An attack on SIDH with arbitrary starting curve (draft)

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Abstract

We present an attack on SIDH which does not require any endomorphism information on the starting curve. Our attack is not polynomial-time, but significantly reduces the security of SIDH and SIKE; our analysis and preliminary implementation suggests that our algorithm will be feasible for the Microsoft challenge parameters $p = 2^{110}3^{67} - 1$ on a regular computer. Our attack applies to any isogeny-based cryptosystem that publishes the images of points under the secret isogeny, for example Séta [26] and B-SIDH [9]. It does not apply to CSIDH [8], CSI-FiSh [3], or SQISign [11].

We are working on an implementation but chose to release our algorithm in this draft in light of the release of the independent work of Castryck and Decru [7], which may lead to time and effort being spent on constructing starting curves over fields that will be too small to resist our attack.

1 Introduction

Supersingular Isogeny Diffie-Hellman (SIDH) [17] is a key exchange proposed in 2011 by Jao and De Feo that makes use of isogenies between elliptic curves. A well-studied hard problem in number theory is to find an unknown high-degree isogeny between two (supersingular) elliptic curves over a finite field, on which many cryptosystems [3, 8, 9, 11, 26] are based. This is a problem that is also believed to be hard for quantum computers, and as such *isogeny-based cryptography* has been one of the frontrunners in developing post-quantum cryptographic algorithms. Arguably the most influential primitive in the field of isogeny-based cryptography is Supersingular Isogeny Key Encapsulation (SIKE) [15], which is the incarnation of SIDH that was submitted to the NIST competition to find a new post-quantum-safe cryptographic standard [22], and which is currently in the Fourth Round to be considered for standardization. In comparison to cryptosystems that rely purely on the isogeny problem, such as CSIDH [8], CSI-FiSh [3], and SQISign [11], the hardness assumption underlying SIKE is weaker, as the image of some torsion points under the secret isogeny are also revealed, giving rise to the *supersingular isogeny with torsion* (SSI-T) problem stated more precisely below. This has been shown to be weaker than the pure isogeny problem in a line of work pioneered by Petit [23] in 2017 and continued and built upon in multiple papers in the last 5 years [5, 14, 25]. However, the SIKE parameters had not been effected by these attacks, which all applied only to variants of SIDH.

In this paper we present an algorithm that solves the supersingular isogeny with torsion (SSI-T) problem in reasonable time for parameters that were believed to be secure, which is the hardness assumption underlying SIKE as well as any other SIDH-related protocols such as B-SIDH [9] and Séta [26]. This is recalled below:

Supersingular Isogeny with Torsion (SSI-T):

Given coprime integers A and B, two supersingular elliptic curves $E_0/\mathbb{F}_{p^2}$ and $E_A/\mathbb{F}_{p^2}$ connected by an unknown degree A-isogeny $\varphi_A : E_0 \to E_A$, and given the restriction of φ to the B-torsion of E_0, recover an isogeny φ matching these constraints.

In particular, note that the SSI-T problem does not assume that E_0 is special in any way, for example that it has known endomorphism ring; our attack applies for any starting curve E_0. As such, it does not have the obvious mitigation that previous torsion-point attacks have had of using a trusted setup. There will be large parameters for which our attacks become infeasible; we leave the construction of such parameter sets for future work.

Finally, our attack makes full use of the public torsion points and as such has no effect on any isogeny-based cryptosystem that does not publish images of points under the secret isogeny, such as CSIDH [8], CSI-FiSh [3], and SQISign [11].

Related work

The inspiration for this attack came from an unrelated collaboration of Luciano Maino with Wouter Castryck and Thomas Decru studying superspecial principally polarized abelian surfaces and (2,2)-isogenies between them, as well as endomorphisms of the form

$$\begin{pmatrix} a & b\widehat{\varphi} \\ c\varphi & d \end{pmatrix}.$$

Upon discovering our attack and sharing our idea with Castryck and Decru we found out that they had independently discovered an attack building on the previous collaboration in a different direction. They had already written their implementation and paper, although it was not yet public, and they were kind enough to share both with us as well as to note our forthcoming independent attack in their paper. In particular we were able to build on their implementation rather than starting from scratch when implementing the (2,2)-isogenies in our algorithm, for which we are very grateful. Their paper is now public [7]; they present a polynomial-time attack and provide an implementation for all the proposed NIST parameter sets for SIKE, but their attack relies on the knowledge of $\mathrm{End}(E_0)$, so can potentially be mitigated by generating a starting curve with no known non-scalar endomorphisms. Unsurprisingly given the common source of inspiration, our attack is similar to that of Castryck and Decru, especially the version they hint at in [7, §8.3].

Further acknowledgements

We would like to thank Lorenz Panny for useful discussions regarding complexity of extension field arithmetic and computations of isogenies with non-rational points in the kernel, and for useful comments on an earlier draft of this paper. We thank Christophe Petit for useful comments regarding methods to compute isogenies with irrational kernel points. We would also like to thank COSIC and KU Leuven for hosting Luciano Maino as an intern, sparking his collaboration that led to this paper. Luciano Maino was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) Centre for Doctoral Training (CDT) in Trust, Identity, Privacy and Security in Large-scale Infrastructures (TIPS-at-Scale) at the Universities of Bristol and Bath.

2 The attack

Let all notation be as in the SSI-T problem statement above. The core idea behind our attack is to construct an elliptic curve E, an isogeny $\varphi_f : E \to E_0$, and an endomorphism Φ of the abelian surface

$E \times E_A$ such that $\varphi_A(P_A)$ and $\varphi_A(Q_A)$ can be recovered from $\Phi(\widehat{\varphi}_f(P_A), 0_{E_A})$ and $\Phi(\widehat{\varphi}_f(Q_A), 0_{E_A})$.[1] This is a natural generalization of previous papers on torsion-point attacks (e.g. [23, 25]) in which a special endomorphism of E_A was constructed that leaked information about the secret isogeny; the success of using endomorphisms of $\mathrm{End}(E \times E_A)$ in place of $\mathrm{End}(E_A)$ essentially stems from there being more choice for good endomorphisms in higher dimension.

We will notate our endomorphisms on $E \times E'$ in matrix notation; the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{End}(E \times E')$$

represents that $\alpha \in \mathrm{End}(E)$, $\beta \in \mathrm{Hom}(E', E)$, $\gamma \in \mathrm{Hom}(E, E')$, and $\delta \in \mathrm{End}(E')$, and maps

$$(P, P') \mapsto (\alpha P + \beta P', \gamma P + \delta P').$$

Our attack is a consequence of the following theorem.

Theorem 1. *Let f, A, and B be pairwise coprime integers such that $B = f + A$ and $-1/fA = c^2 \pmod{B}$. Let $E/\mathbb{F}_{p^2}$ and $E'/\mathbb{F}_{p^2}$ be two supersingular elliptic curves connected by an fA-isogeny $\varphi : E \to E'$, let λ be the product polarization on $E \times E'$, let (P_B, Q_B) be a basis of $E[B]$, and let*

$$K := \langle (P_B, c\varphi(P_B)), (Q_B, c\varphi(Q_B)) \rangle.$$

Then K is the kernel of a (B, B)-isogeny of principally polarized abelian surfaces $(E \times E', B\lambda) \to (E \times E', \lambda)$ represented by the endomorphism

$$\Phi := \begin{pmatrix} B - cfA & \widehat{\varphi} \\ c\varphi & -1 \end{pmatrix} \in \mathrm{End}(E \times E').$$

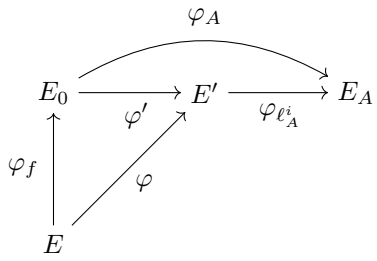
Before proving this theorem, we present our attack. For ease of notation, in all that follows we will assume that $A = \ell_A^a$ and $B = \ell_B^b$. The astute reader will observe that Theorem 1 does not recover the secret isogeny φ_A of SSI-T directly but the composition of φ_A with an isogeny of degree f where satisfies some conditions. The attacker can however choose an isogeny of degree f and an elliptic curve E (satisfying the conditions coming from Theorem 1) such that $\varphi_f : E \to E_0$, then apply the theorem to recover the isogeny $\varphi_A \circ \varphi_f : E \to E_A$. The problem faced by the attacker is that the computation of φ_f is not necessarily easy as there is no reason that $B - A$ would be smooth. To mitigate this, we increase our pool of available cofactors f by brute-forcing the last few steps of φ_A and/or by brute-forcing some extra torsion-point images.

The picture that we should keep in mind when reading through the attack below is the following commutative diagram, where:

- $\varphi_A : E_0 \to E_A$ is the secret key,
- $\varphi_f : E \to E_0$ is a f-isogeny chosen by the attacker,[2]
- $\varphi_{\ell_A^i} : E' \to E_A$ is a guess of the (dual of the) last i steps of φ_A,
- $\varphi' : E_0 \to E'$ is the corresponding first $a - i$ steps of φ_A such that $\varphi_A = \varphi_{\ell_A^i} \circ \varphi'$, and
- $\varphi : E \to E'$ is the $f\ell_A^{a-i}$-isogeny to which we apply Theorem 1.

[1] In practise, we choose a curve E' close to E_A in the ℓ_A-isogeny graph and finish off the attack with brute-force as this is more efficient.

[2] In practise, the attacker computes $\widehat{\varphi}_f$ and deduces φ_f from this.



(1)

The attack is described in Algorithm 1. To discuss the complexity of this attack we should split it into three parts:

1. The precomputation step (Step 1); this can be done once-and-for-all for each parameter set A, B.
2. The cofactor isogeny computation (Step 2); if SIDH is set up with a fixed (arbitrary) E_0, this can be done once-for-all for this E_0.
3. The online steps (Steps 3 to 8); these steps need to be performed for every new public key.

The cost of the cofactor isogeny computation. The cofactor isogeny is deterministic and chosen by the attacker and, as such, does not need to be recomputed at any point because a wrong guess was made when brute-forcing. We compute the isogeny φ_f via a chain of prime-degree $q-$isogenies φ_q. The cost of computing φ_q for the larger factors of q is discussed in detail in Section 2.1.2 and can be approximated by $\tilde{O}(q^{3/2})$ multiplications in $\mathbb{F}_{p^2}$ plus a call to an oracle to find an irreducible polynomial of degree $\approx q$ in $\mathbb{F}_{p^2}$ or $\tilde{O}(q^2)$ multiplications in $\mathbb{F}_{p^2}$ plus a call to an oracle for factoring the $q-$division polynomial over $\mathbb{F}_{p^2}$.

The cost of the online steps. The discussion in Section 2.1.1 approximates the cost of Steps 3 to 8 by $\approx C \cdot e^4 \ell_A^i q_e^4 \log q_e$, where q_e is the largest prime dividing e and C is polynomial in $\log(p)$. We allow i and e to grow to increase the pool of options for f in order to get a smaller q_f.

The precomputation. The cost of the precomputation step to find optimal parameters can be played off against working with suboptimal but still good parameters for the rest of the attack, and furthermore the optimal choice depends on the instantiation of SIDH.

- If SIDH is set up to start every key exchange with a new E_0, the optimal choice of (e, i, j, f) for the attacker ensures that the cost of Step 2 is approximately the same as the cost of Steps 3 to 5. A first search for good parameters would take the form of a brute force search over all parameters (e, i, j, f) such that $q_f^{3/2} \leq e^4 \ell_A^i q_e^4 \log q_e$ and $0 \leq j \leq b$. This may return some choices for f for which q_f-isogenies are impractical to compute; each choice needs to be compared against the potential extension field arithmetic and discarded if so. The definition of this search space could and should be made more precise with better estimates on extension field arithmetic and the complexity of the polynomial-time steps of the attack.
- If SIDH is set up to start at a fixed curve E_0 (which can be arbitrary), the optimal choice of (e, i, j, f) for the attacker ensures that the cost of Step 2 is on the edge of feasible, as this can be done once-and-for-all, at the bonus of decreasing the cost of Steps 3 to 5. As such, a first search for good parameters would take the form of a brute force search over all parameters (e, i, j, f) with $q_f^{3/2} < 2^\lambda$, where the attacker has access to computing power that can perform 2^λ operations in their given timeframe, and with $C \cdot e^4 \ell_A^i q_e^4 \log q_e$ as small as possible (in particular $< 2^\lambda$).

See Section 2.1.1 for details on how to choose these parameters as well as a choice for two SIDH instances.

Algorithm 1: Recovering the secret isogeny.

Input: Coprime integers $A = \ell_A^a$ and $B = \ell_B^b$, two supersingular elliptic curves $E_0/\mathbb{F}_{p^2}$ and $E_A/\mathbb{F}_{p^2}$ connected by an unknown degree-A-isogeny $\varphi_A : E_0 \to E_A$, a basis $\{P_B, Q_B\}$ of $E_0[B]$, a basis $\{P_A, Q_A\}$ of $E_0[A]$, the image points $\varphi_A(P_B)$, $\varphi_A(Q_B)$.

Output: $\varphi_A : E_0 \to E_A$.

1 Compute integers e, j, f, and i such that e is small and smooth, $0 \leq j \leq b$, f is smooth and positive, i is small, $-fA\ell_A^{-i} = c^2 \pmod{eB\ell_B^{-j}}$, and $eB\ell_B^{-j} = f + A\ell_A^{-i}$ For ease of notation, we set $A' = A\ell_A^{-i}$ and $B' = B\ell_B^{-j}$. For more details, see Section 2.1.1.

2 Compute a curve that is f-isogenous to E_0, define the dual of the computed isogeny to be $\varphi_f : E \to E_0$, and compute $\widehat{\varphi}_f(P_A), \widehat{\varphi}_f(Q_A), \widehat{\varphi}_f(P_B), \widehat{\varphi}_f(Q_B)$. For more details, see Section 2.1.2.

3 Compute a basis $\{P_{eB'}, Q_{eB'}\}$ of $E[eB']$ such that $[e]P_{eB'} = [\ell_B^j]\widehat{\varphi}_f(P_B)$ and $[e]Q_{eB'} = [\ell_B^j]\widehat{\varphi}_f(Q_B)$.

4 Choose a guess $\varphi_{\ell_A^i} : E' \to E_A$ for the last i steps of φ_A, recall the definition of the corresponding $\varphi : E \to E'$ from diagram (3), and choose $R, S \in E'[eB']$ such that

$$[e]R = [\ell_A^{-i} f \ell_B^j]\widehat{\varphi}_{\ell_A^i} \circ \varphi_A(P_B)$$

and

$$[e]S = [\ell_A^{-i} f \ell_B^j]\widehat{\varphi}_{\ell_A^i} \circ \varphi_A(Q_B);$$

R, S are a guess for the images $\varphi(P_{eB'}), \varphi(Q_{eB'})$ respectively.

5 Attempt to compute a (eB', eB')-isogeny $\Phi_{\text{guess}} \in \text{End}(E \times E')$ from

$$\ker(\Phi_{\text{guess}}) = \langle (P_{eB'}, cR), (Q_{eB'}, cS) \rangle;$$

this is the kernel of a (eB', eB')-isogeny $E \times E' \to E \times E'$ exactly when $R = \varphi(P_{eB'})$ and $S = \varphi(Q_{eB'})$, in which case

$$\Phi_{\text{guess}} = \Phi = \begin{pmatrix} eB' - cfA' & \widehat{\varphi} \\ c\varphi & -1 \end{pmatrix} \in \text{End}(E \times E')$$

as described in Theorem 1. On failure, return to Step 4 and take a new guess $(\varphi_{\ell_A^i}, R, S)$. For more details see Section 2.2.

6 Set $P_{A'} = [\ell_A^i]\widehat{\varphi}_f(P_A)$ and $Q_{A'} = [\ell_A^i]\widehat{\varphi}_f(Q_A)$ $\{P_{A'}, Q_{A'}\}$. Compute $\varphi(P_{A'})$ and $\varphi(Q_{A'})$ via

$$\Phi(P_{A'}, 0_{E'}) = ([eB']P_{A'}, [c]\varphi(P_{A'}))$$

and

$$\Phi(Q_{A'}, 0_{E'}) = ([eB']Q_{A'}, [c]\varphi(Q_{A'})).$$

7 Compute the action of φ' on the A'-torsion of E_0 via $\varphi'([\ell_A^i]P_A) = [f^{-1}]\varphi(P_{A'})$ and $\varphi'([\ell_A^i]Q_A) = [f^{-1}]\varphi(Q_{A'})$.

8 Set

$$\ker(\widehat{\varphi'}) = \langle \varphi'([\ell_A^i]P_A) + \varphi'([\ell_A^i]Q_A) \rangle$$

and $\varphi' = \widehat{\widehat{\varphi'}}$. Return $\varphi_{\ell_A^i} \circ \varphi'$.

Proof of Theorem 1. Observe that

$$\begin{pmatrix} 1 & \widehat{\varphi} \\ c\varphi & cfA - B \end{pmatrix} \begin{pmatrix} B - cfA & \widehat{\varphi} \\ c\varphi & -1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},$$

this is that the composition of the endomorphism Φ and

$$\Phi_d = \begin{pmatrix} 1 & \widehat{\varphi} \\ c\varphi & cfA - B \end{pmatrix}$$

is the multiplication-by-B endomorphism on $E \times E'$. Also

$$\Phi(P_B, c\varphi(P_B)) = ((B - cfA)P_B + c\widehat{\varphi}\varphi(P_B), -c\varphi(P_B) + c\varphi(P_B)) = (0_E, 0_{E'}),$$

and similarly

$$\Phi(Q_B, c\varphi(Q_B)) = (0_E, 0_{E'}),$$

so K defines a subgroup of the kernel of Φ. One can similarly check that

$$K_d = \langle (fAP_B, -\varphi(P_B)), (fAQ_B, -\varphi(Q_B)) \rangle$$

defines a subgroup of the kernel of Φ_d. As P_B and Q_B are independent order-B points by definition and c and A are coprime to B, we have that $\varphi(P_B)$ and $c\varphi(Q_B)$ are also independent order-B points so by counting degrees we see that $K = \ker(\Phi)$; in particular

$$\ker(\Phi) \cong \mathbb{Z}/B\mathbb{Z} \times \mathbb{Z}/B\mathbb{Z}.$$

To prove that Φ is a principally polarized isogeny, we need to show further that K is a maximal isotropic group with respect to the pairing induced by the product polarization λ. First, we prove that K is isotropic. Let

$$\begin{matrix} (a_1P_B + b_1Q_B, c \cdot a_1\varphi(P_B) + c \cdot b_1\varphi(Q_B)), \\ (a_2P_B + b_2Q_B, c \cdot a_2\varphi(P_B) + c \cdot b_2\varphi(Q_B)) \end{matrix} \in K,$$

then

$$\begin{aligned} e_B^{E\times E'}\Big((a_1P_B + b_1Q_B, c \cdot a_1\varphi(P_B) + c \cdot b_1\varphi(Q_B)), \\ (a_2P_B + b_2Q_B, c \cdot a_2\varphi(P_B) + c \cdot b_2\varphi(Q_B))\Big) = \end{aligned}$$

$$e_B^E(P_B, Q_B)^{\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}} e_B^{E'}(\varphi(P_B), \varphi(Q_B))^{c^2\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}} =$$

$$e_B^E(P_B, Q_B)^{(1+c^2fA)\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}} = 1,$$

where the last step follows from the identity $c^2 = -1/fA \pmod{B}$, hence K is isotropic. As for the maximality of K, let $(P_E, P_{E'}) \in (E \times E')[B]$ such that, for all $T \in K$,

$$e_B^{E\times E'}((P_E, P_{E'}), T) = 1. \tag{2}$$

Let $s, t \in \mathbb{Z}$ such that $P_E = sP_B + tQ_B$. Equation 2 is equivalent to the statement that for all $P' \in E'[B]$ we have

$$e_B^{E\times E'}(P', P_{E'} + c(s\varphi(P_B) + t\varphi(Q_B))),$$

that is $P_{E'} + c(s\varphi(P_B) + t\varphi(Q_B)) = 0_{E'}$, or equivalently $(P_E, P_{E'}) \in K$.

So far, we have proved that $(E \times E')/K = E \times E'$ is endowed with a principal polarization λ'. However, this does not suffice to prove that λ' coincides with the product polarization λ. To do so, we employ Kani's Reducibility Criterion [18, Theorem 2.6] (c.f. similar use in [7, Theorem 1]). In Kani's language, we have set up our parameters so that $(\varphi, A\ker(\varphi), f\ker(\varphi))$ is an *isogeny diamond configuration of order* B; that is, setting $H_1 = A\ker(\varphi)$ and $H_2 = f\ker(\varphi)$ we have the conditions $H_1 \cap H_2 = \{0_E\}$, $\#H_1 + \#H_2 = B$ and $\#H_1 \cdot \#H_2 = \deg(\varphi)$. Therefore by Kani's Reducibility Criterion, $\lambda' = \lambda$. □

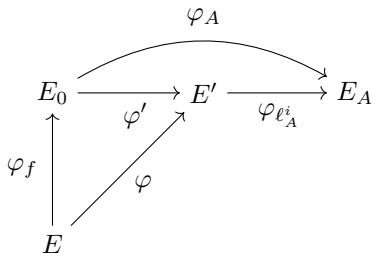
2.1 Complexity of Algorithm 1

Here we give some details on and study the complexity of the first four steps of Algorithm 1 in the case relevant to SIKE, namely $A = 3^a$ and $B = 2^b$, with a focus on the Microsoft challenge parameters $A = 3^{67}$ and $B = 2^{110}$ and the parameters that were proposed for NIST level I $A = 3^{137}$ and $B = 2^{216}$.

2.1.1 Choosing parameters

To understand Step 1, we recall the commutative diagram that we keep in mind during this attack, where:

- $\varphi_A : E_0 \to E_A$ is the secret key,
- $\varphi_f : E \to E_0$ is a f-isogeny chosen by the attacker,
- $\varphi_{\ell_A^i} : E' \to E_A$ is a guess of the last i steps of φ_A,
- $\varphi' : E_0 \to E'$ is the corresponding first $a - i$ steps of φ_A such that $\varphi_A = \varphi_{\ell_A^i} \circ \varphi'$, and
- $\varphi : E \to E'$ is the $f\ell_A^{a-i}$-isogeny to which we apply Theorem 1.



(3)

Choosing f. The shape of f determines the complexity of computing φ_f. The cofactor f does not need to be small as the isogeny is deterministic but it does need to be smooth: Consider the extreme case that f is prime and $\approx A$, computing φ_f directly will be harder than computing φ_A directly (because of the extension field arithmetic). Exactly how smooth we require f to be depends on what we hope we can achieve in complexity for the attack. If q is the largest prime divisor of f then the complexity of Step 2 will be dominated by the cost of the computation of a q-isogeny, which can be performed $O(\sqrt{q})$ multiplications in the field of definition of a generator of the kernel of the isogeny using sqrtVelu [1, Section 4.1.4]. The field of definition is however hard to control, and large field extensions can seriously slow down our arithmetic. It is hard to make this precise; for some values of q the minimal k for which $E(\mathbb{F}_{p^k})$ contains a q-torsion point will be much smaller than q and in some cases it will be much larger.

In order to make an approximation of the complexity of computing φ_f on which we can base our search for good parameters for our attack, we ran some experiments to look at the behaviour of field

extensions for different values of p. As an illustration let us consider $E_{1728}/\mathbb{F}_p$ with $p = 2^{216}3^{137} - 1$ as in the proposed NIST level I parameters for SIKE. Only the even degree extension fields are relevant as the codomain of each isogeny is defined over $\mathbb{F}_{p^2}$. Figures 2, 3, and 4 show the q for which there exists an even $k \leq 1000$ such that there is an $\mathbb{F}_{p^k}$-rational point of order q (only the minimal even k is depicted). In total, we find 72% of the primes $< 10^2$ (c.f. Figure 2), 62.5% of the primes $< 10^3$ (c.f. Figure 3), and 22% of the primes $< 10^4$ (c.f. Figure 4). Based on these experiments, to guide our parameter selection for our attack we make a very crude estimate that we expect the minimal field extension for the maximal q dividing f is about size q. This gives us a very rough estimate of $\tilde{O}(q^{3/2})$ for the complexity of computing a φ_q-isogeny, which in turn is the dominating cost of computing φ_f. If the largest factor of our smoothest f only admits very large extension fields we can of course choose to take a slightly less smooth f (that is, a slightly bigger q) where the field extension is smaller, or use Kohel's algorithm at the expense of increase the complexity to $\tilde{O}(q^2)$; see Section 2.1.2 for more details.

Choosing i and e. The cost coming from i is the cost of brute-forcing all the cyclic $3^i = \ell_A^i$-isogenies from E_A, which costs $\approx 3^i$ multiplications in $\mathbb{F}_{p^2}$. This is however multiplied by the brute-force cost of guessing the images of the e-torsion points in Step 4 and by the cost of computing Φ. Guessing the images of the e-torsion points amounts to checking all the pairs of points of order e on E', which is $\approx e^4$. This is one sense in which e and i have to be 'small': We have to run Steps 3 to 5 of Algorithm 1 $\approx e^4 3^i$ times.

Additionally, the endomorphism Φ (which we will attempt to compute $\approx e^4 3^i$ times) is an (eB', eB')-isogeny; in particular it factors via an (e, e)-isogeny. So, in addition we require e to be q-smooth, where q is the largest prime for which it is feasible to compute (q, q)-isogenies (potentially over an extension field, which again will add a non-negligible cost). The need for the computation of the (e, e)-isogeny is the main barrier to implementing our algorithm for the proposed NIST parameters, as to do so requires a working implementation of (q, q)-isogenies, which while should theoretically be possible and reasonably fast, requires some research to achieve. There exists literature on this topic [4, 6, 20, 21], from which we have made a baseline assumption than computation of a (q, q)-isogeny over $\mathbb{F}_{p^k}$ can be performed in $O(q^3)$ multiplications in $\mathbb{F}_{p^k}$. However, there is very little existing work in the way of practical implementation of supersingular Jacobians and products of elliptic curves. We do note here that it would be possible to avoid implementing the factors of the (e, e)-isogenies to also map to and from products of elliptic curves, as we can ensure to start and finish the computation of Φ with a (2,2)-isogeny, which may make the practical implementation of (e, e)-isogenies with regards to this attack a more achievable goal.

Working with our baseline assumption that a (q, q)-isogeny can be computed in approximately q^3 multiplications over the base field of its kernel, we expect the cost of computing Φ as a (eB, eB)-isogeny to be dominated by the cost of computing a (q, q)-isogeny where q is the largest prime factor of e. We leave a careful analysis of the sizes of the field extensions for genus 2 to later work that includes a practical implementation of (q, q)-isogenies for prime $q \neq 2$, but let us assume for the sake of argument that the slow down for the extension field arithmetic scales with q similarly to the elliptic curve case. Then we approximate the cost of computing the (q, q)-isogeny by $O(q^3 \cdot q \log q)$. This is probably an overestimate: More research is needed into the existence of sqrtVélu-style-algorithms in the case of abelian surfaces. However, if the attack costs 2^λ, note that e is already forced to be relatively small compared to this by the fact that we have to search through $\approx e^4$ pairs of possible images of e-torsion points. Because of this, we can expect e to be fairly smooth compared to f, for example, so q (and the corresponding field extension) need not be particularly large.

In our choice of parameters for our toy example, we have chosen to demonstrate the use of e without the need to delve into (q, q)-isogenies for $q > 2$ by choosing $e = 2^3$. In this case we need a field extension of degree 4 for the 2^{b+1}-torsion points, of degree 8 for the 2^{b+2}-torsion points and of degree 16 for the 2^{b+3}-torsion points. This is not special to this instance but a consequence of the fact that the pull-back of the multiplication-by-2 map contains a square root (and no other rational

Figure 1: Extension field degrees < 1000 needed for $\mathbb{F}_{p^k}$-rational q-torsion

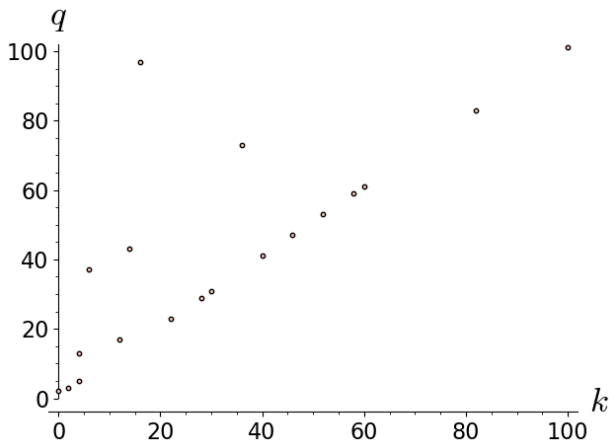


Figure 2: $q < 10^2$

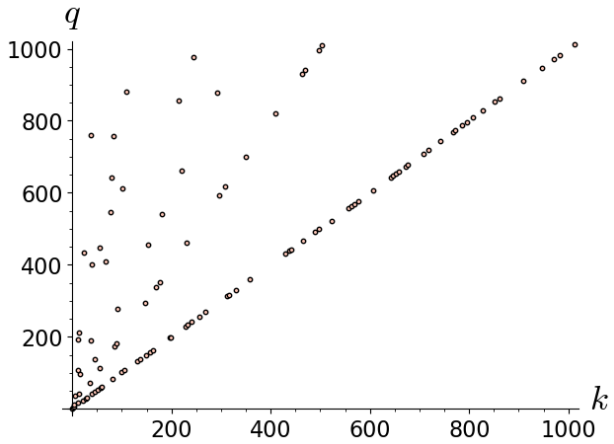


Figure 3: $q < 10^3$

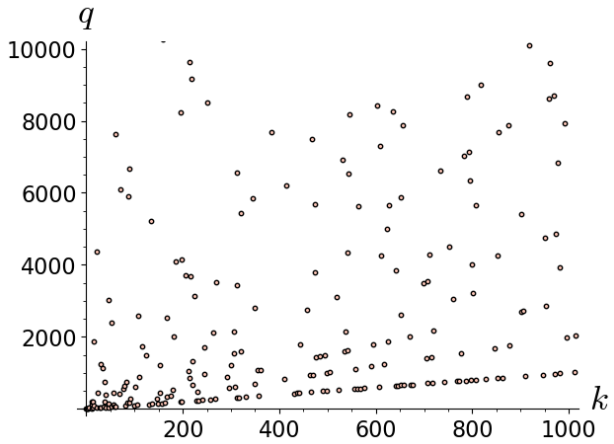


Figure 4: $q < 10^4$

but not integral powers), and so each lift of a point of order 2^i to a point of order 2^{i+1} will either double the degree of the field extension or keep it the same.

Choosing j. The choice of j only effects the precomputation step, Step 1 of Algorithm 1, as we achieve $B' = 2^{-j}B$-torsion points by multiplying the known B-torsion by 2^j; for this reason we have no restrictions on non-negative j. Notice that we do not require e to be coprime to B, so e may contain powers of two, accounting also for the possibility of negative j.

We ensure that $-fA'$ is a square mod eB' as it is a necessary condition in Theorem 1. This is not particularly restrictive and does not cause problems in practise.

Two concrete parameter choices. We present here two parameter choices in a toy example and in one of the cases of the Microsoft challenge parameters. We will add parameter choices for NIST Level I upon completion of the code running on Step 1 of Algorithm 3.

Toy parameters: We consider a small example to test our algorithm: $A = 3^8$, $B = 2^{15}$, $i = 3$, $e = 8$, $j = 0$, $f = 5 \cdot 7^2 \cdot 11 \cdot 97$. The largest field extension that we need for the computation of φ_f is $\mathbb{F}_{p^{20}}$, for the 11-isogeny. The largest field extension for $e = 8$ is 16, for the pullbacks of the order-2^{15} points to order-2^{18} points.

Challenge parameters: We consider one of the sets of challenge parameters put forward by Microsoft [10]: $A = 3^{67}$, $B = 2^{110}$, $i = 7$, $e = 1$, $j = 2$,

$$f = 5 \cdot 7 \cdot 13^3 \cdot 43^2 \cdot 73 \cdot 151 \cdot 241 \cdot 269 \cdot 577 \cdot 613 \cdot 28111 \cdot 321193.$$

The largest field extension we would need for the computation of φ_{321193} using sqrtVélu is of degree 642384; in this case it might be faster to use a variant of Kohel's algorithm to avoid the extension field arithmetic (see Section 2.1.2 for more details). Based on preliminary SageMath experiments, we expect the computation of φ_{321193} to be feasible on a regular laptop. The extension field degrees for all the factors of f are given by

$$\begin{aligned}[k, q] =&[8, 5], [12, 7], [24, 13], [28, 43], [144, 73], [75, 151], [480, 241],\\ &[67, 269], [1152, 577], [1224, 613], [56220, 28111], [642384, 321193].\end{aligned}$$

The choice of $i = 7$ also means that we need to run Steps 3 to 5 of Algorithm 1 up to $3^7 \approx 2^{11}$ times, which we expect to take at most a few hours on a laptop. In particular, if the SIDH instantiation uses a fixed (arbitrary) starting curve, the computation of φ_f can be performed as a precomputation and the attack on an individual public key is relatively fast, just some Richelot isogenies of abelian surfaces and 3-isogenies of elliptic curves, repeated potentially 3^7 times.

An alternative choice. This example can already illustrate the freedom that being able to compute efficiently (ℓ, ℓ)-isogenies for $\ell \neq 2$ can provide: We open up more options for attack parameters, including in this case in which one requires very little brute-force (only repeating Steps 4 to Step 5 up to 4 times): $A = 2^{110}$, $B = 3^{67}$, $A' = 2^{a-j} = 2^{108}$, $B' = 3^{b-i} = 3^{48}$, $e = 1$, and

$$f = 5 \cdot 7 \cdot 13 \cdot 61 \cdot 73 \cdot 431 \cdot 593 \cdot 607 \cdot 881 \cdot 36997 \cdot 139393 \cdot 227233.$$

The extension field degrees for all the factors of f are given by

$$\begin{aligned}[k, q] =&[8, 5], [12, 7], [24, 13], [60, 61], [144, 73],\\ &[860, 431], [1184, 593], [303, 607], [220, 881],\\ &[73992, 36997], [34848, 139393], [56808, 227233].\end{aligned}$$

2.1.2 Computing the cofactor isogeny

The points in the kernel of a factor φ_q of φ_f will not be defined over $\mathbb{F}_{p^2}$ in general. When we choose the value of f, as well as checking that f is smooth, we check, for each prime factor q, the degree k of the field extension that would be required to find a point of order q.

When computing an isogeny $\varphi_q : E_n \to E_{n+1}$ through which φ_f factors, in order to control field extensions in the whole attack, we need to choose φ_q so that

- the codomain E_{n+1} is defined over $\mathbb{F}_{p^2}$, and
- the image points $\varphi_q(P)$ and $\varphi_q(Q)$ are defined over $\mathbb{F}_{p^2}$.

To compute large-degree isogenies, we can use the sqrtVélu method described in [1, Section 4.14], which has complexity $\tilde{O}(q^{1/2}\mathsf{m}_k)$, where m_k is the cost of a multiplication in $\mathbb{F}_{p^k}$. As outlined above we expect $k \approx q$ on average if we allow for some freedom in the choice of f. To guide our choice of attack parameters, we therefore take the complexity of computing our large-degree isogenies and the images of points under these to be $\tilde{O}(q^{3/2})$. However, for large k, finding an irreducible polynomial to generate $\mathbb{F}_{p^k}$ may be a bottleneck. We leave investigation into whether or not this search can be improved using a quantum algorithm to future work.

When the minimal extension degree of the field in which the kernel points of a q-isogeny φ_q are defined is large, it will be faster to instead use a variant of Kohel's algorithm [19, Section 2.4]. Kohel's algorithm computes the isogeny from its kernel polynomial, which, assuming E is defined as $y^2 = f(x)$, is defined by

$$K(x) = \prod_{(x_P, \pm y_P) \in \ker(\varphi)} (x - x_P) \in \mathbb{F}_{p^2}[x];$$

each x_P appears only once so $\deg(K) = (q-1)/2$. Constructing the kernel polynomial from this definition would also require computing the extension field in which the x_P live, but we can also construct a choice for $K(x)$ from the q-division polynomial.

The q-division polynomial for E is defined by

$$\psi_q(x) = \prod_{(x_P, \pm y_P) \in E[q]} (x - x_P) \in \mathbb{F}_{p^2}[x],$$

and can either be precomputed for an E with general coefficients (e.g. a Montgomery coefficient A) or computed recursively for a given E [27, Exercise 3.7]. In [2, Section 9] a careful analysis is given of both approaches to computing division polynomials; evaluation of a precomputed polynomial can be faster if q is fairly small but if q is large enough that multiplying polynomials of degree q^2 will be faster using FFT, then it will be faster to compute them directly for any given E. For these large q, the cost of computing the division polynomial is $O(q^2 \log q)$.

So, suppose we have computed the q-division polynomial, we can then factorize the degree-$(q-1)^2/2$-polynomial ψ_q into irreducible factors over $\mathbb{F}_{p^2}$. If there exists an irreducible factor of φ_q of degree $(q-1)/2$, then we can choose this for $K(x)$, the kernel polynomial of E, and compute φ_q using Kohel's algorithm in time $O(q^2)$. The factorization of division polynomials has been completely described by Verdure [30]. In particular, for large k there exists at least one irreducible factor of degree $(q-1)/2$ over $\mathbb{F}_{p^2}$.

Although factorization of large-degree polynomials is polynomial-time in $\log(p)$ and q, as q is large the complexity of this step can easily grow to infeasible. We hope that our algorithm can be improved by either searching for only one[3] irreducible factor of degree $(q-1)/2$, or by using quantum algorithms (e.g. [12]), or both. We leave the details of this to future work.

[3] Since we only need an irreducible factor of degree $(q-1)/2$, after the "Distinct-degree factorization" stage in [12], we can run "Equal-degree factorization" on a single square-free polynomial. However, this does not improve the asymptotic complexity of the algorithm in our case.

In this case, for large q, the entire cost of computing φ_q is $\tilde{O}(q^2)$ multiplications in $\mathbb{F}_{p^2}$, plus a call to a (quantum?) oracle for factoring.

2.2 Computing (ℓ, ℓ)-isogenies

In order for our algorithm to reach its full potential it is necessary to also consider integers e in Step 1 of Algorithm 1 that do not divide B, and in particular are not necessarily powers of two. It may also be that there is a nice parameter choice (e, i, j, f) with A a power of 2 and B a power of 3, or one may want to consider more general setups. In all of these cases, in Step 5 of Algorithm 1 it will be necessary to compute (ℓ, ℓ)-isogenies for $\ell \neq 2$, which as observed above requires more research to achieve practically (for $\ell = 3$ there is however already some interesting work on this topic [6]). For this reason, we leave all instantiations of the attack that use e not dividing B to future work and focus on the case of $(2, 2)$-isogenies, that is, $B = 2^b$ and $e|B$. Recall that we set $B' = B2^{-j}$, where $0 \leq j \leq b$.

In order to compute the chain of $(2, 2)$-isogenies whose composition is the (eB', eB')-isogeny Φ, we need to able to compute three different flavours of $(2, 2)$-isogenies between principally polarized abelian surfaces:

- A (2,2)-isogeny from a Jacobian of a genus 2 curve to a Jacobian of a genus 2 curve, for which we refer to reader to [29, §2.3.1].
- A (2,2)-isogeny from a Jacobian of a genus 2 curve to a product of elliptic curves, for which we refer the reader to [28, Proposition 8.3.1]. (This is required for the last step of Φ).
- A (2,2)-isogeny from a product of elliptic curves to the Jacobian of a genus 2 curve, for which we refer the reader to [7] for more details. (This is required for the first step of Φ).

Upon sharing our application to breaking SIKE with Castryck and Decru, they were kind enough to share an implementation of this step with us that they had written for their paper [7], which at the time was not publicly available. Their implementation and description of the first step of the (2,2)-isogeny path (ProdToJac) and of the intermediate steps (JacToJac) provided us with useful insights, and the current incarnation of our implementation of our attack now uses Giacomo Pope and collaborator's SageMath implementation [24] for these steps, modified only to include the computation of the images of the 3-power torsion points.

2.2.1 Endomorphism up to isomorphisms

After computing the chain of (2,2)-isogenies Φ with kernel

$$\langle (P_B, c\varphi(P_B)), (Q_B, c\varphi(Q_B)) \rangle,$$

we end up on an elliptic product $F \times F'$ that is isomorphic to $E \times E'$ with respect to the product polarizations. Following [13, §5.2], isomorphism classes of elliptic product are identified via unordered pair of distinct $j-$invariants when the two curves are not isomorphic - which is the case with overwhelming probability. Therefore, to recover the secret key, we only need to study the projection of Φ onto the curve with $j-$invariant equal to the $j-$invariant of E.

3 Future work

Our algorithm to attack SIDH, discovered independently from Castryck and Decru [7] but inspired by an earlier joint project, makes no use of any special endomorphisms on E_0 and as such can be applied to an arbitrary starting curve E_0. It is however not clear how well it scales, as there will

be choices of A and B for which, even with allowing for some brute-force to increase the solution space, the cost of computing the cofactor isogeny will be sufficiently high to protect SIKE against our attack. We leave the construction of such parameters, as well as the search for generalizations of this idea to attack larger parameters, to future work.

An implementation of the attack for the Microsoft challenge parameters will appear in a later version of this paper, and if progress is made on the computation of (ℓ, ℓ)-isogenies also for the proposed NIST level I parameters, to get some benchmark timings.

Finally, we propose some open questions, the answers to which will increase our understanding of the reach of this attack and how to compute rescaled parameters for SIKE achieving the required security levels with respect to this attack, assuming of course that it is possible to construct starting curves with unknown endomorphisms to mitigate the polynomial-time attack of [7].

Open question 1. How can we compute the optimal choice for f, taking into account all speed-ups available from subfield arithmetic? Can we implement an operation counter to find best trade-off?
Open question 2. The algorithm we currently use to select parameters in Step 1 of Algorithm 1 is just a brute-force search over the entire parameter space, which becomes infeasible for large instances, although good parameters may still exist. Can we improve this search, or even construct a smooth f deterministically?
Open question 3. For a given new large parameter set for SIKE, starting from a curve with no known non-integral endomorphisms, can we prove that it has 2^λ-bit security? That is, can we prove that there is no choice (e, i, j, f) for which our attack takes $< 2^\lambda$ multiplications in $\mathbb{F}_{p^2}$ (or equivalent)?

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