{C}$ over $\mathbb{F}_{q^m}$ is a k-dimensional subspace of $\mathbb{F}_{q^m}^n$. The dual code of $\mathcal{C}$, denoted by $\mathcal{C}^\perp$, is the orthogonal space of $\mathcal{C}$ under the usual inner product over $\mathbb{F}_{q^m}^n$. A $k \times n$ matrix G over $\mathbb{F}_{q^m}$ of full row rank is called a generator matrix of $\mathcal{C}$ if its row vectors form a basis of $\mathcal{C}$. A generator matrix H of $\mathcal{C}^\perp$ is called a parity-check matrix of $\mathcal{C}$. For a codeword $\boldsymbol{c} \in \mathcal{C}$, the Hamming support of $\boldsymbol{c}$, denoted by $\text{Supp}_H(\boldsymbol{c})$, is defined to be the set of coordinates of $\boldsymbol{c}$ at which the components are nonzero. The Hamming weight of $\boldsymbol{c}$, denoted by $\text{wt}_H(\boldsymbol{c})$, is the cardinality of $\text{Supp}_H(\boldsymbol{c})$. The minimum Hamming distance of $\mathcal{C}$ is defined as the minimum Hamming weight of nonzero codewords in $\mathcal{C}$. The rank support of $\boldsymbol{c}$, denoted by $\text{Supp}_R(\boldsymbol{c})$, is the linear space spanned by the components of $\boldsymbol{c}$ over $\mathbb{F}_q$. The rank weight of $\boldsymbol{c}$ with respect to $\mathbb{F}_q$, denoted by $\text{wt}_R(\boldsymbol{c})$, is defined to be the dimension of $\text{Supp}_R(\boldsymbol{c})$ over $\mathbb{F}_q$. For a matrix $M \in \mathcal{M}_{k,n}(\mathbb{F}_{q^m})$, the rank support $\text{Supp}_R(M)$ of M is defined to be the linear space spanned by entries of M over $\mathbb{F}_q$. Similarly, the rank weight of M with respect to $\mathbb{F}_q$, denoted by $\text{wt}_R(M)$, is defined as the dimension of $\text{Supp}_R(M)$ over $\mathbb{F}_q$.

2.2 Gabidulin codes

Before presenting the concept of Gabidulin codes, we first introduce the definition of Moore matrices and some related results.

Definition 1 (Moore matrices). For a nonnegative integer i, we introduce the notation $[i] = q^i$. Under this notation, we define $\alpha^{[i]} = \alpha^{q^i}$ to be the i-th Frobenius power of $\alpha \in \mathbb{F}_{q^m}$. For a vector $\boldsymbol{g} = (g_1, \cdots, g_n) \in \mathbb{F}_{q^m}^n$, we define $\boldsymbol{g}^{[i]} = (g_1^{[i]}, \cdots, g_n^{[i]})$ to be the i-th Frobenius power of $\boldsymbol{g}$. A matrix $G \in \mathcal{M}_{k,n}(\mathbb{F}_{q^m})$ is called a Moore matrix generated by $\boldsymbol{g}$ if the i-th row vector of G is exactly $\boldsymbol{g}^{[i-1]}$ for $1 \leqslant i \leqslant k$, namely we have

$$G = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1^{[1]} & g_2^{[1]} & \cdots & g_n^{[1]} \\ \vdots & \vdots & & \vdots \\ g_1^{[k-1]} & g_2^{[k-1]} & \cdots & g_n^{[k-1]} \end{pmatrix}.$$

Remark 1. For an $[n, k]$ linear code $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$, the i-th Frobenius power of $\mathcal{C}$ is defined to be $\mathcal{C}^{[i]} = \{\boldsymbol{c}^{[i]} : \boldsymbol{c} \in \mathcal{C}\}$. Furthermore, it is easy to verify that $\mathcal{C}^{[i]}$ is also an $[n, k]$ linear code over $\mathbb{F}_{q^m}$.

The following proposition describes some simple algebraic properties of Moore matrices.

Proposition 1. *(1) For two Moore matrices $A, B \in \mathcal{M}_{k,n}(\mathbb{F}_{q^m})$, $A + B$ is also a Moore matrix.*

(2) For a vector $\boldsymbol{a} \in \mathbb{F}_{q^m}^n$ and a matrix $Q \in GL_n(\mathbb{F}_q)$, let $A \in \mathcal{M}_{k,n}(\mathbb{F}_{q^m})$ be a Moore matrix generated by $\boldsymbol{a}$, then AQ is a Moore matrix generated by $\boldsymbol{a}Q$.

(3) For positive integers $k \leqslant n \leqslant m$ and a vector $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{F}_{q^m}^n$ with $\mathrm{wt}_R(\boldsymbol{a}) = n$, let $A \in \mathcal{M}_{k,n}(\mathbb{F}_{q^m})$ be a Moore matrix generated by $\boldsymbol{a}$. Then A has full row rank, that is Rank$(A) = k$.

(4) For a vector $\boldsymbol{a} \in \mathbb{F}_{q^m}^n$ with $\mathrm{wt}_R(\boldsymbol{a}) = s$ where $s \leqslant n$ is a positive integer, let $A \in \mathcal{M}_{k,n}(\mathbb{F}_{q^m})$ be a Moore matrix generated by $\boldsymbol{a}$. Then we have Rank$(A) = \min\{s, k\}$.

Proof. Statements (1) and (2) are trivial from a straightforward verification.

(3) We suppose that $\mathrm{Rank}(A) < k$, then there exists $\boldsymbol{\lambda} = (\lambda_0, \cdots, \lambda_{k-1}) \in \mathbb{F}_{q^m}^k$ such that $\boldsymbol{\lambda}A = \boldsymbol{0}$. Let $f(x) = \sum_{j=0}^{k-1} \lambda_j x^{[j]} \in \mathbb{F}_{q^m}[x]$, then we have $f(\alpha_i) = 0$ holds for any $1 \leqslant i \leqslant n$. Following this, it is easy to verify that $f(\alpha) = 0$ for any $\alpha \in \langle \alpha_1, \cdots, \alpha_n \rangle_q$, which conflicts with the fact that $f(x) = 0$ admits at most q^{k-1} roots.

(4) Because of $\mathrm{wt}_R(\boldsymbol{a}) = s$, there exists $Q \in GL_n(\mathbb{F}_q)$ and $\boldsymbol{a}^* \in \mathbb{F}_{q^m}^s$ with $\mathrm{wt}_R(\boldsymbol{a}^*) = s$ such that $\boldsymbol{a} = (\boldsymbol{a}^* | \boldsymbol{0})Q$. Let $A^* \in \mathcal{M}_{k,s}(\mathbb{F}_{q^m})$ be a Moore matrix generated by $\boldsymbol{a}^*$, then it is easy to see that $A = [A^* | 0]Q$. From Statement (3), we have $\mathrm{Rank}(A) = \mathrm{Rank}(A^*) = \min\{s, k\}$.

□

Now we formally introduce the definition of Gabidulin codes.

Definition 2 (Gabidulin codes). For positive integers $k \leqslant n \leqslant m$, let $\boldsymbol{g} \in \mathbb{F}_{q^m}^n$ be a vector such that $\mathrm{wt}_R(\boldsymbol{g}) = n$. Denote by $G \in \mathcal{M}_{k,n}(\mathbb{F}_{q^m})$ the $k \times n$ Moore matrix generated by $\boldsymbol{g}$, then the $[n, k]$ Gabidulin code $\mathcal{G}$ generated by $\boldsymbol{g}$ is defined to be the linear space $\langle G \rangle_{q^m}$.

Gabidulin codes can be seen as an analogue of generalized Reed-Solomon (GRS) codes in the rank metric, both of which have pretty good algebraic properties. Gabidulin codes are optimal in the rank metric, namely an $[n, k]$ Gabidulin code has minimum rank distance $d = n-k+1$ [38] and can therefore correct up to $\lfloor \frac{n-k}{2} \rfloor$ rank errors in theory. Efficient decoding algorithms for Gabidulin codes can be found in [39–41].

2.3 Partial cyclic codes

In the paper [6], Lau and Tan proposed the use of partial cyclic codes to shrink the public-key size in code-based cryptography. Now we formally introduce this family of codes and some related results.

Definition 3 (Partial circulant matrices). Let $\boldsymbol{m} = (m_0, \cdots, m_{n-1}) \in \mathbb{F}_q^n$, then the circulant matrix $M \in \mathcal{M}_{n,n}(\mathbb{F}_q)$ generated by $\boldsymbol{m}$ is defined to be

$$M = \begin{pmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_{n-1} & m_0 & \cdots & m_{n-2} \\ \vdots & \vdots & & \vdots \\ m_1 & m_2 & \cdots & m_0 \end{pmatrix}.$$

For a positive integer $k \leqslant n$, the $k \times n$ partial circulant matrix generated by $\boldsymbol{m}$, denoted by $\mathcal{P}_k(\boldsymbol{m})$, is defined to be the first k rows of M. Particularly, we denote by $\mathcal{P}_n(\boldsymbol{m})$ the circulant matrix generated by $\boldsymbol{m}$. Furthermore, we denote by $\mathcal{P}_n(\mathbb{F}_q)$ the set of all $n \times n$ circulant matrices over $\mathbb{F}_q$.

Remark 2. Chalkley in [23] proved that all circulant matrices in $\mathcal{P}_n(\mathbb{F}_q)$ form a commutative ring under the usual matrix addition and multiplication. Let $\mathbf{1} = (1, 0, \cdots, 0) \in \mathbb{F}_q^n$, then $\mathcal{P}_k(\boldsymbol{m}) = \mathcal{P}_k(\mathbf{1}) \cdot \mathcal{P}_n(\boldsymbol{m})$ for any $\boldsymbol{m} \in \mathbb{F}_q^n$. Following this, we have that for a $k \times n$ partial circulant matrix A and a circulant matrix B of order n, the product matrix AB is also a $k \times n$ partial circulant matrix.

Since we will use invertible circulant matrices as part of the secret key, it is necessary to make clear in what situation a circulant matrix is invertible and how many invertible circulant matrices of order n over $\mathbb{F}_q$ there will be. The following two propositions first describe a sufficient and necessary condition for a circulant matrix being invertible, and then make an accurate estimation on the number of invertible circulant matrices over $\mathbb{F}_q$.

Proposition 2. *[21] For a vector $\boldsymbol{m} = (m_0, \cdots, m_{n-1}) \in \mathbb{F}_q^n$, we define $\boldsymbol{m}(x) = \sum_{i=0}^{n-1} m_i x^i \in \mathbb{F}_q[x]$. A sufficient and necessary condition for $\mathcal{P}_n(\boldsymbol{m})$ being invertible is* $\gcd(\boldsymbol{m}(x), x^n - 1) = 1$.

Proposition 3. *[22] For a monic polynomial $f(x) \in \mathbb{F}_q[x]$ of degree n, let $g_1(x), \cdots, g_s(x) \in \mathbb{F}_q[x]$ be s distinct irreducible factors of $f(x)$, namely we have $f(x) = \prod_{i=1}^{s} g_i(x)^{e_i}$ for some positive integers $e_1, \cdots, e_s$. Let $d_i = \deg(g_i(x))$ for $1 \leqslant i \leqslant s$, then we have*

$$\Phi_q(f(x)) = q^n \prod_{i=1}^{s} (1 - \frac{1}{q^{d_i}}), \tag{1}$$

where $\Phi_q(f(x))$ denotes the number of monic polynomials relatively prime to $f(x)$ of degree less than n.

Now we introduce the concept of partial cyclic codes.

Definition 4 (Partial cyclic codes). For a vector $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{F}_q^n$, let $G = \mathcal{P}_k(\boldsymbol{a})$ be the $k \times n$ partial circulant matrix generated by $\boldsymbol{a}$. An $[n, k]$ linear code $\mathcal{C}$ spanned by the rows of G over $\mathbb{F}_q$ is called a partial cyclic code.

Remark 3. For a basis vector $\boldsymbol{g} = (g^{[n-1]}, g^{[n-2]}, \cdots, g)$ of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, let $G = \mathcal{P}_k(\boldsymbol{g})$ be the $k \times n$ partial circulant matrix generated by $\boldsymbol{g}$. It is easy to verify that G is a $k \times n$ Moore matrix. The linear code $\mathcal{G}$ spanned by the rows of G over $\mathbb{F}_{q^n}$ is called a partial cyclic Gabidulin code.

As for the total number of $[n, k]$ partial cyclic Gabidulin codes over $\mathbb{F}_{q^n}$, or equivalently the total number of normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, we present the following proposition.

Proposition 4. *[22] Normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ are in one-to-one correspondence to circulant matrices in $GL_n(\mathbb{F}_q)$, which implies that the total number of normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ can be evaluated as $\Phi_q(x^n - 1)$.*

3 Hard problems in coding theory

This section mainly discusses some classical hard problems in coding theory on which the security of code-based cryptosystems lies, as well as some attacks on them that will be useful to estimate the security level of our proposal later in this paper.

Definition 5 (Syndrome Decoding (SD) Problem). Given positive integers n, k and t, let H be an $(n-k) \times n$ matrix over $\mathbb{F}_q$ of full rank and $\boldsymbol{s} \in \mathbb{F}_q^{n-k}$. The SD problem with parameters (q, n, k, t) is to search for a vector $\boldsymbol{e} \in \mathbb{F}_q^n$ such that $\boldsymbol{s} = \boldsymbol{e}H^T$ and $\mathrm{wt}_H(\boldsymbol{e}) = t$.

The SD problem, proved to be NP-complete by Berlekamp et al. in [15], plays a crucial role in both complexity theory and code-based cryptography. The NP-completeness implies that the best known algorithm of solving this problem requires exponential time. The first instance of the SD problem being used in code-based cryptography is the McEliece cryptosystem [16] based on Goppa codes. A generalized version of this problem in the rank metric is the rank syndrome decoding problem defined as follows.

Definition 6 (Rank Syndrome Decoding (RSD) Problem). Given positive integers m, n, k and t, let H be an $(n-k) \times n$ matrix of full rank over $\mathbb{F}_{q^m}$ and $\boldsymbol{s} \in \mathbb{F}_{q^m}^{n-k}$. The RSD problem with parameters (q, m, n, k, t) is to search for a vector $\boldsymbol{e} \in \mathbb{F}_{q^m}^n$ such that $\boldsymbol{s} = \boldsymbol{e}H^T$ and $\mathrm{wt}_R(\boldsymbol{e}) = t$.

The RSD problem is an important issue in rank code-based cryptography, which has been used for designing cryptosystems since the proposal of the GPT cryptosystem [2] in 1991. However, the hardness of this problem had never been proved until the work in [14], where the authors gave a randomized reduction of the SD problem to the RSD problem.

Generally speaking, attacks on the RSD problem can be divided into two categories, namely the combinatorial attack and algebraic attack. The main idea of combinatorial attacks consists in solving a linear system obtained from the parity-check equation, whose unknowns are components of $e_i \, (1 \leqslant i \leqslant n)$ with respect to a potential support of $\boldsymbol{e}$. Up to now, the best known combinatorial attacks can be found in [17, 19, 20], as summarized in Table 1.

Attack	Complexity
[17]	$\mathcal{O}\left(\min\left\{m^3t^3q^{(t-1)(k+1)}, (k+t)^3t^3q^{(t-1)(m-t)}\right\}\right)$
[19]	$\mathcal{O}\left((n-k)^3m^3q^{\min\left\{t\left\lceil\frac{mk}{n}\right\rceil, (t-1)\left\lceil\frac{m(k+1)}{n}\right\rceil\right\}}\right)$
[20]	$\mathcal{O}\left((n-k)^3m^3q^{t\left\lceil\frac{m(k+1)}{n}\right\rceil - m}\right)$

Table 1: Best known combinatorial attacks on the RSD problem.

As for the algebraic attack, the main idea consists in converting an RSD instance into a quadratic system and then solving this system using algebraic approaches. Here in this paper, we mainly consider the attacks proposed in [12, 13, 18, 19], whose complexity and applicable condition are summarized in Table 2.

Attack	Condition	Complexity
[19]	$\left\lceil \frac{(t+1)(k+1)-(n+1)}{t} \right\rceil \leqslant k$	$\mathcal{O}\left(k^3 t^3 q^{t\left\lceil \frac{(t+1)(k+1)-(n+1)}{t} \right\rceil}\right)$
[18]		$\mathcal{O}\left(k^3 m^3 q^{t\left\lceil \frac{km}{n} \right\rceil}\right)$
[12]	$m\binom{n-k-1}{t} \geqslant \binom{n}{t} - 1$	$\mathcal{O}\left(m\binom{n-p-k-1}{t}\binom{n-p}{t}^{\omega-1}\right)$, where $\omega = 2.81$ and $p = \min\{1 \leqslant i \leqslant n : m\binom{n-i-k-1}{t} \geqslant \binom{n-i}{t} - 1\}$
[13]		$\mathcal{O}\left(\left(\frac{((m+n)t)^t}{t!}\right)^{\omega}\right)$
[12]	$m\binom{n-k-1}{t} < \binom{n}{t} - 1$	$\mathcal{O}\left(q^{at} m\binom{n-k-1}{t}\binom{n-a}{t}^{\omega-1}\right)$, where $a = \min\{1 \leqslant i \leqslant n : m\binom{n-k-1}{t} \geqslant \binom{n-i}{t} - 1\}$
[13]		$\mathcal{O}\left(\left(\frac{((m+n)t)^{t+1}}{(t+1)!}\right)^{\omega}\right)$

Table 2: Best known algebraic attacks on the RSD problem.

4 Semilinear transformations

Note that $\mathbb{F}_{q^m}$ can be regarded as an $\mathbb{F}_q$-linear space of dimension m. Let $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_m)$ and $\boldsymbol{b} = (\beta_1, \cdots, \beta_m)$ be two basis vectors of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. We define an $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$ as follows

$$\varphi(\alpha_1) = \beta_1, \cdots, \varphi(\alpha_m) = \beta_m.$$

This implies that for any $\alpha = \sum_{i=1}^m \lambda_i \alpha_i \in \mathbb{F}_{q^m}$ with $\lambda_i \in \mathbb{F}_q$, we have $\varphi(\alpha) = \sum_{i=1}^m \lambda_i \beta_i$. Furthermore, we introduce the following notations:

(1) For a vector $\boldsymbol{v} = (v_1, \cdots, v_n) \in \mathbb{F}_{q^m}^n$, we define $\varphi(\boldsymbol{v}) = (\varphi(v_1), \cdots, \varphi(v_n))$;

(2) For a set $\mathcal{V} \subseteq \mathbb{F}_{q^m}^n$, we define $\varphi(\mathcal{V}) = \{\varphi(\boldsymbol{v}) : \boldsymbol{v} \in \mathcal{V}\}$;

(3) For a matrix $M = (M_{ij}) \in \mathcal{M}_{k,n}(\mathbb{F}_{q^m})$, we define $\varphi(M) = (\varphi(M_{ij}))$.

In the remaining part of this section, we will make a further study on this type of transformations. Firstly, we introduce a basic fact about the $\mathbb{F}_q$-linear automorphisms of $\mathbb{F}_{q^m}$.

Proposition 5. *The total number of $\mathbb{F}_q$-linear automorphisms of $\mathbb{F}_{q^m}$ is*

$$\prod_{i=0}^{m-1} (q^m - q^i).$$

Proof. For a basis vector $\boldsymbol{a}$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and an invertible matrix $A \in GL_m(\mathbb{F}_q)$, it is easy to see that $\boldsymbol{a}A$ is also a basis vector. On the contrary, let $\boldsymbol{b}$ be another basis vector, then there exists a unique $B \in GL_m(\mathbb{F}_q)$ such that $\boldsymbol{b} = \boldsymbol{a}B$. This enables us to conclude that, for a given basis vector $\boldsymbol{a}$, all the $\mathbb{F}_q$-linear automorphisms of $\mathbb{F}_{q^m}$ are in one-to-one correspondence to $GL_m(\mathbb{F}_q)$. By evaluating the cardinality of $GL_m(\mathbb{F}_q)$, we obtain the conclusion immediately. □

For a given vector $\boldsymbol{c} \in \mathbb{F}_{q^m}^n$, a natural question is how the Hamming (rank) weight of $\boldsymbol{c}$ changes under the action of an $\mathbb{F}_q$-linear transformation φ. To answer this question, we introduce the following proposition.

Proposition 6. *An $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$ is an isometric transformation in both the Hamming metric and rank metric.*

Proof. Let $\alpha \in \mathbb{F}_{q^m}$ and φ be an $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$, apparently we have $\varphi(\alpha) = 0$ if and only if $\alpha = 0$. Hence $\mathrm{Supp}_H(\varphi(\boldsymbol{v})) = \mathrm{Supp}_H(\boldsymbol{v})$ holds for any $\boldsymbol{v} \in \mathbb{F}_{q^m}^n$, which implies that $\mathrm{wt}_H(\varphi(\boldsymbol{v})) = \mathrm{wt}_H(\boldsymbol{v})$.

As for the rank metric, let $\boldsymbol{v} \in \mathbb{F}_{q^m}^n$ be a vector such that $\mathrm{wt}_R(\boldsymbol{v}) = n$. If $\mathrm{wt}_R(\varphi(\boldsymbol{v})) < n$, then there exists $\boldsymbol{b} \in \mathbb{F}_q^n \backslash \{\boldsymbol{0}\}$ such that $\varphi(\boldsymbol{v})\boldsymbol{b}^T = \varphi(\boldsymbol{v}\boldsymbol{b}^T) = 0$. Following this we have $\boldsymbol{v}\boldsymbol{b}^T = 0$, which conflicts with $\mathrm{wt}_R(\boldsymbol{v}) = n$. More generally, suppose that $\mathrm{wt}_R(\boldsymbol{v}) = r < n$. Then there exist $\boldsymbol{v}^* \in \mathbb{F}_{q^m}^r$ with $\mathrm{wt}_R(\boldsymbol{v}^*) = r$ and $Q \in GL_n(\mathbb{F}_q)$ such that $\boldsymbol{v} = (\boldsymbol{v}^*|\boldsymbol{0})Q$. Apparently we have $\varphi(\boldsymbol{v}) = (\varphi(\boldsymbol{v}^*)|\boldsymbol{0})Q$, and then $\mathrm{wt}_R(\varphi(\boldsymbol{v})) = \mathrm{wt}_R(\varphi(\boldsymbol{v}^*)) = r$. □

Remark 4. Let $\mathbb{K}$ be an extension field of $\mathbb{F}_{q^m}$. Similar to Proposition 6, we come to the conclusion from a direct verification that an $\mathbb{F}_{q^m}$-linear automorphism of $\mathbb{K}$ preserves the rank weight of a vector in $\mathbb{K}^n$ with respect to $\mathbb{F}_q$.

For a linear code $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ and an $\mathbb{F}_q$-linear automorphism φ of $\mathbb{F}_{q^m}$, it is easy to see that $\varphi(\mathcal{C})$ is an $\mathbb{F}_q$-linear space, but generally no longer $\mathbb{F}_{q^m}$-linear. Formally, we classify the $\mathbb{F}_q$-linear automorphisms of $\mathbb{F}_{q^m}$ according to the following definition.

Definition 7. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be an $[n, k]$ linear code, and φ be an $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$. If $\varphi(\mathcal{C})$ is also an $\mathbb{F}_{q^m}$-linear code, we say that φ is linear on $\mathcal{C}$. Otherwise, we say that φ is semilinear on $\mathcal{C}$. If φ is linear on all linear codes over $\mathbb{F}_{q^m}$, we say that φ is fully linear over $\mathbb{F}_{q^m}$. Otherwise, we say that φ is semilinear over $\mathbb{F}_{q^m}$.

The following theorem provides a sufficient and necessary condition for φ being fully linear over $\mathbb{F}_{q^m}$.

Theorem 8. *Let $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_m)$ be a basis vector of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, and φ be an $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$. Let $A = \left[\varphi(\alpha_1\boldsymbol{a})^T, \cdots, \varphi(\alpha_m\boldsymbol{a})^T\right]^T$, then a sufficient and necessary condition for φ being fully linear is that A has rank 1.*

Proof. On the necessity aspect, assume that φ is fully linear over $\mathbb{F}_{q^m}$. Let $\mathcal{C} = \langle \boldsymbol{a} \rangle_{q^m}$ be a linear code over $\mathbb{F}_{q^m}$, and $\boldsymbol{a}_i = \varphi(\alpha_i\boldsymbol{a})$ be the i-th row vector of A. Since φ is linear on $\mathcal{C}$, there must be $\mu\boldsymbol{a}_i \in \varphi(\mathcal{C})$ for any $\mu \in \mathbb{F}_{q^m}$ and $1 \leqslant i \leqslant m$. This implies that there exists $\alpha \in \mathbb{F}_{q^m}$ such that $\mu\boldsymbol{a}_i = \varphi(\alpha\boldsymbol{a})$. Since components of $\boldsymbol{a}$ form a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, there exists $\lambda_j \in \mathbb{F}_q$ $(1 \leqslant j \leqslant m)$ such that $\alpha = \sum_{j=1}^m \lambda_j\alpha_j$. Hence we have

$$\mu\boldsymbol{a}_i = \varphi(\alpha\boldsymbol{a}) = \varphi(\sum_{j=1}^m \lambda_j\alpha_j\boldsymbol{a}) = \sum_{j=1}^m \lambda_j\varphi(\alpha_j\boldsymbol{a}) = \sum_{j=1}^m \lambda_j\boldsymbol{a}_j.$$

Let $\mathcal{V}_i = \{\mu\boldsymbol{a}_i : \mu \in \mathbb{F}_{q^m}\}$ and $\mathcal{V} = \{\sum_{j=1}^m \lambda_j\boldsymbol{a}_j : \lambda_1, \cdots, \lambda_m \in \mathbb{F}_q\}$, apparently we have $\mathcal{V}_i \subseteq \mathcal{V}$. Note that $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$ are linearly independent over $\mathbb{F}_q$, then we have $|\mathcal{V}_i| = |\mathcal{V}| = q^m$ and hence $\mathcal{V}_i = \mathcal{V}$. Particularly, we have $\boldsymbol{a}_j \in \mathcal{V}_i$ for any $1 \leqslant j \leqslant m$. This implies that A has rank 1 over $\mathbb{F}_{q^m}$.

On the sufficiency aspect, let $\mathcal{V}$ and $\mathcal{V}_i$ e defined as above. Note that A has rank 1 over $\mathbb{F}_{q^m}$, then for a fixed $\boldsymbol{a}_i$, there exists $\mu_j \in \mathbb{F}_{q^m}^* = \mathbb{F}_{q^m}\backslash\{0\}$ such that $\boldsymbol{a}_j = \mu_j \boldsymbol{a}_i$ for $1 \leqslant j \leqslant m$. This implies that $\mathcal{V} = \{\sum_{j=1}^m \lambda_j \mu_j \boldsymbol{a}_i : \lambda_j \in \mathbb{F}_q\}$. Apparently $\mathcal{V} \subseteq \mathcal{V}_i$, together with $|\mathcal{V}| = |\mathcal{V}_i|$ we have $\mathcal{V} = \mathcal{V}_i$. Hence for any $\mu \in \mathbb{F}_{q^m}$, there exist $\lambda_1, \cdots, \lambda_m \in \mathbb{F}_q$ such that $\mu \boldsymbol{a}_i = \sum_{j=1}^m \lambda_j \boldsymbol{a}_j$.

Let $\mathcal{C}$ be an arbitrary linear code over $\mathbb{F}_{q^m}^n$. For any $\boldsymbol{c} \in \varphi(\mathcal{C})$, there exists $\boldsymbol{u} \in \mathcal{C}$ such that $\boldsymbol{c} = \varphi(\boldsymbol{u})$. Furthermore, there exists $M \in \mathcal{M}_{m,n}(\mathbb{F}_q)$ such that $\boldsymbol{u} = \boldsymbol{a}M$. Apparently we have

$$\boldsymbol{a}_j M = \varphi(\alpha_j \boldsymbol{a})M = \varphi(\alpha_j \boldsymbol{a}M) = \varphi(\alpha_j \boldsymbol{u}) \in \varphi(\mathcal{C})$$

for any $1 \leqslant j \leqslant m$. Assume that $\sum_{i=1}^m a_i \alpha_i = 1$ where $a_i \in \mathbb{F}_q$, then $\boldsymbol{u} = \boldsymbol{a}M = \sum_{i=1}^m a_i \alpha_i \boldsymbol{a}M$. Hence

$$\mu \boldsymbol{c} = \mu \varphi(\boldsymbol{u}) = \mu \varphi(\sum_{i=1}^m a_i \alpha_i \boldsymbol{a}M) = \mu \sum_{i=1}^m a_i \varphi(\alpha_i \boldsymbol{a})M = \sum_{i=1}^m a_i \mu \boldsymbol{a}_i M.$$

Note that for any $\mu \in \mathbb{F}_{q^m}$ and $1 \leqslant i \leqslant m$, there exists $\lambda_{ij} \in \mathbb{F}_q$ such that $\mu \boldsymbol{a}_i = \sum_{j=1}^m \lambda_{ij} \boldsymbol{a}_j$. Hence we have

$$\mu \boldsymbol{c} = \sum_{i=1}^m a_i (\sum_{j=1}^m \lambda_{ij} \boldsymbol{a}_j) M = \sum_{i=1}^m \sum_{j=1}^m \lambda_{ij} a_i (\boldsymbol{a}_j M) \in \varphi(\mathcal{C})$$

because of $\boldsymbol{a}_j M \in \varphi(\mathcal{C})$ and $\varphi(\mathcal{C})$ being $\mathbb{F}_q$-linear. Following this, we conclude that $\varphi(\mathcal{C})$ is $\mathbb{F}_{q^m}$-linear, and hence then φ is fully linear over $\mathbb{F}_{q^m}$. □

Remark 5. Note that the rank of A is independent of the basis vector. More generally, let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ be another two basis vectors of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, then there exist $Q_1, Q_2 \in \mathcal{P}_n(\mathbb{F}_q) \cap GL_n(\mathbb{F}_q)$ such that $\boldsymbol{a}_1 = \boldsymbol{a}Q_1$ and $\boldsymbol{a}_2 = \boldsymbol{a}Q_2$. Let $A' = \varphi(\boldsymbol{a}_1^T \boldsymbol{a}_2)$, then $A' = \varphi((\boldsymbol{a}Q_1)^T \boldsymbol{a}Q_2) = \varphi(Q_1^T \boldsymbol{a}^T \boldsymbol{a}Q_2) = Q_1^T A Q_2$, which leads to the conclusion that $\text{Rank}(A) = \text{Rank}(A')$.

The following theorem gives an accurate count of fully linear transformations over $\mathbb{F}_{q^m}$.

Theorem 9. *The total number of fully linear transformations over $\mathbb{F}_{q^m}$ with respect to $\mathbb{F}_q$ is $m(q^m - 1)$.*

Proof. For a polynomial element $\alpha \in \mathbb{F}_{q^m}$, let $\boldsymbol{a} = (1, \alpha, \cdots, \alpha^{m-1})$ be a basis vector of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, and φ be an $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$. By Theorem 8, a necessary condition for φ being fully linear is that $\varphi(\alpha \boldsymbol{a}) = \gamma \varphi(\boldsymbol{a})$ or equivalently

$$(\varphi(\alpha), \varphi(\alpha^2), \cdots, \varphi(\alpha^m)) = \gamma(\varphi(1), \varphi(\alpha), \cdots, \varphi(\alpha^{m-1})) \tag{2}$$

holds for some $\gamma \in \mathbb{F}_{q^m}$. Assume that $\varphi(1) = \beta \in \mathbb{F}_{q^m}^*$, then we can deduce from (2) that

$$\varphi(\alpha^i) = \gamma \varphi(\alpha^{i-1}) = \gamma^i \beta \text{ for } 1 \leqslant i \leqslant m.$$

Let $f(x) = x^m + \sum_{i=0}^{m-1} a_i x^i \in \mathbb{F}_q[x]$ be the minimal polynomial of α, then we have

$$f(\alpha) = \alpha^m + \sum_{i=0}^{m-1} a_i \alpha^i = 0. \tag{3}$$

Because of φ being $\mathbb{F}_q$-linear, applying φ to both sides of (3) leads to the equation

$$\varphi(\alpha^m) + \sum_{i=0}^{m-1} a_i \varphi(\alpha^i) = \gamma^m \beta + \sum_{i=0}^{m-1} a_i \gamma^i \beta = 0.$$

This implies that $f(\gamma) = 0$, and there must be $\gamma = \alpha^{[i]}$ for some $0 \leqslant i \leqslant m-1$.

Conversely, let $\Gamma = \{\alpha^{[i]} : 0 \leqslant i \leqslant m-1\}$, then it is easy to verify that for any duple $(\gamma, \beta) \in \Gamma \times \mathbb{F}_{q^m}^*$, the $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^m}$ determined by $\varphi(\alpha^i) = \beta\gamma^i$ $(0 \leqslant i \leqslant m-1)$ forms a fully linear transformation over $\mathbb{F}_{q^m}$. Hence all the fully linear transformations over $\mathbb{F}_{q^m}$ are in one-to-one correspondence to the Cartesian product $\Gamma \times \mathbb{F}_{q^m}^*$, which leads to the conclusion immediately. □

Remark 6. For a polynomial element α of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, let $\Gamma = \{\alpha^{[i]}\}_{i=0}^{m-1}$ be the set of conjugates of α. For any $\gamma \in \Gamma$ and $\beta \in \mathbb{F}_{q^m}^*$, the $\mathbb{F}_q$-linear transformation, determined by $\varphi(\alpha^i) = \beta\gamma^i$ for $0 \leqslant i \leqslant m-1$, forms a fully linear transformation over $\mathbb{F}_{q^m}$ according to Theorem 9. Note that γ is a conjugate of α, then there exists $0 \leqslant j \leqslant m-1$ such that $\gamma = \alpha^{[j]}$. For any $\mu = \sum_{i=0}^{m-1} \lambda_i \alpha^i \in \mathbb{F}_{q^m}$ where $\lambda_i \in \mathbb{F}_q$, we have

$$\varphi(\mu) = \varphi(\sum_{i=0}^{m-1} \lambda_i \alpha^i) = \sum_{i=0}^{m-1} \lambda_i \varphi(\alpha^i) = \sum_{i=0}^{m-1} \lambda_i \beta \gamma^i = \beta \sum_{i=0}^{m-1} \lambda_i (\alpha^{[j]})^i = \beta(\sum_{i=0}^{m-1} \lambda_i \alpha^i)^{[j]} = \beta \mu^{[j]}.$$

This implies that a fully linear transformation over $\mathbb{F}_{q^m}$ can be seen as a composition of the Frobenius transformation and stretching transformation.

Theorem 10. *For two positive integers $k < n$, let $\mathcal{C}$ be an $[n,k]$ linear code over $\mathbb{F}_{q^m}$. Let $G = [I_k|A]$ be the systematic generator matrix of $\mathcal{C}$, where I_k is the $k \times k$ identity matrix and $A = (A_{ij}) \in \mathcal{M}_{k,n-k}(\mathbb{F}_{q^m})$. Let $\mathcal{S} = \{A_{ij} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n-k\}$, then we have the following statements.*

(1) If $\mathcal{S} \subseteq \mathbb{F}_q$, then any $\mathbb{F}_q$-linear automorphism φ of $\mathbb{F}_{q^m}$ is linear on $\mathcal{C}$. Furthermore, we have $\varphi(\mathcal{C}) = \mathcal{C}$;

(2) If there exists $\alpha \in \mathcal{S}$ such that α is a polynomial element of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, then any $\mathbb{F}_q$-linear automorphism φ of $\mathbb{F}_{q^m}$ is fully linear if and only if φ is linear on $\mathcal{C}$.

Proof. (1) Let $\boldsymbol{g}_i$ be the i-th row vector of G, then we have $\varphi(\alpha \boldsymbol{g}_i) = \varphi(\alpha)\boldsymbol{g}_i$ for any $\alpha \in \mathbb{F}_{q^m}$. For any $\boldsymbol{c} \in \mathcal{C}$, there exists $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k) \in \mathbb{F}_{q^m}^k$ such that $\boldsymbol{c} = \boldsymbol{\lambda} G$. Then there will be

$$\varphi(\boldsymbol{c}) = \varphi(\boldsymbol{\lambda} G) = \varphi(\sum_{i=1}^{k} \lambda_i \boldsymbol{g}_i) = \sum_{i=1}^{k} \varphi(\lambda_i \boldsymbol{g}_i) = \sum_{i=1}^{k} \varphi(\lambda_i)\boldsymbol{g}_i \in \mathcal{C}.$$

By the definition of $\varphi(\mathcal{C})$, we have $\varphi(\mathcal{C}) \subseteq \mathcal{C}$. Together with $|\varphi(\mathcal{C})| = |\mathcal{C}|$, there will be $\varphi(\mathcal{C}) = \mathcal{C}$.

(2) The necessity is obvious, then it suffices to prove the sufficiency. Without loss of generality, we consider the first row vector of G and assume that $\boldsymbol{g}_1 = (1, 0, \cdots, 0, \alpha, \star) \in \mathbb{F}_{q^m}^n$, where $\alpha \in \mathbb{F}_{q^m}$ is a polynomial element and "$\star$" represents some vector in $\mathbb{F}_{q^m}^{n-k-1}$. Note that φ is linear on $\mathcal{C}$, or equivalently $\varphi(\mathcal{C})$ is an $\mathbb{F}_{q^m}$-linear code. Apparently $\varphi(\mathcal{C})$ has $\varphi(G)$

as a generator matrix, which implies that there exists $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k) \in \mathbb{F}_{q^m}^k$ such that $\varphi(\beta \boldsymbol{g}_1) = \boldsymbol{\lambda}\varphi(G)$ for any $\beta \in \mathbb{F}_{q^m}$. It is easy to see that $\lambda_1 \in \mathbb{F}_{q^m}^*$ and $\lambda_i = 0$ for $2 \leqslant i \leqslant k$, which means $\varphi(\boldsymbol{g}_1)$ and $\varphi(\beta \boldsymbol{g}_1)$ are linearly dependent over $\mathbb{F}_{q^m}$. Then we can deduce that $(\varphi(\beta), \varphi(\alpha\beta)) = \lambda_1(\varphi(1), \varphi(\alpha))$ and furthermore $\varphi(1)\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$. Let $\gamma = \frac{\varphi(\alpha)}{\varphi(1)}$, then

$$\varphi(\alpha\beta) = \frac{\varphi(\alpha)}{\varphi(1)}\varphi(\beta) = \gamma\varphi(\beta).$$

Because of α being a polynomial element, $\boldsymbol{a} = (1, \alpha, \cdots, \alpha^{m-1}) \in \mathbb{F}_{q^m}^m$ forms a basis vector of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Following this, we have

$$\varphi(\alpha\boldsymbol{a}) = (\varphi(\alpha), \cdots, \varphi(\alpha^m)) = (\gamma\varphi(1), \cdots, \gamma\varphi(\alpha^{m-1})) = \gamma\varphi(\boldsymbol{a}),$$

and furthermore $\varphi(\alpha^i \boldsymbol{a}) = \gamma^i \varphi(\boldsymbol{a})$ for $0 \leqslant i \leqslant m-1$. By Theorem 8, we have that φ forms a fully linear transformation over $\mathbb{F}_{q^m}$.

□

Corollary 1. *Let m be a prime and $\mathcal{S}$ be defined as above in Theorem 10. If there exists $\alpha \in \mathcal{S}$ such that $\alpha \notin \mathbb{F}_q$, then any $\mathbb{F}_q$-linear automorphism φ of $\mathbb{F}_{q^m}$ is fully linear if and only if φ is linear on $\mathcal{C}$.*

Proof. Note that m is a prime, then any $\alpha \in \mathbb{F}_{q^m} \backslash \mathbb{F}_q$ is a polynomial element of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Hence the conclusion is proved immediately from Theorem 10. □

5 Our proposal

In this section, we will first give a formal description of our new proposal and then discuss how to choose the private key to avoid some potential structural weakness.

5.1 Description of our proposal

For a given security level, choose a finite field $\mathbb{F}_q$ and positive integers m, n, k, l, λ_1 and λ_2 such that $n = lm$. Let $\boldsymbol{g} = (g^{[n-1]}, g^{[n-2]}, \cdots, g) \in \mathbb{F}_{q^n}^n$ be a normal basis vector of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, and $G = \mathcal{P}_k(\boldsymbol{g}) \in \mathcal{M}_{k,n}(\mathbb{F}_{q^n})$ be a partial circulant matrix generated by $\boldsymbol{g}$. Denote by $\mathcal{G}$ the $[n, k]$ partial cyclic Gabidulin code over $\mathbb{F}_{q^n}$ that has G as a generator matrix. Now we introduce the cryptosystem with the following three procedures.

- Key generation

  For $i = 1, 2$, randomly choose an $\mathbb{F}_q$-linear space $\mathcal{V}_i \subseteq \mathbb{F}_{q^n}$ such that $\dim_q(\mathcal{V}_i) = \lambda_i$. Randomly choose $\boldsymbol{m}_i \in \mathcal{V}_i^n$ such that $\mathrm{wt}_R(\boldsymbol{m}_i) = \lambda_i$. Let $M_i = \mathcal{P}_n(\boldsymbol{m}_i)$ and check whether or not M_i is invertible. If not, then rechoose $\boldsymbol{m}_i$ and repeat the process above. Randomly choose an $\mathbb{F}_{q^m}$-linear automorphism φ of $\mathbb{F}_{q^n}$ such that φ is not fully linear over $\mathbb{F}_{q^n}$. Compute $\varphi(GM_1^{-1})M_2^{-1} = \mathcal{P}_k(\boldsymbol{g}^*)$, where $\boldsymbol{g}^* = \varphi(\boldsymbol{g}M_1^{-1})M_2^{-1}$. We publish $(\boldsymbol{g}^*, t)$ as the public key where $t = \left\lfloor \frac{n-k}{2\lambda_1\lambda_2} \right\rfloor$, and keep $(\boldsymbol{m}_1, \boldsymbol{m}_2, \varphi)$ as the private key.

- Encryption

For a plaintext $\boldsymbol{x} \in \mathbb{F}_{q^m}^k$, randomly choose $\boldsymbol{e} \in \mathbb{F}_{q^n}^n$ such that $\mathrm{wt}_R(\boldsymbol{e}) = t$. Then the ciphertext corresponding to $\boldsymbol{x}$ is computed as

$$\boldsymbol{y} = \boldsymbol{x}\mathcal{P}_k(\boldsymbol{g}^*) + \boldsymbol{e} = \boldsymbol{x}\varphi(GM_1^{-1})M_2^{-1} + \boldsymbol{e}.$$

- Decryption

For a ciphertext $\boldsymbol{y} \in \mathbb{F}_{q^n}^n$, we first recover $G = \mathcal{P}_k(\boldsymbol{g})$, $M_1 = \mathcal{P}_n(\boldsymbol{m}_1)$ and $M_2 = \mathcal{P}_n(\boldsymbol{m}_2)$. Then compute

$$\boldsymbol{y}M_2 = \boldsymbol{x}\varphi(GM_1^{-1}) + \boldsymbol{e}M_2 = \varphi(\boldsymbol{x}GM_1^{-1}) + \boldsymbol{e}M_2,$$

and

$$\boldsymbol{y}' = \varphi^{-1}(\boldsymbol{y}M_2)M_1 = \boldsymbol{x}G + \varphi^{-1}(\boldsymbol{e}M_2)M_1.$$

Let $\boldsymbol{e}' = \varphi^{-1}(\boldsymbol{e}M_2)M_1$, then we have

$$\mathrm{wt}_R(\boldsymbol{e}') \leqslant \mathrm{wt}_R(\varphi^{-1}(\boldsymbol{e}M_2)) \cdot \lambda_1 = \mathrm{wt}_R(\boldsymbol{e}M_2) \cdot \lambda_1 \leqslant \mathrm{wt}_R(\boldsymbol{e}) \cdot \lambda_2 \cdot \lambda_1 \leqslant \left\lfloor \frac{n-k}{2} \right\rfloor.$$

Applying the fast decoder of $\mathcal{G}$ to $\boldsymbol{y}'$ will lead to the error vector $\boldsymbol{e}'$, then we can recover $\boldsymbol{x}$ by solving the linear system $\boldsymbol{x}G = \boldsymbol{y}' - \boldsymbol{e}'$ with $\mathcal{O}(n^3)$ operations in $\mathbb{F}_{q^n}$.

5.2 A note on the underlying Gabidulin code

Now we explain why the underlying Gabidulin code is not used as part of the secret key. Firstly, we need to introduce the following proposition, which reveals the relationship between two normal basis vectors.

Proposition 7. *Let α be a normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, then $\beta \in \mathbb{F}_{q^n}$ is normal if and only if there exists $Q \in \mathcal{P}_n(\mathbb{F}_q) \cap GL_n(\mathbb{F}_q)$ such that*

$$(\beta^{[n-1]}, \beta^{[n-2]}, \cdots, \beta) = (\alpha^{[n-1]}, \alpha^{[n-2]}, \cdots, \alpha)Q.$$

Proof. Trivial from a straightforward verification. □

Note that keeping the matrix $G = \mathcal{P}_k(\boldsymbol{g})$ secret cannot enhance the security of the cryptosystem. Let $\boldsymbol{g}' \in \mathbb{F}_{q^n}^n$ be another normal basis vector of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. By Proposition 7, there exists a matrix $Q \in \mathcal{P}_n(\mathbb{F}_q) \cap GL_n(\mathbb{F}_q)$ such that $\boldsymbol{g} = \boldsymbol{g}'Q$. Let $G' = \mathcal{P}_k(\boldsymbol{g}')$, then we have $G = G'Q$ and

$$\varphi(GM_1^{-1})M_2^{-1} = \varphi(G'QM_1^{-1})M_2^{-1} = \varphi(G'M_1^{-1})QM_2^{-1} = \varphi(G'M_1^{-1})M_2'^{-1},$$

where $M_2' = M_2Q^{-1} \in \mathcal{P}_n(\mathbb{F}_{q^n}) \cap GL_n(\mathbb{F}_{q^n})$ satisfying $\mathrm{wt}_R(M_2') = \lambda_2$. Furthermore, it is easy to verify that anyone with the knowledge of $\varphi, \boldsymbol{g}'$, M_1 and M_2' can decrypt any ciphertext in polynomial time. This implies that breaking this cryptosystem can be reduced to recovering φ, M_1 and M_2'. Hence we conclude that it does not make a difference to keep the underlying partial cyclic Gabidulin code secret.

5.3 On the choice of φ

First we explain why the secret transformation φ cannot be fully linear over $\mathbb{F}_{q^n}$. Assume that φ is fully linear over $\mathbb{F}_{q^n}$ with respect to $\mathbb{F}_{q^m}$, then by Remark 6 there exist $\beta \in \mathbb{F}_{q^n}^*$ and $0 \leqslant j \leqslant l-1$ such that

$$\varphi(GM_1^{-1}) = \beta(GM_1^{-1})^{[mj]} = \beta G^{[mj]}(M_1^{-1})^{[mj]} = \beta G^{[mj]}(M_1^{[mj]})^{-1}.$$

Following this we have

$$\varphi(GM_1^{-1})M_2^{-1} = \beta G^{[mj]}(M_1^{[mj]})^{-1} \cdot M_2^{-1} = G^{[mj]}(\beta^{-1}M_2M_1^{[mj]})^{-1} = G'M'^{-1},$$

where $G' = G^{[mj]}$ and $M' = \beta^{-1}M_2M_1^{[mj]}$. On the one hand, $\langle G' \rangle_{q^n}$ forms an $[n,k]$ partial cyclic Gabidulin code because of $G' = \mathcal{P}_k(\boldsymbol{g}^{[mj]})$ and Remark 3. On the other hand, entries of M' belong to an $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}$ of dimension at most $\lambda_1\lambda_2$. Furthermore, a straightforward verification shows that one can decrypt any ciphertext with the knowledge of G' and M'. This suggests that this cryptosystem degenerates into a sub-instance of Loidreau's proposal [4], which has been proved to be insecure in some cases [29–31]. Hence choosing φ to be fully linear over $\mathbb{F}_{q^n}$ may bring some security problems.

Furthermore, we hope that $\varphi(\langle GM_1^{-1} \rangle_{q^n})$ does not preserve the $\mathbb{F}_{q^n}$-linearity. According to our experiments on Magma, the systematic form of GM_1^{-1} always has entries that serve as a polynomial element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_{q^m}$. To generate a transformation φ that is semilinear on $\langle GM_1^{-1} \rangle_{q^n}$, it sufficies to choose a semilinear transformation over $\mathbb{F}_{q^n}$ because of Theorem 10.

In what follows, we investigate the equivalence between different semilinear transformations. For any $\beta \in \mathbb{F}_{q^n}^*$ and a semilinear transformation φ over $\mathbb{F}_{q^n}$ with respect to $\mathbb{F}_{q^m}$, it is easy to verify that $\beta\varphi$ is also a semilinear transformation, where $\beta\varphi$ is defined as $\beta\varphi(\alpha) = \beta \cdot \varphi(\alpha)$ for any $\alpha \in \mathbb{F}_{q^n}$. Furthermore, let $\varphi' = \beta\varphi$ and $M_2' = \beta M_2$, then we have $\text{wt}_R(M_2') = \text{wt}_R(M_2) = \lambda_2$ and

$$\varphi(GM_1^{-1})M_2^{-1} = \beta^{-1}\varphi'(GM_1^{-1})M_2^{-1} = \varphi'(GM_1^{-1})(\beta M_2)^{-1} = \varphi'(GM_1{}^{-1})M_2'^{-1}.$$

From the perspective of a brute-force attack, we say that φ and φ' are equivalent. We define $\overline{\varphi} = \{\beta\varphi : \beta \in \mathbb{F}_{q^n}^*\}$, called the equivalent class of φ. Apparently for any two transformations φ_1 and φ_2, we have either $\overline{\varphi_1} = \overline{\varphi_2}$ or $\overline{\varphi_1} \cap \overline{\varphi_2} = \varnothing$.

Now we make an estimation on the number of nonequivalent semilinear transformations. By Proposition 5, the number of $\mathbb{F}_{q^m}$-linear automorphisms over $\mathbb{F}_{q^n}$ can be computed as $\prod_{i=0}^{l-1}(q^n - q^{mi})$. By Theorem 9, the number of fully linear transformations over $\mathbb{F}_{q^n}$ with respect to $\mathbb{F}_{q^m}$ is $l(q^n-1)$. Denote by $\mathcal{N}(\overline{\varphi})$ the number of nonequivalent semilinear transformations, then we have

$$\mathcal{N}(\overline{\varphi}) = \left(\prod_{i=0}^{l-1}(q^n - q^{mi}) - l(q^n-1)\right) \Big/ (q^n-1) \approx q^{(l-1)n}.$$

5.4 On the choice of $\boldsymbol{m}_1, \boldsymbol{m}_2$

Firstly, we investigate how to choose $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ to evade some structural weakness. We point out that neither $\boldsymbol{m}_1$ or $\boldsymbol{m}_2$ should be taken over $\mathbb{F}_{q^m}$, otherwise this cryptosystem will be degenerated. Now we discuss this problem in the following two cases:

(1) If $\boldsymbol{m}_1 \in \mathbb{F}_{q^m}^n$, then there will be $M_1, M_1^{-1} \in GL_n(\mathbb{F}_{q^m})$. Following this, we have

$$\varphi(GM_1^{-1})M_2^{-1} = \varphi(G)M_1^{-1}M_2^{-1} = \varphi(G)(M_1M_2)^{-1} = \varphi(G)M^{-1},$$

where $M = M_1M_2$ satisfying $\text{wt}_R(M) \leqslant \text{wt}_R(M_1) \cdot \text{wt}_R(M_2) = \lambda_1\lambda_2$. A direct verification shows that if one can recover φ and M, then one can decrypt any valid ciphertext in polynomial time. Let $G'_{pub} = \mathcal{P}_n(\boldsymbol{g}^*)$ and $G' = \mathcal{P}_n(\boldsymbol{g})$, then it can be verified that $G'_{pub} = \varphi(G')M^{-1}$. If one can manage to find φ, then one can recover M by computing ${G'_{pub}}^{-1}\varphi(G')$. This indicates that breaking this cryptosystem can be reduced to finding the secret φ.

(2) If $\boldsymbol{m}_2 \in \mathbb{F}_{q^m}^n$, then there will be $M_2, M_2^{-1} \in GL_n(\mathbb{F}_{q^m})$. Furthermore, we have

$$\varphi(GM_1^{-1})M_2^{-1} = \varphi(GM_1^{-1}M_2^{-1}) = \varphi(GM^{-1}),$$

where $M = M_1M_2$ satisfying $\text{wt}_R(M) \leqslant \text{wt}_R(M_1) \cdot \text{wt}_R(M_2) = \lambda_1\lambda_2$. Similarly, a direct verification shows that one can decrypt any valid ciphertext with the knowledge of φ, G and M. If one can manage to find φ, then one can recover GM^{-1} and then M using the same method as above. This indicates that breaking this cryptosystem can be reduced to finding the secret φ.

Secondly, we discuss the equivalence of potential $\boldsymbol{m}_1$'s. For a matrix $Q \in \mathcal{P}_n(\mathbb{F}_q) \cap GL_n(\mathbb{F}_q)$, let $M'_1 = M_1Q$ and $M'_2 = M_2Q$, then $\text{wt}_R(M'_1) = \text{wt}_R(M_1)$ and $\text{wt}_R(M'_2) = \text{wt}_R(M_2)$. Following this, we have

$$\varphi(G{M'_1}^{-1})M_2^{-1} = \varphi(GQ^{-1}M_1^{-1}){M'_2}^{-1} = \varphi(GM_1^{-1})Q^{-1}M_2^{-1} = \varphi(G{M_1}^{-1}){M'_2}^{-1}.$$

From the perspective of a brute-force attack against $\boldsymbol{m}_1$, it does not make a difference to multiply $\boldsymbol{m}_1$ by an invertible circulant matrix over $\mathbb{F}_q$. We define $\overline{\boldsymbol{m}}_1 = \{\boldsymbol{m}_1Q : Q \in \mathcal{P}_n(\mathbb{F}_q) \cap GL_n(\mathbb{F}_q)\}$, called the equivalent class of $\boldsymbol{m}_1$.

Now we make an estimation on the number of nonequivalent $\overline{\boldsymbol{m}}_1$'s. For a positive integer λ, let $\mathcal{V} \subseteq \mathbb{F}_{q^n}$ be an $\mathbb{F}_q$-linear space of dimension λ. For a matrix $M \in \mathcal{P}_n(\mathcal{V}) \cap GL_n(\mathcal{V})$ with $\text{wt}_R(M) = \lambda$, there exists a decomposition $M = \sum_{j=1}^{\lambda} \alpha_jA_j$, where α_j's form a basis of $\mathcal{V}$ over $\mathbb{F}_q$ and A_j's are nonzero matrices in $\mathcal{P}_n(\mathbb{F}_q)$. Denote by $\mathcal{N}(\mathcal{V})$ the number of λ-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^n}$, and by $\mathcal{N}(A)$ the number of nonzero matrices in $\mathcal{P}_n(\mathbb{F}_q)$, then we have $\mathcal{N}(A) = q^n - 1$ and

$$\mathcal{N}(\mathcal{V}) = \prod_{j=0}^{\lambda-1} \frac{q^n - q^j}{q^\lambda - q^j} \approx q^{\lambda n - \lambda^2}.$$

The number of matrices $M \in \mathcal{P}_n(\mathbb{F}_{q^n}) \cap GL_n(\mathbb{F}_{q^n})$ with $\text{wt}_R(M) = \lambda$ can be evaluated as

$$\mathcal{N}(M) = \mathcal{N}(\mathcal{V}) \cdot \mathcal{N}(A)^\lambda \cdot \xi \approx \xi q^{2\lambda n - \lambda^2},$$

where ξ denotes the probability of a random $M \in \mathcal{P}_n(\mathbb{F}_{q^n})$ with $\text{wt}_R(M) = \lambda$ being invertible. As for the probability ξ, we have the following proposition.

Proposition 8. *If $q^\lambda - q^{\lambda-1} \geqslant 2n$, then $\xi \geqslant \frac{1}{2}$.*

Proof. For a λ-dimensional $\mathbb{F}_q$-linear space $\mathcal{V} \subseteq \mathbb{F}_{q^n}$, denote by $\mathcal{M}_\lambda(\mathcal{V})$ the set of all matrices with rank weight λ in $\mathcal{P}_n(\mathcal{V})$. Let U be the set of all singular matrices in $\mathcal{M}_\lambda(\mathcal{V})$, and $V = \mathcal{M}_\lambda(\mathcal{V}) \cap GL_n(\mathcal{V})$. In what follows, we will construct an injective mapping σ from U to V. First, we divide U into a certain number of subsets. For a matrix $M \in U$, let $\boldsymbol{m} = (m_0, m_1, \cdots, m_{n-1}) \in \mathcal{V}^n$ be the first row vector of M, namely $M = \mathcal{P}_n(\boldsymbol{m})$. Let $\overline{M} = \{N \in U : M - N \text{ is a scalar matrix}\}$, a set of matrices in U that resemble M at the last $n-1$ coordinates. Let $\boldsymbol{x} = (x, m_1, \cdots, m_{n-1})$, and $X = \mathcal{P}_n(\boldsymbol{x})$. Denote by $f(x) \in \mathbb{F}_{q^n}[x]$ the determinant of X, then $f(x)$ is a polynomial of degree n. In the meanwhile, we have that $|\overline{M}|$ equals the number of roots of $f(x) = 0$ in $\mathcal{V}$, which indicates that $|\overline{M}| \leqslant n$. Let $\boldsymbol{m}^* = (m_1, \cdots, m_{n-1})$, then it is easy to see that $\mathrm{wt}_R(\boldsymbol{m}^*) \geqslant \lambda - 1$. Now we establish the mapping σ in the following two cases:

(1) $\mathrm{wt}_R(\boldsymbol{m}^*) = \lambda - 1$.

For a matrix $M_1 \in \overline{M}$, let $\boldsymbol{m}_1 = (\delta_1, \boldsymbol{m}^*)$ be the first row vector of M_1. Let $\mathcal{W} = \langle m_1, \cdots, m_{n-1}\rangle_q$, then $\dim_q(\mathcal{W}) = \lambda - 1$. Because of $q^\lambda - q^{\lambda-1} > n$, there exists $\delta_1' \in \mathcal{V}\backslash\mathcal{W}$ such that $f(\delta_1') \neq 0$, where $f(x)$ is defined as above. Let $\boldsymbol{m}_1' = (\delta_1', \boldsymbol{m}^*)$, then we have $M_1' = \mathcal{P}_n(\boldsymbol{m}_1') \in GL_n(\mathcal{V})$, and $\mathrm{wt}_R(\boldsymbol{m}_1') = \lambda$ in the meanwhile. We define $\sigma(M_1) = M_1'$.

For $2 \leqslant i \leqslant n$ and a matrix $M_i \in \overline{M}\backslash\{M_j\}_{j=1}^{i-1}$, if any, let $\boldsymbol{m}_i = (\delta_i, \boldsymbol{m}^*)$ be the first row vector of M_i. Because of $q^\lambda - q^{\lambda-1} - (i-1) > n$, there exists $\delta_i' \in \mathcal{V}\backslash(\mathcal{W} \cup \{\delta_j'\}_{j=1}^{i-1})$ such that $f(\delta_i') \neq 0$. Let $\boldsymbol{m}_i' = (\delta_i', \boldsymbol{m}^*)$, then we have $M_i' = \mathcal{P}_n(\boldsymbol{m}_i') \in GL_n(\mathcal{V})$, and $\mathrm{wt}_R(\boldsymbol{m}_i') = \lambda$ in the meanwhile. We define $\sigma(M_i) = M_i'$.

(2) $\mathrm{wt}_R(\boldsymbol{m}^*) = \lambda$.

For a matrix $M_1 \in \overline{M}$, let $\boldsymbol{m}_1 = (\delta_1, \boldsymbol{m}^*)$ be the first row vector of M_1. Because of $q^\lambda > n$, there exists $\delta_1' \in \mathcal{V}$ such that $f(\delta_1') \neq 0$, where $f(x)$ is defined as above. Let $\boldsymbol{m}_1' = (\delta_1', \boldsymbol{m}^*)$, then we have $M_1' = \mathcal{P}_n(\boldsymbol{m}_1') \in GL_n(\mathcal{V})$, and $\mathrm{wt}_R(\boldsymbol{m}_1') = \lambda$ in the meanwhile. We define $\sigma(M_1) = M_1'$.

For $2 \leqslant i \leqslant n$ and a matrix $M_i \in \overline{M}\backslash\{M_j\}_{j=1}^{i-1}$, if any, let $\boldsymbol{m}_i = (\delta_i, \boldsymbol{m}^*)$ be the first row vector of M_i. Because of $q^\lambda - (i-1) > n$, there exists $\delta_i' \in \mathcal{V}\backslash\{\delta_j'\}_{j=1}^{i-1}$ such that $f(\delta_i') \neq 0$. Let $\boldsymbol{m}_i' = (\delta_i', \boldsymbol{m}^*)$, then we have $M_i' = \mathcal{P}_n(\boldsymbol{m}_i') \in GL_n(\mathcal{V})$, and $\mathrm{wt}_R(\boldsymbol{m}_i') = \lambda$ in the meanwhile. We define $\sigma(M_i) = M_i'$.

It is easy to see that σ forms an injective mapping from U to V. Apparently $\sigma(U) = \{\sigma(M) : M \in U\} \subseteq V$, which implies that $|U| = |\sigma(U)| \leqslant |V|$. Together with $U \cap V = \varnothing$ and $\mathcal{M}_\lambda(\mathcal{V}) = U \cup V$, we have that

$$\xi = \sum_{\mathcal{V}\subseteq\mathbb{F}_{q^n}, \dim_q(\mathcal{V})=\lambda} |V| \Big/ \sum_{\mathcal{V}\subseteq\mathbb{F}_{q^n}, \dim_q(\mathcal{V})=\lambda} |\mathcal{M}_\lambda(\mathcal{V})| \geqslant \frac{1}{2}.$$

□

Remark 7. Proposition 8 provides a sufficient condition for $\xi \geqslant \frac{1}{2}$. Actually, this inequality always holds according to our extensive experiments on Magma, even when the sufficient condition is not satisfied. Hence we suppose that $\xi = \frac{1}{2}$ in practice. Finally, the number of nonequivalent $\overline{\boldsymbol{m}}_1$'s can be evaluated as

$$\mathcal{N}(\overline{\boldsymbol{m}}_1) = \frac{\mathcal{N}(M_1)}{|\mathcal{P}_n(\mathbb{F}_q) \cap GL_n(\mathbb{F}_q)|} \approx \frac{q^{2\lambda_1 n - \lambda_1^2}}{\Phi_q(x^n - 1)},$$

where $\Phi_q(x^n - 1)$ is defined as in (1).

In what follows, we will discuss how to generate in an efficient way the secret matrix $M_i \in \mathcal{P}_n(\mathbb{F}_{q^n}) \cap GL_n(\mathbb{F}_{q^n})$ such that $\text{wt}_R(M_i) = \lambda_i$. Let $M = \sum_{j=1}^{\lambda} \alpha_j A_j$ be defined as above. According to our experiments on Magma, if one of these A_j's is chosen to be invertible, then M is invertible with high probability. For example, let $M = \alpha_1 A_1 + \alpha_2 A_2$ be a circulant matrix of order 20 over $\mathbb{F}_{2^{20}}$. In the case of A_1 being randomly chosen from $\mathcal{P}_{20}(\mathbb{F}_2) \cap GL_{20}(\mathbb{F}_2)$ and A_2 being randomly chosen from $\mathcal{P}_{20}(\mathbb{F}_2)$, only 2 out of 1 million M's turn out to be singular. For the case of A_1 and A_2 being randomly chosen from $\mathcal{P}_{20}(\mathbb{F}_2)$, however, up to 252344 out of 1 million M's turn out to be singular. To efficiently generate the secret M_i, therefore, we adopt the following procedure:

1. Randomly choose $\alpha_1, \cdots, \alpha_{\lambda_i} \in \mathbb{F}_{q^n}$ that are linearly independent over $\mathbb{F}_q$;
2. Randomly choose nonzero matrices $A_1, \cdots, A_{\lambda_i} \in \mathcal{P}_n(\mathbb{F}_q)$ such that $A_1 \in GL_n(\mathbb{F}_q)$;
3. Compute $M_i = \sum_{j=1}^{\lambda_i} \alpha_j A_j$.
4. Check whether or not M_i is invertible. If not, go back to Step 2.

6 Security analysis

In code-based cryptography, there are mainly two types of attacks on a cryptosystem, namely the structural attack and generic attack. Structural attacks aim to recover the secret key from the published information, with which one can decrypt any ciphertext in polynomial time. Generic attacks aim to recover the plaintext directly without the knowledge of the secret key. In what follows, we will investigate the security of our new cryptosystem from these two perspectives.

6.1 Structural attacks

Ever since Gabidulin et al. exploited Gabidulin codes in the design of public-key cryptosystems [2], many variants based on these codes have been proposed one after another. Unfortunately, most of these cryptosystems were completely broken due to the inherent structural vulnerability of Gabidulin codes. The best known structural attacks are the one proposed by Overbeck in [25] and some of its derivations [26, 27].

To prevent these attacks, Loidreau [4] proposed a new cryptosystem based on Gabidulin codes, which can be seen as a rank-metric counterpart of the BBCRS cryptosystem [24] based on GRS codes. In Loidreau's proposal, the secret code is disguised by multiplying a matrix whose inverse is taken over a small $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^m}$. This technique of hiding the information about the secret key, as claimed by Loidreau in the original paper, were able to resist these structural attacks mentioned above. A similar technique is applied in our proposal, which we believe can as well prevent these attacks.

In [29], Coggia and Couvreur proposed an effective method to distinguish the public code of Loidreau's proposal from general ones, and gave a practical key-recovery attack in the case of $\lambda = 2$ and the code rate being greater than $1/2$. Instead of operating the public code directly, Coggia and Couvreur considered the dual of the public code. Specifically, let $G_{pub} = GM^{-1}$ be the public matrix of Loidreau's proposal, where G is a generator matrix of an $[n, k]$ Gabidulin code $\mathcal{G}$ over $\mathbb{F}_{q^m}$ and entries of M are contained in a λ-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^m}$. Denote by H a parity-check matrix of $\mathcal{G}$, then it is easy to verify that $H_{pub} = HM^T$ forms a parity-check matrix of the public code $\mathcal{G}_{pub} = \langle G_{pub} \rangle_{q^m}$. As for the dual code $\mathcal{G}_{pub}^{\perp} = \langle H_{pub} \rangle_{q^m}$, the Coggia-Couvreur distinguisher states a fact that the following equality holds with high probability

$$\dim_{q^m}(\mathcal{G}_{pub}^{\perp} + {\mathcal{G}_{pub}^{\perp}}^{[1]} + \cdots + {\mathcal{G}_{pub}^{\perp}}^{[\lambda]}) = \min\{n, \lambda(n-k) + \lambda\}.$$

For an $[n, k]$ random linear code $\mathcal{C}_{rand}$ over $\mathbb{F}_{q^m}$, however, the following equality holds with high probability

$$\dim_{q^m}(\mathcal{C}_{rand}^{\perp} + {\mathcal{C}_{rand}^{\perp}}^{[1]} + \cdots + {\mathcal{C}_{rand}^{\perp}}^{[\lambda]}) = \min\{n, (\lambda+1)(n-k)\}.$$

Not long after Coggia and Couvreur's work, this attack was generalized to the case of $\lambda = 3$ by Ghatak [30] and then by Duc and Loidreau [31].

In our proposal, the public matrix is computed as $G_{pub} = \mathcal{P}_k(\boldsymbol{g}^*) = \varphi(GM_1^{-1})M_2^{-1}$. Let H_{pub} be a parity-check matrix of the public code $\mathcal{G}_{pub} = \langle G_{pub} \rangle_{q^n}$, then there will be $H_{pub} = HM_2^T$ where H is a parity-check matrix of the code $\mathcal{C} = \langle \varphi(GM_1^{-1}) \rangle_{q^n}$. Apparently one can distinguish $\langle GM_1^{-1} \rangle_{q^n}^{\perp}$ from general linear codes using the Coggia-Couvreur distinguisher, but it does not work on $\mathcal{C}^{\perp}$. For instance, the following equality holds with high probability according to our experiments on Magma for the case of $\lambda_1 = 2, 3$

$$\dim_{q^n}(\mathcal{C}^{\perp} + \mathcal{C}^{\perp[1]} + \cdots + \mathcal{C}^{\perp[\lambda_1]}) = \min\{n, (\lambda_1+1)(n-k)\}.$$

Therefore, we believe that our proposal can resist the Coggia-Couvreur attack.

In a talk [32] at CBCrypto 2021, Loidreau proposed an algorithm to recover a decoder of the public code with a complexity of $\mathcal{O}(((\lambda n + (n-k)^2)m)^{\omega} q^{(\lambda-1)m})$. With this decoder one can decrypt any valid ciphertext in polynomial time. Similar to Coggia and Couvreur's approach, this attack also considered the dual of the public code. However, an applicable condition for this attack is that the public matrix can be decomposed as $G_{pub} = GM^{-1}$, where G is a generator matrix of a Gabidulin code or one of its subcodes and entries of M belong to an $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}$. Obviously the public matrix in our proposal does not satisfy this condition, which implies that this attack does not work on this new cryptosystem.

At the end of this section, we consider a potential brute-force attack against the duple $(\overline{\varphi}, \overline{\boldsymbol{m}}_1)$. Note that for any $\varphi' \in \overline{\varphi}$ and $\boldsymbol{m}_1' \in \overline{\boldsymbol{m}}_1$, there exists $\boldsymbol{m}_2' \in \mathbb{F}_{q^n}^n$ with $\mathrm{wt}_R(\boldsymbol{m}_2') = \lambda_2$ such that $G_{pub} = \varphi(GM_1{}^{-1})M_2^{-1} = \varphi'(GM_1'^{-1})M_2'^{-1}$, where $M_1' = \mathcal{P}_n(\boldsymbol{m}_1')$ and $M_2' = \mathcal{P}_n(\boldsymbol{m}_2')$. Let $G_{pub}' = \mathcal{P}_n(\boldsymbol{g}^*)$ and $G' = \mathcal{P}_n(\boldsymbol{g})$, then $G_{pub}' = \varphi(G'M_1^{-1})M_2^{-1} = \varphi'(G'M_1'^{-1})M_2'^{-1}$. This implies that one can compute $M_2' = {G_{pub}'}^{-1}\varphi'(G'M_1'^{-1})$. Furthermore, a straightforward verification will show that one can decrypt any valid ciphertext with the knowledge of $\varphi', \boldsymbol{m}_1', \boldsymbol{m}_2'$ and the public $\boldsymbol{g}$. Apparently the complexity of this brute-force attack by exhausting $(\overline{\varphi}, \overline{\boldsymbol{m}}_1)$ is $\mathcal{O}(\mathcal{N}(\overline{\varphi}) \cdot \mathcal{N}(\overline{\boldsymbol{m}}_1))$.

6.2 Generic attacks

A legitimate message receiver can always recover the plaintext in polynomial time, while an adversary without the secret key has to deal with the underlying RSD problem presented in Section 3. Attacks that aim to recover the plaintext directly by solving the corresponding RSD problem are called generic attacks, the complexity of which only relates to the parameters of the cryptosystem. In what follows, we will show how to establish a connection between our proposal and the RSD problem.

Let $G_{pub} = \varphi(GM_1^{-1})M_2^{-1} \in \mathcal{M}_{k,n}(\mathbb{F}_{q^n})$ be the public matrix, and $H_{pub} \in \mathcal{M}_{n-k,n}(\mathbb{F}_{q^n})$ be a parity-check matrix of the public code $\mathcal{G}_{pub} = \langle G_{pub} \rangle_{q^n}$. Let $\boldsymbol{y} = \boldsymbol{x}G_{pub} + \boldsymbol{e}$ be the received ciphertext, then the syndrome of $\boldsymbol{y}$ with respect to H_{pub} can be computed as $\boldsymbol{s} = \boldsymbol{y}H_{pub}^T = \boldsymbol{e}H_{pub}^T$. By Definition 6, we obtain an RSD instance of parameters (q, n, n, k, t). Solving this RSD instance by the combinatorial attacks listed in Table 1 and algebraic attacks listed in Table 2 will lead to the error vector $\boldsymbol{e}$, then we can recover the plaintext by solving the liear system $\boldsymbol{y} - \boldsymbol{e} = \boldsymbol{x}G_{pub}$.

7 Parameters and public-key sizes

In this section, we consider the practical security of our proposal against the generic attacks presented in Section 3, as well as a brute-force attack against the duple $(\overline{\varphi}, \overline{\boldsymbol{m}}_1)$ described in Section 6.1, with a complexity of $\mathcal{O}(\mathcal{N}(\overline{\varphi}) \cdot \mathcal{N}(\overline{\boldsymbol{m}}_1))$. The public key in our proposal is a vector in $\mathbb{F}_{q^n}^n$, leading to a public-key size of $n^2 \cdot \log_2(q)$ bits. In Table 3, we give some suggested parameters for the security level of 128 bits, 192 bits and 256 bits. After that, we make a comparison on public-key size with some other code-based cryptosystems in Table 4. It is easy to see that our proposal has an obvious advantage over other variants in public-key representation.

Parameters							Public-Key Size	Security
q	m	n	k	l	λ_1	λ_2		
2	55	110	54	2	2	2	1513	138
2	60	120	64	2	2	2	1800	197
2	72	144	72	2	2	2	2592	257

Table 3: Parameters and public-key size (in bytes) for different security levels.

8 Conclusion

In this paper, we introduced a new transformation in coding theory, which was defined as an $\mathbb{F}_q$-linear automorphism of a finite field $\mathbb{F}_{q^m}$. According to their algebraic properties when acting on linear codes over $\mathbb{F}_{q^m}$, these transformations were divided into two categories, namely the fully linear transformation and semilinear transformation. As an application of semilinear transformations, a new technique was developed to conceal the secret information of code-based cryptosystems. To obtain a small public-key size, we exploited the so-called partial cyclic Gabidulin code to construct an encryption scheme, whose security does not rely on the confidentiality of the underlying linear code. According to our analysis, this cryptosystem can resist the existing structural attacks, and admits quite a small public-key size compared to some other code-based cryptosystems. For instance, only 2592 bytes are enough for our proposal to achieve the security of 256 bits, 403 times smaller than that of Classic McEliece moving onto the third round of the NIST PQC standardization process.

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Instance \ Security	128	192	256
Classic McEliece [1]	261120	524160	1044992
NTS-KEM [33]	319488	929760	1419704
KRW [34]			578025
Guo-Fu II [11]	79358	212768	393422
Guo-Fu I [11]	8021	20149	37005
LIGA [35]	5618	9565	13823
HQC [36]	2249	4522	7245
BIKE [37]	1540	3082	5121
Lau-Tan [6]	2421	3283	4409
Our proposal	1513	1800	2592

Table 4: Comparison on public-key size (in bytes) with other code-based cryptosystems.

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