Subgroup membership testing on elliptic curves via the Tate pairing

Dmitrii Koshelev [1]

Computer sciences and networks department, Télécom Paris

Abstract. This note explains how to guarantee the membership of a point in the prime-order subgroup of an elliptic curve (over a finite field) satisfying some moderate conditions. For this purpose, we apply the Tate pairing on the curve, however it is not required to be pairing-friendly. Whenever the cofactor is small, the new subgroup test is much more efficient than other known ones, because it needs to compute at most two n-th power residue symbols (with small n) in the basic field. More precisely, the running time of the test is (sub-)quadratic in the bit length of the field size, which is comparable with the Decaf-style technique. The test is relevant, e.g., for the zk-SNARK friendly curves Bandersnatch and Jubjub proposed by the Ethereum and Zcash research teams respectively.

Key words: non-prime-order elliptic curves, power residue symbol, subgroup membership testing, Tate pairing.

1 Introduction

Undoubtedly, elliptic curves E over finite fields $\mathbb{F}_q$ are actively used in discrete logarithm cryptography. In fact, elliptic cryptosystems are often deployed not in the entire $\mathbb{F}_q$-point group $E(\mathbb{F}_q)$, but in its subgroup $\mathbb{G}$ of large prime order r and cofactor c. In order not to complicate, preference was initially given only to curves for which $\mathbb{G} = E(\mathbb{F}_q)$.

However, twisted Edwards curves [1] (with a cofactor multiple of 4) have become very popular in the last decade, which is characterized by inclusion of two of them in the draft NIST SP 800-186 [2]. The curves in question are called Curve25519 [3] and Ed448-Goldilocks [4]. Recall that twisted Edwards curves are birationally isomorphic to Montgomery ones (see, e.g., [5, Section 9.12.1]) and vice versa. It is also worth noting twisted Hessian curves [6] whose cofactor is a multiple of 3. Nevertheless, as far as we know, they have not been used in real-world cryptography.

To avoid timing attacks complete addition formulas are often utilized, because they are correctly defined for any pair of $\mathbb{F}_q$-points on E. Such formulas exist even for prime-order curves (in the Weierstrass form), but they are quite inefficient according to [7], [8]. In turn, the twisted Edwards form enjoys the fastest complete addition formulas among all known forms of elliptic curves.

To be protected against fault attacks from [9] [10] in any elliptic cryptography protocol, when receiving a point from a communication channel, it is necessary to make sure that

[1] Email: dimitri.koshelev@gmail.com
ResearchGate: https://www.researchgate.net/profile/dimitri-koshelev
LinkedIn: https://www.linkedin.com/in/dimitri-koshelev
GitHub: https://github.com/dishport

it belongs to E. Fortunately, this can be easily done by substituting the coordinates of the point in the curve equation. However, in the presence of a non-trivial cofactor even more work needs to be done. The seminal article [11] discusses subgroup security for the multiplicative group of a finite field, although its arguments are also valid in the elliptic curve setting.

To thwart the subgroup attack it is often sufficient to just multiply an obtained point by c. Nevertheless, in a series of modern protocols it is required to check whether a point really belongs to the subgroup $\mathbb{G}$. For example, as it turned out [12], multiplication by c in the signature scheme, used in the Monero cryptocurrency and a number of others, could lead to double-spending if any of the malicious users noticed this bug. In addition, clearing the cofactor forces protocol developers to complicate their security proofs.

An obvious way to test membership in $\mathbb{G}$ is to multiply a point by r. Even if the curve enjoys an effectively computable endomorphism, which makes it possible to apply the GLV technique or its variations (see, e.g., [5, Section 11.3.3]), the mentioned test is still laborious. More concretely, it performs $\Theta\bigl(\log_2(r)\bigr)$ additions in $E(\mathbb{F}_q)$. Hence its bit complexity equals $\Theta\bigl(\log_2(r)M\bigr)$ with a non-little constant behind Θ, where M is the bit complexity of a multiplication in $\mathbb{F}_q$. We can safely assume that $M = \Theta(\ell^2)$ at least for the popular choice $\ell \approx 256$, where $\ell := \log_2(q)$. Indeed, it is widely recognized [5, Remark 2.2.5] that for such ℓ the "school" multiplication algorithm is more practical. Since in the given context $c \approx 1$, we eventually get $\Theta(\ell^3)$.

Another way is to determine whether the endomorphism of multiplication by c has a non-empty inverse image in $E(\mathbb{F}_q)$ (cf. [5, Exercise 11.6.1]). This way generally requires sequential finding of the roots of several polynomials over $\mathbb{F}_q$ (see details in [13, Section 3], [14, Section 4]). To be definite, let's confine to square roots, because c is usually a power of two. It is well known that such a root is at best expressed via one exponentiation in the field. The latter operation clearly has $\Theta(\ell M)$ as the bit complexity with a little constant hidden in Θ. And for highly 2-adic fields one has to be content with the slower Tonelli–Shanks algorithm [5, Section 2.9]. More precisely, if $2^\nu \mid\mid q-1$, then its bit complexity amounts to $\Theta\bigl((\ell+\nu^2)M\bigr)$. Inter alia, for ν close to ℓ we obtain $\Theta(\ell^2 M)$ or just $\Theta(\ell^4)$.

Among solutions to the subgroup attack problem, those called Decaf [15] and Ristretto [16] (for $c = 4$, 8 respectively) deserve special mention. We should be aware that their essence extends to other cofactors. The ones 4, 8 are simply the most important in practice. In particular, there is the actual draft [17] about Decaf and Ristretto applied to the curves Ed448-Goldilocks and Curve25519 respectively. If these solutions are deployed without (de)compression techniques, then their bottleneck is the inversion operation in $\mathbb{F}_q$. According to [18], [19] this primitive (as well as the multiplication) enjoys (sub-)quadratic algorithms. Consequently, the overall bit complexity does not exceed $O(\ell^2)$.

In a nutshell, the solutions under consideration find coset representatives (canonical in some sense) of the quotient group $E(\mathbb{F}_q)/E(\mathbb{F}_q)[c] \simeq \mathbb{G}$. These representatives are not required to lie in the subgroup $\mathbb{G}$, which is a more natural system of representatives. The Decaf-style approach undoubtedly provides protection, but it has a number of disadvantages. First, formally speaking, it does not answer the question of belonging to $\mathbb{G}$. Second, at least for now, there is no unified theory generalizing Decaf. Therefore, determining canonical representatives may not be immediate for an arbitrary cofactor. Third, in our opinion, even the authentic Decaf (not to mention Ristretto) manipulates quite cumbersome formulas in comparison with the more laconic Tate pairing.

Finally, the sources [20], [21], [22] contain more information on subgroup membership checking in the case of pairing-friendly curves (see, e.g., [5, Section 26.6]). The author carefully analysed that for such curves the test of the given work does not surpass the state-of-the-art tests in performance. Looking ahead, the reason lies in huge cofactors, which occur for today's pairing groups $\mathbb{G}_1$, $\mathbb{G}_2$. With the permission of the reader, details are omitted. Thus despite the fact that the Tate pairing underlies the new test, it is relevant only for non-pairing-friendly curves, although this property is not utilized anywhere by us.

2 New subgroup check

Consider an elliptic curve $E\colon y^2 = x^3 + a_2x^2 + a_4x + a_6$ (with the point $\mathcal{O} := (0 : 1 : 0)$ at infinity) over a finite field $\mathbb{F}_q$ of characteristic $p > 2$. We know [5, Theorem 9.8.1] that the rational point group $E(\mathbb{F}_q) \simeq \mathbb{Z}/n_0 \times \mathbb{Z}/n_1$, where $n_1 \mid n_0$. As is customary in discrete logarithm cryptography, there is a subgroup $\mathbb{G} \subset E(\mathbb{F}_q)$ of large prime order r such that $r \mid\mid n_0$, but $r \nmid n_1$ (and hence $n_1 \mid e$, but $r \nmid e$, where $e := n_0/r$). In other words, $E(\mathbb{F}_q) = \mathbb{G} \times E(\mathbb{F}_q)[e]$. So the order $N := \#E(\mathbb{F}_q) = n_0n_1$ and the cofactor $c := N/r = en_1$. Throughout the text, we assume that $e \mid q - 1$, but $p \nmid N$. For the sake of uniformity, put $e_0 := e$ and $e_1 := n_1$. Besides, let $E(\mathbb{F}_q)[e] = \langle P_0\rangle \times \langle P_1\rangle$, where $\operatorname{ord}(P_i) = e_i$.

Recall that for any $k \mid q - 1$ the reduced Tate pairing [5, Section 26.3.2], [23, Equality (12)] can be represented in the form

$$t_k\colon E(\mathbb{F}_q)[k] \times E(\mathbb{F}_q)/kE(\mathbb{F}_q) \to \mu_k \qquad t_k(P, Q) := f_{k,P}(Q)^{(q-1)/k}, \tag{1}$$

where $\mu_k \subset \mathbb{F}_q^*$ is the group of all k-th roots of unity, $P \ne Q \ne \mathcal{O}$, and $f_{k,P} \in \mathbb{F}_q(E)$ is a Miller function satisfying the conditions

$$\operatorname{div}(f_{k,P}) = k(P) - k(\mathcal{O}), \qquad \Big(\Big(\frac{x}{y}\Big)^k \cdot f_{k,P}\Big)(\mathcal{O}) = 1.$$

The functions $f_{k,P}$ are recursively constructed, e.g., in [5, Section 26.3.1] by means of Miller's algorithm.

Note that the final exponentiation of the pairing t_k is nothing but the k-th power residue symbol $\left(\frac{\alpha}{q}\right)_k := \alpha^{(q-1)/k}$ when substituting $\alpha := f_{k,P}(Q)$. It is worth saying that we always can batch the inversion and symbol computation, since

$$\Big(\frac{\alpha_0/\alpha_1}{q}\Big)_k = \Big(\frac{\alpha_0\alpha_1^{k-1}}{q}\Big)_k$$

given $\alpha_i \in \mathbb{F}_q^*$. As said in [24], at least for $k \leqslant 11$ the symbol can be determined by Euclidean-type algorithms whose bit complexity amounts to $O(\ell^2)$. In particular, for $k = 2$ we deal with the ordinary Legendre symbol and the latest sources on this topic include [18], [25]. The important cases $k \in \{3, 4, 8\}$ are the subject of study in [26], [27], [28], and [29]. Finally, if k is not small, then the exponentiation is seemingly the best way to compute $\left(\frac{\alpha}{q}\right)_k$.

For compactness of notation, we also define the homomorphisms

$$h_i\colon E(\mathbb{F}_q) \to \mu_{e_i} \qquad h_i(Q) := t_e(P_i, Q) = t_{e_i}(P_i, Q).$$

The last identity is from [5, Exercise 26.3.8]. For our purpose it is unnecessary to know the values $h_i(P_i)$, hence we can benefit from the form (1).

Lemma 1. *There are the equalities* $\mathbb{G} = eE(\mathbb{F}_q) = \ker(h_0) \cap \ker(h_1)$. *In particular,* $\mathbb{G} = \ker(h_0)$ *when* $e_1 = 1$.

Proof. Given a point $Q \in \mathbb{G}$ we see that $Q = eR$ for $R := (e^{-1} \bmod r)Q$. The opposite inclusion $\mathbb{G} \supset eE(\mathbb{F}_q)$ is even more trivial. Further, according to [5, Theorem 26.3.3] the Tate pairing is non-degenerate. Consequently, a point $Q \in E(\mathbb{F}_q)$ in fact belongs to $eE(\mathbb{F}_q)$ if and only if $t_e(P, Q) = 1$ for all $P \in E(\mathbb{F}_q)[e]$ or, equivalently, $h_0(Q) = h_1(Q) = 1$. Finally, it is readily seen that $f_{e,\mathcal{O}} = f_{1,\mathcal{O}} = 1$, so for $e_1 = 1$ the homomorphism h_1 is trivial. □

Of course, the lemma cannot be regarded as a new result. Nevertheless, it gives rise to simple (sub-)quadratic-time Algorithm 1 whose proof-of-concept implementation in Magma is provided in [30]. By the way, since the point Q is always public information, we are not afraid of timing attacks. Therefore, we do not care about how easy it is to implement the algorithm (essentially the power residue symbol) in constant time.

Algorithm 1: New subgroup membership test

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Data: a point Q ∈ E(F_q)
Result: the answer to the question Q ∈ G?
begin
    compute the values β_0 := h_0(Q) and β_1 := h_1(Q) as described above;
    if β_0 = β_1 = 1 then
        return yes
    else
        return no
    end
end
```

Let's take a look at basic examples. For the first two of them we need simple facts from [31, Exercise 1.3.2].

The case $e_0 = 2$, $e_1 = 1$. Without lost of generality, $E\colon y^2 = x(x^2 + a_2x + a_4)$, where $a_2^2 - 4a_4$, $a_4 \notin (\mathbb{F}_q^*)^2$. The curves E are so-called double-odd curves thoroughly studied in [32], [33]. Clearly, $P_0 = (0, 0)$ and $f_{2,P_0} = x$. Lemma 1 states that a point $(x, y) \in E(\mathbb{F}_q)$ lies in $\mathbb{G}$ if and only if $x \in (\mathbb{F}_q^*)^2$. We obtain the subgroup membership test invented in [32, Section 1.2].

The case $e_0 = e_1 = 2$. In this one (valid for Bandersnatch), $E\colon y^2 = x(x - \alpha_1)(x - \alpha_2)$, where $\alpha_1, \alpha_2 \in \mathbb{F}_q^*$, but $\alpha_1\alpha_2 \notin (\mathbb{F}_q^*)^2$. Putting $\alpha_0 := 0$ in addition, we can pick, e.g., $P_i = (\alpha_i, 0)$. Therefore, $f_{2,P_i} = x - \alpha_i$. Under the chosen parameters, Lemma 1 is exactly [31, Theorem 1.4.1], because $x - \alpha_2 \in (\mathbb{F}_q^*)^2$ automatically whenever $x - \alpha_i \in (\mathbb{F}_q^*)^2$ for $i \in \{0, 1\}$. By the way, this result is used in [34, Section 3.2] to speed up compression of SIDH public keys.

The case $e_0 = 2^m$, $e_1 = 1$ for any $m \in \mathbb{N}$. The value $m = 3$ is relevant for Jubjub. As above, let $P_0 \in E(\mathbb{F}_q)$ be a point of order 2^m. For $0 \leqslant j \leqslant m - 1$ we also introduce points $P_j = (x_j, y_j) := [2^j]P_0$ whose $\operatorname{ord}(P_j) = 2^{m-j}$ obviously. Be careful, there is a

Curve	$\lceil \ell \rceil$	e_0	e_1	ν
Curve25519 [3]	255	8	1	2
Ed448-Goldilocks [4]	448	4		1
Jubjub [35]	255	8		32
Bandersnatch [36]		2	2	

Table 1: Some popular elliptic curves of non-prime order

discrepancy with our previous notation P_1. Classical formulas of the double operation $P_{j+1} = [2]P_j$ on E (see, e.g., [5, Section 9.1]) are given as follows (now $0 \leqslant j \leqslant m-2$):

$$\lambda_j := \frac{3x_j^2 + 2a_2x_j + a_4}{2y_j}, \qquad x_{j+1} = \lambda_j^2 - 2x_j - a_2, \qquad y_{j+1} = \lambda_j(x_j - x_{j+1}) - y_j.$$

Moreover, we have [23, Equalities (1), (2), (14)]:

$$\ell_j := (y - y_j) - \lambda_j(x - x_j), \qquad \nu_j := x - x_j, \qquad \mu_j := \frac{\ell_j}{\nu_{j+1}}, \qquad \mu_{m-1} := x - x_{m-1},$$

$$f_{2^{j+1},P_0} = f_{2^j,P_0}^2 \cdot \mu_j, \qquad \text{and hence} \qquad f_{2^m,P_0} = \prod_{j=0}^{m-1} \mu_j^{2^{m-j-1}}.$$

Table 1 contains a series of elliptic curves utilized in real-world cryptography as well as some numerical values of interest to us. As in the introduction, ν is the 2-adicity of $q-1$. We see that the first two curves are unfortunately not appropriate for the new subgroup check. They are added to the table just to be honest about non-universality of the current work. Incidentally, the last two curves are over the same field $\mathbb{F}_q$. Besides, in contrast to the other three curves, Bandersnatch is an incomplete twisted Edwards curve (see details in [37, Section 2]). Nevertheless, at points of its subgroup $\mathbb{G}$ the addition formulas are always defined. So for the mentioned curve the subgroup membership problem is even more relevant. In turn, the webpage [38] is dedicated to the given problem for Jubjub.

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