Anonymous Permutation Routing

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Abstract. The Non-Interactive Anonymous Router (NIAR) model was introduced by Shi and Wu [SW21] as an alternative to conventional solutions to the anonymous routing problem, in which a set of senders wish to send messages to a set of receivers. In contrast to most known approaches to support anonymous routing (e.g. mix-nets, DC-nets, etc.) which rely on a network of routers communicating with users via interactive protocols, the NIAR model assumes a *single* router and is inherently *non-interactive* (after an initial setup phase). In addition to being non-interactive, the NIAR model is compelling due to the security it provides: instead of relying on the honesty of some subset of the routers, the NIAR model requires anonymity even if the router (as well as an arbitrary subset of senders/receivers) is corrupted.

In this paper, we present a protocol for the NIAR model that improves upon the results from [SW21] in two ways:

- Improved computational efficiency (quadratic to near linear): Our protocol matches the communication complexity of [SW21] for each sender/receiver, while reducing the computational overhead for the router to polylog overhead instead of linear overhead.
- Relaxation of assumptions: Security of the protocol in [SW21] relies on the Decisional Linear assumption in bilinear groups; while security for our protocol follows from the existence of any rate-1 oblivious transfer (OT) protocol (instantiations of this primitive are known to exist under DDH, QR and LWE [DGI+19,GHO20]).

Keywords. Anonymous Routing, Private-Information Retrieval, Permutation Routing, Non-Interactive Protocols.

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1 Introduction

As the collection and access of digital information in our daily lives becomes ever-more ubiquitous (internet, local networks, mobile networks, IoT), so too does the need for the development of technologies to protect access and transmission of this data. While protecting the integrity and access to sensitive data remain important tasks, there has been a growing need for *anonymity* in protecting data access and communications between users. Throughout this paper, anonymity will refer to the inability to associate which nodes in a network are communicating with each other; i.e. the unlinkability between one or more senders and the associated receiver(s). The conventional approach to providing such protection (onion routing, mix-nets, and others) relies on a network of routers relaying messages, where anonymity is only guaranteed if there are sufficiently many uncorrupted routers. A markedly different approach to this problem was recently introduced by Shi and Wu [SW21], who proposed using cryptographic techniques to hide connectivity patterns. Namely, they introduce the Non-Interactive Anonymous Router (NIAR) model, in which a set of N receiving nodes wish to receive information from a set of N sending nodes, with all information passing through a central router. Anonymity in their model is defined to be the inability to link any sender to the corresponding receiver, even if the router and (up to $N-2$) various (sender, receiver) pairs are susceptible to attack by an (honest-but-curious) adversary.

Notice that (assuming PKI) an immediate solution to anonymity in the NIAR model is to have each sending node encrypt their message (under the desired recipient node's public key or using a shared secret key with the recipient), send the encrypted message to the center router, and then simply have the router flood all N (encrypted) messages to each of the N receivers. While this naïve approach satisfies anonymity (as well as privacy, in that receivers only receive messages intended for them), it has the pitfall of excessive communication: $O(N)$ for each recipient node; $O(N^2)$ for the router. Shi and Wu present in [SW21] a protocol which, under the Decisional Linear assumption (on certain bilinear groups), achieves anonymity with minimal *communication* overhead.

Having re-framed the goal of anonymity to the NIAR model and with the toolbox of cryptographic techniques at hand, a natural observation is that Private Information Retrieval (PIR) can be used as a potential solution. In (single server) PIR [KO97], a server stores a database DB of N elements, and clients issue queries to the server to retrieve the i^{th} element $DB[i]$. Security in the PIR model means that the server does not learn the index i being queried. Thus, if N senders encrypt their messages and send them to the router, we can view the router as acting as a PIR server with the N concatenated (encrypted) messages forming the contents of the PIR database. Each receiver can then issue a PIR query to fetch the appropriate message, and anonymity follows from the security of PIR. As with the protocol of [SW21], this solution enjoys both the requisite security features, as well as having minimal communication overhead (e.g. $\log N$ overhead, depending on the PIR protocol; see survey of PIR results in [OS07]).

1.1 Motivation and Technical Challenges

An important metric in determining the feasibility of a protocol in the NIAR model is the end-to-end message transmission time, which depends on the computational burden on each user, and especially that of the central router. A significant drawback of both the protocol of [SW21][1] and the naïve PIR solution described above is that they require *quadratic* (in terms of the number of users) computation at the router. As this computation cost is likely prohibitive (or at least extremely inefficient) when there are a large number of users, we set out to explore the possibility of a NIAR protocol that maintained the minimal communication burden of the naïve PIR and [SW21] solutions, but reduced computation overhead (at the router) from $O(N^2)$ closer to the optimal $O(N)$.

Our first observation is that the NIAR model is similar to so-called "permutation routing", but with anonymity. Namely, permutation routing seeks to connect N senders to N receivers through a network, which (from a communication standpoint) is what is required in the NIAR model. Our idea was to leverage the efficient routing (and therefore minimal overhead) of a permutation routing network, but then to administer PIR at each node to keep each routing decision hidden, thereby allowing for the anonymity required by the NIAR model. In particular, we envisioned a solution in which the central router simulates a virtual permutation routing network by itself, where the actual path the messages take (from each of the N senders on one end of the network to the N receivers at the other end) is hidden (from the central router) by using PIR along each edge. Namely, at each node of the (virtual) network, a PIR query is applied to each of the node's outgoing edges, where the PIR query (privately) selects a message from one of the node's incoming edges.

While the above idea captures the spirit of our solution (and indeed, the idea of layering PIR on top of various routing networks/protocols may have other interesting applications for anonymizing communication), there are several complications that required additional consideration:

1. (Virtual) Network Size. Since each outgoing edge is assigned a PIR query, and this PIR query is for a (virtual) database whose size is the number of incoming edges of the node in question, the computation cost of simulating routing in a (virtual) permutation network is roughly $O(E \cdot I)$, where E is the number of edges and I is the number of incoming edges per node. Since E is necessarily at least $O(N)$, having a NIAR protocol with only polylog computation overhead requires that E is at most $O(N \; polylog \; N)$ and I is $O(polylog \; N)$.
2. Standard PIR Won't Work. Even if network size is small ($O(N \; polylog(N))$), if the depth (number of nodes a message passes through from sender to receiver) is not constant, then standard PIR schemes will not work, since

[1] Router computation is not explicitly measured in the protocol of [SW21], our analysis of their protocol yields $O(N^2)$ computation load on the router: their Multi-Client Functional Encryption (MCFE) protocol is invoked N times by the router, with each invocation processing N ciphertexts.

each invocation of PIR typically requires $O(polylog(N))$ bits in the PIR server's response, and hence the message size will incur an exponential blow-up in the depth of the network. For example, even log-depth networks will have messages of size $O(2^{\log N}) = O(N)$ by the time they reach the last layer of the network, in which case we are no better off than the naïve PIR approach mentioned above.

3. Correctness Requires Edge-Disjoint Paths. Since PIR is being used to hide routing decisions made at each node/routing gate in the network, this requires that each outgoing edge forwards the message on (at most) one of the node's incoming edges. In particular, if any two paths connecting the sender-receiver pairs in the permutation network contain a common edge, then correctness is compromised. Since a *random* path selection algorithm will be crucial to proving anonymity, the given (virtual) permutation network must have the property that, with high probability, a random sample of paths connecting the sender-receiver pairs are edge-disjoint.
4. Edge-Disjoint Property is Insufficient for Anonymity. While having edge-disjoint paths is necessary for correctness, it is not sufficient to ensure anonymity. For example, if the central router is colluding with $(N-2)$ sender-receiver pairs (and therefore only needs to determine the linkage amongst the remaining two senders and two receivers), then knowledge that all paths are edge-disjoint can give the router an advantage in identifying the linkage between the remaining two senders and two receivers: Namely, the router knows (via collusion) $N-2$ paths, and thus can eliminate available options for the remaining two paths. Indeed, since permutation routing networks have been studied outside of the context of anonymity, to our knowledge there has not been any research into understanding how network properties and path selection protocols impact anonymity, even in the case where nodes (in the permutation network) do not learn which outgoing edge each of its received messages is forwarded along.

1.2 Overview of Our Results

Our solution to the NIAR problem, which blends techniques from permutation routing with techniques for hiding routing decisions made at each node of the (virtual) permutation network, overcomes the challenges outlined in the previous section as follows. By using familiar networks from permutation routing, which are inherently small ($O(N \cdot polylog(N))$), we are able to ensure the network size is suitably small, thus addressing the first potential issue. Furthermore, a common (and well-studied) feature of many permutation networks is the edge-disjoint property, and in particular this inspired our choice to use an (extended) Beneš permutation Network, thus addressing the third issue. We observe that there is an inherent tension between network topology (number of nodes, edges, and depth) in terms of achieving correctness and anonymity versus low router computation. Our solution includes carefully selecting appropriate network parameters to balance these trade-offs. Meanwhile, recent works

of [DGI+19,GHO20,CGH+21] present so-called *rate-1 PIR* protocols, which can address the second issue of exponential growth of message size per network layer.

Addressing the fourth issue is one of our key technical achievements. In spirit, the edge-disjoint property is related to anonymity, but as mentioned above, it is in general insufficient. Identifying a property that *is* sufficient, and specifically using such a property to formally argue anonymity, requires some thought and careful analysis (Definitions 22 and 25 define this property, and using it to prove security is done in Corollary 26 and in the analysis of (19)).

Assuming rate-1 PIR, we present in Figure 7 a routing protocol for the NIAR model that achieves $O(\log N)$ per-party communication and $O(N \cdot polylog(N))$ router computation. At a high level, our protocol dictates that the central router emulates routing in a permutation network, whereby each routing gate is (virtually) obliviously evaluated using a rate-1 PIR query/response for each outgoing edge. Our protocol consists of a setup phase in which the PIR queries that correspond to all outgoing edges of every routing gate are prepared, and then an online routing phase where a stream of (encrypted) messages are injected by the senders to receivers (re-using the setup).

2 Previous Work

2.1 Permutation Routing

In permutation routing [AKS83,Lei84,Upf89,MS92], messages from a set of N "input" nodes are routed through a network G to a set of N "output" nodes. Such works attempt to identify networks G with various desired properties, and protocols within these networks that can efficiently route these messages, for any possible permutation σ that dictates which input node is connected to which output node. While our work is partially inspired by the routing networks considered in this line of work, the NIAR model is sufficiently distinct from the permutation routing model, both because of the number of routers (one versus $\Theta(N \log N)$) and due to the required privacy of the permutation σ. In other words, we do not route the messages over a physical routing network (which is an iterative process that depends on the "depth" of the network), but rather use a virtual sorting network as a mean to design our non-interactive routing protocol.

2.2 PIR

As there has been extensive work done in the original PIR model and its variants, we discuss here only the works most relevant to us.

Multi-Client PIR As discussed in the introduction, the NIAR model can be viewed as a special case of multi-client PIR. Indeed, a solution to generic multi-client PIR in which the PIR server's work does not scale with the number of users would imply an efficient solution to the NIAR problem. While no such

result exists, we discuss a few relevant areas, and why they are insufficient for the NIAR model.

In [IP07], it is demonstrated how a single user can efficiently issue multiple queries to a PIR server. However, their results rely on a single decoding algorithm, whereas the NIAR model would require decoding keys for each of the N receivers. In [HOWW19], they present a related notion of private anonymous data access; we note that the results in their model do not scale to the full corruption threshold ($N-2$) required in the NIAR security model. Finally, results in the related areas of Batch Codes [IKOS04] and Public-Key Encryption with amortized updates [COS10] address a different model, and consequently do not seem to be directly applicable to the NIAR model.

Rate-1 PIR A recent line of work [DGI+19,GHO20,CGH+21] has demonstrated the viability of rate-1 PIR, in which the server response is comparable in size to the database entry being fetched. Formally, for a database of N elements each of size B, rate-1 PIR means that the ratio of B to the server response size approaches 1 as $N \to \infty$. Stated differently, a rate-1 PIR scheme has a constant-stretch term δ_{PIR}, such that the server's response has size $B + \delta_{PIR}$. [CGH+21] demonstrate a rate-1 PIR scheme based on the SXDH assumption [BGdMM05] that has $\lambda \cdot \log N$ client communication (for security parameter λ).

3 Preliminaries

3.1 Beneš Network

In a butterfly network (Figure 1), N input nodes are connected to N output nodes via a leveled network of $(1 + \log N)$ levels, each with N nodes. A Beneš network appends a second (inverted) butterfly network to the first (Figure 3); and more generally an extended Beneš network appends numerous "blocks" of butterfly networks together. We further expand the complexity of such networks by introducing the notion of color, in which each node and edge is replicated c times (Figure 5). Finally, our protocol will assume wide edges, which means that each edge can simultaneously route w messages (requiring specification of which of the w "slots" each message occupies).

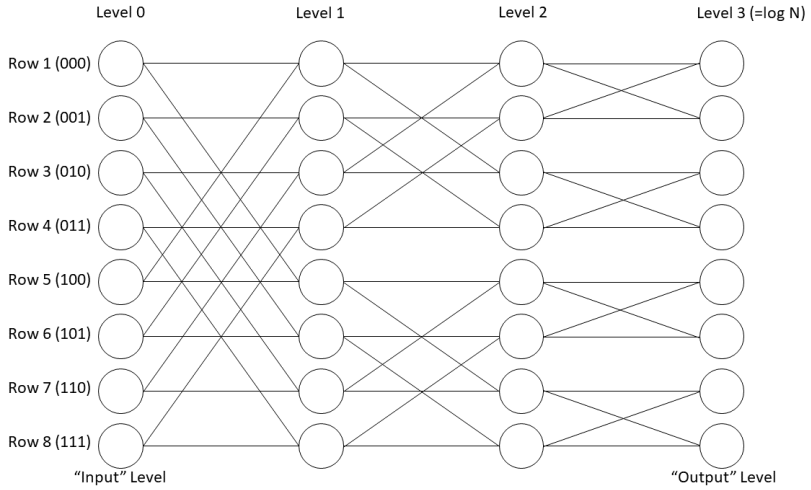


Fig. 1: Butterfly network with $N = 8$ input nodes.

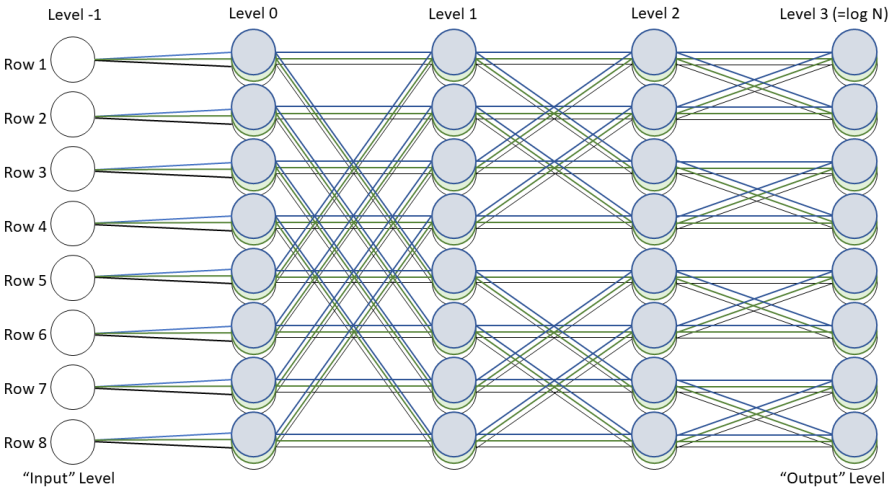


Fig. 2: Colored Butterfly network with N=8 input nodes and replication factor c=3.

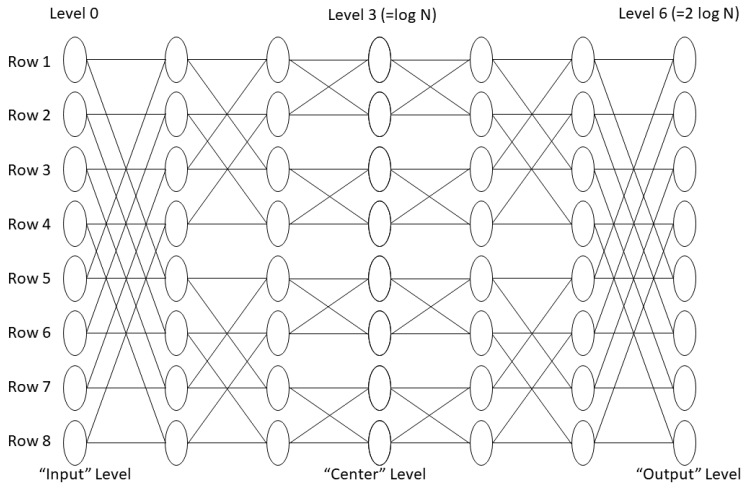


Fig. 3: Beneš network with $N = 8$ input nodes.

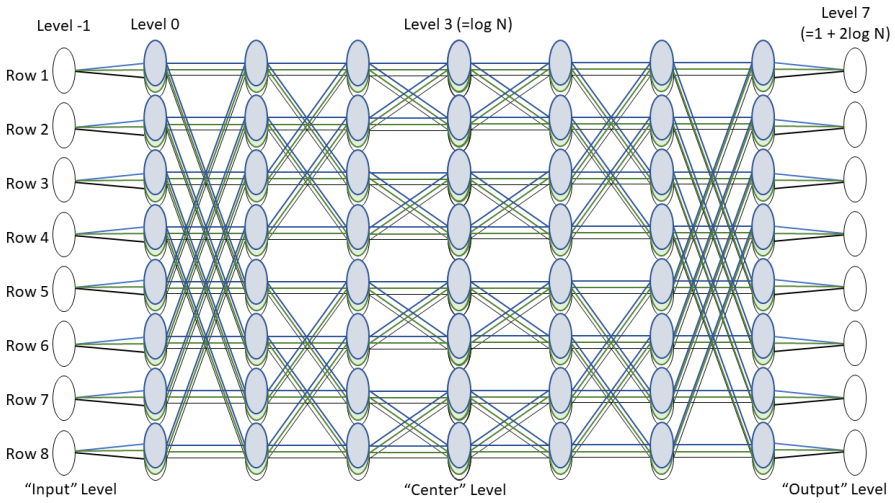


Fig. 4: Colored Beneš network with $N{=}8$ input nodes and replication factor $c{=}3$.

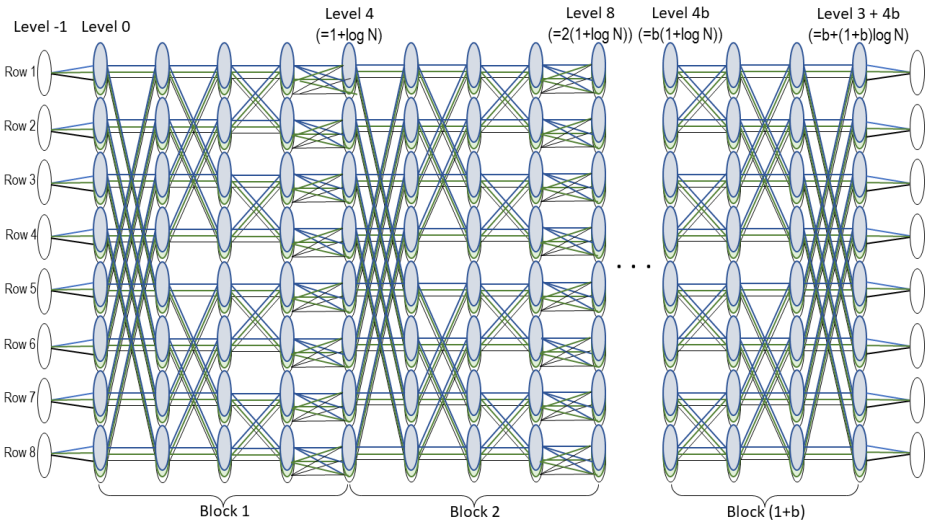
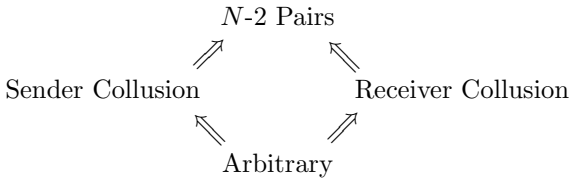


Fig. 5: Extended, Colored Beneš network with b blocks, replication factor $c = 3$, and $N = 8$.

3.2 Non-Interactive Anonymous Routing (NIAR)

We adopt the NIAR model of [SW21], in which N senders each have a series of m (e.g. single-bit) messages they wish to send to a distinct receiver *anonymously*. The anonymity guarantee refers to the unlinkability of each sender-receiver pair, and crucially it must be preserved even if the central router colludes with a subset of the senders/receivers. Depending on the application, there are various

collusion patterns that may be of interest, e.g.:



where all four scenarios include collusion with router C, with the top scenario allowing for arbitrary corruption of up to $N-2$ (sender, receiver) pairs; the left scenario allowing for arbitrary corruption of all senders (plus up to $N-2$ receivers); the right scenario allowing for arbitrary corruption of all receivers (plus up to $N-2$ senders); and the bottom scenario allowing for arbitrary (router C plus up to $2N-2$ of the senders/receivers) corruption. (The implication arrows above indicate that a protocol that is secure in one model is automatically secure in the other.) In this paper, we demonstrate our protocol is secure against the top and left scenario.[2] We do not consider the case of arbitrary receiver collusion in this paper for two reasons: First, one of the main application areas for the NIAR model is when N users wish to connect to N servers to retrieve information from them (e.g. to check email or stream content). In such scenarios, the receivers already know the senders they wish to connect to, so anonymity of the senders (in the case that all receivers are colluding) is irrelevant. The second reason we do not consider the full Receiver Collusion models is because providing such protection in this model requires additional techniques than those considered in this paper. For example in [SW21], the protocol description, performance, and cryptographic hardness assumptions are all made more complex in the Receiver Collusion scenarios.

Formally, the (reformulated) NIAR model of [SW21] is as follows:

(Trusted) Setup. Upon input security parameters $(1^{\lambda_c}, 1^{\lambda_s})$, number of senders/ receivers N, and permutation $\sigma : [N] \to [N]$, the Setup algorithm outputs sender keys $\{pk_i\}_{i\in[N]}$, receiver keys $\{(sk_i, \kappa_i)\}_{i\in[N]}$, and token q for router C: $\big(\{pk_i\}_{i\in[N]}, \{(sk_i, \kappa_i)\}_{i\in[N]}, \mathsf{q}\big) \leftarrow \mathbf{Setup}(1^{\lambda_c}, 1^{\lambda_s}, N, \sigma)$.
Once Setup has been run, the Senders $\{S_i\}$ can communicate arbitrary messages $\{m_i\} = \{m_{i,\alpha}\}$ with the Receivers $\{R_i\}$ through router C.

Send Message. Using key pk_i, each Sender S_i encodes message $m_i = m_{i,\alpha}$ (where α denotes the α^{th} bit of message m_i), and sends the result to router C: $c_{i,\alpha} \leftarrow \mathbf{Enc}_{pk_i}(m_{i,\alpha})$.

Route Message. Upon inputs $\{c_i\}_{i\in[N]}$ from each Sender S_i, and using key q, router C prepares messages $\{z_i\}_{i\in[N]}$, and sends these to each Receiver R_i: $(z_1, z_2, \ldots, z_N) \leftarrow \mathbf{Route}(\mathsf{q}, c_1, c_2, \ldots, c_N)$.

Decode Message. Using keys (sk_i, κ_i), each Receiver R_i decodes the message $z_i = z_{i,\alpha}$ received from router C, and outputs $\widetilde{m}_i = \widetilde{m}_{i,\alpha}$: $\widetilde{m}_{i,\alpha} \leftarrow \mathbf{Dec}_{sk_i}(\kappa_i, z_{i,\alpha})$.

[2] The emphasis in [SW21] was on the top, right, and bottom scenarios.

Correctness. An oblivious permutation routing protocol has:
Perfect Correctness: If each receiver R_i outputs message $\widetilde{m}_i = m_i$ with probability 1.
λ_c - Statistical Correctness: If each receiver R_i outputs message $\widetilde{m}_i = m_i$ with probability at least $\left(1 - \frac{1}{2^{\lambda_c}}\right)$ for security parameter λ_c.

Security. Informally, anonymity means that if a subset of players collude (including router C), the permutation σ (namely, its restriction to non-colluding parties) should remain unknown. Formally, let $\mathcal{A}$ denote a (computationally bounded, honest-but-curious) adversary. Consider the following challenge game:

1. On input security parameter λ, Adversary $\mathcal{A}$ chooses N, two distinct permutations σ_0, σ_1 on $[N]$, a set of sender indices $S_{\mathcal{A}} \subseteq [N]$ to corrupt, and a set of receiver indices $R_{\mathcal{A}} \subseteq [N]$ to corrupt; subject to constraints:
   (a) $|R_{\mathcal{A}}| \leq N - 2$;
   (b) σ_0 and σ_1 match for all receivers in $R_{\mathcal{A}}$: $\forall\, i \in R_{\mathcal{A}} : \sigma_0^{-1}(i) = \sigma_1^{-1}(i)$.
2. Adversary $\mathcal{A}$ sends $\{\sigma_0, \sigma_1\}$ to a Challenger $\mathcal{C}$.
3. Challenger $\mathcal{C}$ chooses $\sigma_b \in \{\sigma_0, \sigma_1\}$ for $b \in \{0, 1\}$ (e.g. by flipping a coin).
4. Challenger $\mathcal{C}$ chooses router token tk_C, encryption keys $\{\mathsf{pk}_i\}_{i\in[N]}$, and decryption keys $\{\mathsf{sk}_i\}_{i\in[N]}$, where key sk_i decrypts ciphertexts created under key $\mathsf{pk}_{\sigma_b(i)}$. $\mathcal{C}$ sends tk_C, $\{\mathsf{pk}_i\}_{i\in S_{\mathcal{A}}}$, and $\{\mathsf{sk}_i\}_{i\in R_{\mathcal{A}}}$ to $\mathcal{A}$.
5. For each round α:
   (a) Based on knowledge of all prior ciphertexts $\{c_{i,\alpha'}\}_{\alpha'<\alpha}$ (see next step), Adversary $\mathcal{A}$ chooses arbitrary messages $\{m_{i,\alpha}^{(0)}\}_{i\in[N]}$ and $\{m_{i,\alpha}^{(1)}\}_{i\in[N]}$, subject to the constraint that all messages bound for a corrupt receiver match: $\forall\, i$ s.t. $i = \sigma_0^{-1}(j)$ for some $j \in R_{\mathcal{A}}$: $m_{i,\alpha}^{(0)} = m_{i,\alpha}^{(1)}$.
   $\mathcal{A}$ sends these messages to $\mathcal{C}$.
   (b) Challenger $\mathcal{C}$ outputs to $\mathcal{A}$ ciphertexts $\{c_{i,\alpha}\}_{i\in[N]}$, where each ciphertext is computed as (with b as chosen in Step 3):
$$c_{i,\alpha} = Enc_{\mathsf{pk}_{\sigma_b(i)}}(m_{i,\alpha}^{(b)})$$
6. Adversary $\mathcal{A}$ outputs a guess $b' \in \{0, 1\}$ of which permutation $\{\sigma_0, \sigma_1\}$ Challenger $\mathcal{C}$ chose.

A NIAR protocol is λ_s-secure if the probability that any computationally bounded adversary $\mathcal{A}$ guesses b correctly is bounded by:

$$\Pr[b' = b] \;\leq\; \frac{1}{2} + \frac{1}{2^{\lambda_s}} \tag{1}$$

3.3 Emulating Oblivious Routing in a Virtual Permutation Routing Network

In this section, we present the main ideas that connect the NIAR model to the permutation routing problem. At a high level, the idea is to have the NIAR router emulate message transmission through a (virtual) routing network that supports permutation routing between N senders and receivers. We begin by formally defining the functionalities of a pair of protocols that will be required for evaluation of a routing gate:

Definition 1 *A routing gate evaluation protocol $\Pi_{RG_{Enc}}$ for routing gate $\mu \in G$ takes as input a set of values $\{v_j\}$ on each incoming edge of μ and a set of encoded indices $\{q_j\}$ on each outgoing edge of μ (where each q_j encodes the index of an incoming edge), and outputs a set of values $\{z_j\}$ for each outgoing edge of μ, where each $z_j = Enc(v_{q_j})$ is an encoding of the value on the appropriate input wire (as specified by q_j). A routing gate reconstruction protocol $\Pi_{RG_{Dec}}$ takes as input a set of reconstruction keys $\{\kappa_{\mu,j}\}$ and a set of values $\{z_{\mu,j}\}$ (one key-value pair $(\kappa_{\mu,j}, z_{\mu,j})$ for each outgoing edge of μ), and outputs values $\widehat{v}_{\mu,j}$ for each outgoing edge of μ.*

The key primitive that is required to support anonymity in a virtual permutation network is the oblivious routing gate, which specifies a procedure for each outgoing edge to *obliviously* select an incoming edge and write (a re-encoding of) the message on this edge to the outgoing edge:

Definition 2 *Let $\mu \in G$ denote a gate in a permutation routing network, and let $\mathcal{I}_\mu$ and $\mathcal{O}_\mu$ denote its incoming and outgoing edges (respectively). The oblivious routing gate (ORG) paradigm for μ takes as input, for each outgoing edge $\widehat{e}_\mu \in \mathcal{O}_\mu$, an associated index $j_{\widehat{e}_\mu} \in \perp \cup [1, |\mathcal{I}_\mu|]$, and outputs:*

i. Encodings $\{q_{\widehat{e}_\mu}\}$ of the input indices $\{j_{\widehat{e}_\mu}\}$;
ii. Specification of a routing gate evaluation protocol $\Pi_{RG_{Enc}}$;
iii. Reconstruction keys $\{\kappa_{\widehat{e}_\mu}\}$;
iv. Specification of a routing gate reconstruction protocol $\Pi_{RG_{Dec}}$ protocol;

with the outputs subject to the constraints:

Message Independence. Outputs $\{q_{\widehat{e}_\mu}\}$ are defined independently from (later-specified) messages $\mathcal{M}_\mu$.

Reusability. Outputs $(\{q_{\widehat{e}_\mu}\}, \{\kappa_{\widehat{e}_\mu}\}, \Pi_{RG_{Enc}}, \Pi_{RG_{Dec}})$ can be reused (indefinitely) for new messages $\{\mathcal{M}_{\mu,1}, \mathcal{M}_{\mu,2}, \ldots\}$, without compromising correctness or security properties for each subsequent run.

*Additionally, the oblivious routing gate paradigm requires **correctness** and **security** properties, which are formally defined in Figure 9, and informally summarized as:*

Correctness. Given arbitrary messages $\mathcal{M}_\mu := \{m_j\}_{j \in [1, |\mathcal{I}_\mu|]}$ assigned to each incoming edge, $\Pi_{RG_{Enc}}(\widehat{e}_\mu, q_{\widehat{e}_\mu}, \mathcal{M}_\mu)$ writes a value $v_{\widehat{e}_\mu}$ on outgoing edge $\widehat{e}_\mu$ that is (a re-encoding of) the input message $m_{j_{\widehat{e}_\mu}} \in \mathcal{M}_\mu$ (where $j_{\widehat{e}_\mu}$ is from the original input, and in particular is the underlying message that $q_{\widehat{e}_\mu}$ encodes). Namely, $\Pi_{RG_{Dec}}(\kappa_{\widehat{e}_\mu}, v_{\widehat{e}_\mu}) = m_{j_{\widehat{e}_\mu}}$ reconstructs to the appropriate message.

Security. The output encodings $\{q_{\widehat{e}_\mu}\}$ reveal nothing about the inputs $\{j_{\widehat{e}_\mu}\}$ that they are encoding.

Figure 6 depicts a possible instantiation of an oblivious routing gate with subprotocols $(\Pi_{RG_{Enc}}, \Pi_{RG_{Dec}})$ set as (PIR-server, PIR-client) protocols (with output values $\{q_{\widehat{e}_\mu}\}$ the PIR queries and $\{\kappa_{\widehat{e}_\mu}\}$ the client's reconstruction keys). With the oblivious routing gate paradigm in hand, we define the emulated permutation routing functionality, which captures how the NIAR router C transmits messages through a (virtual) permutation routing network:

Definition 3 *Let G denote a permutation routing network: N pairs of designated (sender, receiver) nodes and a permutation $\sigma : [N] \to [N]$ connecting them. Let $\{\mathcal{P}_i\}$ denote a set of N paths that link each (sender, receiver) pair with a path through G. For every node $\mu \in G$, and for each outgoing edge $\widehat{e}_\mu \in \mathcal{O}_\mu$, define the index $j_{\widehat{e}_\mu} \in \perp \cup [1,\ |\mathcal{I}_\mu|]$ as follows:*

- *If exactly one path $\mathcal{P}_i$ traverses outgoing edge $\widehat{e}_\mu$: $j_{\widehat{e}_\mu}$ is the index of $\mathcal{P}_i$'s incoming edge to μ.*
- *If either zero or two or more paths $\mathcal{P}_i, \mathcal{P}_j$ traverse outgoing edge $\widehat{e}_\mu$: $j_{\widehat{e}_\mu}$ is defined to be $\perp$.*

Given as input: (i) A set of values $\{v_j\}$ for each outgoing edge e_j that emanates from one of the N source nodes; and (ii) A protocol Π_{ORG} instantiating the oblivious routing gate paradigm; an emulated permutation routing (EPR) *protocol $\Pi_{EPR}(\Pi_{ORG}, \{v_j\})$, is defined as:*

***(One-time) Setup Phase**. For each node $\mu \in G$, run $\Pi_{ORG}(\{j_{\widehat{e}_\mu}\})$, and let $\{q_{\widehat{e}_\mu}\}$ denote the output encodings (from output (i)) and $\Pi_{RG_{Enc}}$ the routing gate evaluation protocol (from output (ii)).*

***(Online) Emulation Phase**. Iterate through the levels l of routing network G:*

- *For each outgoing edge $\widehat{e}_\mu$ of each node $\mu \in G$ at level l, invoke protocol $\Pi_{RG_{Enc}}(\widehat{e}_\mu, q_{\widehat{e}_\mu}, \mathcal{M}_\mu)$ on inputs:*
  - *$q_{\widehat{e}_\mu}$: From Setup Phase (above).*
  - *$\mathcal{M}_\mu = \{m_j\}_{j \in [1,\ |\mathcal{I}_\mu|]}$: Values on the input wires of μ (either from input (i) values $\{v_j\}$ if level $l = 1$ is the first level, or from the output values of $\Pi_{RG_{Enc}}$ in iteration l-1).*
- *For each incoming edge $e_{\mu,j}$ of each sink node μ, output value $z_{\mu,j}$, where $z_{\mu,j}$ denotes the value assigned to this edge (viewed as an output edge of some node on the previous level) by $\Pi_{RG_{Enc}}$ (from the last iteration).*

See Figure 11 for an extended protocol definition of the emulated permutation routing protocol Π_{EPR}.

4 Our Protocol

4.1 Overview of Our Solution

Given N pairs of (sender, receiver) nodes and central router C, our protocol routes messages from the senders to the receivers via a virtual routing network G that C emulates where, for each node in the network, the router C obliviously executes a routing gate by simulating the functionality of a (rate-1) PIR query. Namely, (as part of trusted setup) each outgoing edge of a routing gate will have an assigned PIR query, and each incoming edge will have a value (which represents an (encrypted) message from one of the senders), and the router C obliviously produces a message on each outgoing edge of the routing gate by running the associated PIR query on this wire against the (virtual) PIR database of messages (from the incoming wires). The determination of *which* incoming edge that a given PIR query (on a routing gate's outgoing edge) should specify is established offline during a setup phase, and specifically it is determined

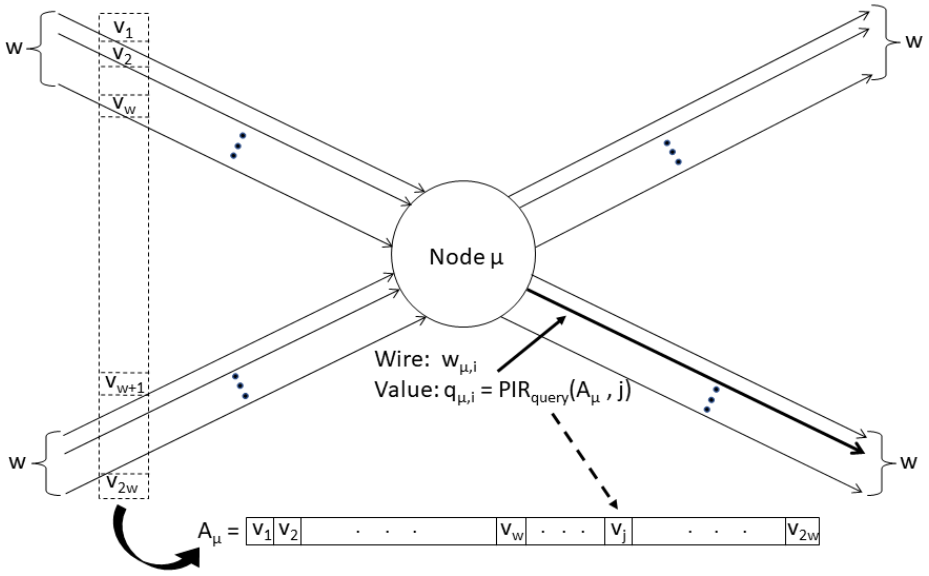


Fig. 6: Oblivious routing gate realization via PIR at node μ with $2w$ incoming and outgoing edges.

by choosing a random path $\mathcal{P}_i$, for each (sender$_i$, receiver$_i$) pair, through the (virtual) routing network G. Notice that once PIR queries are assigned (during an offline setup phase) as per all chosen paths $\{\mathcal{P}_i\}$, they may be reused indefinitely during the online routing phase to continuously route new messages for each (sender, receiver) pair. The key features of our protocol are:

- Correctness. Ensuring each receiver gets every message reduces to showing that the paths $\{\mathcal{P}_i\}$ connecting each (sender, receiver) pair are *edge-disjoint*.
- Privacy. Since each sender encrypts their messages under the intended recipient's public key, receivers can only decipher messages intended for them.
- Anonymity. This property is obtained so long as the paths $\{\mathcal{P}_i\}$ chosen are sufficiently edge-disjoint *over a sub-graph* of G.
- Communication. To limit the expansion of message size through each (virtual) routing gate, we employ **rate-1** PIR, which ensures the final message size is proportional to the length of the chosen path $\mathcal{P}$ through the (virtual) routing network G; and that any such path is small (polylog).
- End-to-End Time. Computation of central router C (which, together with communication, determines end-to-end transmission time) will depend on the size of the virtual graph $G = (V, E)$. Thus, in order to minimize computational overhead, $|E|$ should be close to N (e.g. $N polylog\ N$). Notice that there is inherent tension in minimizing end-to-end time versus satisfying the Correctness and Anonymity properties: the former requires small $|V|$ and $|E|$, while the latter two are readily achieved for larger $|V|$ and $|E|$. Our

protocol finds appropriate (minimal) parameters to achieve correctness and anonymity, while introducing minimal end-to-end overhead.

One final point concerning anonymity: while PIR is the main tool that hides (from central router C and any other parties it colludes with) the linkage between uncorrupted (sender, receiver) pairs, applying it naïvely will not provide the desired protection. Namely, if any two of the paths $\{\mathcal{P}_i\}$ through the virtual routing network have an edge in common, then a PIR query cannot be assigned to that edge, as there will be conflicting input edge indices (and conflicting messages on those edges) to select. Since, in proving anonymity, path selection must be a randomized process (in particular, edge conflicts cannot be deliberately avoided), our protocol will handle edge conflicts by producing garbage PIR queries for such edges. While this approach introduces failures in terms of delivering messages along the conflicting paths that were chosen for any such (sender, receiver) pairs, the threat to correctness is overcome by ensuring enough redundancy in the system to account for (the low probability event of) edge conflicts. However, edge conflicts (and the *lack* of edge conflicts), also threatens anonymity: for example, the router C could observe many messages from (sender, receiver) pairs it has corrupted all pass through a common node, and the router may also know that the message from an *uncorrupted* sender has some probability of passing through this same node. Thus, the presence or absence of an edge conflict on the set of outgoing edges of this node may give the router an advantage in determining if the uncorrupted sender's path goes through this node, and if so, some probabilistic advantage in knowing which outgoing edge the path used; and these advantages then threaten anonymity since the router may be able to have an advantage in guessing the ultimate destination (i.e. receiving node) of this path. Demonstrating that this approach cannot be used to give the router a non-negligible advantage in linking uncorrupted (sender, receiver) pairs will require: (i) Identifying what property a network should have to avoid this attack; (ii) Generating such a network that also supports the desired computation and correction requirements; (iii) An appropriate analysis that this property indeed proves anonymity. For example, the natural candidate property of exhibiting (with high probability on randomly chosen paths) the edge-disjoint property is insufficient, as it is susceptible to the above attack.

Figures 7 and 8 below formally describe our protocol.

4.2 Analysis of Our Protocol

Theorem 4 *Assuming the existence of rate-1 PIR, following trusted setup,*[3] *the protocol presented in Figure 7 is λ_s-secure with λ_c-statistical correctness, $O(\log N)$ per-party communication, and $O(N \text{ polylog } N)$ router computation.*

Remark. Instead of trusted setup, under appropriate cryptographic hardness assumptions the ideal functionality $\Pi_{ORG}(G, \widehat{c}, r, l, \Pi_{1\text{-}PIR})$ could instead be realized via generic secure multiparty computation (MPC) techniques. This would

[3] Trusted setup is required for establishing public/secret key pairs for encryption and for instantiating ideal functionality $\Pi_{ORG}(G, \widehat{c}, r, l, \Pi_{1\text{-}PIR})$.

Anonymous Permutation Routing Protocol Π_{NIAR}

Input. Anonymous Permutation Routing parameters: $N = 2^n$, "central router" party C, "sender" parties $\{S_i\}_{i\in[N]}$ each with arbitrarily many messages $\{m_{i,\alpha}\}$, "receiver" parties $\{R_i\}_{i\in[N]}$, permutation $\sigma\colon[N]\to[N]$.

Output. For each party index $1 \le i \le N$, receiver $R_{\sigma(i)}$ outputs message $\widetilde{m}_i$.

Notation. Let $\lambda := \max(\lambda_c/(2\text{-}\log 3),\ 2\log N + \max(\lambda_s, 2 + \log\log N))$. Let $G = B(\widehat{N}, b, c, w)$ denote a wide-edged, extended and colored Beneš network with parameters $\widehat{N} = N$, $b = \lambda - 1$, $c = 4\cdot a_\lambda$, and $w = 1.2\cdot\lambda\cdot\log N\cdot(1+\log N)$ (for $a_\lambda := \max(2, \lambda^{1/(\log N - 1)})$); see Figure 5. For each $1 \le i \le N$, the parties $\{S_i\}$ and $\{R_i\}$ are assigned to "row" i, associating each sender S_i with I_i (the i^{th} "input" node of G, i.e. the left-most node in row i) and each receiver R_i with O_i (the i^{th} "output" node of G).

Setup Phase.

1. For each $i \in [N]$: let (pk_i, sk_i) denote a public-key/secret-key pair. Output: $S_i \leftarrow pk_i$ and $R_{\sigma(i)} \leftarrow sk_i$.
2. For each $1 \le m \le M = \lambda$:
   (a) Choose random paths through G (as per σ). For each $i \in [N]$: let $\mathcal{P}_i = \mathcal{P}_{m,i}$ denote a random path through G (namely, paths are chosen as per protocol $\Pi_{N,\sigma,G}(i)$; Figure 14).
   (b) Assign rate-1 PIR queries and keys to each edge. For each *internal* node $\mu = \mu_{\widehat{c},r,l}$ at position $(\widehat{c}, r, l)$ of G (i.e. color $\widehat{c} \in [c]$, row $r \in [N]$, and level $l \in [0, (b + (1 + b)\cdot\log N)]$; see Figure 5), invoke the oblivious routing gate protocol $\Pi_{ORG(G,\widehat{c},r,l,\Pi_{1\text{-}PIR})}$ (Fig. 9) with inputs $\{(w_i, w'_i)\}$, where:
      - $w_i = \perp$ if $\mu \notin \mathcal{P}_i$; otherwise $w_i \in [1, |\mathcal{I}_l|]$ is the incoming wire index to μ (as specified by $\mathcal{P}_i$).
      - $w'_i = \perp$ if $\mu \notin \mathcal{P}_i$; otherwise $w'_i \in [1, |\mathcal{O}_l|]$ is the outgoing wire index from μ (as specified by $\mathcal{P}_i$).

      For each $j \in [1, |\mathcal{O}_l|]$, let $\{q_{\mu,j}\}$ denote the rate-1 PIR queries and let $\{(\mu, j, \kappa_{\mu,j})\}$ denote the set of reconstruction keys that are output by the $\Pi_{ORG(G,\widehat{c},r,l,\Pi_{1\text{-}PIR})}$ protocol (by definition, a reconstruction key for output wire j of node μ is output by Π_{ORG} if and only if there exists a unique index $i \in [N]$ such that $w'_i = j$).
   (c) Aggregate reconstruction keys along each path. For each $i \in [N]$: let $\kappa_i = \kappa_{m,i}$ be either $\perp$ (if the reconstruction key for *any* outgoing edge $(\mu, j) \in \mathcal{P}_i$ is not output by Π_{ORG}), and otherwise let κ_i be the collection of reconstruction keys $\{(\mu, j, \kappa_{\mu,j})\}_{\mu\in\mathcal{P}_i}$ for each outgoing edge $(\mu, j) \in \mathcal{P}_i$.

   Output: $C \leftarrow \{q_{\mu,j}\}$, and for each receiver: $R_i \leftarrow \kappa_i$.

Fig. 7: Anonymous Permutation Routing protocol Π_{NIAR}.

contribute $O(N^2\ polylog\ N)$ to the asymptotic cost of the protocol (to deal the $O(N\ polylog\ N)$ rate-1 queries and $O(N^2\ polylog\ N)$ reconstruction keys), but because $\Pi_{ORG}(G, \widehat{c}, r, l, \Pi_{1\text{-}PIR})$ is utilized only in the Setup Phase, this would be incurred as a one-time cost and would not impact cost of the Routing Phase.

Anonymous Permutation Routing Protocol Π_{NIAR} (continued)

Routing Phase.
Repeat the following procedure for each successive message $\{m_{i,\alpha}\}$:

Senders $\{S_i\}$.
1. Sender S_i encrypts $m_i = m_{i,\alpha}$ under pk_i and sends $Enc_{pk_i}(m_i)$ to router C.

Central Router C.
1. For each $1 \leq m \leq M = \lambda$:
   (a) C runs the Π_{EPR_m} protocol with inputs $\{Enc_{pk_i}(m_i)\}$ (from each sender S_i's Step 1 of the Routing Phase) and $\{q_{\mu,j}\}$ (from the Setup Phase). Let $\{z_{m,i,j}\}_{i\in[0,(N\text{-}1)],j\in[1,c]}$ denote the output of Π_{EPR_m}.
   (b) C sends $\{z_{m,i,j}\}_{j\in[c]}$ to receiver $R_{\sigma(i)}$.

Receivers $\{R_{\sigma(i)}\}$.
1. Receiver $R_{\sigma(i)}$ initializes final output value $\widetilde{w}_i = \perp$.
2. For each $1 \leq m \leq M = \lambda$:
   (a) Let $\mathcal{P}_i = \mathcal{P}_{m,i}$, and note that $\mathcal{P}_i$ is edge-disjoint (as per Definition 10) if and only if for every node $\{\mu_l\}$ in $\mathcal{P}_i$, $R_{\sigma(i)}$ received the reconstruction key for μ_l (as per Output (ii) of Π_{ORG}).
   (b) If $\mathcal{P}_i$ *is not* edge-disjoint, or if $\widetilde{w}_i \neq \perp$ (i.e. $\widetilde{w}_i$ was set in a previous iteration $m' < m$): do nothing.
   (c) If $\mathcal{P}_i$ *is* edge-disjoint, then $R_{\sigma(i)}$ uses the reconstruction keys $\kappa_{m,i}$ to traverse $\mathcal{P}_i$ *backwards*, starting with the final value $z_{m,i,j}$ that it received from C (along the appropriate color index $j \in [c]$, as specified by $\mathcal{P}_i$) in Step 1b above. Using the reconstruction keys $\{\kappa_\mu\}$, level by level $R_{\sigma(i)}$ reconstructs to remove one layer of the PIR stretch δ_{PIR} that was added by Π_{ORG}. When $R_{\sigma(i)}$ has traversed backwards to level 0, it will have reconstructed value $Enc_{pk_i}(m_i)$. In this case, $R_{\sigma(i)}$ updates $\widetilde{w}_i$ with value $\widetilde{w}_{m,i} = Dec_{sk_i}(m_i)$.
3. $R_{\sigma(i)}$ outputs $\widetilde{w}_i$.

Fig. 8: Anonymous Permutation Routing protocol Π_{NIAR} (continued).

Proof (Proof of Theorem 4).
Cost. The per-party computation costs for each step of the routing phase are:

Party	**Step**	**Computation**
S_i	1	$Cost(\Pi_{Enc})$
R_i	$2b$	$Cost(\Pi_{Dec}) + (1+b) \cdot (1+\log N) \cdot Cost(\Pi_{PIR\text{-}Rec})$
C	$1a$	$M \cdot \lvert E \rvert \cdot Cost(\Pi_{PIR\text{-}Query})$

and communication costs are:

Party	**Step**	**Communication**
S_i	1	$c_{_{Enc}}$
R_i	$2b$	$c_{_{Enc}} + (1+b) \cdot \delta_{PIR}$
C	$1a$	$N \cdot (2 \cdot c_{_{Enc}} + (1+b) \cdot \delta_{PIR})$

where:
– $|E| = (2 \log N + c) \cdot (c \cdot w \cdot N \cdot (1+b))$ is the number of edges in network $B(N, b, c, w)$.

- $Cost(\Pi_{Enc})$ is the (computation) cost of encrypting a message m.
- $Cost(\Pi_{Dec})$ is the (computation) cost of decrypting a ciphertext $Enc_{pk_i}(m)$.
- c_{Enc} is the size of a ciphertext $Enc_{pk_i}(m)$.
- δ_{PIR} is the constant stretch of the underlying rate-1 PIR protocol $\Pi_{1\text{-}PIR}$.
- $Cost(\Pi_{PIR\text{-}Query})$ is the PIR server cost of running a query for $\Pi_{1\text{-}PIR}(c \cdot w, c_{Enc} + (1+b) \cdot \delta_{PIR})$.
- $Cost(\Pi_{PIR\text{-}Rec})$ is the cost of running the reconstruction algorithm (on a PIR response) for $\Pi_{1\text{-}PIR}(c \cdot w, c_{Enc} + (1+b) \cdot \delta_{PIR})$.

Correctness. The intuition for the proof is as follows: Independent of adversarial presence, we first demonstrate bounds of certain properties of routing in the Beneš network, as per the protocols described in Figures 7 and 14. Namely, we demonstrate in Corollary 18 that, with overwhelming probability, for any row index $i \in [N]$ there will exist (at least) one experiment $m \in [M]$ for which the path $\mathcal{P}_{m,i}$ is edge-disjoint from all other paths $\{\mathcal{P}_{m,j}\}_{j \neq i}$. Then as per protocol Π_{NIAR} specification (Step 2b of the Output Parties portion of the Routing Phase; see Figure 7), the existence of an edge-disjoint path $\mathcal{P}_i$ means that $R_{\sigma(i)}$ will update $\widetilde{w}_i \leftarrow \widetilde{w}_{m,i}$. By the correctness property of the ideal functionality of Π_{ORG}, this value will be *correct* (i.e. it will equal p_i).

Formally, with $\lambda = \max(\lambda_c/(2\text{-}\log 3),\ 2\log N + \max(\lambda_s, 2 + \log\log N) \geq \frac{\lambda_c}{2-\log 3}$, Corollary 18 states that the probability that there exists some row index $i \in [N]$ for which $\mathcal{P}_{m,i}$ is *not* edge-disjoint for *every* experiment $m \in [M]$ is bounded by:

$$\Pr[X = 0] \ < \ \left(\frac{3}{4}\right)^{\lambda} \ \leq \ \left(\left(\frac{3}{4}\right)^{\frac{1}{2-\log 3}}\right)^{\lambda_c} \ = \ \frac{1}{2^{\lambda_c}}$$

Security. As with the Correctness proof, we first demonstrate (probability bounds for) a version of the *edge-disjoint* property. Namely, we demonstrate in Corollary 26 that, using the parameters as per Π_{NIAR} (Figure 7), with overwhelming probability (in λ_s), for any pair of row indices $i, i' \in [N]$ and for every experiment $m \in [M]$, there will exist a block in which the chosen paths $\mathcal{P}_{m,i}$ and $\mathcal{P}_{m,i'}$ *as well as* their alternate paths $\mathcal{P}'_{m,i}$ and $\mathcal{P}'_{m,i'}$ are each edge-disjoint from all other paths in this block. Effectively, this means that for any two *uncorrupted* receiver nodes $i, i' \notin R_{\mathcal{A}}$, that for each experiment there exists some block in which the Adversary will necessarily lose all ability to distinguish between $\mathcal{P}_{m,i}$ and $\mathcal{P}_{m,i'}$ by the time these paths cross through this block. We then use a hybrid argument to show that the existence of an adversary that can distinguish between two arbitrary permutations (as per (1)) implies the existence of an adversary who can distinguish (with a smaller probability) between two permutations that differ only on two points; and then this contradicts the existence of a block in which any two paths become indistinguishable after that block.

Formally, the proof reduces the NIAR security game (with Challenger invoking the NIAR protocol Π_{NIAR} of Figure 7) to Challenge Game 2, and then uses the indistinguishability of Challenge Game 2 (Lemma 28). To match notation of protocol Π_{NIAR} with the communication sent to adversary $\mathcal{A}$ in the NIAR security game:

For Step 4 of the NIAR security game:

- Encryption keys $\{\mathsf{pk}_i\}$: The $\{pk_i\}$ from Step 1 of the Setup Phase (Figure 7).
- Decryption keys $\{\mathsf{sk}_i\}$: The $\{sk_i\}$ from Step 1 of the Setup Phase, together with the reconstruction keys $\{\kappa_i\} = \{(\mu, j, \kappa_{\mu,j})\}$ from Step 2b of the Setup.
- Router token tk_C: The rate-1 PIR queries $\{q_{\mu,j}\}$ from Step 2b of the Setup.

For Step 5b of the NIAR security game:

- Ciphertexts $\{c_{i,\alpha}\}$: The encrypted messages $\{Enc_{pk_i}(m_{i,\alpha})\}$ from Sender's Step 1 of the Routing Phase (Figure 7).

First observe that indistinguishability of the distribution of ciphertexts $\{c_{i,\alpha}\} = \{Enc_{pk_i}(m_{i,\alpha})\}$ under $b = 0$ versus $b = 1$ follows from the security of the encryption scheme, together with the constraint that all messages bound for a corrupt receiver must match for $b = 0$ and $b = 1$ (see the specified constraint in Step 5a of the NIAR security game). Thus, for any ciphertext $c_{i,\alpha}$ for which Adversary $\mathcal{A}$ does *not* hold the decryption key, the security of the encryption scheme ensures indistinguishability of this as a ciphertext of $m^{(0)}_{i,\alpha}$ versus $m^{(1)}_{i,\alpha}$; and for any ciphertext $c_{i,\alpha}$ for which Adversary $\mathcal{A}$ *does* hold the decryption key, the constraint in Step 5a of the NIAR security game dictates that this ciphertext encodes a common message $m^{(0)}_{i,\alpha} = m^{(1)}_{i,\alpha}$.

Next we argue indistinguishability of the encryption keys $\{pk_i\}_{i\in S_{\mathcal{A}}}$ and the decryption keys $\{sk_i\}_{i\in R_{\mathcal{A}}}$. Notice first that due to the constraint in Step 1b of the NIAR security game, the distribution of decryption keys $\{sk_i\}_{i\in R_{\mathcal{A}}}$ looks the same for $b = 0$ and $b = 1$, since σ_0 and σ_1 necessarily agree here (i.e. they each map some index $j \in [N]$ to i. Meanwhile, for the distribution of encryption keys, we focus on indices $i \in [N]$ for which $\sigma_0(i) \neq \sigma_1(i)$. Fix any such i, and define $j = \sigma_0(i)$ and $j' = \sigma_1(i)$, so $j \neq j'$. Again due to the constraint in Step 1b of the NIAR security game, we have that neither j nor j' is in $R_{\mathcal{A}}$. This means that Adversary $\mathcal{A}$ does *not* hold the corresponding decryption key for pk_i regardless of whether $b = 0$ or $b = 1$, and thus by the security of the encryption scheme, the distribution of pk_i for $b = 0$ appears identical as the distribution when $b = 1$.

For indistinguishability of the router token $\mathsf{tk}_C = \{q_{\mu,j}\}$: for a given $q_{\mu,j}$ for which Adversary $\mathcal{A}$ does *not* hold the corresponding reconstruction key $\kappa_{\mu,j}$, indistinguishability follows from the security of the underlying rate-1 PIR scheme. Conversely, for a given $q_{\mu,j}$ for which Adversary $\mathcal{A}$ *does* hold the corresponding reconstruction key $\kappa_{\mu,j}$, Adversary $\mathcal{A}$ learns the input wire index that $q_{\mu,j}$ is selecting. However, since the paths chosen through G are independent of each other and depend only on the given (sender, receiver) indices and coin flips of path selection protocol $\Pi_{N,\sigma,G}(i)$ (Figure 14), and because Adversary $\mathcal{A}$ knows reconstruction key $\kappa_{\mu,j}$ if and only if outgoing edge (μ, j) is on the path leading to a corrupt receiver $i \in R_{\mathcal{A}}$, we again rely on the constraint in Step 1b of the NIAR security game to argue that σ_0 and σ_1 must agree on the (sender, receiver) indices for this path, so the input wire index that $q_{\mu,j}$ is selecting is the same.

It remains to argue indistinguishability of the reconstruction keys $\{\kappa_i\}_{i\in R_{\mathcal{A}}} = \{(\mu, j, \kappa_{\mu,j})\}$. If for a given tuple $(\mu, j, \kappa_{\mu,j})$ the last component is a *valid* reconstruction key (i.e. $\kappa_{\mu,j} \neq \perp$), then indistinguishability follows the same argument as above for the router token. On the other hand, if $\kappa_{\mu,j} \neq \perp$, then as per the

Correctness property of any Π_{ORG} protocol, Adversary $\mathcal{A}$ learns that at least two distinct paths chose outgoing edge (μ, j). Since this is the exact scenario as Challenge Game 2, the proof now follows from Lemma 28.

5 Subprotocol Definitions and Constructions

5.1 Oblivious Routing Gate (ORG) Paradigm

Ideal Functionality for Oblivious Routing Gate in Beneš Network G via (rate-1) PIR

Setup (see Figure 6).

- Parameters for a wide-edged, extended and colored Beneš network $G = B(N, b, c, w)$: $N = 2^n, b, c, w$.
- Color index $\widehat{c} \in [1, c]$, row index $r \in [N]$, level index $l \in [0, (b+(1+b)\cdot \log N))]$.
- Let $\mu = \mu_{\widehat{c},r,l}$ denote the node at position $(\widehat{c}, r, l)$ of G (i.e. color $\widehat{c}$, row r, and level l; see Figure 5).
- Let $\mathcal{I} = \mathcal{I}_l$ denote the set of input wires to node μ, and notice for network G:

$$|\mathcal{I}_l| = \begin{cases} 1 & \text{if } l = 0 \\ c \cdot w & \text{if } l = j \cdot (1 + \log N) \text{ (for arbitrary } j \in [1, b]) \\ 2 \cdot w & \text{otherwise} \end{cases} \quad (2)$$

- Let $\mathcal{O} = \mathcal{O}_l$ denote the set of output wires from node μ, and notice (by definition of Beneš network G):

$$|\mathcal{O}_l| = \begin{cases} 1 & \text{if } l = (1 + b) \cdot (1 + \log N) - 1 \\ c \cdot w & \text{if } l = j \cdot (1 + \log N) - 1 \text{ (for arbitrary } j \in [1, b]) \\ 2 \cdot w & \text{otherwise} \end{cases} \quad (3)$$

- When the Π_{ORG} protocol is invoked, each of the $|\mathcal{I}_l|$ input wires will have an associated value on them. Let $|v| = |v_l|$ denote the size of each value.[a]
- Let $\Pi_{1\text{-}PIR} = \Pi_{1\text{-}PIR}(|v_l|, |\mathcal{I}_l|)$ denote a rate-1 PIR protocol for $|\mathcal{I}_l|$ elements, each of size $|v_l|$.
  - Let β_l denote the size of the reconstruct key (used by PIR client to parse the server's response).
  - Let $\gamma = \gamma_l$ denote the number of bits in a query to $\Pi_{1\text{-}PIR}(|v_l|, |\mathcal{I}_l|)$.
  - Let δ_{PIR} denote the (additive) stretch constant.
- Parties $\{S_1, \ldots, S_N\}$ connected via permutation σ to parties $\{R_1, \ldots, R_N\}$, and central router C.

Input. For each $i \in [N]$, values (w_i, w_i') with "input wire index" $w_i \in (\perp \cup \{1, 2, \ldots, |\mathcal{I}_l|\})$ and "output wire index" $w_i' \in (\perp \cup \{1, 2, \ldots |\mathcal{O}_l|\})$, subject to constraint: $w_i = \perp \Leftrightarrow w_i' = \perp$.

Output. For each output wire index $1 \leq j \leq |\mathcal{O}_l|$:

i. Rate-1 PIR query $q_j = q_{\mu,j} \in \{0,1\}^\gamma$.
ii. $(\mu, j, \kappa_{\mu,j})$ with $\kappa_{\mu,j}$ either $\perp$ or a reconstruction key for q_j (as per Correctness requirement below).

[a] When Π_{ORG} is invoked, $|v_l| = (c_{Enc} + l \cdot \delta_{PIR})$, where c_{Enc} is the ciphertext size of some scheme Enc.

Fig. 9: Definition of Π_{ORG} functionality via (rate-1) PIR in Beneš Network G.

Ideal Functionality for Oblivious Routing Gate in Beneš Network G via (rate-1) PIR (continued)

Correctness. For each $1 \leq j \leq |\mathcal{O}_l|$, output values $(q_j, \kappa_{\mu,j})$ satisfy:

Case 1: $j \notin \{w'_i\}_{S_i}$. In this case (no output wire index w'_i indicated wire index j), there is no correctness requirement for q_j (except that it is a valid rate-1 PIR query), and $\kappa_{\mu,j} = \perp$.
Case 2: $j = w'_i$ **for exactly one** $i \in [N]$. In this case (exactly one output wire index w'_i is equal to j), q_j is a PIR query corresponding to position w_i, and $\kappa_{\mu,j}$ is the corresponding reconstruction key.
Case 3: $j = w'_i = w'_{i'}$. This case (two or more output wires $\{w'_i, w'_{i'}\}$ equal j) is the same as Case 1.

Security. Consider the following security game featuring a (polynomially bounded) adversary $\mathcal{A}$:

1. $\mathcal{A}$ sees all output values $\{q_{\mu,j}\}_{j\in[1,|\mathcal{O}_l|]}$.
2. $\mathcal{A}$ specifies any subset $R_{\mathcal{A}} \subset [N]$, subject only to the constraint that $|R_{\mathcal{A}}| \leq N-2$. Let $\mathcal{T} := [N] \setminus R_{\mathcal{A}}$ denote the set of indices in $[N]$ that are *not* selected as part of $R_{\mathcal{A}}$, and let $\mathcal{O}_{\mathcal{A}} := \{w'_i \mid w'_i \neq \perp\}_{i\in R_{\mathcal{A}}}$ denote the set of (non-$\perp$) output wire indices for all $i \in R_{\mathcal{A}}$. Further partition $\mathcal{T}$ as:
   (a) $\mathcal{T}_1 \subseteq \mathcal{T} = \{i \mid w'_i = \perp\}$: The subset of $\mathcal{T}$ consisting of indices $i \in [N]$ whose output wire index $w'_i = \perp$
   (b) $\mathcal{T}_2 \subseteq \mathcal{T} \setminus \mathcal{T}_1 = \{i \mid w'_i \in \mathcal{O}_{\mathcal{A}}\}$: The subset of $\mathcal{T}$ consisting of indices $i \in [N]$ whose output wire index w'_i (is not equal to $\perp$ and) equals the output wire index $w'_{i'}$ for some index $i' \in R_{\mathcal{A}}$
   (c) $\mathcal{T}_3 = \mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$: The subset of $\mathcal{T}$ consisting of indices $i \in [N]$ whose output wire index $w'_i \notin (\mathcal{O}_{\mathcal{A}} \cup \perp)$
3. For each $i \in (R_{\mathcal{A}} \cup \mathcal{T}_2)$, $\mathcal{A}$ sees inputs $(w_i, w'_i) \in (\perp \cup \{1, \ldots, |\mathcal{I}_l|\}) \times (\perp \cup \{1, \ldots |\mathcal{O}_l|\})$.
4. $\mathcal{A}$ chooses $j \notin (R_{\mathcal{A}} \cup \mathcal{T}_2)$, and outputs guess (w_j, w'_j) for j's input and output wire indices.
5. $\mathcal{A}$ wins if (at least) one of its guesses $\{w_j, w'_j\}$ is both correct and *not*[a] the special value $\perp$.

Protocol Π_{ORG} is secure if any (polynomially bounded) adversary $\mathcal{A}$'s probability of winning the above security game is bounded by:

$$\Pr[\mathcal{A} \text{ wins via correct } w_j] \leq \frac{1}{|\mathcal{I}_l|} \quad \text{and}$$

$$\Pr[\mathcal{A} \text{ wins via correct } w'_j] \leq \frac{1}{|\mathcal{O}_l| - |\mathcal{O}_{\mathcal{A}}|}$$

[a] Observe that Adversary $\mathcal{A}$ *cannot* win if $\mathcal{T}_3 = \varnothing$.

Fig. 10: Definition of Π_{ORG} via (rate-1) PIR in Beneš Network G (continued).

5.2 Emulated Permutation Routing (EPR)

Per-Experiment Emulated Permutation Routing Protocol Π_{EPR_m}

Input. Same parameters as parent protocol Π_{NIAR}. Also, router C has:

1. For each node $\mu = \mu_{\widehat{c},r,l}$ and each of its $1 \le j \le |\mathcal{O}_l|$ outgoing wires: A rate-1 PIR query $q_{\mu,j}$.
2. For each sender S_i: Ciphertext $Enc_{pk_i}(m_i)$.

Output. For each row index $1 \le r \le N$ and for each of the c input wires to the node μ_r that is on row r of the last level in G, output values $\{z_{r,j}\}_{r\in[N],j\in[c]} \in \{0,1\}^{(c_{Enc}+(1+b)\cdot(1+\log N)\cdot\delta_{PIR})}$.

Protocol. This protocol has central router C simulate traverse graph G, acting as the (rate-1) PIR server at each node μ_l on level l. The simulation begins with router C assigning the values $\{Enc_{pk_i}(m_i)\}$ to each input wire of each node at row i of level 0. The values that are written on the outgoing wires of any node μ_l (for $l \ge 0$) are the (rate-1) PIR response that would result from issuing PIR query q_{μ_l}. We now formalize the protocol rules:

1. [Level -1]. For each node $\mu = \mu_{\widehat{c},r,0}$ on level 0 (for any $\widehat{c} \in [c]$ and any $r \in [N]$), C writes value $Enc_{pk_r}(m_r)$ (from Input Step 2) on each of $\mu_{\widehat{c},r,0}$'s input wires (there are $|\mathcal{I}_0| = c$) such input wires. Let A_μ denote the virtual array that holds these $|\mathcal{I}_0| = c$ values, each of size $|v_0| = c_{Enc}$ (see Figure 6).
2. [Levels $0 \le l \le (b + (1 + b) \cdot \log N)$]. For each node $\mu = \mu_{\widehat{c},r,l}$ on level l:
   (a) Let A_μ denote the virtual array holding the $|\mathcal{I}_l|$ values (each value of size $|v_l| = c_{Enc}+l\cdot\delta_{PIR}$) on each input wire leading to μ. Note that this array was constructed in Step (2d) of the previous iteration, or in Step 1 if $l = 0$.
   (b) For each output wire $1 \le j \le |\mathcal{O}_l|$ of μ, C simulates running the (rate-1) PIR query $q_{\mu,j}$ (from Input Step 1) against A_μ. Let $z_{\mu,j}$ denote the PIR response, and notice that (by definition of rate-1) it has size $|z_{\mu,j}| = c_{Enc} + (l + 1) \cdot \delta_{PIR}$.
   (c) (If $l = (b+(1+b)\cdot\log N)$ is the last iteration, skip this step and (2d) below.) For each of the $\{z_{\mu,j}\}$, cluster each contiguous set of w values together:
   $$\{(\underbrace{z_{\mu,1},\ldots,z_{\mu,w}}_{\widehat{A}_{\mu,1},}),\quad (\underbrace{z_{\mu,1+w},\ldots,z_{\mu,2w}}_{\widehat{A}_{\mu,2},}),\ \ldots,\ (\underbrace{z_{\mu,1+bw},\ldots,z_{\mu,(b+1)w}}_{\widehat{A}_{\mu,b+1}})\}$$
   where $b = 1$ if $l = j \cdot (1 + \log N) - 1$ (for any $j \in [b]$; see (3)); else $b = c$-1.
   (d) Construct the PIR input array $A_{\mu'}$ for each μ' on the *next* level by concatenating the appropriate number[a] of arrays $\{\widehat{A}_{\mu,j}\}$ (from Step 2c), choosing the appropriate nodes μ (as dictated by which nodes μ have output wires that lead to μ', as per graph G).
3. [Level $(1 + b) \cdot (1 + \log N)$]. From the $l = (b + (1 + b) \cdot \log N)$ iteration of the previous Step 2.b, each node μ' on level $(b + (1 + b) \cdot \log N)$ had exactly 1 value $z_{\mu',1}$ that was generated (see e.g. (3)), and this value has size $|z_{\mu',1}| = c_{Enc} + (1 + b) \cdot (1 + \log N) \cdot \delta_{PIR}$. For any node μ on the final level $l = (1 + b) \cdot (1 + \log N)$, there are c nodes $\{\mu'_j\}_{j\in[c]}$ from the previous level that each have exactly one output wire leading to μ, and each such output wire has value $z_{\mu'_j,1}$ of size $|z_{\mu'_j,1}| = c_{Enc} + (1 + b) \cdot (1 + \log N) \cdot \delta_{PIR}$. Output these $\{z_{\mu'_j,1}\}$ as the final output for each node μ on the final level of G.

[a] Two arrays $(\widehat{A}_{\widehat{\mu},j}, \widehat{A}_{\widehat{\mu},j})$ are concatenated if $(1 + \log N) \nmid l$; otherwise c such arrays are concatenated; see e.g. (2) (Figure 6).

Fig. 11: Per-Experiment Protocol Π_{EPR_m} for Anonymous Permutation Routing.

6 Correctness and Security

In this section, we present a series of definitions and lemmas that allow us to argue our main protocol (Figures 7 and 8) satisfies the correctness and security properties of the NIAR model (Section 3.2). Due to space limitations, all proofs can be found in the Supplementary Materials.

6.1 Probabilities in a Beneš Network

Lemma 5 *Suppose that for each input node $\{\nu_i\}_{i=1}^{N}$ on the butterfly network of Figure 1, a random path $\mathcal{P}_i$ of $\log N$ steps is performed. For any node μ_l (at level $l \in [0, \log N]$), let X_{μ_l} denote the random variable that indicates how many of the paths $\{\mathcal{P}_i\}$ pass through node μ_l. Then for any integer $k \geq 1$:*

$$Pr[X_{\mu_l} \geq k] \;\leq\; \frac{2^l}{k!} \tag{4}$$

Proof. Fix an arbitrary node μ_l at level $l \in [0, \log N]$. Notice that a random path $\mathcal{P}_i$ originating at input node ν_i (for $i \in [N]$) can pass through μ_l if and only if ν_i is one of the 2^l *ancestors* of μ_l (see Figure 12); and in this case, the probability that a random path $\mathcal{P}_i$ originating at ν_i passes through μ_l is exactly $1/2^l$ (see Figure 13). Since each random path is chosen independently from one another, the random variable X_{μ_l} is exactly described by the probability mass function of the binomial distribution, for $M_l := 2^l$ experiments (corresponding to the M_l ancestors of node μ_l) and probability of success $p = 1/M_l$. Furthermore, since the expected value of X_{μ_l} is $M_l \cdot p = 1$, the probability is maximized when $X_{\mu_l} = 1$. Consequently, for any $k \geq 1$:

$$\begin{aligned}
\Pr[X_\mu \geq k] \;&\leq\; (M_l - k) \cdot \Pr[X_\mu = k] \\
&\leq\; (M_l - k) \cdot \binom{M_l}{k} \cdot p^k \cdot (1-p)^{M_l - k} \\
&\leq\; (M_l - k) \cdot \frac{M_l \cdot (M_l - 1) \cdot \cdots \cdot (M_l - k + 1)}{k!} \cdot p^k \\
&\leq\; \frac{M_l \cdot (M_l - 1) \cdot \cdots \cdot (M_l - k)}{k!} \cdot \left(\frac{1}{M_l}\right)^k \;\leq\; \frac{M_l}{k!} \;=\; \frac{2^l}{k!}
\end{aligned}$$

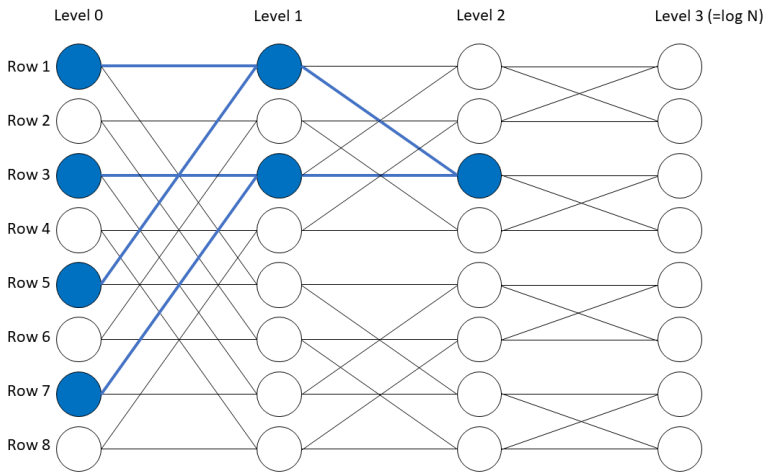


Fig. 12: Ancestors of a node on some level of a Butterfly network with $N = 8$.

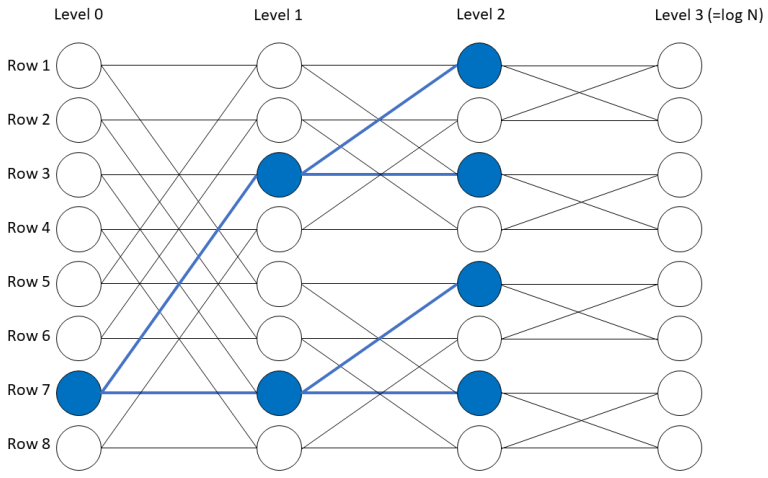


Fig. 13: Descendants of a node on some level of a Butterfly network with $N = 8$.

Lemma 6 *Suppose that for each input node*[4] $\{\nu_i\}_{i=1}^{N}$ *of a* ***colored*** *butterfly network of Figure 2 (with replication factor c), a random path* $\mathcal{P}_i$ *of* $(1+\log N)$ *steps is performed (the first step chooses the color* $\widehat{c} \in [1c]$*). For any node* $\mu_l = \mu_{\widehat{c},r,l}$ *(at level* $l \in [0, \log N]$*, row* $r \in [N]$*, and color* $\widehat{c} \in [c]$*), let* X_{μ_l} *denote the random variable that indicates how many of the paths* $\{\mathcal{P}_i\}$ *pass through node* μ_l*. Then for any integer* $k \geq 1$*:*

$$Pr[X_{\mu_l} \geq k] \;\leq\; \frac{2^l}{k! \cdot c^k} \tag{5}$$

Proof. The argument is analogous to the proof of Lemma 5, with each node $\mu_l = \mu_{\widehat{c},r,l}$ (at level $l \in [0, \log N]$, row $r \in [N]$, and color $\widehat{c} \in [c]$) still having $M_l = 2^l$ ancestor nodes (now at level -1), and with each ancestor having probability $p = 1/(c \cdot M_l)$ of passing through node μ_l (e.g. one of c edges are chosen from level -1 to level 0 that determines the "color"). Therefore, the analysis used in the proof of Lemma 5 can be followed exactly, plugging in $p = 1/(c \cdot M_l)$ in the last step.

Lemma 7 *Suppose that for each input node* $\{\nu_i\}_{i=1}^{N}$ *of a* ***colored*** *butterfly network of Figure 2 (with replication factor c), a random path* $\mathcal{P}_i$ *of* $(1+\log N)$ *steps is performed (the first step chooses the color* $\widehat{c} \in [c]$*). For any integer* $k \geq 1$*, let* X_k *denote an indicator variable on whether there exists* ***any*** *node* μ *(in the entire colored butterfly network) that has more than* k *(of the* N *total) random paths* $\{\mathcal{P}_i\}$ *pass through it. Then:*

$$Pr[X_k = 1] \;\leq\; \frac{2c \cdot N^2}{k! \cdot c^k} \tag{6}$$

Proof. Since $k \geq 1$, there are clearly no nodes on levels -1 or 0 that have more than one path pass through it. The corrolary therefore follows immediately from (5), by applying a union bound over all of the $c \cdot N \log N$ nodes ($c \cdot N$ nodes on each of the levels $l \in [1, \log N]$):

$$\Pr[X_k = 1] \;\leq\; (c \cdot N) \cdot \sum_{l=1}^{\log N} \frac{2^l}{k! \cdot c^k} \;=\; \left(\frac{c \cdot N}{k! \cdot c^k}\right) \cdot \sum_{l=1}^{\log N} 2^l \;\leq\; \frac{2c \cdot N^2}{k! \cdot c^k}$$

We now extend a (colored) butterfly network by concatenating several "blocks," each block consisting of $\log N$ levels, and then finishing with one final level that is the mirror reflection of a butterfly network:

Definition 8 *An extended (colored) Beneš network with b blocks consists of b butterfly networks concatenated together, followed by a single (reflected) butterfly network. Additionally, where each pair of blocks are connected, there is a single level inserted which consists of edges connecting all colors of each node (at each*

[4] A colored butterfly network can be viewed as c disjoint butterfly networks overlaid on top of one another. Alternatively, we can view a colored butterfly network as a single (connected) graph by adding an extra input level (with level index -1) on the far left, consisting of N input nodes. Then there are c edges emanating from each input node, connecting it to each of the c colored nodes in level 0 of the corresponding row.

"row") to each other; see Figure 5. A **block** j, *for* $j \in [1, (1+b)]$, *refers to the* $(1+\log N)$ *levels (and edges) between levels* $(j-1)\cdot(1+\log N)$ *and* $j\cdot(1+\log N)$. *That is, a block corresponds to a contiguous set of* $(1+\log N)$ *levels, whose first* $\log N$ *levels are a butterfly network, and the last level is the "connecting" level that consists of all edges connecting the different colors of all nodes on the same "row."*[5] *The* **input** *level of a block* $j \in [1, 1+b]$ *is level* $(j-1)\cdot(1+\log N)$, *and the* **output** *level is* $j\cdot(1+\log N)$ *(notice the input level of block* b *is the same as the output level of block* $b-1$*).*

The following is analogous to Lemma 7, but bounds the probability with respect to each *block* of an extended, colored Beneš network:

Lemma 9 *Let* $\sigma : [N] \rightarrow [N]$ *be an arbitrary permutation on* N *items. Suppose that for each input node* $\{\nu_i\}_{i=1}^{N}$ *of an extended, colored Beneš network with replication factor* c *and* b *blocks, a random path* $\mathcal{P}_i$ *of* $(1+b\cdot(1+\log N))$ *steps is performed, and then each such path is extended (from level* $(b\cdot(1+\log N))$ *to level* $(1+b)\cdot(1+\log N)$*) by traversing the unique path from the current node (on level* $(b\cdot(1+\log N))$*) to* $\sigma(i)$ *(see Figure 5). For any* $j \in [1,\ (b+1)]$ *and for any integer* $k \geq 1$, *let* $X_{j,k}$ *denote an indicator variable on whether there exists* ***any*** *node* μ_j *within block* j *(i.e. between levels* $[(j-1)\cdot(1+\log N),\ j\cdot(1+\log N)-1]$ *that has more than* k *(of the* N *total) random paths* $\{\mathcal{P}_i\}$ *pass through it. Then:*

$$Pr[X_{1,k}=1] \ = \ Pr[X_{1+b,k}=1] \ \leq \ \frac{2c\cdot N^2}{k!\cdot c^k}$$

$$\forall\ j \in [2,\ b]: \quad Pr[X_{j,k}=1] \ \leq \ \frac{c\cdot N^2\cdot(1+\log N)}{k!\cdot c^k} \tag{7}$$

Proof. For $j=1$, (7) follows immediately from Lemma 7. Similarly for $j=1+b$, since σ is a permutation, then the setup (for continuing each path $\{\mathcal{P}_i\}_{i=1}^{N}$ from level $(b\cdot\log N)$ through the final output level $1+(1+b)\cdot\log N)$ is identical (up to reflection/mirror image) to the hypotheses of Lemma 7. Thus, it remains to demonstrate (7) for each $j \in [2,\ b]$.

The argument for all such blocks j is simply a counting argument: since each path has an equal probability of starting at any of the $(c\cdot N)$ nodes on the "input" level of block j (namely, on level $(j-1)\cdot(1+\log N)$), and from there a uniformly random path is chosen, we have that for any level $l \in [(j-1)\cdot(1+\log N),\ j\cdot(1+\log N)-1]$ contained in the j^{th} block, each path has an equal probability of passing through each of the $(c\cdot N)$ nodes on that level. For any node μ_j in the j^{th} block, let X_{μ_j} denote the random variable that counts the number of paths passing through μ_j. Then X_{μ_j} is exactly described by the probability mass function of the binomial distribution, for N experiments (corresponding to the N paths $\{\mathcal{P}_i\}$) and probability of success $p = 1/(c\cdot N)$. Furthermore, since the expected number of paths passing through μ_j is $N\cdot p = 1/c$, the probability

[5] In the special case of the $(1+b)^{th}$ block, the first $\log N$ levels of this block are a *reflected* butterfly network, and the last level of the block is the final "output" level of the entire network.

is maximized when $X_{\mu_j} = 1/c$. Consequently, for any $k \geq 1$:

$$\begin{aligned}
\Pr[X_{\mu_j} \geq k] &\leq (N-k) \cdot \Pr[X_{\mu_j} = k] \\
&\leq (N-k) \cdot \binom{N}{k} \cdot p^k \cdot (1-p)^{N-k} \\
&\leq (N-k) \cdot \frac{N \cdot (N-1) \cdot \cdots \cdot (N-k+1)}{k!} \cdot p^k \\
&\leq \frac{N \cdot (N-1) \cdot \cdots \cdot (N-k)}{k!} \cdot \left(\frac{1}{cN}\right)^k \leq \frac{N}{k! \cdot c^k} \qquad (8)
\end{aligned}$$

Thus, applying a union bound on the $cN(1+\log N)$ nodes in block j:

$$\Pr[X_{j,k}] \leq (cN(1+\log N)) \cdot \Pr[X_{\mu_j} \geq k] \leq \frac{c \cdot N^2 \cdot (1+\log N)}{k! \cdot c^k}$$

6.2 Permutation Routing Problem

We begin with the definitions that are needed to describe the Permutation Routing Problem and the desired properties that a successful solution must exhibit.

Definition 10 *Given a graph $G = (V, E)$ and a collection of paths $\{\mathcal{P}_i\}$ within the graph, we say that any given path $\mathcal{P}_i$ is edge-disjoint from the others if no edge in $\mathcal{P}_i$ is contained/traversed by any other path. We say the entire collection of paths $\{\mathcal{P}_i\}$ is edge-disjoint if each individual edge is edge-disjoint.*

Definition 11 *A Permutation Routing Problem(N, σ, G) is defined as follows: For input integer $N \in \mathbb{N}$, permutation $\sigma : [N] \to [N]$, and graph G that has N designated "input" nodes $\{I_1, I_2, \ldots, I_N\}$ and N designated "output" nodes $\{O_1, O_2, \ldots, O_N\}$, construct N **edge-disjoint** paths through G that connect each input-output pair $(I_i, O_{\sigma(i)})$.*

We extend the notion of the extended, colored Beneš network to a *wide-edged* variant, in which each edge has been replicated w times (which can equivalently be viewed as each edge having capacity w):

Definition 12 *A wide-edged, extended, colored Beneš network $B(N, b, c, w)$ is an extended and colored Beneš network (Figure 5) in which, for each level $l \in [1, (b+(1+b)\cdot\log N)]$, each edge connecting levels $(l\text{-}1, l)$ is replicated w times.*

Notice that the added *color* and *edge-width* features serve a similar purpose: they each reduce the probability of an edge conflict (i.e. increase the probability of being edge-disjoint, as per Definition 10); but they do so in slightly different ways: the *color* feature not only introduces new edges, but also additional nodes, so that once a path chooses a color for a particular block (which happens only at the start of each block, when there is a transition between levels in which each edge connects the various "colors" corresponding to the nodes on a common "row;" see Figure 5), it will not conflict (on the present block) with paths that chose another color. In contrast, the *edge-width* feature reduces the chances that two paths conflict across a given edge; but those same paths may still end up in the same node at the far end of this edge, and thus may conflict in a later edge.

Definition 13 *Given a wide-edged, extended, colored Beneš network $B(N, b, c, w)$, and given a routing algorithm $\Pi = \Pi_{N,\sigma,G=B(N,b,c,w)}$ that attempts to solve the Permutation Routing Problem (Definition 11), for each $i \in [N]$ and for each block $1 \leq j \leq (1+b)$, let $\mathsf{X}_{\Pi}(\mathsf{i},\mathsf{j})$ denote the boolean random variable that indicates whether Π constructs an edge-disjoint path* ***on the $\mathbf{j^{th}}$ block*** *for the pair $(I_i, O_{\sigma(i)})$. That is, $X_{\Pi}(i,j) = 1$ if the path connecting I_i and $O_{\sigma(i)}$ within the j^{th} block (as specified by Π) is edge-disjoint from all other paths specified by Π.*

The algorithm in Figure 14 formalizes a naïve solution for the Permutation Routing Problem in which random paths are chosen in an extended and colored Beneš network. Namely, this algorithm specifies that each path $\mathcal{P}_i$ emanating from input I_i chooses random edges for each level through the first b blocks in an extended and colored Beneš network $B(N, b, c, w)$, and then follows the unique path from its current node on level $(b \cdot (1 + \log N))$ to the destination node $O_{\sigma(i)}$ (by choosing one of the w replicates of each edge along this path).

We now demonstrate several properties that the naïve routing protocol of Figure 14 satisfies.

Lemma 14 *Let $\Pi = \Pi_{\sigma_N}$ denote the routing algorithm of Figure 14 on a wide-edged, extended, and colored Beneš network $B(N, b, c \geq 2, w)$. Then for any $i \in [0, N]$, for any $1 \leq j \leq (1+b)$, and for any $1 \leq k \leq N$, the probability that $X_{\Pi}(i,j) = 0$ (as per Definition 13) is bounded by:*

$$Pr[X_{\Pi}(i,j) = 0] \;\leq\; (1 + \log N) \cdot \left(\frac{c \cdot N^2 (1 + \log N)}{k! \cdot c^k} + \frac{k}{2w} \right) \tag{9}$$

Proof. Fix any $i \in [N]$, $j \in [(1+b)]$, and $k \in [N]$. Let $\mathcal{P}_i$ denote the path selected by Π_{σ_N} connecting $(I_i, O_{\sigma(i)})$ for a given run of the algorithm in Figure 14. Let e_l be an edge in $\mathcal{P}_i$ that is in block j; that is, e_l connects levels $(l, l+1)$ for some level $l \in [(j-1) \cdot (1 + \log N),\ (j \cdot (1 + \log N) - 1)]$. We first argue that the probability that e_l is **not** edge-disjoint (that is, that some other path $\mathcal{P}_{i'}$ specified by Π_{σ_N} also contains e_l, for $i' \neq i$) is bounded by:

$$\Pr[e_l \in \mathcal{P}_{i'} \text{ for } i' \neq i] \;\leq\; \frac{c \cdot N^2 (1 + \log N)}{k! \cdot c^k} + \frac{k}{2w} \tag{10}$$

To prove (10), let μ_l denote the node on level l that lies on $\mathcal{P}_i$ (so the left-hand node of edge e_l), and let Z_{k,μ_l} denote the boolean random variable indicating whether μ_l has more than k total paths that pass through it (as specified by the given run of Π_{σ_N}). Consider:

$$\begin{aligned}
\Pr[e_l \in \mathcal{P}_{i'} \text{ for } i' \neq i] \;&\leq\; \Pr[e_l \in \mathcal{P}_{i'} \text{ for } i' \neq i \mid Z_{k,\mu_l} = 1] \cdot \Pr[Z_{k,\mu_l} = 1] + \\
&\qquad \Pr[e_l \in \mathcal{P}_{i'} \text{ for } i' \neq i \mid Z_{k,\mu_l} = 0] \cdot \Pr[Z_{k,\mu_l} = 0] \\
&\leq\; \Pr[Z_{k,\mu_l} = 1] \;+\; \Pr[e_l \in \mathcal{P}_{i'} \text{ for } i' \neq i \mid Z_{k,\mu_l} = 0] \\
&\leq\; \frac{c \cdot N^2 (1 + \log N)}{k! \cdot c^k} + \frac{k}{2w}
\end{aligned} \tag{11}$$

Routing Algorithm $\Pi_{N,\sigma,G}$ in a Wide-Edged, Extended, and Colored Beneš Network $B(N, b, c, w)$

Input. Parameters $N = 2^n, b, c, w \in \mathbb{N}$, and a permutation $\sigma_N : [N] \to [N]$. Also, a given wide-edged, extended, and colored Beneš network $B(N, b, c, w)$ with N designated "input" nodes $\{I_1, I_2, \ldots, I_N\}$ (on level -1) and N designated "output" nodes $\{O_1, O_2, \ldots, O_N\}$ (on level $(1 + b) \cdot (1 + \log N)$).

Output. The specification of N paths $\{\mathcal{P}_i\}_{i=1}^{N}$ connecting each $(I_i, O_{\sigma(i)})$.

Protocol. For each row index $1 \leq i \leq N$, run the $\Pi_{N,\sigma,G}(i)$ sub-protocol.

Sub-protocol $\Pi_{N,\sigma,G}(i)$: Outputs path $\mathcal{P}_i$ connecting input node I_i with output node $O_{\sigma(i)}$.

1. [First Step.] Choose a uniformly random value $\widehat{c}_{-1} \in [c]$, and set the first edge in $\mathcal{P}_i$ (which connects node I_i (at level -1) to one of its c neighbors on level 0) accordingly.
2. ["Internal" Blocks.] For each block $1 \leq j \leq b$:
   (a) Choose a uniformly random node μ_j on the output level of block j (level $j \cdot (1 + \log N)$). Namely, choose a uniformly random color $\widehat{c}_j \in [c]$ and uniformly random row $r_j \in [N]$. Notice that choice of μ_j completely determines the "up/down" specification of path $\mathcal{P}_i$ on block j. It remains only to specify, for each expanded edge, which of the w duplicates of this edge to traverse.
   (b) For each $1 \leq l \leq (1 + \log N)$, choose a uniformly random value $w_{j,l} \in [w]$, which specifies which duplicated edge $\mathcal{P}_i$ will traverse between levels $(j - 1) \cdot (1 + \log N) + l - 1$ and $(j - 1) \cdot (1 + \log N) + l$.
3. [Final Block.] For each $1 \leq l \leq \log N$, set the edge connecting level $(b \cdot (1 + \log N) + l - 1,\ b \cdot (1 + \log N) + l)$ in $\mathcal{P}_i$ as follows:
   (a) Choose a uniformly random value $w_l \in [w]$.
   (b) Let $r_l \in \{0, 1\}$ be determined by $\sigma(i)$; namely, $r_l = 0$ iff the $(\log N - l - 1)$ bit of the target node $\sigma(i)$'s row is 0. Set the next edge in $\mathcal{P}_i$ according to (w_l, r_l); i.e. choose the w^{th} duplicate of the "up" edge if $r_l = 0$, and otherwise choose the w^{th} duplicate of the "down" edge.
4. [Final Level.] For $l = (1 + b) \cdot (1 + \log N)$: set the final edge in $\mathcal{P}_i$ to be the unique edge that leads from the current node (on level l) to node $O_{\sigma(i)}$.

Fig. 14: Routing algorithm $\Pi_{N,\sigma,G}$ in a wide-edged, extended and colored Beneš network $B(N, b, c, w)$.

where the first term of (11) comes from (7), and the second comes from applying a union bound: since there are at most k other paths that pass through μ_l (since $Z_{k,\mu_l} = 0$ for this term), the probability that any of these (at most k) paths also traverses edge e_l is bounded by $1/2w$ (since there are the two[6] choices of "up/down," and (expansion factor) w choices for which duplicate of each edge.

[6] In the case that e_l traverses the final level of the block, then instead of two there are c choices for edge (corresponding to each edge leading to each color node), and by assumption $c \geq 2$.

Using (10), we conclude the proof by applying a union bound on the (1+log N) levels in block j.

We now extend Definition 13 (and in particular the indicator random variable $X_{\Pi}(i,j) = 0$) to a statement about a path $\mathcal{P}_i$ being edge-disjoint across the *entire network* G:

Definition 15 *Given a routing algorithm $\Pi = \Pi_{N,\sigma,G}$ that attempts to solve the Permutation Routing Problem (Definition 11), for each $i \in [N]$, let $\mathsf{X}_{\Pi}(\mathsf{i})$ denote the boolean random variable that indicates whether Π constructs an edge-disjoint path for the pair $(I_i, O_{\sigma(i)})$. That is, $X_{\Pi}(i) = 1$ if the path connecting I_i and $O_{\sigma(i)}$ (as specified by Π) is edge-disjoint from all other paths specified by Π.*

Lemma 16 *Let $\Pi = \Pi_{\sigma_N}$ denote the routing algorithm of Figure 14 on a wide-edged, extended, and colored Beneš network $B(N, b, c \geq 2, w)$. Then for any $i \in [N]$ and for any $1 \leq k \leq N$, the probability that $X_{\Pi}(i) = 0$ (as per Definition 15) is bounded by:*

$$Pr[X_{\Pi}(i) = 0] \;\leq\; (1+b) \cdot (1 + \log N) \cdot \left(\frac{c \cdot N^2 \cdot (1 + \log N)}{k! \cdot c^k} + \frac{k}{2w} \right) \qquad (12)$$

Proof. This follows immediately from Lemma 14 by applying a union bound on the $(1 + b)$ blocks of the Beneš network $B(N, b, c, w)$.

We are now ready to present the final definition and corresponding statement that will be required for the correctness property of protocol 7.

Definition 17 *Given an (independent) collection $\{\Pi_m\}$ of M routing algorithms that attempt to solve the Permutation Routing Problem (see Definition 11) in a wide-edged, extended and colored Beneš network $B(N, b, c, w)$, let X denote the boolean random variable that indicates if, for every $i \in [N]$, there exists (at least) one experiment $m \in [M]$ in which $X_{\Pi_m}(i) = 1$ (where $X_{\Pi_m}(i)$ is the random variable in Definition 15).*

Corollary 18 *For any security parameter λ and for any input parameters $2^n = N \geq 64$, $b = \lambda - 1$, $c = 4 \cdot a_{\lambda}$, and $w = 1.2 \cdot \lambda \cdot \log N \cdot (1 + \log N)$ (for[7] $a_{\lambda} := \max(2, \lambda^{1/(\log N - 1)})$), if the Routing Algorithm of Figure 14 is repeated $M := \lambda$ times, then the probability that $X = 0$ (Definition 17) is bounded by:*

$$Pr[X = 0] \;<\; \left(\frac{3}{4}\right)^{\lambda} \qquad (13)$$

[7] Notice $a_{\lambda} = 2$ if $\lambda \leq N/2$.

Proof. First, we examine the RHS of (12) with the parameter values as specified and with $k = \log N$:

$$\begin{aligned}(1+b)\cdot(1+\log N)\cdot\left(\frac{cN^2\cdot(1+\log N)}{k!\cdot c^k}+\frac{k}{2w}\right) &= \lambda\cdot\left(\frac{4a_\lambda N^2\cdot(1+\log N)^2}{(\log N)!\cdot a_\lambda^{\log N}\cdot 4^{\log N}}+\frac{\log N}{96\lambda\log N}\right)\\ &= \lambda\cdot\left(\frac{4\cdot(1+\log N)^2}{(\log N)!\cdot a_\lambda^{\log N-1}}+\frac{1}{2.4\lambda}\right)\\ &< \lambda\cdot\left(\frac{1}{3\lambda}+\frac{5}{12\lambda}\right)\\ &= \frac{3}{4} \qquad (14)\end{aligned}$$

where we have used in the inequality that $\frac{4\cdot(1+\log N)^2}{(\log N)!} < 1/3$ (which holds for any $N \geq 64$) and that $a_\lambda^{\log N-1} \geq \lambda$ (which is immediate by definition of a_λ). Consider:

$$\begin{aligned}\Pr[X=0] \;&\leq\; \Pr[\forall\, m \in [M] :\; X_{\Pi_m}(i) = 0]\\ &\leq\; \left((1+b)\cdot(1+\log N)\cdot\left(\frac{c\cdot N^2\cdot(1+\log N)}{k!\cdot c^k}+\frac{k}{2w}\right)\right)^M\\ &<\; \left(\frac{3}{4}\right)^\lambda\end{aligned}$$

where the first inequality is Lemma 16 and the second inequality is (14) and substituting $M = \lambda$.

Ultimately, Corollary 18 will demonstrate *correctness* of our routing protocol (7). However, for the *security* property, we will need to consider two sets of (input, output) node pairs. The following definition (which extends Definition 13, but for two sets of (input, output) pairs of nodes) will be used to capture the requisite probabilities for our security proof.

Definition 19 *Given a wide-edged, extended, colored Beneš network $B(N, b, c, w)$ and two routing algorithms $\Pi = \Pi_{N,\sigma,G=B(N,b,c,w)}$ and $\Pi' = \Pi'_{N,\sigma,G=B(N,b,c,w)}$ that attempt to solve the Permutation Routing Problem (Definition 11), for any pair of row indices $(i, i') \in [N]$ and for any block $1 \leq j \leq (1+b)$, let $\mathsf{Y}_{\Pi,\Pi'}(\mathsf{i}, \mathsf{i}', \mathsf{j})$ denote the boolean random variable that indicates whether each of the four paths $\{\mathcal{P}_i, \mathcal{P}_{i'}, \mathcal{P}'_i, \mathcal{P}'_{i'}\}$ are edge-disjoint from all other paths on block j.*

Aside. Notice that Definition 19 is only concerned about what happens on a single block of a wide-edged, extended, and colored Beneš network $B(N, b, c, w)$. In particular, we do not actually require two routing algorithms Π, Π' to be defined on the full network $B(N, b, c, w)$ in order to evaluate whether $Y_{\Pi,\Pi'}(i, i', j)$ equals zero or one on a given block $j \in [1, 1+b]$ (as per Definition 19); rather, we only need to know what each algorithm does on block j. Also notice that there is no requirement that the four paths be edge-disjoint *from each other*.

Definition 20 *Given a wide-edged, extended, and colored Beneš network $G = B(N, b, c, w)$, and a routing algorithm $\Pi = \Pi_{N,\sigma,G}$ that attempts to solve the Permutation Routing Problem (Definition 11), and given any pair of row indices $i, i' \in [N]$ and any block index $j \in [1, (1+b)]$, define the* block j alternate routing algorithm $\Pi'_{i,i',j}$ *as follows:*

- *$\Pi'_{i,i',j}$ is identical to Π on the first $(j-1)$ blocks.*
- *On the j^{th} block:*
  - *For all $\widehat{i} \notin \{i, i'\}$: $\Pi'_{i,i',j}$ is identical to Π.*
  - *Let μ_i (respectively $\mu_{i'}$) denote the node on the output level (which has level index $j \cdot (1 + \log N)$) of block j that $\mathcal{P}_i$ (respectively $\mathcal{P}_{i'}$) passes through, as per Π (see Step 2a of Figure 14). Then $\Pi'_{i,i',j}$ is identical to Π except that the choice of μ_i versus $\mu_{i'}$ is swapped in Step 2a for i and i'.*[8]
- *For all blocks beyond the j^{th} block:*
  - *For all $\widehat{i} \notin \{i, i'\}$: $\Pi'_{i,i',j}$ is identical to Π.*
  - *For i, i': $\Pi'_{i,i',j}$ is identical to Π, except that it has swapped paths $\mathcal{P}_i$ and $\mathcal{P}_{i'}$.*[9]

With these definitions in hand, we provide an analogous probability bound for $Y_{\Pi,\Pi'}(i, i', j)$ as Lemma 14 provided for $X_{\Pi}(i, j)$.

Lemma 21 *Let $\Pi = \Pi_{\sigma_N}$ denote the routing algorithm of Figure 14 on a wide-edged, extended, and colored Beneš network $B(N, b, c \geq 2, w)$. Fix any pair of row indices $i, i' \in [N]$ and any block index $j \in [1, (1+b)]$, and let $\Pi' = \Pi'_{i,i',j}$ denote the "block j alternate routing protocol" (Definition 20). Then for any $1 \leq k \leq N$, the probability that $Y_{\Pi,\Pi'}(i, i', j) = 0$ (as per Definition 19) is bounded by:*

$$Pr[Y_{\Pi,\Pi'}(i, i', j) = 0] \;\leq\; 4 \cdot (1 + \log N) \cdot \left(\frac{c \cdot N^2 \cdot (1 + \log N)}{k! \cdot c^k} + \frac{k}{2w} \right) \qquad (15)$$

Proof. We can use a similar argument as was done for Lemma 14 to bound the probability that any one of the four paths $\{\mathcal{P}_i, \mathcal{P}_{i'}, \mathcal{P}'_i, \mathcal{P}'_{i'}\}$ is *not* edge-disjoint. Specifically, observe that:

- $\forall \widehat{i} \notin \{i, i'\}$: $\mathcal{P}_{\widehat{i}} = \mathcal{P}'_{\widehat{i}}$ (by construction of Π'), and hence any of the four paths in $\{\mathcal{P}_i, \mathcal{P}_{i'}, \mathcal{P}'_i, \mathcal{P}'_{i'}\}$ is edge-disjoint from $\mathcal{P}'_{\widehat{i}}$ if and only if it is edge-disjoint from $\mathcal{P}_{\widehat{i}}$.
- For any of the four paths in $\{\mathcal{P}_i, \mathcal{P}_{i'}, \mathcal{P}'_i, \mathcal{P}'_{i'}\}$, the setup is identical to the setup in the hypotheses of Lemma 14, *except* that there are now $N + 2$ total paths through block j (instead of just N paths). Namely, $N-2$ of the N paths specified by $\mathcal{P}'$ exactly overlap with the corresponding $N - 2$ paths specified by $\mathcal{P}$, plus the (up to) four distinct paths: $\{\mathcal{P}_i, \mathcal{P}_{i'}, \mathcal{P}'_i, \mathcal{P}'_{i'}\}$.

[8] Notice that if $\mu_i = \mu_{i'}$, then $\Pi'_{i,i',j}$ is identical to Π (for *all* paths $\{\mathcal{P}_i\}$) on all blocks through j (including block j).

[9] Swapping paths is only necessary for the sake of making sure the paths link up/ connect between blocks (since output node μ_i and $\mu_{i'}$ were swapped in block j). However, as was noted in the Aside note following Definition 19, the details of what $\Pi'_{i,i',j}$ does beyond block j will be irrelevant for the context of Lemmas 21 and 24.

However, notice that Definition 19 does not require the four paths $\{\mathcal{P}_i, \mathcal{P}_{i'}, \mathcal{P}'_i, \mathcal{P}'_{i'}\}$ to be edge-disjoint from each other. Therefore, when counting the number of paths that pass through a given node (see e.g. the analysis leading to (8)), we can ignore these four paths. More specifically, the bounds from Lemma 9 (and more specifically, in (7)) can be used as-is (they still provide an upper-bound, although they could be slightly tightened to leverage that there are only $N-2$ relevant paths instead of the N paths that were used in the analysis leading to (8)) when bounding $\Pr[Z_{k,\mu_l} = 1]$, as per the analysis in the proof of Lemma 14 (see (11)). Consequently, the analysis used in the proof of Lemma 14 remains valid, and hence, for any of the four paths, the probability that the path is edge-disjoint (from the $N-2$) other paths in $\{\mathcal{P}_{\widehat{i}}\}_{\widehat{i} \neq i,i'}$ on block j is bounded by $(1+\log N) \cdot \left(\frac{c \cdot N^2 \cdot (1+\log N)}{k! \cdot c^k} + \frac{k}{2w}\right)$. We conclude the proof by applying a union bound on the individual probabilities, for the four paths $\{\mathcal{P}_i, \mathcal{P}_{i'}, \mathcal{P}'_i, \mathcal{P}'_{i'}\}$.

Just as $\mathsf{X}_{\Pi}(\mathsf{i},\mathsf{j})$ (Definition 13) and the corresponding bound for it (Lemma 14) were extended from variables/statements about *blocks* to variables/statements about the *entire network* (in the corresponding Definition 15 and Lemma 16), we likewise extend $Y_{\Pi,\Pi'}(i,i',j)$ (Definition 19) and the corresponding Lemma 21 to variables/statements about the entire network. However, these extensions differ slightly from before, as ultimately we only need the *existence* of a block that satisfies the key property, as opposed to requiring that *all blocks* satisfy some property.

Definition 22 *Given a wide-edged, extended, and colored Beneš network $G = B(N,b,c,w)$, and given two routing algorithms $\Pi = \Pi_{N,\sigma,G}$ and $\Pi' = \Pi'_{N,\sigma,G}$ that attempt to solve the Permutation Routing Problem (Definition 11), for any pair of row indices $(i,i') \in [N]$, let $\mathsf{Y}_{\Pi,\Pi'}(\mathsf{i},\mathsf{i}')$ denote the boolean random variable that indicates whether there exists some block $j \in [1,(1+b)]$ in which the four paths $\{\mathcal{P}_i, \mathcal{P}_{i'}, \mathcal{P}'_i, \mathcal{P}'_{i'}\}$ are each edge-disjoint from all other paths on block j.*

Definition 23 *Given a wide-edged, extended, and colored Beneš network $G = B(N,b,c,w)$, and a routing algorithm $\Pi = \Pi_{N,\sigma,G}$ that attempts to solve the Permutation Routing Problem (Definition 11), and given any pair of row indices $i,i' \in [N]$, define the alternate routing algorithm $\Pi'_{\mathsf{i},\mathsf{i}'}$ as follows:*

1. *$\forall\, j \in [1,(1+b)]$, let $\Pi'_j = \Pi'_{i,i',j}$ denote the block j alternate routing algorithm (Definition 20).*
2. *Construct $\Pi'_{i,i'}$ from the family of alternate routing algorithms $\{\Pi_j\}$ as follows:*
   a. *If there exists an index $j \in [1,(1+b)]$ such that $Y_{\Pi,\Pi'_j}(i,i',j) = 1$ (as per Definition 13), then let $\Pi'_{i,i'} = \Pi'_j$ (for the minimal j satisfying $Y_{\Pi,\Pi'_j}(i,i',j) = 1$).*
   b. *Otherwise, define $\Pi'_{i,i'} = \Pi$.*

Lemma 24 *Let $\Pi = \Pi_{\sigma_N}$ denote the routing algorithm of Figure 14 on a wide-edged, extended, and colored Beneš network $B(N,b,c \geq 2,w)$, let $i,i' \in [N]$ be any two row indices, and let $\Pi' = \Pi'_{i,i'}$ be the alternate routing algorithm (as*

per Definition 23). Then for any $1 \leq k \leq N$, the probability that $Y_{\Pi,\Pi'}(i,i') = 0$ (as per Definition 22) is bounded by:

$$Pr[Y_{\Pi,\Pi'}(i,i') = 0] \;\leq\; \left(4 \cdot (1+\log N) \cdot \left(\frac{c \cdot N^2(1+\log N)}{k! \cdot c^k} + \frac{k}{2w}\right)\right)^{(1+b)} \tag{16}$$

Proof. For any index $\widehat{j} \in [1,(1+b)]$, let $\Pi_{\widehat{j}}$ denote the "block j alternate routing algorithm" as per Definition 20, with corresponding random variable $Y_{\Pi,\Pi'_{\widehat{j}}}(i,i',\widehat{j})$ (as per Definition 19). Then by Lemma 21, the probability that $Y_{\Pi,\Pi'_{\widehat{j}}}(i,i',\widehat{j}) = 0$ is bounded by $4 \cdot (1+\log N) \cdot \left(\frac{c \cdot N^2 \cdot (1+\log N)}{k! \cdot c^k} + \frac{k}{2w}\right)$. Therefore, the probability that $Y_{\Pi,\Pi'_j}(i,i',j) = 0$ for *all* block indices $j \in [1,(1+b)]$ is bounded by $\left(4 \cdot (1+\log N) \cdot \left(\frac{c \cdot N^2 \cdot (1+\log N)}{k! \cdot c^k} + \frac{k}{2w}\right)\right)^{(1+b)}$. The lemma therefore follows from the observation that $Y_{\Pi,\Pi'}(i,i') = 1$ if and only if Π' is defined as per Step (2a) in Definition 23; i.e. if there exists $j \in [1,(1+b)]$ such that $Y_{\Pi,\Pi'_j}(i,i',j) = 1$.

We are now ready to present the final definition and corresponding statement that will be required for the security proof of protocol 7.

Definition 25 *Given an (independent) collection $\{\Pi_m\}$ of M routing algorithms that attempt to solve the Permutation Routing Problem (Definition 11) in a wide-edged, extended and colored Beneš network $B(N,b,c,w)$, let Y denote the boolean random variable that indicates if, for every Π_m and every pair of row indices $i,i' \in [N]$, that $Y_{\Pi_m,\Pi'_m}(i,i') = 1$ (where $\Pi'_m = \Pi'_{m,i,i'}$ is the alternate routing algorithm (Definition 23) and $Y_{\Pi_m,\Pi'_m}(i,i')$ is the corresponding random variable (Definition 22)).*

Corollary 26 *For*[10] *any security parameter $\lambda \geq 8$ and any input parameters $2^n = N \geq 64$, $b = \lambda - 1$, $c = 4 \cdot a_\lambda$, and $w = 1.2 \cdot \lambda \cdot \log N \cdot (1+\log N)$ (for $a_\lambda := \max(2, \lambda^{1/(\log N - 1)})$), if the Routing Algorithm of Figure 14 is repeated $M := \lambda$ times, then the probability that $Y = 0$ (Definition 25) is bounded by:*

$$Pr[Y = 0] \;<\; \frac{\lambda \cdot N^2}{4^\lambda} \tag{17}$$

Proof. First, notice that with the parameters as in the hypothesis and with $k = \log N$, the inner-term (everything except the exponent) on the RHS of (16)

[10] Notice that these parameter values all match those in the hypothesis of Corollary 18.

becomes:

$$\begin{aligned}
4\cdot(1+\log N)\cdot\left(\frac{cN^2\cdot(1+\log N)}{k!\cdot c^k}+\frac{k}{2w}\right) &= 4\cdot\left(\frac{4a_\lambda N^2\cdot(1+\log N)^2}{(\log N)!\cdot a_\lambda^{\log N}\cdot 4^{\log N}}+\frac{\log N}{2.4\lambda\log N}\right)\\
&= 4\cdot\left(\frac{4\cdot(1+\log N)^2}{(\log N)!\cdot a_\lambda^{\log N-1}}+\frac{5}{12\lambda}\right)\\
&< 4\cdot\left(\frac{1}{3\cdot N/2}+\frac{5}{96}\right)\\
&\leq 4\cdot\left(\frac{1}{96}+\frac{5}{96}\right)\\
&= \frac{1}{4} \qquad (18)
\end{aligned}$$

where we have used in the first inequality that $\frac{4\cdot(1+\log N)^2}{(\log N)!} < 1/3$ (which holds for any $N \geq 64$), that $a_\lambda^{\log N-1} \geq N/2$ (which is immediate from the definition of a_λ), and that $\lambda \geq 8$ (by hypothesis); and in the second inequality that $N \geq 64$. Consider:

$$\begin{aligned}
\Pr[Y=0] &= \Pr[\exists(m,i,i') \in [M]\times[N]\times[N] : Y_{\Pi_m,\Pi'_m}(i,i')=0]\\
&\leq M\cdot N^2\cdot\left(4\cdot(1+\log N)\cdot\left(\frac{c\cdot N^2(1+\log N)}{k!\cdot c^k}+\frac{k}{2w}\right)\right)^{(1+b)}\\
&\leq \lambda\cdot N^2\cdot\left(\frac{1}{4}\right)^{\lambda}
\end{aligned}$$

where the first inequality is from a union bound combined with Lemma 24; and the second inequality is from (18) and substituting $M=\lambda$ and $b=\lambda-1$.

6.3 Security

Succinctly, security (anonymity) will follow for the routing protocol of Figure 7 from:

$$\begin{aligned}
\text{Corollary 26} &\Rightarrow (!\exists\mathcal{A} \text{ with non-negligible advantage in Challenge Game 1})\\
&\Rightarrow (!\exists\mathcal{A} \text{ with non-negligible advantage in Challenge Game 2})\\
&\Rightarrow (\text{Routing Protocol of Figure 7 is secure (per Definition 11)})
\end{aligned} \qquad (19)$$

In this section, we define Challenge Games 1 and 2, and then demonstrate the first two implications in (19) (the third implication was already presented in the proof of Theorem 4).

Challenge Game 1

Input Parameters:

- Number of input/output nodes $2^n = N \geq 64$.
- Security parameter $\lambda \geq 8$.
- A wide-edged, extended and colored Beneš network $G = B(N, b, c, w)$, with parameters as per Corollaries 18 and 26: $b = \lambda - 1$, $c = 4 \cdot a_\lambda$, and $w = 1.2 \cdot \lambda \cdot \log N \cdot (1 + \log N)$ (for $a_\lambda := \max(2, \lambda^{1/(\log N - 1)})$).
- There are N "global input nodes" on level -1 of the Beneš network $G = B(N, b, c, w)$, which are denoted: $\mathcal{I} = \{I_1, I_2, \ldots, I_N\}$, and N global output nodes $\mathcal{O} = \{O_1, O_2, \ldots, O_N\}$.
- Set the experiment replication amount $M = \lambda$.

Challenge Game:

1. Challenger $\mathcal{C}$ chooses a permutation σ on N elements $\sigma : [N] \to [N]$.
2. For each experiment $m \in [M]$: Challenger $\mathcal{C}$ performs the routing algorithm $\Pi_m = \Pi_{m,N,\sigma_b,G}$ (for $G = B(N, b, c, w)$) of Figure 14. For each $i \in [N]$, let $\mathcal{P}_{m,i}$ denote the path chosen (by Π_m) that connects nodes $(I_i, \sigma_b(I_i))$.
3. Let Y be the boolean random variable from Definition 25. If $Y = 0$, Challenger $\mathcal{C}$ aborts (Adversary $\mathcal{A}$ wins).
4. Challenger $\mathcal{C}$ chooses any two distinct indices $i, i' \in [N]$, and gives[11] $\sigma|_{[N]\setminus\{i,i'\}}$ to Adversary $\mathcal{A}$, which is the mapping of σ on all indices *except* i and i'. Notice that since σ is a permutation, Adversary $\mathcal{A}$ now has complete knowledge of σ, *except* for what σ does to i and i'. In particular, there are two range indices $\sigma(i), \sigma(i') \in [N]$ that are *not* mapped to (based on what $\mathcal{C}$ gives to $\mathcal{A}$). Let τ denote the permutation that is identical to σ, except that it swaps where i and i' are mapped to (so $\tau(i) = \sigma(i')$ and $\tau(i') = \sigma(i)$). Notice that after this step, Adversary $\mathcal{A}$ knows that the permutation chosen by Challenger $\mathcal{C}$ is either σ or τ.
5. (If this step is reached) Since $Y = 1$, for each run $1 \leq m \leq M$ of the experiment, we have that alternate routing algorithm $\Pi'_{m,i,i'}$ must have been constructed as per Step 2a of Definition 23 (as opposed to Step 2b). Therefore, let $j_m \in [1, (1 + b)]$ denote the block index for which $\Pi'_{m,i,i'}$ is defined as in Step 2a; i.e. j_m (is the minimal index that) satisfies $Y_{\Pi_m,\Pi'_m,j_m}(i, i', j_m) = 1$. Then, for each experiment $m \in [M]$:
   (a) [Block Index]: Challenger $\mathcal{C}$ gives Adversary $\mathcal{A}$ the block index j_m (recall this is the first block for which $Y_{\Pi_m,\Pi'_m,j_m}(i, i', j_m) = 1$).
   (b) [All Non-Interesting Paths]: Challenger $\mathcal{C}$ gives Adversary $\mathcal{A}$ all paths $\{\mathcal{P}_{m,\widehat{i}}\}_{\widehat{i} \notin \{i,i'\}}$.
   (c) [Interesting Paths *Before* Block j_m]: Challenger $\mathcal{C}$ gives Adversary $\mathcal{A}$, *through the first* (j_m-1) *blocks only*, paths $\mathcal{P}_{m,i}$ and $\mathcal{P}_{m,i'}$.
   (d) [Interesting Paths + Alternate Paths for Block j_m]: Denote the two sub-paths of $\mathcal{P}_{m,i}$ and $\mathcal{P}_{m,i'}$ that are restricted to block j_m (i.e. just the edges of these paths within block j_m) and their two alternate sub-paths (as specified by alternate routing protocol $\Pi'_{i,i'}$ (Definition 23)) as: $\{\mathcal{P}_{m,i,j_m}, \mathcal{P}_{m,i',j_m}, \mathcal{P}'_{m,i,j_m}, \mathcal{P}'_{m,i',j_m}\}$.Then Challenger $\mathcal{C}$ gives Adversary $\mathcal{A}$ the *unordered* set $\{\mathcal{P}_{m,i,j_m}, \mathcal{P}_{m,i',j_m}, \mathcal{P}'_{m,i,j_m}, \mathcal{P}'_{m,i',j_m}\}$.

[11] Notice that this information is also available indirectly from what $\mathcal{C}$ gives to $\mathcal{A}$ in Step 5a below.

(e) [(Unordered) Interesting Paths *Beyond* Block j_m]: For each level with index $j_m \cdot (1+\log N) \leq l \leq (1+b) \cdot (1+\log N)$ in $G = B(N, b, c, w)$ that lies *after* block j_m, Challenger $\mathcal{C}$ gives Adversary $\mathcal{A}$ the *unordered* set of edges $\{\mathcal{P}_{m,i,l}, \mathcal{P}_{m,i',l}\}_l$, where $\mathcal{P}_{m,i,l}$ (resp. $\mathcal{P}_{m,i',l}$) denotes the l^{th} edge on the path $\mathcal{P}_{m,i}$ (resp. on the path $\mathcal{P}_{m,i'}$). In other words, Adversary $\mathcal{A}$ learns the edges (beyond block j_m) traversed by the paths $\mathcal{P}_{m,i}$ and $\mathcal{P}_{m,i'}$, but $\mathcal{A}$ is *not* explicitly told which edges belong to which path ($\mathcal{P}_{m,i}$ versus $\mathcal{P}_{m,i'}$).

6. Adversary $\mathcal{A}$ outputs a guess of whether Challenger's permutation was σ or τ (see Step 4).

The Adversary $\mathcal{A}$ wins Challenge Game 1 either if Challenger $\mathcal{C}$ aborts in Step 3, or if $\mathcal{A}$'s output guess in Step 6 is correct.

The main result for Challenge Game 1 (which is the first implication in (19)) is:

Lemma 27 *The probability that an (unbounded) Adversary $\mathcal{A}$ wins Challenge Game 1 is bounded by:*

$$Pr[\mathcal{A} \text{ wins Challenge Game 1}] \leq \frac{1}{2} + \frac{\lambda \cdot N^2}{4^\lambda} \tag{20}$$

Proof. By Corollary 26, the probability that Adversary $\mathcal{A}$ wins in Step 3 is bounded by $\frac{1}{2 \cdot 2^\lambda}$. Thus, we need only show that the probability that Adversary $\mathcal{A}$ wins in Step 6 (i.e. that $\mathcal{A}$ correctly chooses between σ and τ) equals 1/2. This in turn follows immediately from the definition of $Y_{\Pi,\Pi'}(i, i', j)$ (Definition 19): Since Adversary $\mathcal{A}$ was given an *unordered* set $\{\mathcal{P}_{m,i,j_m}, \mathcal{P}_{m,i',j_m}, \mathcal{P}'_{m,i,j_m}, \mathcal{P}'_{m,i',j_m}\}$ in Step 5d, there is no way for the Adversary $\mathcal{A}$ to distinguish the actual path $\mathcal{P}_{m,i,j_m}$ (as per Π_m) from its alternate path $\mathcal{P}'_{m,i,j_m}$ (as per Π'_m); and ditto for $\mathcal{P}_{m,i',j_m}$ and its alternate path $\mathcal{P}'_{m,i',j_m}$. Namely, if we let $\mu_{m,i}, \mu_{m,i'}$ denote the two (not necessarily distinct) "block j_m *input* level" nodes (i.e. the nodes at level $(j_m - 1) \cdot (1 + \log N)$) that $\mathcal{P}_{m,i}$ and $\mathcal{P}_{m,i'}$ pass through, and similarly let $\nu_{m,i}, \nu_{m,i'}$ denote the two (not necessarily distinct) "block j_m *output* level" nodes (i.e. the nodes at level $(j_m \cdot (1+\log N))$) that $\mathcal{P}_{m,i}$ and $\mathcal{P}_{m,i'}$ pass through, then notice:

- Within block j_m: $\mathcal{P}_{m,i,j_m}$ starts at $\mu_{m,i}$ and ends at $\nu_{m,i}$.
- Within block j_m: $\mathcal{P}_{m,i',j_m}$ starts at $\mu_{m,i'}$ and ends at $\nu_{m,i'}$.
- Within block j_m: $\mathcal{P}'_{m,i,j_m}$ starts at $\mu_{m,i}$ and ends at $\nu_{m,i'}$.
- Within block j_m: $\mathcal{P}'_{m,i',j_m}$ starts at $\mu_{m,i'}$ and ends at $\nu_{m,i}$.

Also, define the two sets of paths: "primary" paths $\mathcal{P}_{m,0} := \{\mathcal{P}_{m,i,j_m},\ \mathcal{P}_{m,i',j_m}\}$ and "alternate" paths $\mathcal{P}_{m,1} := \{\mathcal{P}'_{m,i,j_m},\ \mathcal{P}'_{m,i',j_m}\}$. Then notice that the Adversary $\mathcal{A}$ can guess the correct permutation σ versus τ in Step 6 if and only if it can distinguish between $\mathcal{P}_{m,0}$ versus $\mathcal{P}_{m,1}$, in terms of which correspond to the "primary" paths and which is the "alternate" paths. Thus, it is sufficient to show that this probability is 1/2.

Consider the following facts:

F1.a The probability that Challenger $\mathcal{C}$ chose $\mathcal{P}_{m,i,j_m}$ (and hence $\mathcal{P}'_{m,i,j_m}$ was the "alternate") equals the probability that Challenger $\mathcal{C}$ instead chose $\mathcal{P}'_{m,i,j_m}$ (and hence $\mathcal{P}_{m,i,j_m}$ would have been the "alternate").

F1.b The probability that Challenger $\mathcal{C}$ chose $\mathcal{P}_{m,i',j_m}$ (and hence $\mathcal{P}'_{m,i',j_m}$ was the "alternate") equals the probability that Challenger $\mathcal{C}$ instead chose $\mathcal{P}'_{m,i',j_m}$ (and hence $\mathcal{P}_{m,i',j_m}$ would have been the "alternate").

F2.a Amongst the two paths in $\mathcal{P}_{m,0}$: One starts at $\mu_{m,i}$ and the other starts at $\mu_{m,i'}$; and one ends at $\nu_{m,i}$ and the other ends at $\nu_{m,i'}$.

F2.b Amongst the two paths in $\mathcal{P}_{m,1}$: One starts at $\mu_{m,i}$ and the other starts at $\mu_{m,i'}$; and one ends at $\nu_{m,i}$ and the other ends at $\nu_{m,i'}$.

F3.a Both paths in $\mathcal{P}_{m,0}$ are edge-disjoint (on block j_m) from all other paths $\{\mathcal{P}_{m,\widehat{i}}\}_{\widehat{i}\notin\{i,i'\}}$.

F3.b Both paths in $\mathcal{P}_{m,1}$ are edge-disjoint (on block j_m) from all other paths $\{\mathcal{P}_{m,\widehat{i}}\}_{\widehat{i}\notin\{i,i'\}}$.

Given the above facts, there is no way for (even an unbounded) Adversary $\mathcal{A}$ to distinguish whether the two paths in $\mathcal{P}_{m,0}$ are the "primary" paths (selected by Π_m) with $\mathcal{P}_{m,1}$ as the "alternate" paths (selected by Π'_m), or vice-versa, since:

- A priori, the two paths in $\mathcal{P}_{m,0}$ are equally likely to be chosen by Π_m as the "primary" paths as the two paths in $\mathcal{P}_{m,1}$ (Fact F1).
- The information of block index j_m (dealt in Step 5a) is no help in distinguishing,
- The information of non-interesting paths (dealt in Step 5b) is no help in distinguishing, since $\mathcal{P}_{m,0}$ and $\mathcal{P}_{m,1}$ are similarly edge-disjoint (on block j_m) from all other paths $\{\mathcal{P}_{m,\widehat{i}}\}_{\widehat{i}\notin\{i,i'\}}$ (Fact F3).
- The information of paths $\mathcal{P}_{m,i,j_m}$ and $\mathcal{P}_{m,i',j_m}$ *before* block j_m (dealt in Step 5c) is no help in distinguishing, since both $\mathcal{P}_{m,0}$ and $\mathcal{P}_{m,1}$ have one path that starts at $\mu_{m,i}$ and one that starts at $\mu_{m,i'}$ (Fact F2).
- The information dealt in Step 5d is no help in distinguishing, since the provided list of paths $\{\mathcal{P}_{m,i,j_m}, \mathcal{P}_{m,i',j_m}, \mathcal{P}'_{m,i,j_m}, \mathcal{P}'_{m,i',j_m}\}$ is *unordered*.
- The (*unordered*) information of which edges lie in $\mathcal{P}_{m,i,j_m}$ and $\mathcal{P}_{m,i',j_m}$ *beyond* block j_m (dealt in Step 5e) is no help in distinguishing, since both $\mathcal{P}_{m,0}$ and $\mathcal{P}_{m,1}$ have one path that ends at $\nu_{m,i}$ and one that ends at $\nu_{m,i'}$ (Fact F2).

Therefore, none of the information provided by the Challenger $\mathcal{C}$ (nor any other a priori bias) provides any information to Adversary $\mathcal{A}$ that allows it to distinguish between $\mathcal{P}_{m,0}$ and $\mathcal{P}_{m,1}$ (in terms of which are the "primary" paths and which are the "alternate" paths). Consequently, the probability that Adversary $\mathcal{A}$ can distinguish between $\mathcal{P}_{m,0}$ and $\mathcal{P}_{m,1}$ (in terms of which are the "primary" paths and which are the "alternate" paths) is 1/2.

Challenge Game 2

Input Parameters:

- Number of input/output nodes $2^n = N \geq 64$.

- Security parameter λ_s. Let $\lambda := 2\log N + \max(\lambda_s, 2 + \log\log N)$.
- A wide-edged, extended and colored Beneš network $G = B(N, b, c, w)$, with parameters as per Corollaries 18 and 26: $b = \lambda - 1$, $c = 4 \cdot a_\lambda$, and $w = 1.2 \cdot \lambda \cdot \log N \cdot (1 + \log N)$ (for $a_\lambda := \max(2, \lambda^{1/(\log N - 1)})$).
- There are N "global input nodes" on level -1 of the Beneš network $G = B(N, b, c, w)$, which are denoted: $\mathcal{I} = \{I_1, I_2, \ldots, I_N\}$, and N global output nodes $\mathcal{O} = \{O_1, O_2, \ldots, O_N\}$.
- Set the experiment replication amount $M = \lambda$.

Challenge Game:

1. On input security parameter λ, Adversary $\mathcal{A}$ chooses N, two distinct permutations σ_0, σ_1 on $[N]$, a set of sender indices $S_\mathcal{A} \subseteq [N]$ to corrupt, and a set of receiver indices $R_\mathcal{A} \subseteq [N]$ to corrupt; subject to constraints:
   (a) $|R_\mathcal{A}| \leq N - 2$;
   (b) σ_0 and σ_1 match for all receivers in $R_\mathcal{A}$: $\forall\, i \in R_\mathcal{A} : \sigma_0^{-1}(i) = \sigma_1^{-1}(i)$.
2. Adversary $\mathcal{A}$ sends $\{\sigma_0, \sigma_1\}$ to a Challenger $\mathcal{C}$.
3. Challenger $\mathcal{C}$ chooses $b \in \{0, 1\}$ (e.g. by flipping a coin) and selects permutation $\sigma_b \in \{\sigma_0, \sigma_1\}$.
4. For each experiment $m \in [M]$:
   (a) Challenger $\mathcal{C}$ performs the routing algorithm $\Pi_m = \Pi_{m,N,\sigma_b,G}$ (for $G = B(N, b, c, w)$) of Figure 14. For each $i \in [N]$, let $\mathcal{P}_{m,i}$ denote the path chosen (by Π_m) that connects nodes $(I_i, O_{\sigma_b(i)})$.
   (b) Adversary $\mathcal{A}$ is given the following information:
      – For each $i \in R_\mathcal{A}$: all edges $e \in \mathcal{P}_{m,i}$ that are edge-disjoint from all other paths $\mathcal{P}_{m,j}$ (for $j \neq i$).
      – The list of edges $\{e\} \in G$ that have at least two distinct paths $\mathcal{P}_{m,i}, \mathcal{P}_{m,i'}$ pass through them, with $i' \neq i$ and $i \in R_\mathcal{A}$. Notice that $\mathcal{A}$ is given only the identity of the set of edges $\{e\}$; in particular, $\mathcal{A}$ *is not* given the information of *which* (nor even *how many*) indices in $[N] \setminus R_\mathcal{A}$ traverse each such edge.
5. Let Y be the boolean random variable from Definition 25. If $Y = 0$, Challenger $\mathcal{C}$ aborts (Adversary $\mathcal{A}$ wins).
6. Adversary $\mathcal{A}$ outputs a guess $b' \in \{0, 1\}$ of which permutation $\{\sigma_0, \sigma_1\}$ Challenger $\mathcal{C}$ chose.

We say that the Adversary wins the above challenge if its output is correct.

The main result for Challenge Game 2 (which is the second implication in (19)) is:

Lemma 28 *The probability that an (unbounded) Adversary $\mathcal{A}$ wins Challenge Game 2 is bounded by:*

$$Pr[\mathcal{A} \text{ wins Challenge Game 2}] \;\leq\; \frac{1}{2} + \frac{1}{2^{\lambda_s}} \tag{21}$$

Proof. (Proof by contradiction.)
Suppose there exists a (computationally unbounded) adversary $\mathcal{A}$ that can win the security challenge game with probability greater than $\frac{1}{2} + \frac{1}{2^{\lambda_s}}$. We will show

this leads to a contradiction.

Reduction. Without loss of generality, we may assume that σ_0 and σ_1 differ in exactly two indices $i \neq i'$, so that $\sigma_0(i) = \sigma_1(i')$, and $\sigma_0(i') = \sigma_1(i)$, and for all other indices $j \neq i, i'$, $\sigma_0(j) = \sigma_1(j)$ (this reduction gives the adversary an extra factor of N advantage).

Proof (Proof of Reduction.). Consider the following chain of permutations:

$$(\sigma_0 =)\tau_0,\ \tau_1,\ \tau_2,\ \ldots,\ \tau_N(=\sigma_1), \tag{22}$$

where each τ_{i+1} is defined iteratively from τ_i, starting with $\tau_0 = \sigma_0$. Then given τ_i, define τ_{i+1} to be the permutation that matches τ_i at all indices *except* (possibly) in positions i and $j \geq i$, where $j := \sigma_0^{-1}(\sigma_1(i))$. Namely, $\tau_{i+1}(i) := \sigma_1(i)$, and $\tau_{i+1}(j) := \sigma_0(i)$ $(= \tau_i(i))$. Then it is easy to demonstrate the for any $1 \leq i \leq N$, permutations τ_{i-1} and τ_i differ in (at most) two places.

We now apply a standard hybrid argument, to show that the existence of an Adversary who wins the security challenge game with σ_0, σ_1 implies the existence of an Adversary $\mathcal{A}'$ who can distinguish between τ_i and τ_{i+1} (for some $i \in [0, N]$), with advantage:

$$\begin{aligned} \Pr[\mathcal{A} \text{ outputs } b \text{ correctly}] \ &> \ \frac{1}{2} + \frac{1}{2^{\lambda_s}} \quad \Rightarrow \\ \Pr[\mathcal{A}' \text{ distinguishes between } \tau_i \text{ and } \tau_{i+1}] \ &> \ \frac{1}{2} + \frac{1}{N \cdot 2^{\lambda_s}} \ = \ \frac{1}{2} + \frac{1}{2^{\lambda}} \end{aligned} \tag{23}$$

The proof by contradiction is now complete, as the existence of adversary $\mathcal{A}'$ with advantage as per (23) violates (20), since for $\lambda = 2\log N + \max(\lambda_s, 2 + \log\log N)$, we have:

$$\frac{\lambda \cdot N^2}{4^{\lambda}} \ = \ \frac{1}{2^{\lambda}} \cdot \frac{N^2}{2^{2\log N}} \cdot \frac{\lambda}{2^{\alpha}} \ \leq \ \frac{1}{2^{\lambda}}$$

where $\alpha := \max(\lambda_s, 2 + \log\log N)$, and we have used that $2^{\alpha} \geq \lambda$ since:

If $\lambda_s \geq 2\log N$: Then $\lambda \leq 2\lambda_s \ \Rightarrow\ \log\lambda \leq \lambda_s \leq \alpha$.
If $\lambda_s \leq 2\log N$: Then $\lambda \leq 4\log N \ \Rightarrow\ \log\lambda \leq 2 + \log\log N \leq \alpha$.

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