Key lifting : Multi-key Fully Homomorphic Encryption in plain model without noise flooding

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Abstract. Multi-key Fully Homomorphic Encryption (MKFHE), based on the Learning With Error assumption (LWE), usually lifts ciphertexts of different users to new ciphertexts under a common public key to enable homomorphic evaluation. The efficiency of the current Multi-key Fully Homomorphic Encryption (MKFHE) scheme is mainly restricted by two aspects:

1. **Expensive ciphertext expansion operation**: In a boolean circuit with input length N, multiplication depth L, security parameter λ, the number of additional encryptions introduced to achieve ciphertext expansion is $O(N\lambda^6 L^4)$.
2. **Noise flooding technology resulting in a large modulus** q: In order to prove the security of the scheme, the noise flooding technology introduced in the encryption and distributed decryption stages will lead to a huge modulus $q = 2^{O(\lambda L)} B_\chi$, which corrodes the whole scheme and leads to sub-exponential approximation factors $\gamma = \tilde{O}(n \cdot 2^{\sqrt{nL}})$.

This paper solves the first problem by presenting a framework called Key-Lifting Multi-key Fully Homomorphic Encryption (KL-MKFHE). With this *key lifting* procedure, the number of encryptions for a local user is reduced to $O(N)$, similar to single-key fully homomorphic encryption (FHE). For the second problem, based on Rényi divergence, we propose an optimized proof method that removes the noise flooding technology in the encryption phase. Additionally, in the distributed decryption phase, we prove that the asymmetric nature of the DGSW ciphertext ensures that the noise after decryption does not leak the noise in the initial ciphertext, as long as the depth of the circuit is sufficient. Thus, our initial ciphertext remains semantically secure even without noise flooding, provided the encryption scheme is leakage-resilient. This approach significantly reduces the size of the modulus q (with $\log q = O(L)$) and the computational overhead of the entire scheme.

Keywords: Multi-key homomorphic encryption · Rènyi divergence · Noise flooding · Leakage resilient cryptography.

1 Introduction

Fully Homomorphic Encryption (FHE**).** The concept of FHE was proposed by Rivest et al. [41] within a year of publishing the RSA scheme [42]. Gentry proposed the first truly fully homomorphic scheme in his doctoral dissertation [22] 2009. Based on Gentry's ideas, a series of FHE schemes have been proposed [23] [44] [11] [21] [24] [16] [15], and their security and efficiency have been continuously improved. FHE is suitable for the problem of unilaterally outsourcing computations. However, for multiple data providers, data must be encrypted by a common public key to support homomorphic evaluation. Due to privacy concerns, it is unreasonable to require participants to use other people's public keys to encrypt their data.

Threshold fully homomorphic encryption (Th-FHE**).** After introducing the first fully homomorphic encryption scheme, Gentry [22] also provided a corresponding strategy for multiple participants: first, all participants execute a secure multi-party computation protocol to obtain a common public key that encrypts all data. Then, ciphertext evaluation is performed. After completing the evaluation, all participants execute another secure MPC protocol to obtain the result. Initially, the threshold was added to FHE only to support multiple users. At the same time, the latter Th-FHE was more concerned with the flexibility of the access strategy in order to cope with different application scenarios.

In addition, two main ways exist to initialize the common public key of Th-FHE. First, assuming that there is a central authority which generates the common public key and disperses the private key (using a Secret Sharing scheme) to each participant [26] [9]. Encryption and evaluation of data are all under the common public key. When decryption is required, the set of participants that satisfy the access control structure obtains the result through a round of interactive decryption. Boneh et al. [9] further proposed the concept of the Universal Thresholdizer, which can convert any fully homomorphic encryption scheme into a threshold fully homomorphic encryption supporting monotonic access control structure in a black-box manner.

The second method is for the parties to generate the common public key in a distributed manner without a central authority. For example, Myers et al. [35] added a threshold functionality to the integer homomorphic scheme [20] and used a distributed manner to generate the common public key and private key without a central setup. Although adopting a black box method for the construction process, the distributed key generation process was quite complicated, which consisted of three steps: generating the private key, then the private key of the squeezed circuit, and finally, the common public key. These three processes are all needed to invoke the distributed bit generation repeatedly, the comparison, and the multiplication protocols. Based on the key homomorphic property, Asharov et al. [6] generated the common public key through two rounds of interaction in a distributed manner, and the common private key was the sum of the individual private keys. In order to match the public and private keys and ensure the security of the private key, a common reference string (CRS) needed to be

introduced. Decryption required everyone to provide the private key, which was actually a (n-n)-threshold fully homomorphic encryption (Th-FHE). Damgård et al. [19] introduced homomorphic encryption in order to optimize the preprocessing stage (such preprocessing was typically based on the classic circuit randomization technique of Beaver [8]). A common reference string also needed to be introduced.

Multi-key Fully Homomorphic Encryption (MKFHE**).** To deal with the privacy of multiple data providers, López-Alt et al. [27] proposed the concept of MKFHE and constructed the first MKFHE scheme based on modified-NTRU [43]. Conceptually, it enhances the functionality of FHE by allowing data providers to encrypt data independently from other participants. Key generation and data encryption is done locally. To obtain the evaluated result, all participants are required to execute a round of threshold decryption protocol.

After López-Alt et al. proposed the concept of MKFHE, many schemes were developed. In 2015, Clear and McGoldrick [17] constructed the LWE-based MKFHE scheme. This scheme defines the common private key as concatenating all private keys. It constructs a masking scheme to convert ciphertext under individual public keys to the common public key by introducing CRS and circular-LWE assumptions. However, this scheme only supports single-hop computation. In 2016, Mukherjee and Wichs [34], Peikert and Shiehian [38], and Brakerski and Perlman [13] constructed MKFHE schemes based on GSW, respectively. Mukherjee and Wichs [34] simplified the masking scheme of [17] and focused on constructing a two-round MPC protocol. Different methods in [38] and [13] were proposed delicately to construct a multi-hop MKFHE. Brakerski and Perlman [13] introduced bootstrapping to realize ciphertext expansion, thereby achieving the multi-hop functionality. Peikert and Shiehian [38] realized the multi-hop function through an ingenious construction. It is worth mentioning that all MKFHE schemes constructed based on LWE require a ciphertext expansion procedure.

1.1 Motivation

The biggest difference between Th-FHE and MKFHE in form is that MKFHE allows participants to encrypt data with their public keys and does not require interaction during the initialization phase. At the same time, Th-FHE needs to introduce a dealer or generate the common key pair in a distributed manner. Conceptually, it is clear that MKFHE is more concise, and a series of work [5, 12, 34] showed that MKFHE was an excellent base tool for building round-optimal MPC. However, despite looking attractive, the construction of MKFHE involves some cumbersome operations and unavoidable assumptions. Below we describe some details of the MKFHE scheme and state our goal in the last paragraph of this subsection.

Ciphertext expansion is expensive: Although the MKFHE based on LWE can use LHL to remove CRS, in order to convert the ciphertext under different

keys to the ciphertext under the same key (ciphertext expansion procedure), participants and the computing server need to do much preparatory work. For ciphertext expansion, it is necessary to encrypt the random matrix $\mathbf{R} \in \mathbb{Z}_q^{m \times m}$ of each ciphertext. For a boolean circuit with an input length of N, multiplication depth of L, and security parameter of λ, where $m = n \log q + \omega(\log \lambda)$, the additional encryption operation introduced is $O(N\lambda^6 L^4)$, in contrast to $O(N)$ for single-key FHE. For computing-sensitive participants, this is much overhead.

CRS looks inevitable: Due to the compact structure of the polynomial ring and some fascinating parallel algorithms such as SIMD, it is generally believed that FHE scheme based on RLWE is more efficient than FHE based on LWE. This is why most current MKFHE schemes, such as [14,33], are constructed based on RLWE.

Leftover Hash Lemma (LHL) over integer ring $\mathbb{Z}$ enjoys the leakage resilient property: It can transform an average quality random sources into higher quality [25] which can be used to get rid of CRS as [12] does. However, regularity lemma [29] over polynomial rings do not have corresponding properties, as [18] mentioned if the j-th Number theoretical Transfer (NTT) coordinate of each ring element in $\mathbf{x} = (x_1, \ldots, x_l)$ is leaked, then the j-th NTT coordinate of $a_{l+1} = \sum a_i x_i$ is defined, so a_{l+1} is very far from uniform, yet this is only a 1/n leakage rate. Therefore, it seems to be more difficult to remove CRS for RLWE-based MKFHE.

Noise flooding leads to extremely large module q : As far as we know so far, whether it is MKFHE or Th-FHE, such as [12] [34] [17] [13] [6], a great noise needs to be introduced in encryption phase to ensure the security or the distributed decryption phase to cover up the partial decryption result, otherwise, the private key may be leaked. In order to make the result of partial decryption simulatable, assuming that the noise accumulated after the evaluation is $\mathbf{e}_{eval}$ and the private key is $\mathbf{s}$, the flooding noise e_{sm} must satisfy $\langle \mathbf{e}_{eval}, \mathbf{s} \rangle / e_{sm} = \mathsf{negl}(\lambda)$. At this time, in order to ensure the correctness of the decryption result, module q needs to satisfy $q \geq 4e_{sm}$. Thus noise flooding results in a q exponentially larger than the q in a single-key FHE. Typically, in [34], the flooding noise $e_{sm} = 2^{O(L\lambda \log \lambda)} B_\chi$, the modulus $q = 2^{\omega(L\lambda \log \lambda)} B_\chi$, and the corresponding approximation factor of GapSVP_γ is $\gamma = \tilde{O}(n \cdot 2^{\lambda L})$ (which is sub-exponential in n by replacing $\lambda = \sqrt{n/L}$)[3].

Therefore, although conceptually attractive, MKFHE as a general framework is not suitable for some specific scenarios. Especially in the mobile Internet era, data providers often do not trust others, and sometimes it is not easy to convince them there is a dealer or the randomness of common reference string generated by a third party. At the same time, it is unreasonable to require the data provider to do $O(N\lambda^6 L^4)$ such a large number of encryption on the personal terminal.

[3] To achieve 2^λ security against known lattice attacks, one must have $n = \Omega(\lambda \log q / B_\chi)$

Our goal : In response to the above problems, we propose our goal: we consider *trust-sensitive* and *computationally-sensitive* scenarios with multi-users.

- Without CRS : we **do not assume** the existence of a dealer or a common reference string
- Data providers does **as many encryptions as the single-key homomorphic scheme**($O(N)$ for the circuit with input length N).
- $q = 2^{O(L)}B_\chi$ of **the same size as the single-key homomorphic scheme**, while $q = 2^{O(\lambda L)}B_\chi$ for those schemes introduced noise flooding.

1.2 Related works

Except sum type of key structure [6], concatenation structure were studied in [17] [38] [34] [13] [14] together with CRS. Ananth et al. [4] removed CRS from a higher dimension; instead of using LHL or regularity lemma, they based on Multiparty Homomorphic Encryption and modified the initialization method of its root node to achieve this purpose; more details, please refer to [4]. Brakerski et al. [12] was the first scheme using the leakage resilient property of LHL to get rid of CRS, which had the concatenation common private key structure, and ciphertext expansion was essential. All of the above schemes introduced noise flooding technology in distributed decryption phase.

Recently, the work [3] has proposed an alternative approach: instead of removing it, they proposed the concept of accountability of CRS, that is, the generator of CRS should be responsible for its randomness; otherwise, the challenging party can provide a publicly verifiable proof that certifies the authority's misbehaviour. We believe this could be an effective means of balancing authority.

We present a comparison of some properties in related work in Table 1.

Table 1. Scheme property comparison

Scheme	Key structure	CRS	Noise flooding	Interaction(setup phase)
THFHE [6]	S	✓	✓	✓
MKFHE [14]	C	✓	✓	×
MKFHE [34]	C	✓	✓	×
MKFHE [12]	C	✓	✓	✓
Our scheme	S	×	×	✓

S" and "C" in the column of Key structure represent the sum and concatenated key structure, respectively. ✓ indicates that the corresponding operation or assumption needs to be introduced, or × indicates that it is not required.

1.3 Our Results

For *trust-sensitive* and *computationally-sensitive* scenarios, we introduce the concept of KL-MKFHE, which is more suitable for such scenarios. Following this

concept, we construct the first KL-MKFHE scheme based on LWE in the plain model.

We briefly introduce the new concept and our scheme below and explain how we remove noise flooding in the encryption and distributed decryption phases, respectively.

The concept of KL-MKFHE : Different from previous definition [34], we abandon ciphertext expansion procedure, instead, introducing a *key lifting* procedure which at a lower cost. Informally, key-lifting is an interactive protocol. The input is the key pair of all participants. After the protocol is executed, the "lifted" key pair is output, called the hybrid key, which has such properties:

- *Everyone's hybrid key is different.*
- *The ciphertext encrypted by different hybrid keys supports homomorphic evaluation.*

In addition to the properties that are required by MKFHE, such as *Correctness, Compactness, Semantic security*, KL-MKFHE should satisfy the following three additional properties :

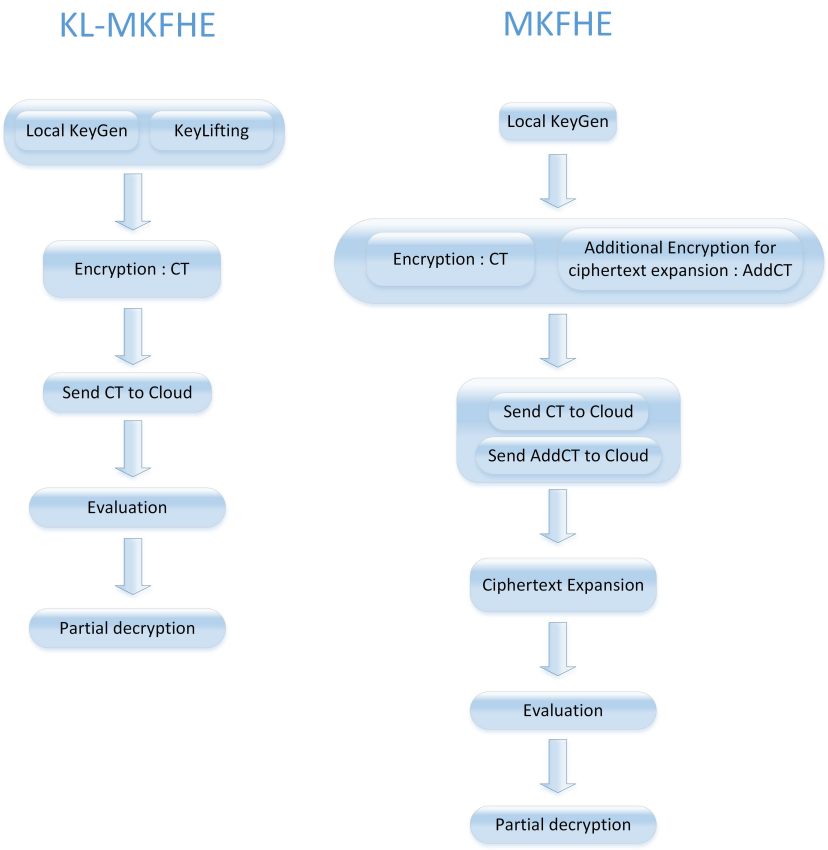
- **Plain model :** *No trusted setup or Common Reference String*
- **Locally Computationally Compactness :** *For a computational task corresponds to a Boolean circuit with an input length of N, a KL-MKFHE scheme is locally computationally compact if the participants do $O(N)$ encryptions as the single-key* FHE *scheme.*
- **Low round complexity :** *Only two round interaction is allowed in the Key lifting procedure.*

For comparison with MKFHE, we describe the procedure of MKFHE and KL-MKFHE in Fig 1. Please refer to Section 4 for more detailed definitions.

Optimized security proof method based on Rényi divergence : In order to prove the security of a scheme, a routine is to construct an instance of the scheme from a well-known hard problem instance. Unfortunately, sometimes, the process does not go so smoothly. To make the constructed distribution statistically indistinguishable from the target distribution, you need to add noise distribution to smooth the gap between the two; this is where noise flooding comes into play. For example, [6] [12] adopted this method to prove security. Unfortunately, the added noise tends to be significant, reducing the scheme's efficiency.

Shi et al. [7] pointed out that Rényi divergence can also be used to distinguish problems: they proved that, under certain conditions, if there is an algorithm that can distinguish problem P, then there is an algorithm that can distinguish problem P'. Note that it does not require that the P problem is indistinguishable from P', which is where the Rényi divergence comes into play. Based on the result of [7, Theorem 4.2], our proof method is as follows :

Fig. 1. The procedures of MKFHE and KL-MKFHE



1. Define the P problem as distinguishing our scheme's ciphertext from a uniform distribution.
2. Prove that for a given hard problem instance I, there exists a distribution $\mathcal{D}$, and a sample x of $\mathcal{D}$ can be constructed from this instance I(It is not easy to construct a sample x of distribution $\mathcal{D}$ from instance I. For this reason, we also introduce a new hardness problem, called the "interactive LWE " problem, and give a reduction(see Section 3). We find this problem interesting and believe it will also be useful in other ways.)
3. Define the P' problem as distinguishing $\mathcal{D}$ from a uniform distribution

Thus, if there is an adversary who can distinguish the P problem, then he can distinguish the P' problem and can also distinguish the hard problem instance I from the uniform distribution.

We believe that this Rényi divergence-based proof method provides an alternative idea for those proofs that must introduce strong assumptions and large noise to ensure security. For example, we give the optimal proof method for the leakage-resilient of the DGSW scheme in Appendix C(without introducing large noise). For more details, please refer to Section 5.4.

Leakage resistance implies a smaller q : We noticed that the distributed decryption of the MKFHE will leak the noise accumulated after the homomorphic evaluation and the decryptor's private key. In order to ensure security, previous MKFHE, such as [6] [14] [34] [12], will add some additional noise to the distributed decryption results to cover up this part of the information. Because we only care about the security of the initial ciphertext (note that the noise after the homomorphic evaluation will leak the privacy of the circuit), as long as it can be proved that the noise of distributed decryption is independent of the noise in the initial ciphertext, and our scheme is anti-leakage, then even without adding additional noise, the semantic security of the initial ciphertext can be guaranteed.

For the Dual GSW-like scheme, we noticed that the noise after its homomorphic multiplication is very regular: let $\mathbf{C}_{\mathsf{mult}} = \mathbf{C}_1 \cdot \mathbf{G}^{-1}(\mathbf{C}_2)$, the noise in $\mathbf{C}_{\mathsf{mult}}$ hardly contains the noise in $\mathbf{C}_2$. Assuming that the initial ciphertext is $\{\mathbf{C}_i\}_{i\in[N]}$, and the circuit multiplication depth is L, as long as $L \geq \log N$, then the noise in the ciphertext $\mathbf{C}_L$ of the L-th layer only contains the noise of a certain initial ciphertext. At this point, just multiply $\mathbf{C}_L$ by a ciphertext $Enc(1)$ whose plaintext is 1, let $\mathbf{C}_{\mathsf{clear}} = Enc(1)\mathbf{G}^{-1}(\mathbf{C}_L)$, then the noise in $\mathbf{C}_{\mathsf{clear}}$ does not contain any noise information in the initial ciphertext $\{\mathbf{C}_i\}_{i\in[N]}$. At this point, the distributed decryption of $\mathbf{C}_{\mathsf{clear}}$ will only leak the decryptor's private key.

Suppose our scheme is anti-leakage and predicts the amount of private key leakage in the distributed decryption process in advance. In that case, we only need to cover this part of the leakage amount when the parameters are initialized. Even if no noise is added in the distributed decryption process, it can guarantee the semantic security of the initial ciphertext. The disadvantage is that the complexity of our scheme could be more circuit-dependent. However,

there is no noise flooding in encryption and distributed decryption, so we can set $q = 2^{O(L)}B_\chi$ to be the same size as the single-key homomorphic scheme, where $q = 2^{O(\lambda L)}B_\chi$ in [6] [34] with noise flooding technology(Correspondingly, the approximation factor of Gapsvp_γ is reduced to $\gamma = \tilde{O}(n \cdot 2^L)$). Refer to Section 5.5 for a detailed discussion.

Our scheme: LWE-based KL-MKFHE under plain model :

Our scheme is based on the LWE assumption. The common private key is the sum of the private keys of all participants. Previous MKFHE or Th-FHE schemes [32] [6] adopt this key, all based on the CRS model. Without CRS, our solution is simpler and more efficient in construction. For a circuit with an input length N, our scheme requires local users to perform $O(N)$ encryption operations, while it is $O(N\lambda^6 L^4)$ for those schemes that require ciphertext expansion.

We give a comparison with schemes [12] [38] [6] in Table 2. Please refer to Section 5 for detailed security and parameters.

Table 2. Scheme complexity comparison

Scheme	Space			Time	Interaction(setup phase)	CRS
	PubKey + EvalKey	Ciphertext	Module q	Extra encryption		
MKFHE [38]	$\tilde{O}(\lambda^6 L^4(k + N\lambda^3 L^2))$	$\tilde{O}(Nk^2\lambda^6 L^4)$	$2^{O(\lambda L)}B_\chi$	$\tilde{O}(N\lambda^{14}L^9)$	×	✓
MKFHE [12]	$\tilde{O}(k^4\lambda^{15}L^{11})$	$\tilde{O}(Nk^4\lambda^8 L^6)$	$2^{O(\lambda L)}B_\chi$	$\tilde{O}(Nk^3\lambda^{15}L^{10})$	2 rounds	×
Th-FHE [6]	$\tilde{O}(\lambda^6 L^4)$	$\tilde{O}(N\lambda^6 L^4)$	$2^{O(\lambda L)}B_\chi$	×	1 rounds	✓
Our scheme	$\tilde{O}((k\lambda L + W)\lambda L^3)$	$\tilde{O}(N(k\lambda L + W)^2 L^4)$	$2^{O(L)}B_\chi$	×	2 rounds	×

The notation $\tilde{O}$ hides logarithmic factors. The "Space" column denotes the bit size of public, evaluation key and ciphertext; the "Extra encryption" column denotes the number of multiplication operations over $\mathbb{Z}_q$; λ denotes the security parameter, k participants number, B_χ the initial LWE noise; N, L, W denotes the input length, depth, and output length of the circuit respectively. In [38] [12] [6], n represents the dimension of the LWE problem in order to compare under the same security level, we replace n with the expression in terms of λ and L. To achieve 2^λ security against known lattice attacks, one must have $n = \Omega(\lambda \log q/B_\chi)$. For our parameter settings $q = 2^{O(L)}B_\chi$, thus we would have $n = \Omega(\lambda L)$, while $n = \Omega(\lambda^2 L)$ for the previous scheme with noise flooding.

1.4 Roadmap:

In Section2, we define some symbols and list some commonly used definitions and results. In Section3, we define a new problem. In Section4, we define KL-MKFHE. In Section5, we construct the first KL-MKFHE scheme based on LWE.

2 Preliminaries

2.1 Notation:

We define the relevant notations in Table 3. Let $\mathsf{negl}(\lambda)$ be a negligible function

Table 3.

λ	security parameter	n	dimension of LWE problem
k	number of participants	d	degree of RLWE problem
N	circuit input length	q	module base
L	circuit multiplicative depth		
W	circuit output length		

parameterized by λ. Lowercase bold letters such as $\mathbf{v}$, unless otherwise specified, represent vectors. Vectors are row vectors by default, and matrices are represented by uppercase bold letters such as $\mathbf{M}$. $[k]$ denotes the set of integers $\{1, \ldots, k\}$. If X is a distribution, then $a \leftarrow X$ denotes that value a is chosen according to the distribution X, or a finite set, then $a \leftarrow U(X)$ denotes that the value of a is uniformly sampled from X. Let $\Delta(X, Y)$ denote the statistical distance of X and Y. For two distributions X, Y, we use $X \overset{\mathsf{stat}}{\approx} Y$ to represent X and Y are statistically indistinguishable, while $X \overset{\mathsf{comp}}{\approx} Y$ are computationally indistinguishable.

In order to decompose elements in $\mathbb{Z}_q$ into binary, we review the Gadget matrix [30] [2] here. Let $\mathbf{G}^{-1}(\cdot)$ be the computable function that for any $\mathbf{M} \in \mathbb{Z}_q^{m \times n}$, it holds that $\mathbf{G}^{-1}(\mathbf{M}) \in \{0,1\}^{ml \times n}$, where $l = \lceil \log q \rceil$. Let $\mathbf{g} = (1, 2, \ldots, 2^{l-1}) \in \mathbb{Z}_q^l$, $\mathbf{G} = \mathbf{I}_m \otimes \mathbf{g} \in \mathbb{Z}_q^{m \times ml}$, it satisfies $\mathbf{G}\mathbf{G}^{-1}(\mathbf{M}) = \mathbf{M}$.

2.2 Some background in probability theory

Definition 1 *A distribution ensemble $\{\mathcal{D}_n\}_{n \in [N]}$ supported over integer, is called B-bounded if :*

$$\Pr_{e \leftarrow \mathcal{D}_n}[\,|e| > B\,] = \mathsf{negl}(n).$$

Lemma 1 (Smudging lemma [6]) *Let $B_1 = B_1(\lambda)$, and $B_2 = B_2(\lambda)$ be positive integers and let $e_1 \in [-B_1, B_1]$ be a fixed integer, let $e_2 \in [-B_2, B_2]$ be chosen uniformly at random, Then the distribution of e_2 is statistically indistinguishable from that of $e_2 + e_1$ as long as $B_1/B_2 = \mathsf{negl}(\lambda)$.*

Theorem 1 ([28, Theorem 5.3.2]) *Let $0 \leq t \leq m$. Then the probability that out of $2m$ coin tosses, the number of heads is less than $m-t$ or large than $m+t$, is at most $e^{-t^2/(m+t)}$.*

The Rènyi divergence (in [7]) : For any two discrete probability distributions P and Q such that $\mathsf{Supp}(P) \subseteq \mathsf{Supp}(Q)$ where $\mathsf{Supp}(P) = \{x : P(x) \neq 0\}$ and $a \in (1, +\infty)$, The Rènyi divergence of order a is defined by :

$$R_a(P||Q) = \left(\sum_{x \in \mathsf{Supp}(P)} \frac{P(x)^a}{Q(x)^{a-1}} \right)^{\frac{1}{a-1}}$$

Omitting the a subscript when $a = 2$, defining the The Rènyi divergence of order 1 and $+\infty$ by :

$$R_1(P||Q) = \exp\left(\sum_{x \in \mathsf{Supp}(P)} P(x) \log \frac{P(x)}{Q(x)} \right)$$

$$R_\infty(P||Q) = \max_{x \in \mathsf{Supp}(P)} \frac{P(x)}{Q(x)}.$$

The definitions are extended naturally to continuous distributions. The divergence R_1 is the (exponential of) the Kullback-Leibler divergence.

Theorem 2 ([7, Theorem 4.2]) *Let Φ, Φ' denote two distribution with $\mathsf{Supp}(\Phi) \subseteq \mathsf{Supp}(\Phi')$, and $D_0(r)$ and $D_1(r)$ denote two distributions determined by some parameter $r \in \mathsf{Supp}(\Phi')$. Let P, P' be two decision problems defined as follows :*

- *Problem P: distinguish whether input x is sampled from distribution X_0 or X_1, where*

$$X_0 = \{x : r \hookleftarrow \Phi, x \hookleftarrow D_0(r)\}, \qquad X_1 = \{x : r \hookleftarrow \Phi, x \hookleftarrow D_1(r)\}.$$

- *Problem P': distinguish whether input x is sampled from distribution X'_0 or X'_1, where*

$$X'_0 = \{x : r \hookleftarrow \Phi', x \hookleftarrow D_0(r)\}, \qquad X'_1 = \{x : r \hookleftarrow \Phi', x \hookleftarrow D_1(r)\}.$$

Assume that $D_0(\cdot)$ and $D_1(\cdot)$ satisfy the following public sampleability property: there exists a sampling algorithm S with run-time T_S such that for all (r, b), given any sample x from $D_b(r)$:

- *$S(0, x)$ outputs a fresh sample distributed as $D_0(r)$ over the randomness of S,*
- *$S(1, x)$ outputs a fresh sample distributed as $D_1(r)$ over the randomness of S.*

Then, given a T-time distinguisher $\mathcal{A}$ for problem P with advantage ϵ, we can construct a distinguisher $\mathcal{A}'$ for problem P' with run-time and distinguishing advantage, respectively, bounded from above and below by (for any $a \in (1, +\infty]$):

$$\frac{64}{\epsilon^2} \log\left(\frac{8R_a(\Phi||\Phi')}{\epsilon^{a/(a-1)+1}} \right) \cdot (T_S + T) \qquad \text{and} \qquad \frac{\epsilon}{4 \cdot R_a(\Phi||\Phi')} \cdot \left(\frac{\epsilon}{2}\right)^{\frac{a}{a-1}}.$$

2.3 Gaussian distribution on Lattice

Definition 2 *Let $\rho_s(\mathbf{x}) = \exp(-\pi||\mathbf{x}/s||^2)$ be a Gaussian function scaled by a factor of $s > 0$. Let $\Lambda \subset \mathbb{R}^n$ be a lattice, and $\mathbf{c} \in \mathbb{R}^n$. The discrete Gaussian distribution $D_{\Lambda+\mathbf{c},s}$ with support $\Lambda + \mathbf{c}$ is defined as :*

$$D_{\Lambda+\mathbf{c},s}(\mathbf{x}) = \frac{\rho_s(\mathbf{x})}{\rho_s(\Lambda + \mathbf{x})}$$

Smoothing parameter : We recall the definition of the smoothing parameter from [31].

Definition 3 *For a lattice Λ and positive real $\epsilon > 0$, the smoothing parameter $\eta_\epsilon(\Lambda)$ is the smallest real $r > 0$ such that $\rho_{1/r}(\Lambda^* \backslash \{\mathbf{0}\}) \leq \epsilon$.*

Lemma 2 (Special case of [31, Lemma 3.3]) *For any $\epsilon > 0$,*

$$\eta_\epsilon(\mathbb{Z}^n) \leq \sqrt{\ln(2n(1 + 1/\epsilon))/\pi}.$$

In particular, for any $\omega(\sqrt{\log n})$ function, there is a negligible $\epsilon = \epsilon(n)$ such that $\eta_\epsilon(\mathbb{Z}^n) \leq \omega(\sqrt{\log n})$.

Lemma 3 (Simplified version of [37, Theorem 3.1]) *Let $\epsilon > 0, r_1, r_2 > 0$ be two Gaussian parameters, and $\Lambda \subset \mathbb{Z}^m$ be a lattice. If $\frac{r_1 r_2}{\sqrt{r_1^2+r_2^2}} \geq \eta_\epsilon(\Lambda)$, then*

$$\Delta(\mathbf{y}_1 + \mathbf{y}_2, \mathbf{y}') \leq 8\epsilon$$

where $\mathbf{y}_1 \leftarrow \mathcal{D}_{\Lambda,r_1}$, $\mathbf{y}_2 \leftarrow \mathcal{D}_{\Lambda,r_2}$, and $\mathbf{y}' \leftarrow \mathcal{D}_{\Lambda,\sqrt{r_1^2+r_2^2}}$.

Lemma 4 ([1]) *Let χ denote the Gaussian distribution with standard deviation σ and mean zero. Then, for all $C > 0$, it holds that:*

$$\Pr[e \leftarrow \chi : |e| > C \cdot \sigma] \leq \frac{2}{C\sqrt{2\pi}} \exp\{-\frac{C^2}{2}\}.$$

2.4 The Learning With Error(LWE) Problem

The Learning With Error problem was introduced by Regev [40].

Definition 4 (Decision-LWE) *Let λ be security parameter, for parameters $n = n(\lambda)$ be an integer dimension, $q = q(\lambda) > 2$ be an integer, and a distribution $\chi = \chi(\lambda)$ over $\mathbb{Z}$, the $\mathsf{LWE}_{n,q,\chi}$ problem is to distinguish the following distribution:*

- *$\mathcal{D}_0$: the jointly distribution $(\mathbf{A}, \mathbf{z}) \in (\mathbb{Z}_q^{m\times n} \times \mathbb{Z}_q^n)$ is sampled by $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m\times n})$ $\mathbf{z} \leftarrow U(\mathbb{Z}_q^n)$*
- *$\mathcal{D}_1$: the jointly distribution $(\mathbf{A}, \mathbf{b}) \in (\mathbb{Z}_q^{m\times n} \times \mathbb{Z}_q^n)$ is computed by $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m\times n})$ $\mathbf{b} = \mathbf{sA} + \mathbf{e}$, where $\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$ $\mathbf{e} \leftarrow \chi^m$*

As shown in Regev [40] [36], the $\mathsf{LWE}_{n,q,\chi}$ problem with χ being discrete Gaussian distribution with parameter $\sigma = \alpha q \geq 2\sqrt{n}$ is at least as hard as approximating the shortest independent vector problem(SIVP) to within a factor of $\gamma = \tilde{O}(n/\alpha)$ in *worst case* dimension n lattices. It leads to the Decision-$\mathsf{LWE}_{n,q,\chi}$ assumption $\mathcal{D}_0 \overset{\mathsf{comp}}{\approx} \mathcal{D}_1$.

2.5 Dual-GSW(DGSW) Encryption scheme

The DGSW scheme [12] and GSW scheme are similar to the Dual-Regev scheme and Regev scheme resp. Which is defined as follows:

- $\mathsf{pp} \leftarrow \mathsf{Gen}(1^\lambda, 1^L)$: For a given security parameter λ, circuit depth L, choose an appropriate lattice dimension $n = n(\lambda, L)$, $m = n\log q + \omega(\lambda)$, a discrete Gaussian distribution $\chi = \chi(\lambda, L)$ over $\mathbb{Z}$, which is bounded by B_χ, module $q = \mathsf{poly}(n) \cdot B_\chi$, Output $\mathsf{pp} = (n, m, q, \chi, B_\chi)$ as the initial parameters.
- $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{KeyGen}(\mathsf{pp})$: Let $\mathsf{sk} = \mathbf{t} = (-\mathbf{s}, 1)$, $\mathsf{pk} = (\mathbf{A}, \mathbf{b})$, where $\mathbf{s} \leftarrow U(\{0,1\}^{m-1})$, $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m-1\times n})$, $\mathbf{b} = \mathbf{sA} \mod q$.
- $\mathbf{C} \leftarrow \mathsf{Enc}(\mathsf{pk}, u)$: Input public key pk and plaintext $u \in \{0,1\}$, choose a random matrix $\mathbf{R} \leftarrow U(\mathbb{Z}_q^{n\times w})$, $w = ml$, $l = \lceil \log q \rceil$ and an error matrix $\mathbf{E} \leftarrow \chi^{m\times w}$, Output the ciphertext :

$$\mathbf{C} = \begin{pmatrix} \mathbf{A} \\ \mathbf{b} \end{pmatrix} \mathbf{R} + \mathbf{E} + u\mathbf{G}$$

  where $\mathbf{G}$ is a gadget Matrix.
- $u \leftarrow \mathsf{Dec}(\mathsf{sk}, \mathbf{C})$: Input private key sk, ciphertext $\mathbf{C}$, let $\mathbf{w} = (0, \ldots, \lceil q/2 \rceil) \in \mathbb{Z}_q^m$, $v = \langle \mathbf{tC}, \mathbf{G}^{-1}(\mathbf{w}^T) \rangle$, output $u' = \lceil \frac{v}{q/2} \rceil$.

Leak resistance : Brakerski *et al* proved in [12] that DGSW is leak-resistant. Informally, even if part of the private key of the DGSW scheme is leaked, the DGSW ciphertext is still semantically secure. As Lemma 5 says :

Lemma 5 ([12]) *Let χ be LWE noise distribution bounded by B_χ, χ' a distribution over $\mathbb{Z}$ bounded by $B_{\chi'}$, satisfying $B_\chi / B_{\chi'} = \mathsf{negl}(\lambda)$. Let $\mathbf{A}_i \in \mathbb{Z}_q^{(m-1)\times n}$ be uniform, and let $\mathbf{A}_j$ for all $j \neq i$ be chosen by a rushing adversary after seeing $\mathbf{A}_i$. Let $\mathbf{s}_i \leftarrow \{0,1\}^{m-1}$ and $\mathbf{b}_{i,j} = \mathbf{s}_i\mathbf{A}_j$. Let $\mathbf{r} \in \mathbb{Z}_q^n$ be uniform, $\mathbf{e} \leftarrow \chi^{m-1}$, $e' \leftarrow \chi'$. Then under the LWE assumption, the vector $\mathbf{c} = \mathbf{A}_i\mathbf{r} + \mathbf{e}$ and number $c' = \langle \mathbf{b}_{i,i}, \mathbf{r} \rangle + e'$ are (jointly) pseudorandom, even given the $\mathbf{b}_{i,j}$'s for all $j \in [k]$ and the view of the adversary that generated the $\mathbf{A}_j$'s.*

Remark : Note that in the proof of [12], the condition for the establishment of Lemma 5 is $|\mathbf{e}/e'| = \mathsf{negl}(\lambda)$. We point out that this condition is not required with our analytical method. We prove it in the Appendix C.

2.6 Multi-Key Fully Homomorphic Encryption

We review the definition of MKFHE in detail here, the main purpose of which is to compare with the definition of KL-MKFHE proposed later.

Definition 5 *Let λ be the security parameter, L be the circuit depth, and k be the number of participants. A levelled multi-key fully homomorphic encryption scheme consists of a tuple of efficient probabilistic polynomial time algorithms* MKFHE=(Init, Gen, Enc, Expand, Eval, Dec) *which defines as follows.*

- $\mathsf{pp} \leftarrow \mathsf{Init}(1^\lambda, 1^L)$ *: Input security parameter λ, circuit depth L, output system parameter* pp*. We assume that all algorithms take* pp *as input.*
- $(\mathsf{pk}_i, \mathsf{sk}_i) \leftarrow \mathsf{Gen}(\mathsf{pp}, \mathsf{crs})$ *: Input* pp*, common reference string* crs *(generated by a third party or random oracle), output a key pair for participant i.*
- $c_i \leftarrow \mathsf{Enc}(\mathsf{pk}_i, u_i)$ *: Input pk_i and plaintext u_i, output ciphertext c_i.*
- $v_i \leftarrow \mathsf{Enc}(\mathsf{pk}_i, r_i)$*: Input pk_i and the random r_i used in ciphertext c_i, output auxiliary ciphertext v_i.*
- $\bar{c}_i \leftarrow \mathsf{Expand}(\{\mathsf{pk}_i\}_{i\in[k]}, v_i, c_i)$*:Input the ciphertext c_i of participant i, the public key set $\{\mathsf{pk}_i\}_{i\in[k]}$ of all participants, auxiliary ciphertext v_i, output expanded ciphertext $\bar{c}_i$ which is under $f(\mathsf{sk}_i, \ldots \mathsf{sk}_k)$ whose structure is undefined.*
- $\bar{c}_{eval} \leftarrow \mathsf{Eval}(\mathcal{S}, \mathcal{C})$*:Input circuit $\mathcal{C}$, the set of all ciphertext $\mathcal{S} = \{\bar{c}_i\}_{i\in[N]}$ while N is the input length of circuit $\mathcal{C}$, output evaluated ciphertext $\bar{c}_{eval}$*
- $u \leftarrow \mathsf{Dec}(\bar{c}_{eval}, f(\mathsf{sk}_1 \ldots \mathsf{sk}_k))$ *: Input evaluated ciphertext $\bar{c}_{eval}$, private key function $f(\mathsf{sk}_1 \ldots \mathsf{sk}_k)$, output u (This is usually a distributed process).*

Remark : Although the definition of MKFHE in [27] does not contain auxiliary ciphertext v_i and ciphertext expansion procedure, in fact, the works [34] [39] [17] include this procedure to support homomorphic operations. This procedure seems essential; we list it here for comparison with KL-MKFHE. The common private key depends on $\{\mathsf{sk}_i\}_{i\in[k]}$, f is a certain function, which is not unique; for example, it can be the concatenation of all keys or the sum of all keys.

Properties implicated in the definition of MKFHE : For the above definition, each participant is required in the key generation and encryption phase independently to generate their keys and complete the encryption operation without interaction between participants. These two phases are similar to single-key homomorphic encryption; the computational overhead is independent of k and only related to λ and L. Only in the decryption phase interaction is involved when participants perform a round of decryption protocol.

3 The "interactive LWE " problem

This section introduces a new hardness problem called the "interactive LWE " problem. This problem can be seen as a variant of the standard LWE problem, where the interactive process reveals the linear relationship between the secret and the noise in standard LWE. We prove this problem is similar to the low

dimensional LWE problem. This new problem is introduced because we will use it in the optimization proof method based on Rènyi divergence. Furthermore, as a standalone result, it will also be useful elsewhere.

Definition 6 (Interactive LWE) *Let λ be security parameter, for parameters $n = n(\lambda)$ be an integer dimension, $q = q(\lambda) > 2$ be an integer, and a distribution $\chi = \chi(\lambda)$ over $\mathbb{Z}$, the matrix version of the $\mathsf{LWE}_{n,q,\chi}$ samples is $(\mathbf{A}, \mathbf{B} = \mathbf{A} \cdot \mathbf{R} + \mathbf{E})$, where $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m \times n})$, $\mathbf{R} \leftarrow U(\mathbb{Z}_q^{n \times w})$, $\mathbf{E} \leftarrow \chi^{m \times w}$ satisfying $n|w$. Consider the following* Game.

1. *Challenger generates the matrix version $\mathsf{LWE}_{n,q,\chi}$ samples $(\mathbf{A}, \mathbf{B})$ and an uniform samples $(\mathbf{A}, \mathbf{U})$, send it to an adversary $\mathcal{A}$.*
2. *After receiving $(\mathbf{A}, \mathbf{B})$ and $(\mathbf{A}, \mathbf{U})$, $\mathcal{A}$ adaptively chooses $\mathbf{s} \in \mathbb{Z}_q^m$, and send it to Challenger.*
3. *After receiving $\mathbf{s}$, challenger computes $\{\mathbf{v}_i\}_{i \in [g]}$ by $\{\mathbf{v}_i \mathbf{R}_i = \mathbf{s}\mathbf{E}_i\}_{i \in [g]}$, where $\mathbf{R}_i \in \mathbb{Z}_q^{n \times n}$ and $\mathbf{E}_i \in \mathbf{Z}_q^{m \times n}$ are i-th block of $\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_g)$ and $\mathbf{E} = (\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_g)$ respectively, send $\{\mathbf{v}_i\}_{i \in [g]}$ to $\mathcal{A}$.*
4. *After receiving $\{\mathbf{v}_i\}_{i \in [g]}$, $\mathcal{A}$ try to distinguish $(\mathbf{A}, \mathbf{B}, \{\mathbf{v}_i\}_{i \in [g]}, \mathbf{s})$ and $(\mathbf{A}, \mathbf{U}, \{\mathbf{v}_i\}_{i \in [g]}, \mathbf{s})$*

Theorem 3 *For any probabilistic polynomial time adversary $\mathcal{A}$, If he can distinguish $(\mathbf{A}, \mathbf{B}, \{\mathbf{v}_i\}_{i \in [g]}, \mathbf{s})$ with $(\mathbf{A}, \mathbf{U}, \{\mathbf{v}_i\}_{i \in [g]}, \mathbf{s})$, then he can also distinguish $n - 1$ dimensional* LWE *samples. (we note that $(\mathbf{A}, \mathbf{B})$ are n dimensional* LWE *samples.)*

Proof. Note that exposing $\{\mathbf{v}_i\}_{i \in [g]}$ to $\mathcal{A}$ will reveal the linear relationship between $\mathbf{R}_i$ and $\mathbf{E}_i$. We need to ensure that after $\mathcal{A}$ gets $\{\mathbf{v}_i\}_{i \in [g]}$, $(\mathbf{A}, \mathbf{B})$ and $(\mathbf{A}, \mathbf{U})$ are still indistinguishable. Let us take a look at what is the distribution of $\{\mathbf{v}_i\}_{i \in [g]}$. For $\mathbf{v}_i \mathbf{R}_i = \mathbf{s}\mathbf{E}_i$,thus $\mathbf{v}_i = \mathbf{s}\mathbf{E}_i \mathbf{R}_i^{-1}$(We discuss in the Appendix A the probability that $\mathbf{R}_q^{n \times n}$ is reversible). For $\mathbf{R}_i$ is uniform and $\mathbf{E}_i$ is discrete Gaussian, so $\mathbf{E}_i \mathbf{R}_i^{-1}$ is uniform over $\mathbb{Z}_q^{m \times n}$. Therefore, when $\mathbf{s} \neq \mathbf{0}$, $\mathbf{v}_i$ is uniform random over $\mathbb{Z}_q^n$, that is, $\mathbf{v}_i$ and $\mathbf{s}$ are independent except at zero.

Let $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$, where $\mathbf{a}_i$ is the i-th column of $\mathbf{A}$. Let $\mathbf{r} = (r_1, \cdots, r_n)^T$, $\mathbf{e} = (e_1, \cdots, e_m)^T$ be the first column of $\mathbf{R}_1$, $\mathbf{E}_1$ respectively. Let $\mathbf{b} = (b_1, \cdots, b_m)^T$ be the first column of the $\mathbf{B}$. Let $\mathbf{v}_1 = (v_1, \cdots, v_n)$, $\mathbf{s} = (s_1, \cdots, s_m)$. Consider the first column of the $\mathbf{B}$ and the first element of vector $\mathbf{v}_1 \mathbf{R}_1 = \mathbf{s}\mathbf{E}_1$, it holds that :

$$(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) \begin{pmatrix} r_1 \\ r_2 \\ \cdots \\ r_n \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}$$

$$v_1 r_1 + v_2 r_2 + \cdots + v_n r_n = s_1 e_1 + s_2 e_2 + \cdots + s_m e_m$$

We show that for any m LWE samples of $n - 1$ dimensions, After getting $\mathbf{s} = (s_1, s_2, \cdots, s_m)$ from $\mathcal{A}$. We can always construct a sample of $(\mathbf{A}, \mathbf{b}, \mathbf{v}_1, \mathbf{s})$.

For m LWE samples of $n-1$ dimensional and $\mathbf{s}$:

$$(\mathbf{a}'_1, \mathbf{a}'_2, \cdots, \mathbf{a}'_{n-1}) \begin{pmatrix} r'_1 \\ r'_2 \\ \cdots \\ r'_{n-1} \end{pmatrix} + \begin{pmatrix} e'_1 \\ e'_2 \\ \cdots \\ e'_m \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \\ \cdots \\ b'_m \end{pmatrix}$$
$$\mathbf{s} = (s_1, s_2, \cdots, s_m)$$

Let $w_0 = t\sum_{i=1}^{m} s_i e'_i$, where $t \leftarrow U(\mathbb{Z}_q)$, $\{w_i\}_{i\in[n-1]} \leftarrow U(\mathbb{Z}_q)$, $r_n = w_0 - \sum_{i=1}^{n-1} w_i r'_i$, $\mathbf{a}_n \leftarrow U(\mathbb{Z}_q^m)$, $\{\mathbf{a}_i = \mathbf{a}'_i + w_i \mathbf{a}_n\}_{i\in[n-1]}$

$$\begin{pmatrix} r_1 \\ r_2 \\ \cdots \\ r_{n-1} \end{pmatrix} = \begin{pmatrix} r'_1 \\ r'_2 \\ \cdots \\ r'_{n-1} \end{pmatrix} \qquad \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_m \end{pmatrix} = \begin{pmatrix} e'_1 \\ e'_2 \\ \cdots \\ e'_m \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \\ \cdots \\ b'_m \end{pmatrix} + w_0 \mathbf{a}_n$$

It holds that :

$$(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) \begin{pmatrix} r_1 \\ r_2 \\ \cdots \\ r_n \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_m \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_m \end{pmatrix}$$
$$t^{-1}w_1 r_1 + t^{-1}w_2 r_2 + \cdots + t^{-1}w_{n-1}r_{n-1} + t^{-1}r_n = s_1 e_1 + s_2 e_2 + \cdots + s_m s_m.$$

We note that r_n is independent of $\{r_i\}_{i\in[n-1]}$ for the reason that w_0 is uniform over $\mathbb{Z}_q$, and similarly $\{\mathbf{a}_i\}_{i\in[n]}$ are independent. Thus :

$$(\{\mathbf{a}_i\}_{i\in[n]}, \{b_i\}_{i\in[m]}, (t^{-1}w_1, t^{-1}w_2, \cdots, t^{-1}w_{n-1}, t^{-1}), \mathbf{s})$$

is a sample of $(\mathbf{A}, \mathbf{b}, \mathbf{v}_1, \mathbf{s})$. If there is an adversary can distinguish $(\mathbf{A}, \mathbf{b}, \mathbf{v}_1, \mathbf{s})$, then he can also distinguish $n-1$ dimensional LWE samples. Above, we have given the reduction of the first column of $\mathbf{B}$, the hardness of the matrix version $(\mathbf{A}, \mathbf{B}, \{\mathbf{v}_i\}_{i\in[g]}, \mathbf{s})$ for any $w = \mathsf{poly}(n)$ can be established from $(\mathbf{A}, \mathbf{b}, \mathbf{v}_1, \mathbf{s})$ via a routine hybrid-argument. ■

4 Key Lifting Multi-key Fully Homomorphic Encryption

In order to cope with *computationally-sensitive* and *trust-sensitive scenarios*, we avoid expensive ciphertext expansion procedures and introduce a relatively simple *Key lifting* procedure to replace it. In addition, a tighter bound is required on the amount of local computation and parameter size; as a compromise, we allow a small amount of interaction during *Key lifting*.

Definition 7 *A KL-MKFHE scheme is a tuple of probabilistic polynomial time algorithm* (Init, Gen, KeyLifting, Enc, Eval, Dec)*, which can be divided into two phases, online phase:* KeyLifting *and* Dec*, where interaction is allowed between*

participants, but the rounds should be constant, local phase: Init, Gen, Enc, *and* Eval, *whose operations do not involve interaction. These five algorithms are described as follows :*

- $\mathsf{pp} \leftarrow \mathsf{Init}(1^\lambda, 1^L)$*:Input security parameter* λ*, circuit depth* L*, output public parameters* pp.
- $(\mathsf{pk}_i, \mathsf{sk}_i) \leftarrow \mathsf{Gen}(\mathsf{pp})$*:Input public parameter* pp*, output the key pair of participant* i
- $\{\mathsf{hk}_i\}_{i\in[k]} \leftarrow \mathsf{KeyLifting}(\{\mathsf{pk}_i, \mathsf{sk}_i\}_{i\in[k]})$*: Input key pair* $\{\mathsf{pk}_i, \mathsf{sk}_i\}_{i\in[k]}$ *of all participants, output the hybrid key* $\{\mathsf{hk}_i\}_{i\in[k]}$ *of all* i*. (online phase: two-round interaction)*
- $c_i \leftarrow \mathsf{Enc}(\mathsf{hk}_i, u_i)$*: Input plaintext* u_i *and* hk_i*, output ciphertext* c_i
- $\hat{c} \leftarrow \mathsf{Eval}(\mathcal{C}, S)$*: Input circuit* $\mathcal{C}$*, ciphertext set* $S = \{c_i\}_{i\in[N]}$ *, output ciphertext* $\hat{c}$
- $u \leftarrow \mathsf{Dec}(\hat{c}, f(\mathsf{sk}_1 \dots \mathsf{sk}_k))$*: Input evaluated ciphertext* $\hat{c}$*,* $f(\mathsf{sk}_1 \dots \mathsf{sk}_k)$*, output* $\mathcal{C}(u_i)_{i\in[N]}$*.(online phase: one round interaction)*

Remark : KL-MKFHE does not need ciphertext expansion procedure; indeed, the input ciphertext set S in $\mathsf{Eval}(\cdot)$ is encrypted by participants under their own hybrid key hk_i which are different among participants, however, the resulting ciphertext c_i supports homomorphic evaluation without extra modification.

we require KL-MKFHE to satisfy the following properties :

Plain model : *No trusted setup or Common Reference String*

Locally Computationally Compactness : *For a computational task corresponds to a Boolean circuit with an input length of* N*, a KL-MKFHE scheme is locally computationally compact if the participants do* $O(N)$ *encryptions as the single-key* FHE *scheme.*

Two round interaction : *Only two round interaction is allow in* $\mathsf{KeyLifting}(\cdot)$ *procedure.*

The indistinguishable of initial ciphertext : *Let* N *and* W *be the input and out length of a circuit, respectively. Let* $\{c_i\}_{i\in[N]}$*,* $\{\gamma_i\}_{i\in[W]}$ *be the initial ciphertext and partial decryption result respectively. The following two distributions are computationally indistinguishable for any probabilistic polynomial time adversary* $\mathcal{A}$*.*

$$(\mathsf{pp}, \{\mathsf{pk}_i\}_{i\in[k]}, \{\mathsf{hk}_i\}_{i\in[k]}, \{c_i\}_{i\in[N]}, \{\gamma_i\}_{i\in[W]}) \overset{\mathsf{comp}}{\approx} (\mathsf{pp}, \{\mathsf{pk}_i\}_{i\in[k]}, \{\mathsf{hk}_i\}_{i\in[k]}, \mathbf{U}, \{\gamma_i\}_{i\in[W]})$$

where $\mathbf{U}$ *is uniform random*

Correctness and Compactness : *A KL-MKFHE scheme is correct if for a given security parameter* λ*, circuit depth* L*, participants* k*, we have the following*

$$\Pr\left[\,\mathsf{Dec}(f(\mathsf{sk}_1 \dots \mathsf{sk}_k), \hat{c}) \neq \mathcal{C}(u_1 \dots u_N)\,\right] = \mathsf{negl}(\lambda).$$

probability is negligible, where $\mathcal{C}$ *is a circuit with input length* N *and depth length less than or equal to* L*. A KL-MKFHE scheme is compact if the size* $\hat{c}$ *of evaluated ciphertext is bounded by* $\mathsf{poly}(\lambda, L, k)$*, but independent of circuit size.*

5 A KL-MKFHE scheme based on DGSW in plain model without noise flooding

Our scheme is based on DGSW. In this section, we first introduce the *key lifting* process, describe the entire scheme, and finally give parameter analysis and security proof.

5.1 Key lifting procedure

Following the definition of KL-MKFHE, the hybrid keys $\{\mathsf{hk}_i\}_{i\in[k]}$ which are obtained by $\mathsf{KeyLifting}(\cdot)$ algorithm are different from each other. Each participant encrypts his plaintext u_i by hk_i and gets $\mathbf{C}_i$. The ciphertexts $\{\mathbf{C}_{i\in[N]}\}$ can be used to evaluation without extra computation by Claim 1. We achieve this property by allowing two-round interaction between participants.

$\{\mathsf{hk}_i\}_{i\in[k]} \leftarrow \mathsf{KeyLifting}(\{\mathsf{pk}_i, \mathsf{sk}_i\}_{i\in[k]})$: Input the DGSW key pair $\{\mathsf{pk}_i, \mathsf{sk}_i\}_{i\in[k]}$ of all participants, where $\mathsf{pk}_i = (\mathbf{A}_i, \mathbf{b}_{i,i})$, $\mathbf{A}_i \leftarrow U(\mathbb{Z}_q^{(m-1)\times n})$, $\mathbf{s}_i \leftarrow U\{0,1\}^{m-1}$, $\mathbf{b}_{i,i} = \mathbf{s}_i\mathbf{A}_i \mod q$. Assuming there is a broadcast channel, all participants are engaged in the following two interactions:

- First round : i broadcasts pk_i and receives $\{\mathsf{pk}_j\}_{j\in[k]\setminus i}$ from other participants.
- Second round : i generates and broadcasts $\{\mathbf{b}_{i,j} = \mathbf{s}_i\mathbf{A}_j\}_{j\in[k]\setminus i}$, and receives $\{\mathbf{b}_{j,i} = \mathbf{s}_j\mathbf{A}_i\}_{j\in[k]\setminus i}$ from other participants.

After above two round interaction, i receives $\{\mathbf{b}_{j,i} = \mathbf{s}_j\mathbf{A}_i\}_{j\in[k]/i}$. Let $\mathbf{b}_i = \sum_{j=1}^{k}\mathbf{b}_{j,i}$, i obtains hybrid key $\mathsf{hk}_i = (\mathbf{A}_i, \mathbf{b}_i)$

Claim 1 *Let* $\bar{\mathbf{t}} = (-\mathbf{s}, 1)$, $\mathbf{s} = \sum_{i=1}^{k}\mathbf{s}_i$, *for ciphertext* $\mathbf{C}_i$, $\mathbf{C}_j$ *encrypted by hybrid key* hk_i, hk_j *respectively :*

$$\mathbf{C}_i = \begin{pmatrix}\mathbf{A}_i\\ \mathbf{b}_i\end{pmatrix}\mathbf{R}_i + \mathbf{E}_i + u_i\mathbf{G}, \qquad \mathbf{C}_j = \begin{pmatrix}\mathbf{A}_j\\ \mathbf{b}_j\end{pmatrix}\mathbf{R}_j + \mathbf{E}_j + u_j\mathbf{G},$$

it holds that(omit small error) :

$$\bar{\mathbf{t}}\mathbf{C}_i \approx u_i\bar{\mathbf{t}}\mathbf{G}, \quad \bar{\mathbf{t}}\mathbf{C}_j \approx u_j\bar{\mathbf{t}}\mathbf{G}$$
$$\bar{\mathbf{t}}(\mathbf{C}_i + \mathbf{C}_j) \approx (u_i + u_j)\bar{\mathbf{t}}\mathbf{G}, \quad \bar{\mathbf{t}}\mathbf{C}_i\mathbf{G}^{-1}(\mathbf{C}_j) \approx (u_iu_j)\bar{\mathbf{t}}\mathbf{G}$$

Proof. According to the construction of $\mathsf{KeyLifting}(\cdot)$, it holds that :

$$\bar{\mathbf{t}}\mathbf{C}_i = \left(\sum_{i=1}^{k} -\mathbf{s}_i, 1\right)\left[\begin{pmatrix}\mathbf{A}_i\\ \sum_{j=1}^{k}\mathbf{b}_{j,i}\end{pmatrix} + \mathbf{E}_i + u_i\mathbf{G}\right] = \bar{\mathbf{t}}\mathbf{E}_i + u_i\bar{\mathbf{t}}\mathbf{G} \approx u_i\bar{\mathbf{t}}\mathbf{G}.$$

Similarly, $\bar{\mathbf{t}}\mathbf{C}_j \approx u_j\bar{\mathbf{t}}\mathbf{G}$, and $\bar{\mathbf{t}}(\mathbf{C}_i + \mathbf{C}_j) \approx (u_i + u_j)\bar{\mathbf{t}}\mathbf{G}$

$$\bar{\mathbf{t}}\mathbf{C}_i\mathbf{G}^{-1}(\mathbf{C}_j) \approx u_i\bar{\mathbf{t}}\mathbf{G}\mathbf{G}^{-1}(\mathbf{C}_j) \approx u_i\bar{\mathbf{t}}\mathbf{C}_j \approx (u_iu_j)\bar{\mathbf{t}}\mathbf{G}$$

■

Therefore, although $\mathbf{C}_i$ and $\mathbf{C}_j$ are encrypted by different hybrid keys, they correspond to the same decryption key $\bar{\mathbf{t}}$ and support homomorphic evaluation without extra modification.

Two hidden dangers for semi-malicious adversaries : There are two main security concerns about $\mathsf{KeyLifting}(\cdot)$. First, a semi-malicious adversary may generate matrix $\mathbf{A}$ with trapdoor, then $\mathbf{s}_i$ is leaked. More specifically, our scheme leaks the key $\mathbf{s}_i$ in two phases: in the $\mathsf{KeyLifting}(\cdot)$ phase, $\{\mathbf{b}_{i,j} = \mathbf{s}_i\mathbf{A}_j\}_{j\in[k]}$ will lose $\mathbf{s}_i$ at most $kn\log q$ bits, in the distributed decryption phase, since we do not introduce noise flooding, for a circuit with output length W, distributed decryption lose $\mathbf{s}_i$ at most $W\log q$ bits, so the total leaked amount of $\mathbf{s}_i$ is $(kn+W)\log q$ bits. According to the proof of Lemma 5, the length of $\mathbf{s}_i$ must be at least $(kn+W)\log q+2\lambda$ to ensure the indistinguishable of the ciphertext, which is why we set $m = (kn+W)\log q + 2\lambda$ in the scheme. Second, semi-malicious adversary j may generate $\mathbf{b}_{j,i}$ adaptively after seeing $\mathbf{b}_{i,i}$, then the hybrid key $\mathbf{b}_i$ of participant i may not distributed as requirement. The general solution is to assume that $\mathbf{b}_{j,i}$ generated by adversary j satisfies the linear relationship $\mathbf{b}_{j,i} = \mathbf{s}_j\mathbf{A}_i$, $\mathbf{s}_j \in \{0,1\}^{m-1}$, and introduce a large noise in the encryption phase to ensure security. Large encryption noise leads to large modulus q and high computational and communication overhead. In order to alleviate this problem, we proposed an analysis method based on Rényi divergence that neither introduces the above assumptions nor a large noise in the encryption process. For more details, please refer to Section 5.4.

5.2 The entire scheme

Our scheme is based on the DGSW scheme, containing the following five algorithms (Init, Gen, $\mathsf{KeyLifting}$, Enc, Eval, Dec)

- $\mathsf{pp} \leftarrow \mathsf{Init}(1^\lambda, 1^L, 1^W)$: Let λ be security parameter, L circuit depth, W circuit output length, lattice dimension $n = n(\lambda, L)$, noise distribution χ over $\mathbb{Z}$, $e \leftarrow \chi$, where $|e|$ is bounded by B_χ with overwhelming probability, modulus $q = 2^{O(L)}B_\chi$, $k = \mathsf{poly}(\lambda)$, $m = (kn+W)\log q + \lambda$, suitable choosing above parameters to make $\mathsf{LWE}_{n,m,q,B_\chi}$ is infeasible. Output $\mathsf{pp} = (k, n, m, q, \chi, B_\chi)$
- $(\mathsf{pk}_i, \mathsf{sk}_i) \leftarrow \mathsf{Gen}(\mathsf{pp})$: Input pp, output the DGSW key pair $(\mathsf{pk}_i, \mathsf{sk}_i)$ of participants i, where $\mathsf{pk}_i = (\mathbf{A}_i, \mathbf{b}_{i,i})$, $\mathbf{A}_i \leftarrow U(\mathbb{Z}_q^{(m-1)\times n})$, $\mathbf{s}_i \leftarrow U\{0,1\}^{m-1}$, $\mathbf{b}_{i,i} = \mathbf{s}_i\mathbf{A}_i \mod q$.
- $\mathsf{hk}_i \leftarrow \mathsf{KeyLifting}(\{\mathsf{pk}_i, \mathsf{sk}_i\}_{i\in[k]})$: All participants are engaged in the *Key lifting procedure* 5.1, output the hybrid key hk_i.
- $\mathbf{C}_i \leftarrow \mathsf{Enc}(\mathsf{hk}_i, u_i)$: Input hybrid key hk_i, plaintext $u_i \in \{0,1\}$, output ciphertext $\mathbf{C}_i = \begin{pmatrix}\mathbf{A}_i\\ \mathbf{b}_i\end{pmatrix}\mathbf{R} + \mathbf{E} + u_i\mathbf{G}$, where $\mathbf{R} \leftarrow U(\mathbb{Z}_q^{n\times ml})$, $l = \lceil \log q \rceil$, $\mathbf{E} \leftarrow \chi^{m\times ml}$, $\mathbf{G} = \mathbf{I}_m \otimes \mathbf{g}$ is a gadget matrix.

– $\mathbf{C}^{(L)} \leftarrow \mathsf{Eval}(S, \mathcal{C})$: Input the ciphertext set $S = \{\mathbf{C}_i\}_{i\in[N]}$ which are encrypted by hybrid key $\{\mathsf{hk}_i\}_{i\in[k]}$, circuit $\mathcal{C}$ with input length N, depth L, output $\mathbf{C}^{(L)}$.

Remark : In the security proof in Section **??**, we require that χ be a discrete Gaussian distribution $\mathcal{D}_{\mathbb{Z},\sigma}$ over $\mathbb{Z}$ with $\sigma > \sqrt{2}\eta_\epsilon(\mathbb{Z})$. and $ml > 4\lambda$.

Homomorphic addition and multiplication : Let $\mathbf{C}_i$, $\mathbf{C}_j$ be ciphertext under hybrid key hk_i and hk_j respectively, by claim 1, we have the following results.

– $\mathbf{C}_{\mathsf{add}} \leftarrow \mathsf{Add}(\mathbf{C}_i, \mathbf{C}_j)$: Input ciphertext $\mathbf{C}_i$, $\mathbf{C}_j$, output $\mathbf{C}_{\mathsf{add}} = \mathbf{C}_i + \mathbf{C}_j$, which $\bar{\mathbf{t}}\mathbf{C}_{\mathsf{add}} \approx (u_i + u_j)\bar{\mathbf{t}}\mathbf{G}$

– $\mathbf{C}_{\mathsf{mult}} \leftarrow \mathsf{Mult}(\mathbf{C}_i, \mathbf{C}_j)$: Input ciphertext $\mathbf{C}_i$, $\mathbf{C}_j$, output $\mathbf{C}_{\mathsf{mult}} = \mathbf{C}_i\mathbf{G}^{-1}(\mathbf{C}_j)$, which $\bar{\mathbf{t}}\mathbf{C}_{\mathsf{mult}} \approx u_i u_j \bar{\mathbf{t}}\mathbf{G}$

Distributed decryption Similar to [34], the decryption procedure is a distributed procedure :

– $\gamma_i \leftarrow \mathsf{LocalDec}(\mathbf{C}^{(L)}, \mathbf{s}_i)$: Input $\mathbf{C}^{(L)}$, let $\mathbf{C}^{(L)} = \begin{pmatrix} \mathbf{C}_{up} \\ \mathbf{c}_{low} \end{pmatrix}$, where $\mathbf{C}_{up}$ is the first $m-1$ rows of $\mathbf{C}^{(L)}$, and $\mathbf{c}_{low}$ is last row of $\mathbf{C}^{(L)}$. i computes $\gamma_i = \langle -\mathbf{s}_i,\ \mathbf{C}_{up}\mathbf{G}^{-1}(\mathbf{w}^T)\rangle$, where $\mathbf{w} = (0, \ldots, 0, \lceil q/2 \rceil) \in \mathbb{Z}_q^m$, then i broadcast γ_i

– $u_L \leftarrow \mathsf{FinalDec}(\{\gamma_i\}_{i\in[k]})$: After receiving $\{\gamma_i\}_{i\in[k]}$, let $\gamma = \sum_{i=1}^{k}\gamma_i + \langle \mathbf{c}_{low},\ \mathbf{G}^{-1}(\mathbf{w}^T)\rangle$, output $u_L = \lceil \frac{\gamma}{q/2} \rceil$

5.3 Correctness analysis

To illustrate the correctness of our scheme, we first study the accumulation of noise. For fresh ciphertext $\mathbf{C} = \begin{pmatrix} \mathbf{A}_i \\ \mathbf{b}_i \end{pmatrix}\mathbf{R} + \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{e}_1 \end{pmatrix} + u\mathbf{G}$ under $\bar{\mathbf{t}}$, it holds that $\bar{\mathbf{t}}\mathbf{C} = \mathbf{e}_1 - \mathbf{s}\mathbf{E}_0 + u\bar{\mathbf{t}}\mathbf{G}$. Let $\mathbf{e}_{init} = \mathbf{e}_1 - \mathbf{s}\mathbf{E}_0$, after L depth circuit evaluation :

$$\bar{\mathbf{t}}\mathbf{C}^{(L)} = \mathbf{e}_L + u_L\bar{\mathbf{t}}\mathbf{G} \tag{1}$$

According to the noise analysis of GSW in [24], the noise $\mathbf{e}_L$ in $\mathbf{C}^{(L)}$ is bounded by $(ml)^L\mathbf{e}_{init}$. By the distributed decryption of our scheme, it holds that :

$$\begin{aligned}\gamma &= \sum_{i=1}^{k}\gamma_i + \langle \mathbf{c}_{low}, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle = \langle \sum_{i=1}^{k} -\mathbf{s}_i, \mathbf{C}_{up}\mathbf{G}^{-1}(\mathbf{w}^T)\rangle + \langle \mathbf{c}_{low}, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle \\ &= \bar{\mathbf{t}}\mathbf{C}^{(L)}\mathbf{G}^{-1}(\mathbf{w}^T) = \langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle + u_L\lceil\frac{q}{2}\rceil\end{aligned}$$

In order to decrypt correctly, it requires $\langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle < \frac{q}{4}$. For our parameter settings :

$$\begin{aligned}\langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle &\leq l \cdot ||\mathbf{e}_L||_\infty \\ &\leq l \cdot (ml)^L \cdot ||\mathbf{e}_{init}||_\infty \\ &\leq l \cdot (ml)^L \cdot (km+1)B_\chi\end{aligned}$$

Thus, $\log(\langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle) = \tilde{O}(L)$. For those $q = 2^{O(L)}B_\chi \geq 4\langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T\rangle$, requirements are fulfilled.

5.4 Semantic Security of Encryption against Semi-Malicious Adversary

The concept of a semi-malicious adversary was proposed by Asharov et al. in [6], which is formalized as a polynomial capability Turing machine with an additional witness tape. It must explain the "legality" of the record on the output tape. Please refer to [6] for a more formal definition.

The semantic security of our scheme: For an honest player i, he generates $\mathbf{A}_i \leftarrow U(\mathbb{Z}_q^{(m-1)\times n})$, $\mathbf{b}_{i,j} = \mathbf{s}_i\mathbf{A}_j$ as the protocol specification, but a semi-malicious adversary may generate it adaptively. Under the semi-malicious adversary model, a common method to prove security is as follows: Assume that $\mathbf{b}_{i,j}$ satisfies the linear relationship $\mathbf{b}_{i,j} = \mathbf{s}_i\mathbf{A}_j$, and $\mathbf{s}_i \in \{0,1\}^{m-1}$, and introduce large noise during encryption. In the following, we introduce this general method and then give an optimization proof method based on Rényi divergence.

A common approach : We complete the simulation by constructing a reduction from our scheme to the DGSW scheme. Consider the following Game:

1. Challenger generates $\mathsf{pk}_1 = (\mathbf{A}_1, \mathbf{b}_{1,1} = \mathbf{s}_1\mathbf{A}_1)$ where $\mathbf{A}_1 \leftarrow U(\mathbb{Z}_q^{(m-1)\times n})$, $\mathbf{s}_1 \leftarrow U\{0,1\}^{m-1}$ and send pk_1 to adversary $\mathcal{A}$
2. After receiving pk_1, $\mathcal{A}$ generates $\{\mathsf{pk}_i\}_{i\in[k]/1}$, where $\mathsf{pk}_i = (\mathbf{A}_i, \mathbf{b}_{i,i} = \mathbf{s}_i\mathbf{A}_i)$, and send it to Challenger.
3. After receiving $\{\mathsf{pk}_i\}_{i\in[k]/1}$, Challenger sets $\{\mathbf{b}_{1,i} = \mathbf{s}_1\mathbf{A}_i\}_{i\in[k]/1}$(the leakage of $\mathbf{s}_1$), and send it to $\mathcal{A}$
4. After receiving $\{\mathbf{b}_{1,i}\}_{i\in[k]/1}$, $\mathcal{A}$ adaptively chooses $\{\mathbf{s}'_i\}_{i\in[k]/1}$, where $\mathbf{s}'_i \in \{0,1\}^{m-1}$, set $\{\mathbf{b}_{i,1} = \mathbf{s}'_i\mathbf{A}_1\}_{i\in[k]/1}$, and send it to Challenger.
5. After receiving $\{\mathbf{b}_{i,1}\}_{i\in[k]/1}$, Challenger sets $\mathsf{hk}_1 = (\mathbf{A}_1, \sum_{i=1}^k \mathbf{b}_{i,1})$.
6. $\mathcal{A}$ chooses a bit $u \leftarrow \{0,1\}$, send it to Challenger.
7. Challenger chooses a bit $\alpha \leftarrow \{0,1\}$, if $\alpha = 0$ sets $\mathbf{C} \leftarrow \mathsf{Enc}(\mathsf{hk}_1, u)$, otherwise $\mathbf{C} \leftarrow U(\mathbb{Z}_q^{m\times ml})$, send $\mathbf{C}$ to $\mathcal{A}$.
8. After receiving $\mathbf{C}$, $\mathcal{A}$ output bit $\bar{\alpha}$, if $\bar{\alpha} = \alpha$, then $\mathcal{A}$ wins.

Claim 2 *Let* $\mathsf{Adv} = |Pr[\bar{\alpha} = \alpha] - \frac{1}{2}|$ *denote* $\mathcal{A}$*'s advantage in winning the game If* $\mathcal{A}$ *can win the game with advantage* Adv*, then* $\mathcal{A}$ *can distinguish between the ciphertext distribution of* DGSW *and the uniform random distribution with the same(up to negligible) advantage.*

Proof. After the third step of the above game, $\mathcal{A}$ obtained pk_1 and $\{\mathbf{b}_{1,i}\}_{i\in[k]/1}$(the leakage of $\mathbf{s}_1$). Next, we use the ciphertext of DGSW to construct $\mathbf{C}$. Consider the following sequence :

1. Challenger chooses a bit $\alpha \leftarrow \{0,1\}$, if $\alpha = 0$ sets $\mathbf{C}_{\mathsf{DGSW}} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{b}_{1,1} \end{pmatrix} \mathbf{R} + \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{e}_1 \end{pmatrix}$, otherwise $\mathbf{C}_{\mathsf{DGSW}} \leftarrow U(\mathbb{Z}_q^{m\times ml})$, send it to $\mathcal{A}$.
2. After receiving $\mathbf{C}_{\mathsf{DGSW}}$, $\mathcal{A}$ adaptively chooses $\{\mathbf{s}'_i\}_{i\in[k]/1}$, a bit $u \leftarrow \{0,1\}$, send it to Challenger.
3. After receiving $\{\mathbf{s}'_i\}_{i\in[k]/1}$ and u, let $\mathbf{C}_{\mathsf{DGSW}} = \begin{pmatrix} \mathbf{C}_0 \\ \mathbf{c}_1 \end{pmatrix}$, $\mathbf{s}' = \sum_{i=2}^{k} \mathbf{s}'_i$,

$$\mathbf{C}' = \mathbf{C}_{\mathsf{DGSW}} + \begin{pmatrix} \mathbf{0} \\ \mathbf{s}'\mathbf{C}_0 \end{pmatrix} + u\mathbf{G}$$

Obviously, if $\alpha = 1$, $\mathbf{C}'$ is uniform, otherwise it holds that :

$$\begin{aligned} \mathbf{s}'\mathbf{C}_0 = \mathbf{s}'(\mathbf{A}_1\mathbf{R} + \mathbf{E}_0) &= \sum_{i=2}^{k} \mathbf{b}_{i,1}\mathbf{R} + \mathbf{s}'\mathbf{E}_0 \\ \mathbf{C}' = \mathbf{C}_{\mathsf{DGSW}} + \begin{pmatrix} \mathbf{0} \\ \mathbf{s}'\mathbf{C}_0 \end{pmatrix} &+ u\mathbf{G} \\ = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{b}_{1,1} \end{pmatrix} \mathbf{R} + \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{e}_1 \end{pmatrix} &+ \begin{pmatrix} \mathbf{0} \\ \mathbf{s}'\mathbf{C}_0 \end{pmatrix} + u\mathbf{G} \\ = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{b}_1 \end{pmatrix} \mathbf{R} + \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{e}_1 + \mathbf{s}'\mathbf{E}_0 \end{pmatrix} &+ u\mathbf{G} \end{aligned}$$

If $||\mathbf{e}_1||_\infty$ is bounded by $2^\lambda B_\chi$, and $||\mathbf{s}'\mathbf{E}_0||_\infty < kmB_\chi$, thus $\mathbf{s}'\mathbf{E}_0/\mathbf{e}_1 = \mathsf{negl}(\lambda)$. By Lemma 1, it holds that $\mathbf{C}' \overset{\mathsf{stat}}{\approx} \mathbf{C}$, if $\mathcal{A}$ can distinguish between $\mathbf{C}$ and uniform random distribution by advantage Adv, then he can distinguish between $\mathbf{C}_{\mathsf{DGSW}}$ and the uniform random distribution with the same advantage. We note that the above sequence handles the leakage of $\mathbf{s}_1$, for $\mathbf{C}_{\mathsf{DGSW}}$ is a ciphertext generated by pk_1, which security is guaranteed by Lemma 5. ■

Remark: When $||\mathbf{e}_1||_\infty$ is bounded by $2^\lambda B_\chi$, according to the correctness analysis in Section 5.3, the initial noise $\mathbf{e}_{init} = \mathbf{e}_1 - \mathbf{s}\mathbf{E}_0$ is bounded by $(2^\lambda + km)B_\chi$. After L-level evaluation, $\langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle$ is bounded by $l\cdot(ml)^L\cdot(2^\lambda + km)B_\chi$, $\log(\langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle) = \tilde{O}(\lambda + L)$. Thus result in a $q = 2^{O(\lambda+L)}B_\chi$

Rényi divergence-based optimization : The work of Shi et al. [7] pointed out that Rényi divergence can also be applied in distinguish problems, and in some cases, it can lead to better parameters than statistical distance. Based on these results, they obtained better parameters of the Regev encryption scheme. Theorem 2 states: if there is an algorithm that can distinguish the P problem, then there is an algorithm that can distinguish the P' problem. Our proof method is as follows :

- Define the P problem as distinguishing our ciphertext from a uniform distribution
- Prove that for a given DGSW ciphertext, there exists a distribution X_0', and a sample x of X_0' can be constructed from this DGSW ciphertext,
- Define the P' problem as distinguishing X_0' from a uniform distribution

Thus, if there is an adversary who can distinguish the P problem, then he can distinguish the P' problem and can also distinguish the DGSW ciphertext from the uniform distribution.

Let $\mathbf{0}^{1\times ml}$ be a zero vector of length ml, Φ be the distribution of hybrid key of Challenger followed by $\mathbf{0}^{1\times ml}$:

$$(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml}) \leftarrow \Phi$$

which determined by $\mathsf{KeyLifting}(\cdot)$ procedure. Let $\mathcal{D}_0(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})$ be the joint distribution of $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})$ and the ciphertext $\begin{pmatrix}\mathbf{A}_1\\ \mathbf{b}_1\end{pmatrix}\mathbf{R}+\begin{pmatrix}\mathbf{E}_0\\ \mathbf{e}_1\end{pmatrix}$ encrypted by $(\mathbf{A}_1, \mathbf{b}_1)$ over the randomness $\mathbf{R}$, $\mathbf{E}_0$, $\mathbf{e}_1$:

$$(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml}, \begin{pmatrix}\mathbf{A}_1\\ \mathbf{b}_1\end{pmatrix}\mathbf{R}+\begin{pmatrix}\mathbf{E}_0\\ \mathbf{e}_1\end{pmatrix}) \leftarrow \mathcal{D}_0(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})$$

Let $\mathcal{D}_1(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})$ be the joint distribution of $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})$ and $\mathbf{U} \leftarrow U(\mathbb{Z}_q^{m\times ml})$:

$$(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml}, \mathbf{U}) \leftarrow \mathcal{D}_1(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})$$

Let P be the decision problems defined as follows :

- Problem P : distinguish whether input x is sampled from distribution X_0 or X_1, where

$$X_0 = \{x : (\mathbf{A_1}, \mathbf{b}_1, \mathbf{0}^{1\times ml}) \leftarrow \Phi, \quad x = (\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml}, \begin{pmatrix}\mathbf{A}_1\\ \mathbf{b}_1\end{pmatrix}\mathbf{R}+\begin{pmatrix}\mathbf{E}_0\\ \mathbf{e}_1\end{pmatrix}) \leftarrow \mathcal{D}_0(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})\}.$$

$$X_1 = \{x : (\mathbf{A_1}, \mathbf{b}_1, \mathbf{0}^{1\times ml}) \leftarrow \Phi, \quad x = (\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml}, \mathbf{U}) \leftarrow \mathcal{D}_1(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})\}.$$

In the above common approach, we showed how to construct $\mathbf{C}' = \begin{pmatrix}\mathbf{A}_1\\ \mathbf{b}_1\end{pmatrix}\mathbf{R}+\begin{pmatrix}\mathbf{E}_0\\ \mathbf{e}_1+\mathbf{s}'\mathbf{E}_0\end{pmatrix}$ with a given DGSW ciphertext $\mathbf{C}_{\mathsf{DGSW}} = \begin{pmatrix}\mathbf{A}_1\\ \mathbf{b}_{1,1}\end{pmatrix}\mathbf{R}+\begin{pmatrix}\mathbf{E}_0\\ \mathbf{e}_1\end{pmatrix}$ and $\{\mathbf{s}_i'\}_{i\in[k]/1}$, which generated by the adversary $\mathcal{A}$. Next, we show that each

such $\mathbf{C}'$ is sampled from some distribution. For the random $\mathbf{R} \in \mathbb{Z}_q^{n \times ml}$ used in $\mathbf{C}_{\mathsf{DGSW}}$, without loss of generality, assuming $\frac{ml}{n} = g$, we can divide $\mathbf{R}$ into g square matrices :

$$\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_g)$$

where $\mathbf{R}_i \in \mathbb{Z}_q^{n \times n}$. Similarly, for $\mathbf{E}_0 \in \mathbb{Z}_q^{(m-1) \times ml}$, $\mathbf{e}_1 \in \mathbb{Z}_q^{ml}$:

$$\mathbf{E}_0 = (\mathbf{E}_{0,1}, \mathbf{E}_{0,2}, \cdots, \mathbf{E}_{0,g})$$
$$\mathbf{e}_1 = (\mathbf{e}_{1,1}, \mathbf{e}_{1,2}, \cdots, \mathbf{e}_{1,g})$$

where $\mathbf{E}_{0,i} \in \mathbb{Z}_q^{(m-1) \times n}$, $\mathbf{e}_{1,i} \in \mathbb{Z}_q^n$. Then, $\mathbf{C}'$ can be expressed as :

$$\mathbf{C}' = \begin{pmatrix} \mathbf{A}_1\mathbf{R} + \mathbf{E}_0 \\ \mathbf{b}_1\mathbf{R}_1 + \mathbf{s}'\mathbf{E}_{0,1} + \mathbf{e}_{1,1}, \mathbf{b}_1\mathbf{R}_2 + \mathbf{s}'\mathbf{E}_{0,2} + \mathbf{e}_{1,2}, \cdots, \mathbf{b}_1\mathbf{R}_g + \mathbf{s}'\mathbf{E}_{0,g} + \mathbf{e}_{1,g} \end{pmatrix}$$

Let $\{\mathbf{v}_i \in \mathbb{Z}_q^n\}_{i \in [g]}$ be the solution of equation :

$$\{\mathbf{v}_i\mathbf{R}_i = \mathbf{s}'\mathbf{E}_{0,i}\}_{i \in [g]}$$

Obviously, if $\mathbf{R}_i$ is random over $\mathbb{Z}_q^{n \times n}$, then $\mathbf{v}_i$ has a unique solution with an overwhelming probability(See Appendix A). Define set V :

$$V = \{\mathbf{0}^{1 \times ml}, \quad (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g)\}$$

Define the distribution $\mathcal{D}(V)$ over set V :

$$\mathbf{d} \leftarrow \mathcal{D}(V) : \begin{cases} \Pr(\mathbf{d} = \mathbf{0}^{1 \times ml}) = p \\ \Pr(\mathbf{d} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g)) = 1 - p \end{cases}$$

Let Φ' be the joint distribution of the hybrid key of our scheme and $\mathcal{D}(V)$:

$$(\mathbf{A}', \mathbf{b}', \mathbf{d}) \leftarrow \Phi'$$

Let P' be the decision problems defined as follows :

– Problem P' : distinguish whether input x is sampled from distribution X_0' or X_1', where

$$X_0' = \{x : (\mathbf{A}', \mathbf{b}', \mathbf{d}) \leftarrow \Phi',$$
$$x = (\mathbf{A}', \mathbf{b}', \mathbf{d}, \begin{pmatrix} \mathbf{A}'\mathbf{R}' + \mathbf{E}_0' \\ (\mathbf{b}' + \mathbf{d}_1)\mathbf{R}_1' + \mathbf{e}_1', (\mathbf{b}' + \mathbf{d}_2)\mathbf{R}_2' + \mathbf{e}_2', \cdots, (\mathbf{b}' + \mathbf{d}_g)\mathbf{R}_g' + \mathbf{e}_g' \end{pmatrix}) \leftarrow \mathcal{D}_0(\mathbf{A}', \mathbf{b}', \mathbf{d})\}$$
$$X_1' = \{x : (\mathbf{A}', \mathbf{b}', \mathbf{d}) \leftarrow \Phi', \quad x = (\mathbf{A}', \mathbf{b}', \mathbf{d}, \mathbf{U}) \leftarrow \mathcal{D}_1(\mathbf{A}', \mathbf{b}', \mathbf{d})\}.$$

where $\mathbf{R}' = (\mathbf{R}_1', \mathbf{R}_2', \cdots, \mathbf{R}_g')$, $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_g)$

Thus for any $\mathbf{C}'$:

$$\mathbf{C}' = \begin{pmatrix} \mathbf{A}_1\mathbf{R} + \mathbf{E}_0 \\ \mathbf{b}_1\mathbf{R}_1 + \mathbf{s}'\mathbf{E}_{0,1} + \mathbf{e}_{1,1}, \mathbf{b}_1\mathbf{R}_2 + \mathbf{s}'\mathbf{E}_{0,2} + \mathbf{e}_{1,2}, \cdots, \mathbf{b}_1\mathbf{R}_g + \mathbf{s}'\mathbf{E}_{0,g} + \mathbf{e}_{1,g} \end{pmatrix}$$

it is a sample :

$$x = (\mathbf{A}', \mathbf{b}', \mathbf{d}, \begin{pmatrix} \mathbf{A}'\mathbf{R}' + \mathbf{E}'_0 \\ (\mathbf{b}' + \mathbf{d}_1)\mathbf{R}'_1 + \mathbf{e}'_1, (\mathbf{b}' + \mathbf{d}_2)\mathbf{R}'_2 + \mathbf{e}'_2, \cdots, (\mathbf{b}' + \mathbf{d}_g)\mathbf{R}'_g + \mathbf{e}'_g \end{pmatrix})$$

of X'_0 with $\mathbf{A}' = \mathbf{A}_1$, $\mathbf{b}' = \mathbf{b}_1$, $\mathbf{R}' = \mathbf{R}$, $\mathbf{E}'_0 = \mathbf{E}_0$, $\mathbf{d} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g)$, $\mathbf{e}'_i = \mathbf{e}_{1,i}$.

We note that $\mathbf{C}'$ only forms part of the sample of X'_0. The completed sample also contains $\{\mathbf{v}_i\}_{i\in[g]}$ which are determined by $\{\mathbf{v}_i\mathbf{R}_i = \mathbf{s}'\mathbf{E}_{0,i}\}_{i\in[g]}$ where $\mathbf{R}_i$, $\mathbf{E}_{0,i}$ is generated by Challenger, $\mathbf{s}'$ is generated by adversary $\mathcal{A}$. Consider the following sequence :

1. Challenger generates DGSW ciphetext $\mathbf{C}_{\mathsf{DGSW}} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{b}_{1.1} \end{pmatrix} \mathbf{R} + \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{e}_1 \end{pmatrix}$ and send it to adversary
2. After receiving $\mathbf{C}_{\mathsf{DGSW}}$, $\mathcal{A}$ adaptively generates $\mathbf{s}'$, and send it to Challenger.
3. Challenger computes $\{\mathbf{v}_i\}_{i\in[g]}$ by $\{\mathbf{v}_i\mathbf{R}_i = \mathbf{s}'\mathbf{E}_{0,i}\}_{i\in[g]}$, and then constructs a complete X'_0 sample from $\mathbf{C}'$ and $\{\mathbf{v}_i\}_{i\in[g]}$

Note that exposing $\{\mathbf{v}_i\}_{i\in[g]}$ to adversary will reveal the linear relationship between $\mathbf{R}_i$ and $\mathbf{E}_{0,i}$. We need to ensure that after $\mathcal{A}$ gets $\{\mathbf{v}_i\}_{i\in[g]}$, $\mathbf{C}_{\mathsf{DGSW}}$ is still indistinguishable. By leftover hash lemma, we can replace $\mathbf{b}_{1,1}$ by $\mathbf{u} \leftarrow U(\mathbb{Z}_q^n)$. Thus, distinguish

$$(\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{b}_{1.1} \end{pmatrix}, \mathbf{C}_{\mathsf{DGSW}}, \{\mathbf{v}_i\}_{i\in[g]}, \mathbf{s}')$$

with uniform corresponding to distinguish the "interactive LWE " problem :

$$(\mathbf{A}, \mathbf{AR} + \mathbf{E}, \{\mathbf{v}_i\}_{i\in[g]}, \mathbf{s}')$$

with $(\mathbf{A}, \mathbf{U}, \{\mathbf{v}_i\}_{i\in[g]}, \mathbf{s}')$ which by Theorem 3 is not simpler than the low dimensional LWE problem.

So far, we have completed the construction of X'_0 samples: that is, for each given DGSW ciphertext $\mathbf{C}_{\mathsf{DGSW}}$, after getting $\mathbf{s}'$ from $\mathcal{A}$, Challenger can convert it into a sample of X'_0. Since the outputs of our distributions of $\mathcal{D}_0(\cdot)$ and $\mathcal{D}_1(\cdot)$ contain the samples $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})$ or $(\mathbf{A}', \mathbf{b}', \mathbf{d})$ of the prior distributions Φ and Φ', thus $\mathcal{D}_0(\cdot)$ and $\mathcal{D}_1(\cdot)$ satisfy the publicly sampleable property(see Theorem 2) required by Theorem 2. The sampling algorithm S is just the encryption operation of our scheme with hybrid key $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{0}^{1\times ml})$ or $(\mathbf{A}', \mathbf{b}', \mathbf{d})$. Then, by Theorem 2, if given a T- time distinguisher $\mathcal{A}$ for problem P with advantage ϵ, we can construct a distinguisher $\mathcal{A}'$ for problem P' with run-time and distinguishing advantage, respectively, bounded from above and below by(for any $a \in (1, +\infty]$) :

$$\frac{64}{\epsilon^2}\log\left(\frac{8R_a(\Phi||\Phi')}{\epsilon^{a/(a-1)+1}}\right)\cdot(T_S + T) \quad and \quad \frac{\epsilon}{4\cdot R_a(\Phi||\Phi')}\cdot\left(\frac{\epsilon}{2}\right)^{\frac{a}{a-1}}.$$

For convenience, we take $R_\infty(\Phi||\Phi')$ analysis, let :

$$R_\infty(\Phi||\Phi') = \max_{Y\in\mathsf{Supp}(\Phi)}\frac{\Phi(Y)}{\Phi'(Y)} = \frac{\Phi(\mathbf{A}_0, \mathbf{b}_0, \mathbf{0}^{1\times ml})}{\Phi'(\mathbf{A}_0, \mathbf{b}_0, \mathbf{0}^{1\times ml})}.$$

$$= \frac{\Pr(\mathbf{A} = \mathbf{A}_0, \mathbf{b} = \mathbf{b}_0)}{\Pr(\mathbf{A} = \mathbf{A}_0, \mathbf{b} = \mathbf{b}_0, \mathbf{d} = \mathbf{0}^{1\times ml})} \tag{2}$$

Because $(\mathbf{A}, \mathbf{b})$ and $\mathcal{D}(V)$ are independent, thus :

$$(2) = \frac{\Pr(\mathbf{A} = \mathbf{A}_0, \mathbf{b} = \mathbf{b}_0)}{\Pr(\mathbf{A} = \mathbf{A}_0, \mathbf{b} = \mathbf{b}_0) \Pr(\mathbf{d} = \mathbf{0}^{1\times ml})} = \frac{1}{p}$$

Then, given a T- time distinguisher $\mathcal{A}$ for problem P with advantage ϵ, we can construct a distinguisher $\mathcal{A}'$ for problem P' with run-time and distinguishing advantage, respectively, bounded from above and below by :

$$\frac{64}{\epsilon^2} \log\left(\frac{8}{p \cdot \epsilon^2}\right) \cdot (T_S + T) \quad \textit{and} \qquad \frac{p \cdot \epsilon^2}{8}.$$

Remark : Under the semi-honest adversary model, $\{\mathbf{A}_i\}_{i\in[k]}$ and $\{\mathbf{s}_i\}_{\in[k]}$ are sampled as specified by the protocol, and the security is obvious. Under the semi-malicious adversary model, the common approach assumes $\mathbf{b}_{j,i} = \mathbf{s}_j\mathbf{A}_i$ and$\{\mathbf{s}_{j\in[k]/1}\} \in \{0,1\}^{m-1}$ is chosen adaptively, and introduces large noise in the encryption process to ensure security. In our proof method based on Rényi divergence, we need to introduce neither the above assumptions nor large encryption noise.

This Rényi divergence-based proof method provides an alternative idea for those proofs that must introduce strong assumptions and large noise to ensure security.

5.5 Noise flooding technology VS Leakage resilient property in partial decryption

We note that introducing noise flooding in the partial decryption phase is essential to guarantee the semantic security of fresh ciphertext, and noise flooding achieves this by masking the private key information in the partial decryption noise. For partial decryption to be simulatable, the magnitude of the noise introduced needs to be exponentially larger than the noise after the homomorphic evaluation. At the same time, as mentioned in [34], masking techniques based on noise flooding can only guarantee weak simulatable properties: input all private keys $\{\mathsf{sk}_j\}_{j\in[k]/i}$ except sk_i, evaluated result u_L, ciphertext $\mathbf{C}^{(L)}$, it can simulate the local decryption result γ_i, while for stronger security requirements: input any private key set $\{\mathsf{sk}_j\}_{j\in S}$ for any subset S of $[k]$, evaluated result u_{eval} and ciphertext $\mathbf{C}^{(L)}$, to simulate $\{\gamma_i\}_{i\in U,\ U=[k]-S}$, it do not know how to achieve it.

With noise flooding : To illustrate how our approach works, let us first review the noise flooding technique. Let $\mathbf{C}^{(L)} = \begin{pmatrix} \mathbf{C}_{up} \\ \mathbf{c}_{low} \end{pmatrix}$ be the ciphertext after L-layer homomorphic multiplication. With a flooding noise $e_i'' \leftarrow U[-B_{smdg}, B_{smdg}]$, introduced in $\mathsf{LocalDec}(\cdot)$, we have :

$$\gamma_i = \langle -\mathbf{s}_i, \mathbf{C}_{up}\mathbf{G}^{-1}(\mathbf{w}^T)\rangle + e_i''$$

By Equation (1) and $\mathsf{FinalDec}(\cdot)$:

$$\gamma_i = u_L\lceil\frac{q}{2}\rceil + \langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle + e_i'' - \langle \mathbf{c}_{low}, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle + \langle \sum_{j\neq i}^{k} \mathbf{s}_j, \mathbf{C}_{up}\mathbf{G}^{-1}(\mathbf{w}^T)\rangle$$

For a simulator $\mathcal{S}$, input $\{\mathsf{sk}_j\}_{j\in[k]/i}$, evaluated result u_L, ciphertext $\mathbf{C}^{(L)}$, output simulated γ_i'

$$\gamma_i' = u_L\lceil\frac{q}{2}\rceil + e_i'' - \langle \mathbf{c}_{low}, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle + \langle \sum_{j\neq i}^{k} \mathbf{s}_j, \mathbf{C}_{up}\mathbf{G}^{-1}(\mathbf{w}^T)\rangle$$

In order to make the partial decryption process simulatable, it requires :

$$\langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle + e_i'' \overset{\mathsf{stat}}{\approx} e_i''$$

For the parameter settings in [34] : $B_{smdg} = 2^{L\lambda\log\lambda}B_\chi$, $q = 2^{\omega(L\lambda\log\lambda)}B_\chi$, obviously :

$$|\langle \mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle / e_i''| = \mathsf{negl}(\lambda)$$
$$\text{thus } \gamma_i \overset{\mathsf{stat}}{\approx} \gamma_i'.$$

In short, the noise e_i'' is introduced to cover up some information(private key $\mathbf{s}_i$ and the noise $\mathbf{E}_i$ in initial ciphertext) of participant i contained in $\mathbf{e}_L$(Noise obtained by decrypting the ciphertext of level L, $\bar{\mathbf{t}}\mathbf{C}^{(L)} = \mathbf{e}_L + u_L\bar{\mathbf{t}}\mathbf{G}$) . Thus the partial decryption result of participant i can be simulated, providing other parties with information.

Without noise flooding : Through the above analysis, we point out that as long as our encryption scheme is leakage-resilient and covers the initial noise $\{\mathbf{E}_i\}_{i\in[N]}$ in $\mathbf{e}_L$, there is no need to introduce noise flood in the partial decryption stage. To illustrate what information is contained in $\mathbf{e}_L$, let us look at how $\mathbf{e}_L$ is generated. For the initial ciphertext :

$$\mathbf{C}_1 = \begin{pmatrix}\mathbf{A}_1\\ \mathbf{b}_1\end{pmatrix}\mathbf{R}_1 + \mathbf{E}_1 + u_1\mathbf{G}, \qquad \mathbf{C}_2 = \begin{pmatrix}\mathbf{A}_2\\ \mathbf{b}_2\end{pmatrix}\mathbf{R}_2 + \mathbf{E}_2 + u_2\mathbf{G},$$

After performing a homomorphic multiplication operation, we obtain:

$$\mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2) = \left[\begin{pmatrix}\mathbf{A}_1\\ \mathbf{b}_1\end{pmatrix}\mathbf{R}_1 + \mathbf{E}_1 + u_1\mathbf{G}\right]\mathbf{G}^{-1}(\mathbf{C}_2)$$
$$= \begin{pmatrix}\mathbf{A}_1\\ \mathbf{b}_1\end{pmatrix}\mathbf{R}_1\mathbf{G}^{-1}(\mathbf{C}_2) + \mathbf{E}_1\mathbf{G}^{-1}(\mathbf{C}_2) + u_1\begin{pmatrix}\mathbf{A}_2\\ \mathbf{b}_2\end{pmatrix}\mathbf{R}_2 + u_1\mathbf{E}_2 + u_1u_2\mathbf{G}$$
$$= \Pi_1 + \delta_1 + u_1u_2\mathbf{G}$$

where :

$$\Pi_1 = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{b}_1 \end{pmatrix} \mathbf{R}_1 \mathbf{G}^{-1}(\mathbf{C}_2) + u_1 \begin{pmatrix} \mathbf{A}_2 \\ \mathbf{b}_2 \end{pmatrix} \mathbf{R}_2$$
$$\delta_1 = \mathbf{E}_1 \mathbf{G}^{-1}(\mathbf{C}_2) + u_1 \mathbf{E}_2$$

and $\bar{\mathbf{t}}\Pi_1 = 0$, δ_1 is the noise after the first homomorphic evaluation. For the ciphertexts $\mathbf{C}_3, \mathbf{C}_4$ of the same level, we have $\mathbf{C}_3\mathbf{G}^{-1}(\mathbf{C}_4) = \Pi_1' + \delta_1' + u_3u_4\mathbf{G}$, where Π_1', δ_1' and Π_1, δ_1 have the same structure.
Let $\mathbf{C}^{(2)}$, $\mathbf{C}^{(2)'}$ be the ciphertext at level 2 :

$$\mathbf{C}^{(2)} = \mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2), \qquad \mathbf{C}^{(2)'} = \mathbf{C}_3\mathbf{G}^{-1}(\mathbf{C}_4)$$
$$\delta_2 = \delta_1\mathbf{G}^{-1}(\mathbf{C}^{(2)'}) + u_1u_2\delta_1'$$

we have $\mathbf{C}^{(2)}\mathbf{G}^{-1}(\mathbf{C}^{(2)'}) = \Pi_2 + \delta_2 + u_1u_2u_3u_4\mathbf{G}$. For the ciphertext at level L, we have :

$$\mathbf{C}^{(L)} = \mathbf{C}^{(L-1)}\mathbf{G}^{-1}(\mathbf{C}^{(L-1)'}) = \Pi_{L-1} + \delta_{L-1} + u_{L-1}u'_{L-1}\mathbf{G}$$
$$\delta_{L-1} = \delta_{L-2}\mathbf{G}^{-1}(\mathbf{C}^{(L-1)'}) + u_{L-1}\delta'_{L-2}$$

To find out what information δ_{L-1} contains, first, we observe $\delta_1 = \mathbf{E}_1\mathbf{G}^{-1}(\mathbf{C}_2) + u_1\mathbf{E}_2$.

Lemma 6 *For the DGSW ciphertext* $\mathbf{C}_1$, $\mathbf{C}_2$, *let* $\mathbf{C}^{(2)} = \mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2)$, *the noise* δ_1 *obtained by decrypting* $\mathbf{C}^{(2)}$ *is dominated by the noise* $\mathbf{E}_1$ *in* $\mathbf{C}_1$ *:*

$$\delta_1 \overset{\mathsf{stat}}{\approx} \mathbf{E}_1\mathbf{G}^{-1}(\mathbf{C}_2) \tag{3}$$

To prove the above statement, we first prove that the distribution of the sum of multiple independent and identically distributed(*iid*) discrete Gaussian is close to discrete Gaussian. The work [37] has already proved the case of two discrete Gaussian summations, while we generalize this result to the case of multiple summations.

Lemma 7 *Let* $\epsilon = 2^{-\lambda}$, $\sigma > \sqrt{2}\eta_\epsilon(\mathbb{Z})$, $m = (kn+W)l$, $l = \lceil \log q \rceil$, $\{y_i\}_{i\in[ml]} \leftarrow \mathcal{D}_{\mathbb{Z},\sigma}$, $y' \leftarrow \mathcal{D}_{\mathbb{Z},\sqrt{ml}\sigma}$. *we have :*

$$\Delta(\sum_{i=1}^{ml} y_i, y') \leq 8ml\epsilon.$$

Proof. Let $\{y_i^{(1)}\}_{i\in[ml/2]} \leftarrow \mathcal{D}_{\mathbb{Z},\sqrt{2}\delta}$, by lemma 3 :

$$\Delta(y_1 + y_2, y_1^{(1)}) < 8\epsilon$$
$$\Delta(y_3 + y_4, y_2^{(1)}) < 8\epsilon$$
$$\cdots$$
$$\Delta(y_{ml-1} + y_{ml}, y_{\frac{ml}{2}}^{(1)}) < 8\epsilon$$

By the subadditivity of statistical distances (we proved it in Appendix B), we have :

$$\Delta(\sum_{i=1}^{ml} y_i, \sum_{i=1}^{\frac{ml}{2}} y_i^{(1)}) < \frac{ml}{2} \cdot 8\epsilon.$$

Let $\{y_i^{(2)}\}_{i\in[ml/4]} \leftarrow \mathcal{D}_{\mathbb{Z},2\delta}$, again by lemma 3 :

$$\Delta(y_1^{(1)} + y_2^{(1)}, y_1^{(2)}) < 8\epsilon$$

Thus:

$$\Delta(\sum_{i=1}^{\frac{ml}{2}} y_i^{(1)}, \sum_{i=1}^{\frac{ml}{4}} y_i^{(2)}) < \frac{ml}{4} \cdot 8\epsilon.$$

Iterating the above process, we have :

$$\Delta(\sum_{i=1}^{ml} y_i, y') \leq \frac{ml}{2} \cdot 8\epsilon + \frac{ml}{4} \cdot 8\epsilon +, \cdots, +8\epsilon = 8ml\epsilon.$$

we complete the proof.

■

Remark: We point out that the result here is certainly not sharp since we directly exploit the results of Lemma 3, which already satisfies our needs. For the case of summing multiple discrete Gaussian, if one follows the path of [37], a smaller statistical distance bound should be obtained.

Here, we prove Lemma 6:

Proof. First, according to the LWE assumption, replace $\mathbf{G}^{-1}(\mathbf{C}_2)$ with $\mathbf{M} \leftarrow U\{0,1\}^{ml\times ml}$. When $u_1 = 0$, it is proved. Assuming $u_1 = 1$, let $\delta_1(i,j), \mathbf{E}_1\mathbf{M}(i,j)$ be the i-th row, j-th column element of δ_1, $\mathbf{E}_1\mathbf{M}$ respectively. We have :

$$\delta_1(1,1) = z_1e_1 + z_2e_2 + \cdots + z_{ml}e_{ml} + e_{ml+1}$$
$$\mathbf{E}_1\mathbf{M}(1,1) = z_1e_1 + z_2e_2 + \cdots + z_{ml}e_{ml}$$

where $\{z_i\}_{i\in[ml]}$ is the first column of $\mathbf{M}$, $\{e_i\}_{i\in[ml]} \leftarrow D_{\mathbb{Z},\sigma}$ is the first row of $\mathbf{E}_1$, $\mathbf{E}_2(1,1) = e_{ml+1} \leftarrow D_{\mathbb{Z},\sigma}$. Suppose, the number of 1s in $\{z_i\}_{i\in[ml]}$ is r. By lemma 7 we have :

$$\Delta(\delta_1(1,1), \mathcal{D}_{\mathbb{Z},\sqrt{r+1}\sigma}) \leq 8(r+1)\epsilon.$$
$$\Delta(\mathbf{E}_1\mathbf{M}(1,1), \mathcal{D}_{\mathbb{Z},\sqrt{r}\sigma}) \leq 8r\epsilon$$

For our parameter setting, $8r\epsilon \leq 8ml\epsilon = \mathsf{poly}(\lambda) \cdot 2^{-\lambda} = \mathsf{negl}(\lambda)$. Thus :

$$\delta_1(1,1) \sim \mathcal{D}_{\mathbb{Z},\sqrt{r+1}\sigma}$$
$$\mathbf{E}_1\mathbf{M}(1,1) \sim \mathcal{D}_{\mathbb{Z},\sqrt{r}\sigma}$$

The statistical distance of $\delta_1(1,1)$ and $\mathbf{E}_1\mathbf{M}(1,1)$ is :

$$\Delta(\delta_1(1,1),\mathbf{E}_1\mathbf{M}(1,1)) = \frac{1}{2}\sum_{-\infty}^{+\infty}\left|\frac{\rho_{\sqrt{r}\sigma}(x)}{\rho_{\sqrt{r}\sigma}(\mathbb{Z})} - \frac{\rho_{\sqrt{r+1}\sigma}(x)}{\rho_{\sqrt{r+1}\sigma}(\mathbb{Z})}\right| = \sum_{-x}^{x}\left(\frac{\rho_{\sqrt{r}\sigma}(x)}{\rho_{\sqrt{r}\sigma}(\mathbb{Z})} - \frac{\rho_{\sqrt{r+1}\sigma}(x)}{\rho_{\sqrt{r+1}\sigma}(\mathbb{Z})}\right)$$
$$= 2\sum_{-\infty}^{-x}\left(\frac{\rho_{\sqrt{r+1}\sigma}(x)}{\rho_{\sqrt{r+1}\sigma}(\mathbb{Z})} - \frac{\rho_{\sqrt{r}\sigma}(x)}{\rho_{\sqrt{r}\sigma}(\mathbb{Z})}\right) < 2\sum_{-\infty}^{-x}\frac{\rho_{\sqrt{r+1}\sigma}(x)}{\rho_{\sqrt{r+1}\sigma}(\mathbb{Z})}.$$

where $x = \sqrt{r(r+1)\ln\frac{r+1}{r}}\sigma$ is the root of equation :

$$\frac{\rho_{\sqrt{r+1}\sigma}(x)}{\rho_{\sqrt{r+1}\sigma}(\mathbb{Z})} = \frac{\rho_{\sqrt{r}\sigma}(x)}{\rho_{\sqrt{r}\sigma}(\mathbb{Z})}$$

Let $C = \sqrt{r(r+1)\ln\frac{r+1}{r}}$, By the Lemma 4 in [1], We have :

$$2\sum_{-\infty}^{-x}\frac{\rho_{\sqrt{r+1}\sigma}(x)}{\rho_{\sqrt{r+1}\sigma}(\mathbb{Z})} < \frac{2}{C\sqrt{2\pi}}\exp\{-\frac{C^2}{2}\}$$
$$= \frac{2}{C\sqrt{2\pi}}\exp\{-\frac{1}{2}r(r+1)\ln\frac{r+1}{r}\}$$
$$= \frac{2}{C\sqrt{2\pi}}\exp\{-\frac{r+1}{2}\}$$

Generally, r is distributed like the summation of ml independent identically distributed 0-1 distribution, thus $r \sim B(ml, \frac{1}{2})$. By Theorem 1,

$$\Pr(r < \lambda) \leq e^{-\frac{(\frac{1}{2}ml-\lambda)^2}{ml-\lambda}} = \mathsf{negl}(\lambda)$$

for $ml > 4\lambda$. Thus, the statistical distance of $\delta_1(1,1)$ and $\mathbf{E}_1\mathbf{M}(1,1)$:

$$\Delta(\delta_1(1,1),\mathbf{E}_1\mathbf{M}(1,1)) < \frac{2}{C\sqrt{2\pi}}\exp\{-\frac{\lambda+1}{2}\} = \mathsf{negl}(\lambda).$$

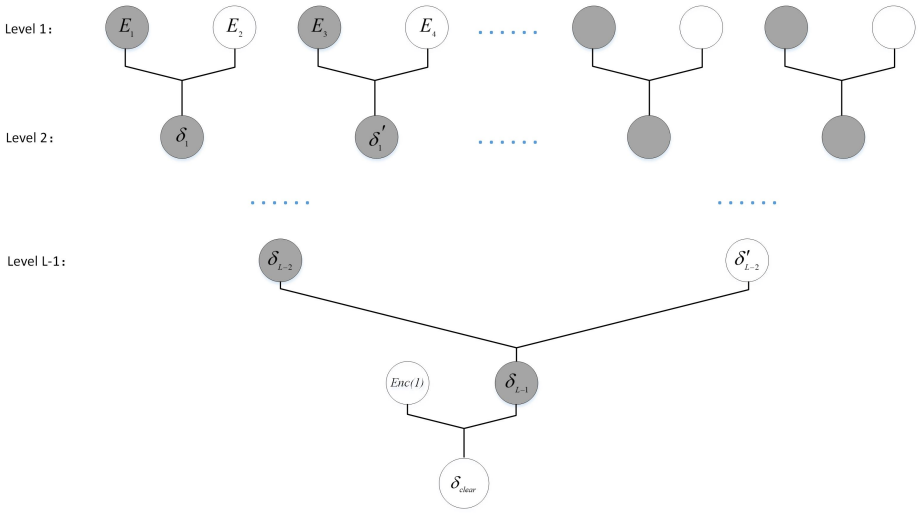
We completed the proof, for other item of $\delta_1(i,j)$ and $\mathbf{E}_1\mathbf{M}(i,j))$ the statement also holds.

■

According to the results we proved above, the noise $\mathbf{E}_2$ of the right ciphertext $\mathbf{C}_2$ in the ciphertext $\mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2)$ is masked by the noise $\mathbf{E}_1$ in the left ciphertext $\mathbf{C}_1$. Similarly, the noise $\mathbf{E}_4$ of $\mathbf{C}_4$ in $\mathbf{C}_3\mathbf{G}^{-1}(\mathbf{C}_4)$ is masked by the noise $\mathbf{E}_3$ of $\mathbf{C}_3$ on the leftside. For the noise $\delta_2 = \delta_1\mathbf{G}^{-1}(\mathbf{C}^{(2)'}) + u_1u_2\delta_1'$ of the third level, δ_1' is masked by δ_1, and similarly the noise $\delta_{L-1} = \delta_{L-2}\mathbf{G}^{-1}(\mathbf{C}^{(L-2)'}) + u_{L-2}\delta_{L-2}'$ of the L-th level, δ_{L-2}' is masked by δ_{L-2}. We illustrate this continuous process in Figure 2.

If the circuit with input length N and depth L, as long as $L > \log N$, then the noise δ_{L-1} of the ciphertext $\mathbf{C}^{(L)}$ of the L-th level only contains the information of noise $\mathbf{E}_t(t \in [N])$ in a certain initial ciphertext. At this point, we only

Fig. 2. Circuit



need to left-multiply $\mathbf{C}^{(L)}$ by a ciphertext $Enc(1)$ whose plaintext is 1, and let $\mathbf{C}_{clear} = Enc(1)\mathbf{G}^{-1}(\mathbf{C}^{(L)})$. Thus, the noise δ_{clear} in $\mathbf{C}_{clear}$ does not contain any information about the noise $\{\mathbf{E}_i\}_{i\in[N]}$ in the initial ciphertext $\{\mathbf{C}_i\}_{i\in[N]}$. Decrypting $\mathbf{C}_{clear}$, we have :

$$\bar{\mathbf{t}}\mathbf{C}_{clear}\mathbf{G}^{-1}(\mathbf{w}^T) = \bar{\mathbf{t}}\delta_{clear}\mathbf{G}^{-1}(\mathbf{w}^T) + u_L\lceil\frac{q}{2}\rceil.$$

Let $\mathbf{e}_L = \bar{\mathbf{t}}\delta_{clear}$, therefore, $\langle\mathbf{e}_L, \mathbf{G}^{-1}(\mathbf{w}^T)\rangle \in \mathbb{Z}_q$ leaks participant i's private key $\mathbf{s}_i$ with at most $\log q$ bits. For a circuit with output length W, the partial decryption leaks $W \log q$ bits of $\mathbf{s}_i$. Because our scheme is leakage-resilient, as long as we set the key length reasonably $m = (kn + W)\log q + \lambda$, the initial ciphertext $\{\mathbf{C}_i\}_{i\in[N]}$ is semantically secure.

Remark : We point out that the asymmetric nature of noise in GSW ciphertext has been noted in [10] before us, but their aims and results are completely different from ours. Their purpose is to preserve the privacy of the circuit, i.e. to ensure that the final decrypted noise is independent of the circuit $\mathcal{C}$. In contrast, our purpose is to be independent of the initial noise. They show a discrete Gaussian version of the leftover hash lemma. In contrast, we show that the statistical distances of the distributions $\sum_{i=1}^{m} e_i$ and $\sum_{i=1}^{m+1} e_i$ are exponentially close to zero with m.

Here, the reader might think that doing so would result in a longer key than noise flooding. We point out that as long as the output length W of the circuit satisfies $W < kn(\lambda - 1)$, the length of the private key will not be longer than

when using noise flooding. For $m = (kn+W)\log q+\lambda$, $q = 2^{O(L)}B_\chi$, while with noise flooding $m' = kn\log q'+\lambda$, $q' = 2^{O(\lambda L)}B_\chi$. In order to make $m < m'$, only $W < kn(\lambda-1)$ is required; thus, for circuits with small output fields, our scheme does not lead to longer keys.

5.6 Bootstrapping

In order to eliminate the dependence on the circuit depth to achieve full homomorphism, we need to use Gentry's bootstrapping technology. It is worth noting that the bootstrapping procedure of our scheme is the same as the single-key homomorphic scheme: After *Key lifting* procedure, participant i uses hybrid key hk_i to encrypt $\mathbf{s}_i$ to obtain evaluation key evk_i. Because evk_i and $\mathbf{C}^{(L)}$ are both ciphertexts under $\bar{\mathbf{t}} = (-\sum_{i=1}^{k}\mathbf{s}_i, 1)$, homomorphic evaluation of the decryption circuit could be executed directly as $\mathbf{C}^{(L)}$ are need to be refresh. Therefore, to evaluate any depth circuit, we only need to set the initial parameters to satisfy the homomorphic evaluation of the decryption circuit.

However, for those MKFHE schemes that require ciphertext expansion, additional ciphertext expansion is required, for the reason that $\mathbf{C}^{(L)}$ is the ciphertext under $\bar{\mathbf{t}}$, but $\{\mathsf{evk}_i\}_{i\in[k]}$ are the ciphertext under $\{\mathbf{t}_i\}_{i\in[k]}$. In order to expand $\{\mathsf{evk}_i\}_{i\in[k]} \to \{\widehat{\mathsf{evk}_i}\}_{i\in[k]}$, participant i needs to encrypt the random matrix of the ciphertext corresponding to evk_i. The extra encryption of i needs to be done locally are $O(\lambda^9 L^6)$.

6 Conclusions

For the LWE-based MKFHE, in order to alleviate the overhead of the local participants, we proposed the concept of KL-MKFHE, which introduced a *Key lifting* procedure, getting rid of expensive ciphertext expansion operation and constructing a DGSW style KL-MKFHE under the plain model. Our scheme is more friendly to local participants than the previous scheme, for which the local encryption $O(N\lambda^6 L^4)$ are reduced to $O(N)$. By abandoning noise flooding, it compresses q from $2^{O(\lambda L)}B_\chi$ to $2^{O(L)}B_\chi$, reducing the computational scale of the entire scheme. However, the key length depends on the number of participants and the amount of leakage, which limits the scheme's application to some extent. Further work will focus on compressing the key length.

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Appendix

A Probability that $\{v_i\}_{i\in[g]}$ has a solution

Random Matrices : For a prime q, the probability that a uniformly random matrix $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{n\times m})$(with $m \geq n$) has full rank is given by :

$$\Pr[\mathsf{rank}(\mathbf{A}) < n] = 1 - \prod_{i=0}^{n-1}(1 - q^{i-m}).$$

For equations :

$$\{\mathbf{v}_i\mathbf{R}_i = \mathbf{s}'\mathbf{E}_{0,i}\}_{i\in[g]}$$

if $\{\mathbf{R}_i\}_{i\in[g]}$ are all invertible, obviously $\{\mathbf{v}_i\}_{i\in[g]}$ has a solution. For a random matrix $\mathbf{R}$ over $\mathbb{Z}_q^{n\times n}$, the probability that it is invertible is $\prod_{i=0}^{n-1}(1-q^{i-n})$. For the parameter settings in our scheme, $q = 2^{O(L)}B_\chi$, $m = (kn+W)\log q + 2\lambda$, $g = mL/n$, the probability that $\{\mathbf{R}_i\}_{i\in[g]}$ are all invertible is :

$$\Pr = (\prod_{i=0}^{n-1}(1-(2^L)^{i-n})^{\frac{(kn+W)L^2+2\lambda L}{n}} \geq (1-2^{-L})^{(kn+W)L^2+2\lambda L}$$

This probability is close to 1, for 2^{-L} decreases faster than L^2. We tested the probability on Maple18 by set $q = 2^{100}$, $k = 50$, $n = 500$, $W = 1000$, $\lambda = 128$(which should be able to cover the actual application) obtained $\Pr > 0.99999999999999999999979487596980336327269120254193$.

B the additivity of statistical distances

Claim 3 *For discrete random variables X, Y, Z with measurable space E, the statistical distance $\Delta(X, Z)$, $\Delta(X, Y)$, $\Delta(Y, Z)$ satisfy: (triangular inequality)*

$$\Delta(X, Z) \leq \Delta(X, Y) + \Delta(Y, Z).$$

Proof.

$$\begin{aligned}
\Delta(X, Z) &= \frac{1}{2} \sum_{k \in E} |(\Pr(X = k) - \Pr(Z = k))| \\
&\leq \frac{1}{2} \sum_{k \in E} (|\Pr(X = k) - \Pr(Y = k)| + |\Pr(Y = k) - \Pr(Z = k)|) \\
&\leq \Delta(X, Y) + \Delta(Y, Z).
\end{aligned}$$

■

Claim 4 *For discrete random variables X, Y, Z with measurable space E, if X, Y, Z are independent, then :*

$$\Delta(X + Y, Y + Z) \leq \Delta(X, Z)$$

Proof.

$$\begin{aligned}
\Delta(X + Y, Y + Z) &= \frac{1}{2} \sum_{k \in E} |\Pr(X + Y = k) - \Pr(Z + Y = k)| \\
&= \frac{1}{2} \sum_{k \in E} |\Pr(X = k - Y) - \Pr(Z = k - Y)| \\
&= \frac{1}{2} \sum_{k \in E} |\sum_{b \in E} (\Pr(Y = b) \Pr(X = k - b) - \Pr(Y = b) \Pr(Z = k - b)| \\
&= \frac{1}{2} \sum_{k \in E} |\sum_{b \in E} \Pr(Y = b)(\Pr(X = k - b) - \Pr(Z = k - b))| \\
&\leq \frac{1}{2} \sum_{k \in E} \sum_{b \in E} |\Pr(Y = b)(\Pr(X = k - b) - \Pr(Z = k - b))| \\
&= \frac{1}{2} \sum_{b \in E} \Pr(Y = b) \sum_{k \in E} |\Pr(X = k - b) - \Pr(Z = k - b)| \\
&\leq \sum_{b \in E} \Pr(Y = b) \cdot \Delta(X, Z) \\
&= \Delta(X, Z)
\end{aligned}$$

■

Claim 5 *For discrete random variables X, Y, Z, W with measurable space E, if X, Y, Z, W are independent, then :*

$$\Delta(X+Y, Z+W) \leq \Delta(X, Z) + \Delta(Y, W).$$

Proof. by Claim 3, We have :

$$\Delta(X+Y, Z+W) \leq \Delta(X+Y, Z+Y) + \Delta(Z+Y, Z+W)$$

then, by Claim 4, We have :

$$\Delta(X+Y, Z+Y) + \Delta(Z+Y, Z+W) \leq \Delta(X, Z) + \Delta(Y, W).$$

■

C The proof of DGSW leakage-resilient in [12], and our improved method.

For a given DGSW ciphertext :

$$\mathbf{C} = \begin{pmatrix} \mathbf{A} \\ \mathbf{b} \end{pmatrix} \mathbf{R} + \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{e}_1 \end{pmatrix}$$

Let $\mathbf{C}_0 = \mathbf{AR} + \mathbf{E}_0$, $\mathbf{c}_1 = \mathbf{bR} + \mathbf{e}_1$. Because $\mathbf{b} = \mathbf{sA}$, thus $\mathbf{C}$ can be rewritten as :

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_0 \\ \mathbf{sC}_0 + \mathbf{e}_1 - \mathbf{sE}_0 \end{pmatrix} \tag{4}$$

The proof in [12] required $\mathbf{sE}_0/\mathbf{e}_1 = \mathsf{negl}(\lambda)$, thus $\mathbf{C} \overset{\mathsf{stat}}{\approx} \begin{pmatrix} \mathbf{C}_0 \\ \mathbf{sC}_0 + \mathbf{e}_1 \end{pmatrix}$. Using the leftover hash lemma with $\mathbf{C}_0$ as a seed and $\mathbf{s}$ as a source, they had that $(\mathbf{C}_0, \mathbf{sC}_0)$ were jointly statistically indistinguishable from uniform, which Lemma 5 followed.

Our method : Below we show that $\mathbf{sE}_0/\mathbf{e}_1 = \mathsf{negl}(\lambda)$ is not necessary to prove that DGSW is leakage-resilient. Through the above analysis, we know that for any DGSW ciphertext, we can always write it in the form of (4). For random $\mathbf{R} \in \mathbb{Z}_q^{n \times ml}$, without loss of generality, assuming $\frac{ml}{n} = g$, we can divide $\mathbf{R}$ into g square matrices :

$$\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_g)$$

where $\mathbf{R}_i \in \mathbb{Z}_q^{n \times n}$. Similarly, for $\mathbf{E}_0 \in \mathbb{Z}_q^{(m-1) \times ml}$, $\mathbf{e}_1 \in \mathbb{Z}_q^{ml}$:

$$\mathbf{E}_0 = (\mathbf{E}_{0,1}, \mathbf{E}_{0,2}, \cdots, \mathbf{E}_{0,g})$$
$$\mathbf{e}_1 = (\mathbf{e}_{1,1}, \mathbf{e}_{1,2}, \cdots, \mathbf{e}_{1,g})$$

where $\mathbf{E}_{0,i} \in \mathbb{Z}_q^{(m-1) \times n}$, $\mathbf{e}_{1,i} \in \mathbb{Z}_q^n$. Let $\{\mathbf{v}_i \in \mathbb{Z}_q^n\}_{i \in [g]}$ be the solution of equation :

$$\{\mathbf{v}_i \mathbf{R}_i = \mathbf{sE}_{0,i}\}_{i \in [g]}$$

Obviously, if $\mathbf{R}_i$ is random over $\mathbb{Z}_q^{n\times n}$, then $\mathbf{v}_i$ has a unique solution with an overwhelming probability. Let $\mathbf{0}^{1\times ml}$ be a zero vector of length ml, Φ be the distribution of public key of DGSW followed by $\mathbf{0}^{1\times ml}$:

$$(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml}) \leftarrow \Phi$$

Let $\mathcal{D}_0(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml})$ be the joint distribution of public key and ciphertext of DGSW, over the randomness $\mathbf{R}$, $\mathbf{E}_0$, $\mathbf{e}_1$:

$$(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml}, \begin{pmatrix}\mathbf{A}\\ \mathbf{b}\end{pmatrix}\mathbf{R} + \begin{pmatrix}\mathbf{E}_0\\ \mathbf{e}_1\end{pmatrix}) \leftarrow \mathcal{D}_0(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml})$$

Let $\mathcal{D}_1(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml})$ be the joint distribution of $(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml})$ and $\mathbf{U} \leftarrow U(\mathbb{Z}_q^{m\times ml})$:

$$(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml}, \mathbf{U}) \leftarrow \mathcal{D}_1(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml})$$

Let P be the decision problems defined as follows :

– Problem P : distinguish whether input x is sampled from distribution X_0 or X_1, where

$$X_0 = \{x : (\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml}) \leftarrow \Phi, \quad x = (\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml}, \begin{pmatrix}\mathbf{A}\\ \mathbf{b}\end{pmatrix}\mathbf{R} + \begin{pmatrix}\mathbf{E}_0\\ \mathbf{e}_1\end{pmatrix}) \leftarrow \mathcal{D}_0(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml})\}.$$

$$X_1 = \{x : (\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml}) \leftarrow \Phi, \quad x = (\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml}, \mathbf{U}) \leftarrow \mathcal{D}_1(\mathbf{A}, \mathbf{b}, \mathbf{0}^{1\times ml})\}.$$

Define set V :

$$V = \{\mathbf{0}^{1\times ml}, \quad (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g)\}$$

Define the distribution $\mathbf{d} \leftarrow \mathcal{D}(V)$ over set V :

$$\{\mathbf{d} \leftarrow \mathcal{D}(V) : \quad \Pr(\mathbf{d} = \mathbf{0}^{1\times ml}) = p \quad \Pr(\mathbf{d} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g)) = 1 - p\}$$

Let Φ' be the joint distribution of DGSW public key and $\mathcal{D}(V)$:

$$(\mathbf{A}', \mathbf{b}', \mathbf{d}) \leftarrow \Phi'$$

Let P' be the decision problems defined as follows :

– Problem P' : distinguish whether input x is sampled from distribution X_0' or X_1', where

$$X_0' = \{x : (\mathbf{A}', \mathbf{b}', \mathbf{d}) \leftarrow \Phi',$$
$$x = (\mathbf{A}', \mathbf{b}', \mathbf{d}, \begin{pmatrix}\mathbf{A}'\mathbf{R}' + \mathbf{E}_0'\\ (\mathbf{b}' + \mathbf{d}_1)\mathbf{R}_1' + \mathbf{e}_1', \cdots, (\mathbf{b}' + \mathbf{d}_g)\mathbf{R}_g' + \mathbf{e}_g'\end{pmatrix}) \leftarrow \mathcal{D}_0(\mathbf{A}', \mathbf{b}', \mathbf{d})\}.$$

$$X_1' = \{x : (\mathbf{A}', \mathbf{b}', \mathbf{d}) \leftarrow \Phi', \quad x = (\mathbf{A}', \mathbf{b}', \mathbf{d}, \mathbf{U}) \leftarrow \mathcal{D}_1(\mathbf{A}', \mathbf{b}', \mathbf{d})\}.$$

where $\mathbf{R}' = (\mathbf{R}_i', \cdots, \mathbf{R}_g') \leftarrow U(\mathbb{Z}_q^{n\times ml})$, $\mathbf{e}_i' \leftarrow \chi^n$, $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_g)$. Thus , for $\mathbf{C}'$:

$$\mathbf{C}' = \begin{pmatrix}\mathbf{C}_0\\ \mathbf{s}\mathbf{C}_0 + \mathbf{e}_1\end{pmatrix} = \begin{pmatrix}\mathbf{A}\mathbf{R} + \mathbf{E}_0\\ \mathbf{b}\mathbf{R} + \mathbf{e}_1 + \mathbf{s}\mathbf{E}_0\end{pmatrix} = \begin{pmatrix}\mathbf{A}\mathbf{R} + \mathbf{E}_0\\ \mathbf{b}\mathbf{R}_1 + \mathbf{e}_{1,1} + \mathbf{s}\mathbf{E}_{0,1}, \cdots, \mathbf{b}\mathbf{R}_g + \mathbf{e}_{1,g} + \mathbf{s}\mathbf{E}_{0,g}\end{pmatrix}$$

it is a sample of X_0' :

$$\begin{pmatrix} \mathbf{A}'\mathbf{R}' + \mathbf{E}_0' \\ (\mathbf{b}' + \mathbf{d}_1)\mathbf{R}_1' + \mathbf{e}_1', \cdots, (\mathbf{b}' + \mathbf{d}_g)\mathbf{R}_g' + \mathbf{e}_g' \end{pmatrix}$$

with $\mathbf{A}' = \mathbf{A}$, $\mathbf{b}' = \mathbf{b}$, $\mathbf{R}' = \mathbf{R}$, $\mathbf{E}_0' = \mathbf{E}_0$, $(\mathbf{e}_1', \mathbf{e}_2', \cdots, \mathbf{e}_g') = (\mathbf{e}_{1,1}, \mathbf{e}_{1,2}, \cdots, \mathbf{e}_{1,g})$, $\mathbf{d} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g)$.

The following process is the same as we showed in Section 5.4. By Theorem 2, if there is an adversary who can distinguish the DGSW ciphertext with uniform distribution(Problem P) that leaks part of the private key, then he can distinguish$(\mathbf{C}_0, \mathbf{s}\mathbf{C}_0 + \mathbf{e}_1)$ with uniform distribution which is jointly statistically indistinguishable by leftover hash lemma.