Less is more: refinement proofs for probabilistic proofs

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Abstract

There has been intense interest over the last decade in implementations of *probabilistic proofs* (IPs, SNARKs, PCPs, and so on): protocols in which an untrusted party proves to a verifier that a given computation was executed properly, possibly in zero knowledge. Nevertheless, implementations still do not scale beyond small computations. A central source of overhead is the *front-end*: translating from the abstract computation to a set of equivalent arithmetic constraints. This paper introduces a general-purpose framework, called Distiller, in which a user translates to constraints not the original computation but an abstracted *specification* of it. Distiller is the first in this area to perform such transformations in a way that is provably safe. Furthermore, by taking the idea of "encode a check in the constraints" to its literal logical extreme, Distiller exposes many new opportunities for constraint reduction, resulting in cost reductions for benchmark computations of 1.3–50×, and in some cases, better asymptotics.

1 Introduction

Probabilistic proofs [5–7, 38–40]—PCPs, IPs, NIZKs, SNARKs, SNARGs, and so on—are fundamental in complexity theory and cryptography. They enable an untrusted *prover* to convince a *verifier* of some statement (for example, that a given computation Ψ, on specific input x, produces an alleged output y). In these protocols, the verifier does not inspect a classical witness to the truth of the statement (or re-execute Ψ) but instead checks an encoded proof *probabilistically*. Zero-knowledge variants allow the prover to keep some of the input to the computation—and the proof itself—hidden from the verifier. Astonishingly, the verifier's checks are (in some protocols) constant-time, regardless of the size of the computation [5, 6, 37]. The appeal of these properties in emerging application areas (most notably, outsourced computation, blockchains, and their intersection) has fueled intense interest in implementations over the last 13 years. The results have included 20 orders of magnitude reduction in costs, deployment of SNARKs in cryptocurrencies [31, 52, 68, 84], and an explosion of frameworks [75, 78, 81].

Yet, probabilistic proofs are heavily limited in scalability, making them impractical for general-purpose use (the hype notwithstanding). One source of costs is the *back-end*, which is the complexity-theoretic and cryptographic proving machinery. The other source of costs is the *front-end*, which translates high-level computations into the format that the back-end works over. In most probabilistic proof implementations, that format is some variant of *arithmetic constraints*: equations over a finite field.

Unfortunately, not only must the prover perform cryptographic operations proportional to the number of constraints (often with memory requirements that scale similarly), but also constraints are a verbose way to represent computations (§2). For example, every iteration of a loop requires separate constraints—likewise with all branches of conditional statements. Inequality tests, when translated into constraints, are expensive. So is RAM.

The question that we ask and answer in this paper is: *if back-end costs are here to stay and we are stuck translating computations to constraints, what can we do to mitigate costs?* Any such technique should achieve:

- *Conciseness.* Compared to a naive translation of a computation Ψ, we want to produce a smaller set of constraints.
- *Coupling.* There should be a way for the prover to actually satisfy the alternate constraints, which is non-trivial, since they may not correspond to the individual program steps that the prover takes to execute Ψ.

These two requirements have been addressed, at least partially. The authors of almost all front-ends observe that translation from a high-level computation Ψ need not result in constraints that simulate execution [23, 24, 27, 28, 44, 53, 63, 65, 66, 70, 72, 76, 79, 92] (§2). Rather, it suffices if the constraints are satisfiable iff the execution is valid. For example, consider a computation that invokes a quicksort subroutine. The naive approach is to compile quicksort into constraints. As an alternative [44, Appx. C], the prover can sort "outside the constraints", with the constraints enforcing that (a) the output is a permutation of the input [18, 80], and (b) this permutation is sorted. The naive approach requires $O(n \log n)$ inequality tests while the alternative requires only $O(n)$ inequality tests (for adjacent elements in part (b)). As inequality tests dominate this computation, the improvement is substantial.

We call such a checker of required properties a *widget*. This generalizes "gadget" [53], which refers to constraints that have been written by hand; widgets can additionally be encoded in a higher-level language and then compiled to constraints. Widgets have been proposed for arithmetic and bitwise operations [70], multi-precision operations [44], storage [24, 62], concurrent access to state [72], cryptographic operations [13, 22, 24, 28, 52, 62, 68], recursive composition [22, 28, 46], and optimization problems [4].

Yet, to the extent that these works make arguments about the correctness of substituting a computation with a widget, none of them provides formal justification: it is entirely possible that there are wrong widgets out there! Note that any such bug destroys the soundness of the end-to-end application. Thus, we add a third requirement:

- *Correctness.* This is not about correctness of the translation to constraints, which is crucial and complementary, and has

been studied [33]. Our focus on correctness in this paper is on substitutions (of computations by a widget) that happen "upstream" of compilation-to-constraints.

There is work addressing this third requirement [76] but at the expense of the first two (§7).

This paper's contribution is a framework, *Distiller*, that addresses all three of these desiderata (§4). Distiller takes the goals in reverse order: it *starts* with Correctness. For each class of computations, the user writes down a specification of the computation, and proves a formal relationship between the implementation and the specification. This justifies compiling the specification (rather than the implementation) to constraints. This relationship is ensured by representing both the implementation and the specification as *transition systems* and adapting ideas from the theory of *refinement* [29, 47, 48, 54, 85]. A refinement relates the externally observable behaviors of two transition systems, formalizing the notion of correct substitution. A proof of refinement then yields a blueprint for the prover to satisfy the abstract constraints (Coupling). For Conciseness, a consequence of Distiller and its generality is to expose new opportunities for constraint reduction: Distiller lets us take the idea of widgets to its literal logical extreme.

We apply Distiller to a series of examples, including binary search, convex hull, maximal strongly connected components, and minimum spanning tree (§5). We also implement and evaluate Distiller (§6). To demonstrate the importance of Correctness, we encode the refinement proofs for our examples and verify them using the program verifier Viper [60]. The verification unveiled bugs that would have compromised Correctness in initial versions of two widgets. For Coupling, we build on the Pequin toolchain [66] to simulate an end-to-end system implementing Distiller where the prover executes the implementation whose result is then checked against the widget. Finally, we also evaluate Conciseness. Distiller achieves reductions in constraint size ranging from small constant factors to asymptotic improvements for some problems, which for small problem instances already result in double-digit factors. Qualitatively, the more complex a computation, the more improvement Distiller generally yields. Computations with many memory accesses or searches of memory see particular benefit under Distiller.

To be clear, replacing an implementation by its specification does not guarantee more concise constraints. However, as we explain (§4–§5), we can often use Distiller to find, and establish the correctness of, an intermediate point between specification and implementation that does yield a substantial improvement.

Distiller is not perfect (§8). As a built system, its trusted computing base includes the encoding of computations in Viper as well as the Pequin toolchain; also, it stops short of fully automating Coupling. However, none of these restrictions are fundamental.

The bottom line is that Distiller has taken a crucial step in improving front-ends: it has exhibited the logically most general way to exploit non-determinism in arithmetic constraints, while doing so soundly, with performance improvements that range from good constants to orders of magnitude.

2 Background: applied probabilistic proofs

This section is intended to give just enough context for the rest of the paper. For a full, rigorous treatment of probabilistic proof implementations, see Thaler [75].

Back-end. In these setups, a *back-end* is a cryptographic or complexity-theoretic protocol between an untrusted prover $\mathcal{P}$ and a verifier $\mathcal{V}$ in which $\mathcal{P}$ convinces $\mathcal{V}$ that a given set of equations $\mathcal{C}$ has a solution.

In more detail, $\mathcal{V}$ and $\mathcal{P}$ (which are possibly probabilistic) agree on $\mathcal{C}$, as defined by a protocol, or defined by a *user* who invokes $\mathcal{V}$ and $\mathcal{P}$. The variables in $\mathcal{C}$ are elements in a finite field, typically $\mathbb{F}_p$ (the integers mod p), where p is a large prime (128 bits or more). For many back-ends, $\mathcal{C}$ is required to be in R1CS format [15, 16, 37, 65, 71]. R1CS generalizes arithmetic circuits, which generalize Boolean circuits. We refer to such a set of equations as *constraints*.

$\mathcal{V}$ does not trust anything $\mathcal{P}$ says; $\mathcal{P}$ can follow an arbitrarily malicious strategy (though some protocols presume a computational bound on $\mathcal{P}$ and cryptographic hardness assumptions of one kind or another).

$\mathcal{P}$ wants to prove to $\mathcal{V}$ that $\mathcal{P}$ holds a solution, or *satisfying assignment*, z to $\mathcal{C}$—but $\mathcal{V}$ does not want to receive z, and $\mathcal{P}$ may wish to keep z hidden. Instead, $\mathcal{P}$ gives $\mathcal{V}$ a certificate, possibly revealed interactively, which $\mathcal{V}$ checks. The guarantees are:

- *Completeness*: If $\mathcal{C}$ is satisfiable, then a correct $\mathcal{P}$ makes $\mathcal{V}$ accept, always (regardless of random choices made by $\mathcal{P}$, $\mathcal{V}$, or by the user in an offline phase).
- *Soundness*: If $\mathcal{C}$ is not satisfiable, the probability that $\mathcal{V}$'s checks pass is negligible (the probability is over random choices made by the verifier or by the user in an offline phase). Some applications require a more general property, *Proof of Knowledge* (PoK): if $\mathcal{P}$ does not have access to a satisfying z (even if $\mathcal{C}$ is satisfiable), then $\mathcal{V}$ accepts with negligible probability. Note that these properties hold regardless of $\mathcal{P}$'s strategy.
- *Zero knowledge*: $\mathcal{V}$ gets no information about z other than what can be deduced from the fact that $\mathcal{C}$ can be satisfied.

Examples of recent back-ends are [20, 25, 26, 35, 36, 45, 46, 50, 55, 69, 82, 83, 88, 89]. These trade off different properties, including the nature of the cryptographic assumptions, noninteractivity, whether there is an offline phase, whether that phase has to be repeated each time the structure of $\mathcal{C}$ changes, and so on. However, in all of these works, the costs have a major dependence on the number of constraints, $|\mathcal{C}|$, and thus all of these works will benefit from improvements to front-ends.

Pipeline. Posit a user who cares about verifying the execution of some high-level computation Ψ, on some input x. $\mathcal{P}$ supplies y that is purportedly $\Psi(x)$, and wants to convince some $\mathcal{V}$, which is trusted by the user, that $y{=}\Psi(x)$. As a generalization, Ψ can be a relation, so the goal is to prove that $y \in \Psi(x)$. Existing implementations have the following pipeline:

Offline (one-time for Ψ):

0. The user writes down the computation Ψ.

1. The user compiles Ψ to constraints, $\mathcal{C}$, over variables X, Y, Z, where X and Y are vectors of variables that represent the inputs and outputs.
2. The user runs any setup procedure required by the back-end.

Online (for each x, y):

3. Given a specific input x, $\mathcal{P}$ identifies a satisfying assignment z to $\mathcal{C}_{X=x,Y=y}$, perhaps by executing Ψ. Here, $\mathcal{C}_{X=x,Y=y}$ means $\mathcal{C}$ with X bound to x ($\mathcal{V}$'s requested input) and Y bound to y (the purported output).
4. $\mathcal{P}$ convinces $\mathcal{V}$ that it has, or knows, a satisfying assignment to $\mathcal{C}_{X=x,Y=y}$.

One property that we need from a pipeline is *End-to-end Soundness*: if $y \neq \Psi(x)$, then $\mathcal{V}$ rejects with overwhelming probability. This property relies on the back-end's properties and the correctness of the compilation from Ψ (see below). Another essential property is *End-to-end Completeness*: if $y = \Psi(x)$, then a correct $\mathcal{P}$ makes $\mathcal{V}$ accept with probability 1. This property relies on the back-end's properties and the mechanics of Step 3.

Front-end. The front-end is Steps 1 and 3. We detail these steps below, incurring some textual debts to Buffet [79]. We focus on a compilation approach that we call the "ASIC approach". The alternative is the "CPU approach", which represents the execution of a CPU in constraints [13, 15–17, 91]. This results in much higher overhead [79].

The key correctness condition needed from the compiler is: for any x and y, $\mathcal{C}_{X=x,Y=y}$ is satisfiable (by some Z=z) if and only if y=$\Psi(x)$. As a simple example, consider a computation whose output is its input plus three. The corresponding constraints are: $\mathcal{C} = \{Z - X = 0, Z + 3 - Y = 0\}$. Notice that for all pairs (x, y), $\mathcal{C}_{X=x,Y=y}$ is satisfiable iff $y = x + 3$.

Given a program, the compiler unrolls loops (each iteration gets its own variables), and converts the code to an intermediate form, for example static single assignment. The compiler then transforms each line into one or more constraints [23, 24, 27, 28, 61, 63, 65, 66, 70, 79, 92]. Arithmetic and logical operations are concise. For example, the line of code `z3 = z2 + z1` becomes $\{Z3 = Z2 + Z1\}$. By contrast, each inequality test and bitwise operation costs $\approx w$ constraints, where w is the bit width of the relevant variables (these operations work by separating a finite field element into bits [70, Appx.C]; see also [15, 65, 71, 79]).

RAM operations (`LOAD` and `STORE`) translate into variables that feed into a separate RAM-checking computation, for example based on permutation networks and coherence checks [14, 16, 67, 79], Merkle trees and memory checking [13, 21, 24], polynomial identity testing [91], or set accumulators [62]. Each `LOAD` and `STORE` is costly, as the RAM-checking computation has a number of constraints proportional to $\Omega(n \cdot r)$, where n is the number of operations, and r is the address width (log of memory size).

Solving. To produce a satisfying assignment, $\mathcal{P}$ in most pipelines (but not all [63]; see §7) goes constraint by constraint. The solution to some constraints is immediate; for example, given the constraint $Z_3 = Z_2 + Z_1$, the setting to Z_3 is mechanically derived if Z_1 and Z_2 are already determined. Other constraints require non-deterministic input from the prover. As examples, in inequality tests, $\mathcal{P}$ supplies the values of each bit and in the RAM-checking sub-computation, $\mathcal{P}$ supplies the settings for switches in the permutation network, so that the coherence checks succeed. In these cases, the constraints have to be annotated to connect them to the original Ψ, thereby telling $\mathcal{P}$ how to solve them.

Widgets. Instead of representing certain operations directly in constraints, one can sometimes represent a validity check of that operation, as with the sorting example in the Introduction. Notice that the pipeline given so far handles such substitution: one treats the checker as the computation that is compiled into constraints (Step 1), and then the solving step (Step 3) runs the actual computation; the aforementioned annotations tell $\mathcal{P}$ which external computations to run to satisfy which constraints. For this substitution to preserve End-to-end Soundness, one not only needs a correct compiler (as stated earlier in this section), one also needs Correctness (as described in Section 1).

Costs and accounting. This paper's primary metric is $|\mathcal{C}|$. That is for two reasons. First, all back-ends in the literature impose costs on $\mathcal{P}$ (and, depending on the protocol, on $\mathcal{V}$) that are at least linear in the number of constraints, $|\mathcal{C}|$. Second, these costs typically dominate the cost to $\mathcal{P}$ of executing and solving (Step 3); thus, even though $\mathcal{P}$ executes the underlying computation, doing so contributes only negligibly to costs.

For concreteness, we sometimes assume the widely-used Groth16 backend [42, 53]. In Groth16, certificate size is constant (128 bytes) and $\mathcal{V}$ runs in constant time. However, the running time for $\mathcal{P}$ and for the setup phase are $O(|\mathcal{C}| \cdot \log |\mathcal{C}|)$. Because of this and memory bottlenecks (the prover needs to perform an FFT of size $|\mathcal{C}|$), single-machine Groth16 provers are highly limited in the size of the computation that they can handle. There are works that take advantage of multiple machines [86] and heterogeneous hardware [90] to try to overcome these bottlenecks, but they too are limited. The bottom line is that *every* work in this research area will benefit from constraint sets with fewer constraints.

3 Motivating example: merging sorted lists

Consider a computation, `merge`, that takes as input multiple sorted lists with unique elements (unique across all lists) and outputs a sorted union of the elements. An example implementation of `merge`, which we denote T_I, is in Figure 1. When translated, `merge` comprises a number of constraints proportional to $L \cdot (\sum_k A_k.\texttt{len})$, because of the nested loops on lines 9 and 11.

Observe that `merge` is *computing* its result. But in the setup of probabilistic proofs, the goal is to provide a proof about some alleged, exogenously-computed output. Thus, the set of constraints could instead *check* that a specification is met. We are interested in how to perform such a substitution systematically, meaning that the requirements in Section 1 are met.

A natural starting point is to translate the weakest logical specification (WLS) of `merge` that still expresses functional correctness: intuitively, one expects that logically weaker specifications "enforce less" and thus should yield smaller constraints when translated. Informally, the WLS is: "$\texttt{merge}(L, A_0, \ldots, A_{L-1}, B)$ terminates and, upon termination, B is monotonically increasing and holds just each element from $\{A_k\}$ exactly once." Pseudocode to check this specification, which we denote T_S, is depicted in Figure 2. Its complexity is $2 \cdot (\sum_k A_k.\texttt{len})$, which is an asymptotic

```
1   void merge(L,A0,..,AL-1,B) {
2   ℓ0 : int[L] curr = {0};
3        int len, running_min, kstar; bool found;
4        len = 0;
5   ℓ1 : for (int k = 0; k < L; k++) {
6         len += Ak.len;
7        }
8        B.len = len;
9   ℓ2 : for (int i = 0; i < len; i++) {
10         found = false
11  ℓ3 :   for (int k = 0; k < L; k++) {
12           if (curr[k] < Ak.len && (!found ||
13                Ak[curr[k]] < running_min)) {
14             running_min = Ak[curr[k]];
15             // running_min is the current min element
16             kstar = k;
17             // kstar indexes the list that contains
18             // running_min
19             found = true;
20             // indicates that branch has been taken
21           }
22         }
23         B[i] = running_min;
24         curr[kstar]++;
25       }
26  ℓ4 : return;
27  }
```

Figure 1: Pseudocode for the computation $\texttt{merge}(L, A_0, \ldots, A_{L-1}, B)$ (T_I). The precondition of `merge` requires that the $\texttt{A}_k$ are strictly sorted and their elements pairwise distinct. Also, there must be enough physical space in `B` to store the elements of all $\texttt{A}_k$.

```
1  void merge_spec_naive(L,A0,..,AL-1,B) {
2    int k_i, j_i, i_kj;
3    havoc B.len;
4    for (int i = 0; i < B.len; i++) {
5      havoc B[i];
6      assume i == 0 || B[i-1] < B[i];
7      havoc k_i;
8      assume 0 <= k_i && k_i < L;
9      havoc j_i;
10     assume 0 <= j_i && j_i < Ak_i.len;
11     assume B[i] == Ak_i[j_i];
12   }
13   for (int k = 0; k < L; k++) {
14     for (int j = 0; j < Ak.len; j++) {
15       havoc i_kj;
16       // each element in some A_k is in B
17       assume 0 <= i_kj && i_kj < B.len;
18       assume Ak[j] == B[i_kj];
19     }
20   }
21   return;
22 }
```

Figure 2: Pseudocode for the weakest logical specification (T_S) of the `merge` computation. The precondition only requires that `B` has enough physical space for the elements of all $\texttt{A}_k$.

improvement over $L \cdot (\sum_k A_k.\texttt{len})$ from earlier.

To read the pseudocode, note that the keyword **`havoc`** denotes a nondeterministic choice, while **`assume`** constrains choices. Concretely, when this pseudocode is compiled to $\mathcal{C}(T_S)$ (§2), **`havoc`** statements become free variables that the prover supplies while **`assume`** statements become constraints that enforce the given statement. The specification uses **`for`**, which (logically) means bounded universal quantification, and (mechanically) unrolls and repeats the enclosed requirements.

In Figure 2, lines 4–12 constrain B to be sorted (in increasing order), and enforce that $B \subseteq \bigcup_k A_k$. In particular, for each position i in B, the prover nondeterministically supplies *which list* (`k_i`) contributes to the ith position, and *which index* in that list (`j_i`) holds the contributed element. For the other direction, lines 13–20 specify that $\bigcup_k A_k \subseteq B$.

But how does the prover supply these values? Ideally they would result from simply executing the original computation.

This brings us to the Correctness and Coupling requirements (§1). We must prove a relationship between T_S and the actual code executed by the prover (T_I). The basic technique is to capture this relationship formally in terms of *refinement* [47, 48, 54]. A refinement proof coupling T_I and T_S not only establishes the correctness of the substitution, it also tells us how to augment T_I. The prover then executes the augmented implementation, which yields the values for the non-deterministically assigned variables in the specification.

A further improvement is possible. Notice that the implementation T_I (Fig. 1) uses the facts that the input lists are unique and sorted, whereas T_S (Fig. 2) uses neither fact. In the framework that we lay out in the sections ahead, we will have the freedom to choose a specification that refines the WLS yet still abstracts the computation. For example, by taking advantage of the uniqueness of the input lists, we obtain a less general but more concise specification than T_S. Specifically, we discard the lines in Figure 2 (13–20) that enforce $\bigcup_k A_k \subseteq B$, resulting in Figure 3, which we call T_E. When translated, T_E now yields a number of constraints proportional to $\sum_k A_k.\texttt{len}$, which saves a factor of two compared to T_S.

4 Framework

We formalize our framework in terms of *transition systems*, which provide a uniform formalism for representing both implementations and their specifications. From a semantic perspective, a transition system T defines a *language* $\mathcal{L}(T)$, which contains for each execution trace σ of T, a sequence of observations $o(\sigma)$ made about how T interacts with its environment during the execution. These observations may for instance encompass I/O, network traffic, etc.

We relate transition systems in terms of their languages. This allows us to formally capture when the execution of one transition system behaves like the execution of another, from the perspective of an external observer.

```
1  void merge_spec_efficient(L,A0,..,AL-1, B) {
2  ℓ'0 : int k_i, j_i;
3  ℓ'1 : havoc B.len;
4      assume B.len == ∑_{k=0}^{L-1} Ak.len;
5  ℓ'2 : for (int i = 0; i < B.len; i++) {
6        havoc B[i];
7        assume i == 0 || B[i-1] < B[i];
8        havoc k_i;
9        assume 0 <= k_i && k_i < L;
10       havoc j_i;
11       assume 0 <= j_i && j_i < Ak_i.len;
12       assume B[i] == Ak_i[j_i];
13     }
14 ℓ'4: return;
15 }
```

Figure 3: Pseudocode for the efficient specification (T_E) of the `merge` computation. The precondition is the same as for `merge` itself.

4.1 Transition systems and refinement

In our formalization, we adapt the classical setup of Abadi and Lamport [1]. A *transition system* $T = \langle \Sigma, \theta, \Delta, O, \alpha \rangle$ consists of a set of *states* Σ, a nonempty set of *initial states* $\theta \subseteq \Sigma$, a set of *transitions* $\Delta \subseteq \Sigma \times \Sigma$, a set of *observations* O, and an *observation function* $\alpha : \Sigma \to O$. Intuitively, the function α formalizes which aspects of a given state are observable. When T is known, we denote a transition $(s, s') \in \Delta$ by $s \to s'$ and say s *steps to* s'. We also call s' a *successor* of s.

Example 4.1. We illustrate with our motivating example (§3). We can regard T_I (Figure 1) as defining a transition system $(\Sigma, \theta, \Delta, O, \alpha)$, as follows. The states Σ of T_I are mappings from program variables to values. For $s \in \Sigma$, we denote by $s.\mathsf{x}$ the value of program variable x in s. We sometimes write x for a value of the program variable x when the state s is unspecified. We write $s[\mathsf{x} \mapsto v]$ to denote the new state obtained from s by updating the value of x to v and keeping the values of all other program variables unchanged. The program variables include a dedicated variable pc storing the value of the program counter, which ranges over the control locations $\ell_0, \ldots, \ell_4$. (For simplicity of exposition, we are treating the execution of a basic block, such as one iteration of a non-nested loop, as a single transition.)

The observations O of T_I are the values of the input arrays and output array at the program start and return. Intuitively, these are the values that an external user can observe from the program. All intermediate program states of the computations are unobservable, which we denote by the special observation τ. Formally, we define O using the following grammar:

$$O ::= \mathsf{in}(L, A_0, \ldots, A_{L-1}, B) \mid \mathsf{out}(L, A_0, \ldots, A_{L-1}, B) \mid \tau \ .$$

The observation function $\alpha : \Sigma \to O$ is then defined as follows:

$$\alpha(s) = \begin{cases} \mathsf{in}(s.\mathsf{L}, s.\mathsf{A}_0, \ldots, s.\mathsf{A}_{(s.\mathsf{L}-1)}, s.\mathsf{B}) & \text{if } s.\mathsf{pc} = \ell_0 \\ \mathsf{out}(s.\mathsf{L}, s.\mathsf{A}_0, \ldots, s.\mathsf{A}_{(s.\mathsf{L}-1)}, s.\mathsf{B}) & \text{if } s.\mathsf{pc} = \ell_4 \\ \tau & \text{otherwise} \ . \end{cases}$$

The transitions Δ of T_I are obtained from the program description in the expected way. For instance, the body of the for loop at control location ℓ_1 yields all transitions $s \to s'$ such that $s.\mathsf{pc} = \ell_1$, $s.\mathsf{k} < s.\mathsf{L}$, and

$$s' = s[\mathtt{len} \mapsto s.\mathtt{len} + s.\mathsf{A}_{(s.k)}.\mathtt{len}][\mathsf{k} \mapsto s.\mathsf{k} + 1] \ .$$

The set of initial states θ consists of all states s that satisfy the precondition of T_I (Figure 1). We assume that this precondition is specified by a formula φ_{pre}. That is, φ_{pre} states that $\mathsf{pc} = \ell_0$, and that the arrays A_k are sorted in strictly increasing order and its elements pairwise distinct. We write $s \models \varphi_{\mathsf{pre}}$ to indicate that s satisfies φ_{pre}.

An infinite sequence of states σ is called an *(execution) trace* of T if it starts in an initial state and respects T's transition relation: formally, $\sigma_0 \in \theta$ and for all $i \geq 0$, either σ_i steps to σ_{i+1} or $\sigma_i = \sigma_{i+1}$ and σ_i has no successors in Δ. If $\sigma_i = \sigma_{i+1}$, we say that σ *stutters* in step i. A terminating execution of T corresponds to a trace that stutters forever in its final state. By abuse of notation, we write $\alpha(\sigma)$ to denote the sequence of observations obtained by applying α pointwise to the states in σ. We denote the set of all traces of T by $\mathsf{traces}(T)$.

Let $\sharp$ be the function that maps a sequence σ to the sequence obtained from σ by replacing all repeated consecutive copies of elements by a single copy, for example, $\sharp(\langle 0,0,1,1,1,2,3,3,3,3 \rangle) = \langle 0,1,2,3 \rangle$.

The *language* of T, denoted $\mathcal{L}(T)$, is defined by applying α pointwise to each trace in $\mathsf{traces}(T)$ and then removing stutters. The intuition for removing stuttering is that we want to capture only the observable behavior: stuttering steps correspond to unobservable internal computation steps. Formally, we define the sequence of observations $o(\sigma)$ made from a trace σ as $o(\sigma) \stackrel{\text{def}}{=} \sharp(\alpha(\sigma))$ and then let

$$\mathcal{L}(T) \stackrel{\text{def}}{=} \{ o(\sigma) \mid \sigma \in \mathsf{traces}(T) \} \ .$$

Example 4.2. In the motivating example (§3), the language of the transition system T_I is simply

$$\mathcal{L}(T_I) = \left\{ \langle \alpha(s), \tau, \alpha(s') \rangle \;\middle|\; s \models \varphi_{\mathsf{pre}} \wedge s' \models \varphi_{\mathsf{post}} \right\} \ .$$

Here, the precondition φ_{pre} is as defined above. The postcondition φ_{post} states that $\mathsf{pc} = \ell_4$, B is sorted in strictly increasing order, and the set of elements of B is equal to the union of the set of elements of the arrays A_k. The single τ in each observation sequence in $\mathcal{L}(T_I)$ summarizes all intermediate states of the computation.

A transition system T_I *refines* another transition system T_S iff $\mathcal{L}(T_I) \subseteq \mathcal{L}(T_S)$. This definition captures the idea that from the perspective of an external observer, every execution of T_I behaves like some execution of T_S. Typically, we think of T_S as *the specification* and T_I as *the implementation*. We denote a refinement relationship by $T_I \leq T_S$.

A classical approach to proving refinement relationships is to construct a *refinement mapping*. Formally, a refinement mapping between T_I and T_S is a function $r : \Sigma_I \to \Sigma_S$ such that

1. $r(\theta_I) \subseteq \theta_S$,
2. $\forall s \in \Sigma_I,\ \alpha_I(s) = \alpha_S(r(s))$, and

3. $\forall s, s' \in \Sigma_I$, if $s \to_I s'$, then $r(s) \to_S r(s')$ or $r(s) = r(s')$.

The first property states that r maps the initial states of T_I to those of T_S. The second property states that the observations computed from states are preserved by r. The third property states that every transition in Δ_I is matched by a corresponding transition in Δ_S under r or by a stuttering step. Together, these properties capture the intuition that the relationship between a refinement and its specification is that the specification abstracts steps that are "internal" to the implementation.

Once it has been established that $r : \Sigma_I \to \Sigma_S$ is a refinement mapping, $T_I \leq T_S$ follows: given a trace σ_I of T_I, the sequence $r(\sigma_I)$ is a trace of T_S (modulo stuttering). Moreover, $r(\sigma_I)$ makes the same observations as σ_I, i.e., $\sharp(\alpha_I(\sigma_I)) = \sharp(\alpha_S(r(\sigma_I)))$. Hence, the existence of r establishes that T_I refines T_S.

We write $T_I \leq_r T_S$ to indicate that r is a refinement mapping between T_I and T_S. An important property that we will use freely later is that refinement mappings compose: $T_1 \leq_r T_2$ and $T_2 \leq_q T_3$ implies $T_1 \leq_{q \circ r} T_3$.

4.2 Refinement-based widgets

We can now use the language of transition systems to recast Steps 0, 1, and 3 in Section 2 and explain how widgets are conventionally used to modify these steps. We start from a given transition system T_I and a property $\phi \subseteq O^\omega$ specifying the observation sequences of interest (Step 0). The problem is for the prover $\mathcal{P}$ to convince the verifier $\mathcal{V}$ that $\mathcal{L}(T_I) \cap \phi$ is nonempty. Here, the property ϕ will, in particular, ensure that the considered observations are restricted to those that are bound to the specific input x and alleged output y. However, ϕ may impose additional requirements on the observation sequences that are of interest to $\mathcal{V}$. The conventional approach is then to first translate T_I into constraints $\mathcal{C}(T_I, \phi) = \mathcal{C}(T_I) \wedge \mathcal{C}(o_I^{-1}(\phi))$. We elide the definition of $\mathcal{C}(o_I^{-1}(\phi))$. In the context of the steps in Section 2, it is simply $X = x \wedge Y = y$. The translation guarantees that $\mathcal{C}(T_I, \phi)$ is satisfiable iff $o(\sigma_I) \in \mathcal{L}(T_I) \cap \phi$ for some σ_I (Step 1). The prover then executes T_I on the specified input x to obtain such a σ_I and derives from it the desired satisfying assignment (Step 3).

The conventional use of a widget is then to replace the constraints $\mathcal{C}(T_I)$ by a smaller set of constraints $\mathcal{C}(T_W)$. The prover still executes T_I to yield σ_I, but uses σ_I to compute a satisfying assignment for $\mathcal{C}(T_W, \phi)$. A crucial shortcoming of this approach is that replacing T_I by T_W is not formally justified. In particular, there is no guarantee that the existence of a satisfying assignment for $\mathcal{C}(T_W, \phi)$ implies the nonemptiness of $\mathcal{L}(T_I) \cap \phi$, potentially compromising the soundness of the proof system. Moreover, there is no systematic approach to compute a satisfying assignment for $\mathcal{C}(T_W, \phi)$ from σ_I. We use the notion of refinement to address both of these shortcomings.

First, we change the problem setup as follows. The new Step 0 is to write down transition systems T_S, T_E, and T_I, as well as refinement mappings r and q such that $T_I \leq_r T_E \leq_q T_S$. Now, the problem is for the prover $\mathcal{P}$ to convince the verifier $\mathcal{V}$ that $\mathcal{L}(T_S) \cap \phi$ is nonempty. That is, $\mathcal{V}$ is only interested in T_S, the weakest specification; the transition systems T_E and T_I are merely a means to an end to solve the problem. T_E then plays the role of T_W above. The new Step 1 is to translate T_E and ϕ into constraints $\mathcal{C}(T_E, \phi)$. The new Step 3 is for $\mathcal{P}$ to execute T_I on x (obtaining $o(\sigma_I) \in \mathcal{L}(T_I) \cap \phi$), to use r to compute a trace $r(\sigma_I)$, and finally to use $r(\sigma_I)$ to compute a satisfying assignment for $\mathcal{C}(T_E, \phi)$.

Observe that $T_E \leq_q T_S$ implies that if $o(\sigma_E) \in \mathcal{L}(T_E) \cap \phi$ for some σ_E, then $o(q(\sigma_E)) \in \mathcal{L}(T_S) \cap \phi$. Hence, assuming the correctness of the translation to constraints, if $\mathcal{C}(T_E, \phi)$ is satisfiable, then $\mathcal{L}(T_S) \cap \phi$ is nonempty. This ensures the soundness of the approach. Similarly, $T_I \leq_r T_E$ means that if $o(\sigma_I) \in \mathcal{L}(T_I) \cap \phi$, then $o(r(\sigma_I)) \in \mathcal{L}(T_E) \cap \phi$ and, hence, $\mathcal{C}(T_E, \phi)$ is satisfiable. This ensures the completeness of the approach.

A difference between proving $T_I \leq_r T_E$ and $T_E \leq_q T_S$ is that q need not be explicit. That is, although End-to-end Soundness requires that if $\mathcal{C}(T_E, \phi)$ is satisfiable then so is $\mathcal{C}(T_S, \phi)$, the actual satisfying assignment to $\mathcal{C}(T_S, \phi)$ is not used explicitly. Consequently, $T_E \leq T_S$ can be established by means other than refinement mappings, for example a proof based on simulation relations [57, 64, 77].

We note that the approach also applies in the special case where $T_S = T_E$. Though, generally, the crux is to find a suitable T_E *in between* T_S and T_I that yields a reduction in the constraint size relative to both T_S and T_I.

Constructing refinement mappings. It remains to show how to construct refinement mappings. We demonstrate this with the merge computation as a guiding example, using general principles inspired by refinement calculi such as [56, 58, 59]. These principles apply broadly (Section 5 contains further examples).

Our first step is to construct a refinement mapping r between the transition system T_I of the merge computation (Fig. 1) and its intermediate specification T_E (Fig. 3). As we will explain below, r can then be used to obtain a satisfying assignment for the constraints $\mathcal{C}(T_E)$ from a given execution of T_I, enabling more efficient verification of that execution. In a second step, we then show that the intermediate specification T_E refines the naive specification T_S.

To prove $T_I \leq T_E$, we divide the construction of the refinement mapping into three steps by deriving two auxiliary transition systems T_{IE} and $\hat{T}_{IE}$ that yield a refinement sequence $T_I \leq T_{IE} \leq \hat{T}_{IE} \leq T_E$. Intuitively, the auxiliary transition systems couple T_I and T_E so that they are executed together.

A transition system T_E refined by an implementation T_I will typically involve nondeterministic (**havoc**, see §3) assignments to program variables that do not appear in the implementation. In our example of the merge computation, these are the assignments to `k_i` and `j_i` in Fig. 3 (lines 8 and 10). The execution of T_E can proceed only if the value chosen by each nondeterministic assignment satisfies the constraints imposed by the subsequent **assume** statements. A key step in the refinement proof is therefore to show that such values can be obtained from the trace σ_I. We make this step explicit in the construction of the intermediate transition system T_{IE}. This transition system augments T_I with those variables unique to T_E as well as assignments to these variables that determine the desired values to be chosen for the nondeterministic assignments in T_E. In our example, this augmentation can be seen in lines 3, 21, and 23 shown in blue in Fig. 4.

Observe that the assignments to `k_i` and `j_i` in T_{IE} of our example depend only on the original program variables of T_I. Moreover, the variables do not interfere with the other parts of the transition system in any way. Such auxiliary variables that

```
1   void merge (L,A0,..,AL-1,B) {
2   ℓ0 : int[L] curr = {0};
3        int len, running_min, kstar, k_i, j_i;
4        bool found;
5        len = 0;
6   ℓ1 : for (int k = 0; k < L; k++) { len += Ak.len; }
7        B.len = len;
8        assume B.len == ∑_{k=0}^{L-1} Ak.len;
9   ℓ2 : for (int i = 0; i < B.len; i++) {
10         found = false;
11  ℓ3 :   for (int k = 0; k < L; k++) {
12           if (curr[k] < Ak.len && (!found ||
13                Ak[curr[k]] < running_min)) {
14             running_min = Ak[curr[k]];
15             kstar = k;
16             found = true;
17           }
18         }
19         B[i] = running_min;
20         assume i == 0 || B[i-1] < B[i];
21         k_i = kstar;
22         assume 0 <= k_i && k_i < L;
23         j_i = curr[kstar];
24         assume 0 <= j_i && j_i < Ak_i.len &&
25                B[i] == Ak_i[j_i];
26         curr[kstar]++;
27       }
28  ℓ4: return;
29  }
```

Figure 4: Pseudocode for the transition system $\hat{T}_{IE}$. The prover will execute the black and blue code (T_{IE}) instead of T_I. The values in blue are used to create an assignment to the nondeterministic variables occurring in the constraints obtained from T_E (the red **assume** statements).

are used only for the purpose of proving a refinement relation are sometimes referred to as *ghost variables*. A transition system T refines any transition system T' that is obtained from T by adding ghost variables [56]. What is more, the refinement mapping between T and T' can be constructed in a generic fashion. Thus, we obtain $T_I \leq T_{IE}$ for free.

In the context of program translation for probabilistic proofs, augmenting an implementation with ghost variables is not only useful for proving the refinement between T_I and T_E. The system T_{IE} also instructs the prover how to obtain the satisfying assignment for the constraints $\mathcal{C}(T_E)$. That is, the prover will actually execute T_{IE} instead of T_I.

The next step in our construction is to augment T_{IE} with the **assume** statements in T_E that constrain the values chosen for the nondeterministic assignments. We call the resulting transition system $\hat{T}_{IE}$. In our merge example, $\hat{T}_{IE}$ is shown in Fig. 4 with the added **assume** statements highlighted in red (lines 8, 20, 22, and 24).

Establishing the refinement $T_{IE} \leq \hat{T}_{IE}$ follows another generic construction. We first show that the added **assume** statements express invariants of T_{IE}. That is, the assumed expressions must always evaluate to `true` in T_{IE}, at the appropriate program points. In Appendix A, we discuss this step of the proof in more detail with regards to the merge computation. Once the invariants have been established, $T_{IE} \leq \hat{T}_{IE}$ follows, by simply using the identity function on the states of T_{IE} as the refinement mapping.

Finally, we observe that T_E can be obtained from $\hat{T}_{IE}$ by abstracting all program variables that appear in T_I but not in T_E. For our merge example, this amounts to removing the loops at locations ℓ_1 and ℓ_3 in $\hat{T}_{IE}$, and replacing the assignments on lines 7, 19, 21, and 23 that depend on the abstracted program variables by **havoc** commands.

Abstracting program variables in this systematic manner again yields a refinement by construction. The refinement mapping changes the value of the program counter in the expected way. For instance, the refinement mapping in our example coalesces locations ℓ_2 and ℓ_3 to ℓ'_3 and maps all other locations ℓ_i to ℓ'_i. The values of the remaining program variables that are common to $\hat{T}_{IE}$ and T_E are preserved by the refinement mapping. This concludes the proof of $T_I \leq T_E$.

It remains to argue that T_E refines T_S. One can generally apply the above technique again, to construct an appropriate refinement mapping. In particular, one can show that the following property is an invariant at the end of the for loop in T_E (Fig. 3):

```
∀ k,j :: 0 <= k && k < L && 0 <= j && j < Ak.len ==>
  ∃ i :: 0 <= i && i < B.len && Ak[j] == B[i]
```

The second for loop at lines 13 to 20 of T_S (Fig. 2) establishes exactly the same property.

Systems view. An end-to-end system view of Distiller is as follows. At compile-time the user provides a weakest specification T_S, an effective specification T_E, and a computation T_I (the new Step 0 in Section 2). The user then shows that the refinement relationships $T_I \leq T_E$ and $T_E \leq T_S$ hold. The proof of $T_I \leq T_E$ yields $\hat{T}_{IE}$ as a byproduct. The $\hat{T}_{IE}$ is a *coupling* of T_I and T_E that makes explicit how the **havoc**ed ghost variables in T_E are computed from T_I.

$\hat{T}_{IE}$ is the input to an augmented front-end that splits $\hat{T}_{IE}$ into T_E and T_{IE}. It then compiles T_E to constraints $\mathcal{C}(T_E)$ (the new Step 1). For each invocation of the probabilistic proof protocol (the new Steps 3 and 4), the prover runs T_{IE} and feeds its values back in to get a satisfying assignment to $\mathcal{C}(T_E)$. This step can be implemented using existing front-end infrastructure (see §6).

Note that $T_E \leq T_S$ is needed for End-to-end Soundness (every satisfying assignment to $\mathcal{C}(T_E)$ encodes an element of $\mathcal{L}(T_S)$) while $T_I \leq T_E$ is needed for End-to-end Completeness (a satisfying assignment to $\mathcal{C}(T_E)$ can be obtained from T_I).

5 Examples

We have applied the Distiller framework to the problems discussed in Dijkstra's classic book *A Discipline of Programming* [30]. We chose this source for two reasons. First, it discusses algorithms for a diverse set of problems. Second, Dijkstra develops his algorithms iteratively, starting from a formal problem specification. This approach helps to identify suitable intermediate transition systems T_E that yield an efficient translation to constraints.

Our evaluation considers 11 of the 14 problems discussed in Dijkstra's book. The three problems we have omitted are "Updat-

Example		Improvement
Merging	(Ch. 16)	$\Theta(L)$
Find Min	(Ch. 12)	$1.5\times$
Binary Search	(Ch. 12)	$\Theta(\log(n)\log(\log(n)))$
Pattern Matching	(Ch. 18)	$3\times$
Next Permutation	(Ch. 13)	$1.5\times$
Dutch Flag	(Ch. 14)	$1.66\times$
RR Sequence	(Ch. 17)	$2\times$
Sum of Powers	(Ch. 19)	$2\times$
2D Convex Hull	(Ch. 24)	$5\times$
2D Convex Hull*	(Ch. 24)	$\Theta(\log(n))$
MSC	(Ch. 25)	$17.5\times$
MST	(Ch. 22)	$52.2\times$

Figure 5: Improvement for all examples based on theoretical analysis on large inputs. For improvements where T_E has asymptotically fewer constraints than T_I, we provide the complexity of the improvement; otherwise we provide a constant factor. L in Merging is the number of lists. n in Binary Search is the length of the array. n in 2D Convex Hull is the total number of nodes, and 2D Convex Hull* is the case where the nodes in the convex hull are marked instead of returned in a list.

```
1   int find_min(n, A, B) {
2     int min = A[0]; int p = 0;
3   ℓ0: for (int i = 0; i < n; i++) {
4       if (A[i] < min) {
5         min = A[i]; p = i;
6       }
7     }
8     bool found = false;
9   ℓ1: for (int i = 0; i < n; i++) {
10      assume min <= A[i];
11      if (A[i] == min) {
12        B[i] = 1; assume B[i] == 1;
13        found = true;
14      } else {
15        B[i] = 0; assume B[i] == 0;
16      }
17    }
18    assume found;
19    return min;
20  }
```

Figure 6: Pseudocode for T_I of Find Min. The code in red is the augmentation needed for proving $T_I \leq T_E$.

ing a sequential file" (Chapter 15), "The problem of the smallest prime factor of a large number" (Chapter 20), and "The problem of the most isolated villages" (Chapter 21). We also have simplified the problem of computing the convex hull in three dimensions (Chapter 24) to the two-dimensional case.

For all the problems that we have considered, we are able to obtain significant reductions in the size of the generated constraints (Figure 5). In some cases, the scale factor of the reduction grows asymptotically with the problem instance size.

In the following, we discuss a selected subset of the considered problems in detail. We explain T_S, T_E, and T_I for these problems, provide a qualitative analysis that explains the expected reduction in constraint size, and explain the key insights behind the refinement proofs.

5.1 Find Min

Given a non-empty array A of length n, the problem is to find its smallest element, min, and mark all occurrences of the minimum using another array B. More precisely, there must exist an index p such that the following conditions hold:

1. $0 \leq p < n$ and $\texttt{min} = \texttt{A}[p]$,
2. for each $i \in [0,n)$, $\texttt{min} \leq \texttt{A}[i]$,
3. for each $i \in [0,n]$, $\texttt{B}[i] = (\texttt{A}[i] = \texttt{min}\,?\,1 : 0)$.

T_S encodes this specification by nondeterministically choosing min, each $\texttt{B}[i]$, and p. It uses two loops that iterate over A to enforce conditions 2 and 3.

T_I is shown in Fig. 6 (without the code in red, which we will discuss later). It also requires two loops: ℓ_0 to compute min, and ℓ_1 to compute the $\texttt{B}[i]$. Comparing T_I and T_S, we note that the two loops in T_I and T_S have exactly the same costs. However, T_S performs an additional dynamic LOAD, namely $\texttt{A}[p]$, to enforce Condition 1. Hence, $\mathcal{C}(T_S)$ incurs the extra cost of RAM initialization, which performs n STOREs to write A into the memory, and is therefore larger than $\mathcal{C}(T_I)$.

However, we can do better than either T_S or T_I. First, observe that unlike in T_I, we can merge the two loops in T_S for conditions 2 and 3 into a single loop because min can be chosen nondeterministically upfront. Compared to T_I, this saves one of the two inequality tests $i < n$ that $\mathcal{C}(T_I)$ would otherwise include for each iteration of the two loops. Furthermore, we can eliminate the $\texttt{LOAD A}[p]$ in Condition 1 of T_S by introducing an auxiliary variable found that indicates whether min has been encountered at least once in the loop that checks conditions 2 and 3. The pseudocode of the resulting T_E is shown in Fig. 7 (excluding the blue code, which we will use later to establish that T_E refines T_S).

Thus, $\mathcal{C}(T_E)$ needs only $2 \cdot n$ inequality tests, saving $1/3$ over $\mathcal{C}(T_I)$. Since the encoding of inequality tests dominates the size of the generated constraints, we observe a similar constant factor improvement in the overall constraint size.

Turning to the refinement proofs, if we add the red code in Fig. 6 to T_I, we obtain the augmented transition system $\hat{T}_{IE}$ for showing $T_I \leq T_E$ (see §4.2). Recall that the main part of the refinement proof is to show that the added **assume** statements in $\hat{T}_{IE}$ coming from T_E always succeed. We focus on the **assume** on Line 18, which is the most interesting one. Observe that the loop at ℓ_0 ensures $0 \leq p < n \wedge \texttt{A}[p] = \texttt{min}$ after the loop has terminated. Using this fact, we can then establish the loop invariant $i < p \vee \texttt{found}$ for the second loop at ℓ_1. This then allows us to prove that the **assume** statement on Line 18 is safe.

Next consider the refinement $T_E \leq T_S$. Adding the blue code in Fig. 7 to T_E yields an augmented transition system $\hat{T}_{ES}$ for the refinement proof $T_E \leq T_S$. We focus on showing that T_E ensures Condition 1. (The other two conditions follow immediately from the loop in T_E.) To this end, we can establish the loop invariant $\texttt{found} = 0 \vee (0 \leq p < n \wedge \texttt{A}[p] = \texttt{min})$ for the loop at ℓ_1'. Together

```
1   int find_min_efficient(n, A, B) {
2       int min, p;
3   ℓ'0 : havoc min;
4       bool found = false;
5   ℓ'1 : for (int i = 0; i < n; i++) {
6         assume min <= A[i];
7         if (min == A[i]) {
8           havoc B[i]; assume B[i] == 1;
9           found = true; p = i;
10        } else {
11          havoc B[i]; assume B[i] == 0;
12        }
13      }
14      assume found;
15      assume 0 <= p < n && A[p] == min;
16      return min;
17  }
```

Figure 7: Pseudocode for T_E of Find Min. The code in blue is the augmentation needed for proving $T_E \leq T_S$.

with Line 14, this implies that adding the assume on Line 15 is safe. This line then establishes Condition 1.

We note that we would not be able to improve over T_I if the array A was guaranteed to have a single minimum, or if we were satisfied with finding any of the minimums in A. The loops at ℓ_1 and ℓ'_1 would be unnecessary.

5.2 Binary Search

Given a sorted array A, the bounds l, r of a possibly empty segment in A, and a value x, the problem is to compute i such that $l \leq i \leq r$ and $A[i] = x$. If no such i exists, return $i = -1$.

T_S for this problem checks i according to the specification above. That is, if $i \neq -1$, T_S checks that $l \leq i < r$ and $x = A[i]$, otherwise it iterates over $A[l \dots r]$ and checks that the segment does not contain x. T_I is based on standard binary search.

For our cost analysis we focus on the number of LOAD operations, which is the largest contributor to the size of the generated constraints. In the worst case, T_I performs $\log(n)$ LOAD operations to search through the segment $A[l \dots r]$ where $n = r - l$. In contrast, T_S performs $n+1$ LOAD operations in the worst case. That is, T_S is asymptotically worse than T_I.

We can do better by exploiting that A is sorted. Introducing an auxiliary value s, we divide the specification for the case when x is not present ($i = -1$) into four subcases:

1. If $i = -1$, then $l = r$ or $x < A[l]$ or $x > A[r-1]$ or $(l \leq s < r-1 \ \wedge \ A[s] < x < A[s+1])$,
2. else $l \leq i < r \ \wedge \ A[i] = x$.

T_E is the direct encoding of this case analysis. It performs a constant number of LOAD operations, achieving an asymptotic improvement over T_I. We note that if the search is viewed as a standalone program, then this improvement is overshadowed by the cost of storing the array segment into RAM, which is linear in n. However, if the search is executed many times or viewed as a subroutine, then the RAM initialization can be amortized.

For proving $T_E \leq T_S$, observe that each of the subcases of Condition 1 implies that x cannot be present anywhere in the segment. For the last three cases, the proof relies on the precondition that A is sorted in strictly increasing order. For proving $T_I \leq T_E$, recall that binary search iteratively shrinks a subsegment $A[l' \dots r']$ of $A[l \dots r]$ that may still contain x. This process continues until the subsegment converges to a single point $l' = r'$, which is the index of the least element larger than x. In the nontrivial case where $l \neq r$ and x is not present in the segment but within the range of values defined by $A[l]$ and $A[r-1]$, we define $s = l' - 1$ for the final point $l' = r'$. Then s is the index that satisfies the last disjunct in Condition 1.

5.3 2D Convex Hull

Given a set of points $P = \{p_0, \dots, p_{n-1}\} \subseteq \mathbb{Z}^2$ with $n > 1$, assume no three points are on the same line, the problem is to find all points in P that lie on the convex hull of the set.

We additionally require P to satisfy the precondition of Graham Search [41], a popular algorithm that solves the 2D Convex Hull problem. Specifically, p_0 has the smallest y coordinate among all points in P, and the greatest x coordinate among all points in p with the same y coordinate as p_0. The remaining points are sorted in counterclockwise order when using p_0 as a reference point. In other words, for each $i \in (0,n)$, let L_i be the line passing through p_0 and p_i. Then intersect L_i with a horizontal line at p_0 and define $\angle_i$ to be the *top-right* angle of the intersection. P is ordered so that $\angle_i < \angle_{i+1}$ for all i.

With these assumptions, C defines the convex hull of P iff the following conditions hold:

1. $C \subseteq P$.
2. $p_0 \in C$ and for all $i \in (0,n)$, $p_i \in C$ or the angle defined by the points $(\texttt{prv}_i, p_i, \texttt{nxt}_i)$ bends inwards, where $\texttt{prv}_i$ and $\texttt{nxt}_i$ are the first points before and after p_i in P that are also in C. If no such $\texttt{nxt}_i$ exists, then $\texttt{nxt}_i = p_0$.

Condition 1 ensures that C contains no points outside P. Since P is sorted, Condition 2 guarantees that C contains all the points of P that lie on the convex hull of P.

T_S nondeterministically chooses C and then checks the above conditions. The size of $\mathcal{C}(T_S)$ is in $O(n^2)$. (In particular, for each p_i, T_S needs to iterate through P again to find $\texttt{prv}_i$ and $\texttt{nxt}_i$.).

We use Graham Search as the T_I for this problem. For each of the n points, T_I needs two STORE operations and two dynamic LOAD operations.

T_E is shown in Fig. 8. It nondeterministically chooses k, C, and the points $\texttt{nxt}_i$, then it iterates over the p_i and checks all relevant conditions in constant time for each i. The refinement proof showing $T_I \leq T_S$ uses the fact that T_I computes the points in C in the order in which they appear in P. Moreover, the $\texttt{nxt}_i$ can be computed by T_{IE} using a simple linear scan of the final C.

To see that T_E yields smaller constraints than T_I, observe that only the array access of C[count] on Line 15 is dynamic and incurs the cost of two LOADs (one for each coordinate of the point). Also, there are no STORE operations. (Recall that a **havoc** command stands for augmented code in T_{IE}. Hence, it does not contribute to $\mathcal{C}(T_E)$.) So T_E only performs two dynamic LOAD operations per iteration. The cost of a STORE depends on how deeply it is nested in conditionals whereas the cost of a LOAD does

```
int X_PROD(p, q, r) {
  return (q.x-p.x) * (r.y-p.y) - (q.y-p.y) * (r.x-p.x)
}
void 2d_convex_hull_efficient(n, P, C) {
  int k;
  havoc k;
  point nxt, prv;
  havoc nxt; // nxt_0
  prv = P[0]; // prv_1
  havoc C[0]; assume C[0] == prv;
  int count = 1;
  for (int i = 1; i < n; i++) {
    point cur = P[i];
    if (nxt == cur) { // P[i] in C
      havoc C[count]; assume C[count] == cur;
      // get nxt_i because nxt_{i-1} ≠ nxt_i
      havoc nxt; // nxt_i
      // angle (prv, cur, nxt) must bend inwards
      assume X_PROD(prv, cur, nxt) > 0;
      prv = cur; // prv_{i+1}
      count++;
    } else { // P[i] !in C
      // nxt_i = nxt_{i-1}; prv_{i+1} = prv_i
      assume X_PROD(prv, cur, nxt) < 0;
    }
  }
  assume nxt == P[0];
  assume k == count;
}
```

Figure 8: Pseudocode of T_E for 2D Convex Hull.

not [79, §3.1]. Specifically, each STORE in T_I is four times more expensive than a LOAD in T_E. We therefore expect that the size of $\mathcal{C}(T_E)$ is about five times smaller than that of $\mathcal{C}(T_I)$.

If we consider the variant where the problem is not to enumerate C but to compute its characteristic function on the indices of P (that is, mark the points in P that belong to C), then we can eliminate all dynamic LOAD operations from T_E and achieve an asymptotic $\log(n)$ factor improvement over T_I.

5.4 Maximal Strong Components

Given a directed graph $G = (V, E)$ with nodes V and edges $E \subseteq V \times V$, the problem is to partition V into the maximal strongly connected components $C_0, \ldots, C_{k-1}$ of G. We represent the C_i implicitly using an array rank that maps every node $v \in V$ to the index of its maximal strongly connected component. That is, we define for all $i \in [0,k)$, $C_i = \{v \in V \mid \mathsf{rank}[v] = i\}$. Given this, the formal problem statement is to find k and rank such that the following three conditions hold:

1. For all $v \in V$, $0 \leq \mathsf{rank}[v] < k$.
2. For all $i \in [0,k)$, there exists a cycle c_i in G that visits exactly the nodes in C_i.
3. For all $i \in [0,k)$ and all cycles c in G, if c visits some node in C_i then c visits only nodes in C_i.

T_S encodes the above specification by nondeterministically choosing k, $\mathsf{rank}[v]$ for each node $v \in V$, and the cycles c_i for each component $i \in [0,k)$. Condition 3 quantifies over the set of all cycles in G, which is in general an infinite set. However, it can be shown that restricting the quantification to *simple* cycles in G yields an equivalent condition. A simple cycle is a path where only the first and last node are equal and all other nodes are distinct. The condition that quantifies over simple cycles can be encoded using a nested loop that iterates over all partial permutations p of nodes in G and then checks that if p forms a simple cycle in G and intersects with a C_i, then it is fully contained in C_i. As the number of partial permutations grows exponentially with $|V|$, so does $|\mathcal{C}(T_S)|$.

We use Dijkstra's MSC algorithm [30, Chapter 25] as our T_I. The algorithm is similar to Tarjan's algorithm [74] but lends itself more directly to an efficient translation into constraints. The algorithm iterates over E and V. In each iteration, it performs up to 13 LOAD and 8 STORE operations. These operations dominate the size of the generated constraints.

However, we can again construct a T_E that improves over both T_I and T_S. The key idea for T_E comes from Dijkstra's correctness argument for his algorithm. Dijkstra observed that a set of connected components $C_0, \ldots, C_{k-1}$ is maximal iff it can be ordered so that all edges leaving a C_i target only nodes in components preceding C_i. Given Dijkstra's observation, we can replace Condition 3 in T_S with the following condition in T_E:

3^*. For all $(v,w) \in E$, $\mathsf{rank}[w] \leq \mathsf{rank}[v]$.

Replacing Condition 3 by 3^* yields a refinement of T_S.

Additionally, Condition 2 can be reformulated as a Condition 2^* that no longer relies on the construction of explicit cycles c_i connecting the nodes in each component. We observe that the nodes in each C_i can be arranged in a tree that implicitly witnesses the existence of an appropriate c_i (which we use in the $T_E \leq T_S$ proof). The tree reflects the way T_I traverses the nodes in V and collapses candidate components whenever a node is revisited. These trees can be obtained from T_I using an augmentation that does not increase T_I's asymptotic complexity. We use this augmentation to establish Condition 2^* when proving that T_I refines T_E.

We defer the details of how Condition 2^* is expressed to Appendix B. What is important is that the combined size of these trees is linear in $|V|$ and so is checking their correctness. As a result, for dense graphs ($|E| \approx |V|^2$), the cost to enforce Condition 2^* is insignificant. A detailed cost analysis yields an expected reduction in total constraint size for dense graphs by a factor of 17.5 for sufficiently large $|E|$. For shallow graphs ($|E| \approx |V|$), we still obtain a reduction by a factor of two. The principal savings come from the fact that conditions 3^* and 2^* can be checked by T_E (in the sense of validated inside an **assume**) with many fewer LOAD and STORE operations versus T_I.

5.5 Minimum Spanning Tree

Given a connected graph $G = (V, E)$ with $|V|$ vertices and $|E|$ undirected edges with unique positive weights, its minimum spanning tree (MST) is the minimum weight subgraph, M, of G which contains all of the vertices in G. The MST problem is to find M

given G. By definition, because our graph is connected, M is a tree with $|V|-1$ edges. There are several equivalent alternative definitions of an MST. The one that we are concerned with is that for every cycle in G, the heaviest edge in that cycle is not in M. T_S encodes this specification by nondeterministically choosing $|V|-1$ edges to be M and $|E|-|V|+1$ cycles attesting to the non-membership of every edge in E not in M. As each cycle contains at most $|V|$ edges, the size of the constraints generated from T_S grows $O(|V|\cdot|E|\cdot\log(|V|\cdot|E|))$.

Our T_I is Kruskal's amortized MST algorithm, a procedure that sorts the edges by weight and then greedily builds M one edge at a time, only adding an edge to M when the edge does not form a cycle. The key to Kruskal's efficiency is the use of a disjoint set data structure S to efficiently detect cycles. In S, vertices are arranged in a forest of inverted trees with individual sets identified by the root vertices of these trees. S supports 3 operations: `INSERT`, `FIND-SET`, and `UNION`. `INSERT` adds an element to S in its own set in $O(1)$ memory operations. `FIND-SET` traverses up the tree to the root to get the unique set identifier for a queried element in amortized $O(\alpha(|V|))$ and worst case $O(\log|V|)$ memory operations (where α is the Inverse Ackermann function). `UNION` takes two sets and combines them into one in $O(1)$ memory operations.

When deciding whether or not to add an edge e to M, the algorithm calls `FIND-SET` at the endpoints of e to efficiently test if that edge would form a cycle. If the edge does not form a cycle, it is added to M, and S is updated by calling `UNION` on the sets for its respective endpoints. The amortization in `FIND-SET` comes from the careful implementation of `FIND-SET` with path compression (updating all pointers along the path to the root to be direct to the root) and `UNION` by rank (to minimize the height of the combined trees). When implementing `FIND-SET` we can either use the amortized version which results in $O(|E|\cdot\alpha(|V|)\log(|E|\cdot\alpha(|V|)))$ constraints with a large overhead or the worst-case version which results in $O(|E|\cdot\log|V|\cdot\log(|E|\cdot\log|V|))$ constraints but a much smaller constant. However, by performing a refinement on the underlying memory model, we can get the near-optimal asymptotic behavior of the amortized version with the small constant of the worst-case version, outperforming both T_I implementations for all input sizes.

The amortized T_I needs to use multiple nested loops and conditionals to maintain asymptotic complexity. The large quantity of branches created by these loops and conditionals generate a massive constant overhead. We can avoid this overhead in our nested find set operations by replacing RAM with a new DS-SET memory construct. We combine the `LOAD`s and `STORE`s into a single `UPDATE` operation, and we define a `FIND-SET` operation that completely eliminates the nested loops. In our model, there is a fixed buffer of nondeterministically supplied `UPDATE` operations to handle the amortized complexity of `FIND-SET` as every `FIND-SET` call is a series of `UPDATE` operations which result in a fixed-point (the root).

As a further reduction, we observe that the main loop, which iterates through the sorted edges, has a conditional that is true exactly $|V|-1$ times; however, when we unroll the loop to constraints we incur its cost for every iteration. Since there is a direct correspondence between the current edge weight, the current iteration of the while loop, and the program counter, we can replace time order in our memory transcript with edge weight. This eliminates the need to sort the edges and allows us to reorder iterations freely. We then split the main loop into two smaller loops without branching. The first loop asserts that the condition is true and executes the contents of the conditional block unconditionally; the second simply asserts that the condition is false. Combined with other small improvements, these RAM refinements completely eliminate all branching and state machine overhead (4 `while` loops and a nested `if` statement) and results in a set of constraints that is several orders of magnitude smaller for small inputs and converges to a $52.2\times$ constant improvement for large inputs.

6 Experimental evaluation

This section answers the following questions:

(1) Does Distiller increase confidence in the correctness of widgets?

(1) How difficult is it to build an end-to-end system for probabilistic proof checking based on Distiller?

(1) Can we empirically achieve a constraint size reduction when using Distiller with existing front-ends?

Mechanizing refinement proofs. To answer Question ((1)) we use the deductive verification tool Viper [60] to mechanize the $T_I \le T_E$ and $T_E \le T_S$ proofs for 10 of our 11 benchmark problems (§5). For MST (§5.5), we perform the refinement proofs only on paper. In this case, T_S also encompasses the specification of a refined RAM, making the proof mechanization more elaborate.

For the proofs showing $T_I \le T_E$, we encode the transition system $\hat{T}_{IE}$ into Viper's programming language, replacing all **`assume`** statements coming from T_E by **`assert`** statements. Then we annotate all loops with candidate invariants that are intended to imply the safety of the **`assert`**s (§4.2). The tool automatically checks that the annotated invariants are inductive and indeed imply the safety of the **`assert`**s.We proceed similarly for the proofs of $T_E \le T_S$.

Viper verifies all programs. However, for two of the benchmarks we discovered bugs in the initial version of T_E. These bugs would have compromised End-to-end Soundness (§2). One bug was a missing array bounds check in T_E of the merge computation (§3). The other one was a subtle omission of a check in T_E for the Sum of Powers problem [30, Chapter 19]. We discovered these bugs when trying to annotate the respective T_E to prove $T_E \le T_S$.

End-to-end system. As outlined in §4.2, an end-to-end system would take as input $\hat{T}_{IE}$ and split it into T_E and T_{IE}. Then it would compile T_E to constraints $\mathcal{C}(T_E)$, run T_{IE}, and feed the values into $\mathcal{C}(T_E)$. We simulate these three steps (split, compile, and feed) using existing mechanisms of the Pequin toolchain [66]. Pequin's front-end accepts programs written in a subset of C, extended with domain-specific constructs. In particular, Pequin allows the input C program to be annotated with assertions. Each assertion

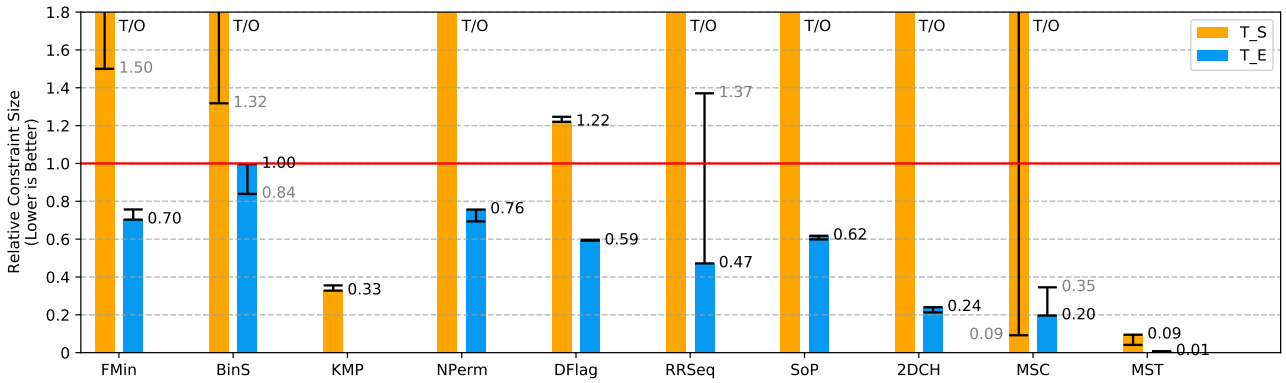
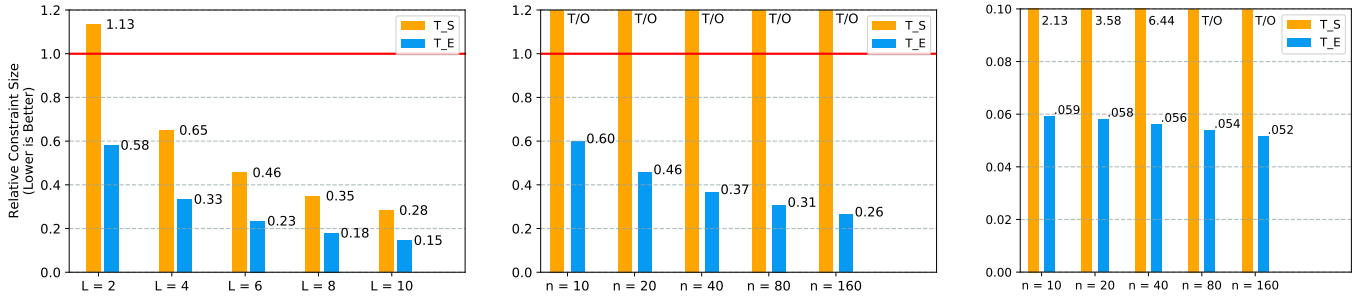


Figure 9: Relative $|\mathcal{C}|$ for T_S (orange) and T_E (blue) compared to the baseline T_I (red). The graph shows the problems where $\mathcal{C}(T_E)$ improves over $\mathcal{C}(T_I)$ by a constant factor (in the limit). The columns show the measurements obtained for the largest problem instances for which Pequin is able to compile T_I without timing out. In many benchmarks, the run time of T_S of the largest problem instance exceeds the timeout threshold. We use "T/O" to denote these cases. The error bars indicate the spread of the measurements obtained for smaller problem instances. For the Pattern Matching problem (KMP) we have $T_S = T_E$.



(a) Merge

(b) Binary Search (w/o RAM init.)

(c) 2D Convex Hull (marking nodes)

Figure 10: Relative $|\mathcal{C}|$ for T_S (orange) and T_E (blue) compared to the baseline T_I (red) for the problems where $\mathcal{C}(T_E)$ improves asymptotically over $\mathcal{C}(T_I)$ with increasing input size. The baseline T_I (red) is omitted for the variant of the 2D Convex Hull problem considered here because the improvement is so vast.

translates to R1CS constraints checking that the assertion holds. We use this construct to compile the `assume` statements in T_E. Another construct is `exo_compute`, a hook allowing the prover to execute another program that produces values for arbitrary nondeterministic inputs to the generated constraints. This feature enables the prover to run T_{IE} and supply the auxiliary inputs to $\mathcal{C}(T_E)$ when solving the constraints. While the coupling of T_E and T_{IE} via `exo_compute` is done manually, the process is mechanical and only takes a few minutes per example, suggesting that it can be automated.

We perform the split, compile, and feed steps for a select subset of our benchmarks. To test End-to-end Completeness of the approach, we apply the Pequin toolchain to the encoded benchmarks and successfully run it on a range of inputs. We note that the overhead of executing T_{IE} versus T_I is negligible compared to the rest of the pipeline (recall that Step 3 contributes negligibly to costs in the first place; §2). To test End-to-end Soundness, we also run Pequin with buggy versions of the T_{IE}. In these cases, the back-end correctly rejects the computation.

Constraint size reduction. Recall that our primary performance metric is $|\mathcal{C}|$ (§2). Our final experiment assesses Distiller against this metric. For all of our benchmarks, we express T_I, T_E, and T_S as input programs for Pequin. Then, with the exception of MST, we run Pequin's front-end on all three programs for a range of values for the loop unrolling bound that determines the maximal input size for each benchmark problem. As MST relies on a refined RAM construct that is not available in Pequin, we calculate the size of the constraints generated by all RAM operations by hand and run Pequin on the rest of the program. We then combine the result of the two parts to obtain the final constraint size. We enforce a timeout threshold of 240s per run, with the exception of MSC, where a 2000s threshold is chosen to enable computations on larger problem instance sizes that demonstrate the exponential behavior of T_S. For each successful run, we measure the size of the generated R1CS constraints and compute the relative sizes of $\mathcal{C}(T_S)$ and $\mathcal{C}(T_E)$ compared to $\mathcal{C}(T_I)$.

Figure 9 shows the results obtained for those benchmarks where our theoretical analysis yields an improvement of T_E over

T_I that converges to a constant factor with increasing problem instance size (Figure 5). The results closely match our analysis. We note that for the MSC problem (§5.4), the relative improvement between T_E and T_I increases with the problem instance size. The maximal MSC instance size for which the translation of T_I does not time out is $n = 20, m = 400$. This is still too small to observe the $17.5\times$ theoretical improvement that we predict for dense graphs. Conversely, for the MST problem T_I has a large constant overhead that causes the improvement achieved with T_E to be $3\times$ larger on small instances than the predicted $52.5\times$ improvement for large problem instances. Finally, for binary search (BinS) we observe that the cost of storing the input array A into RAM, which is linear in the size of the array, dwarfs the $\log(n)$ improvement achieved for a single invocation of the binary search (§5.2).

Figure 10 shows the results obtained for the three problems where our theoretical analysis predicts that T_E performs asymptotically better than T_I with increasing problem instance size. The experiment again confirms our predictions. In particular, for the merge problem discussed in §3, Fig. 10a shows that $|C(T_E)|/|C(T_I)|$ is approximately hyperbolic, which we expect because the predicted improvement for each point is $L\times$, where L is the number of merged arrays. Also, if we discount the RAM initialization cost for binary search, then we see the expected $\log(n)$ factor improvement (Fig. 10b).

7 Other related work

Probabilistic proofs. Section 2 gave an overview of probabilistic proof implementations, covering back-ends and front-ends; see also [75, 78, 81]. Unlike Distiller, none of the front-ends achieves all three requirements stated in the Introduction; in fact, none creates a framework for proving the correctness of widgets.

Distiller combines formal methods and probabilistic proofs. Very few works live at this intersection. Some notable exceptions are as follows. CirC [63] is a toolkit for building compilers to a family of constraint formalisms, including R1CS and SMT instances. Its architecture takes advantage of the rich SMT toolbox, allowing users to build powerful analyses and optimizations. The two works are complementary: one could compile a Distiller-verified widget in CirC, to get further reductions.

The Orbis Specification Language (OSL) [76] has a similar ideology to ours: replace a computation with its formal specification, and compile the latter to constraints, in the hope of gaining more concise constraints. However, as our examples (§5) make clear, the cost of a naive specification is often exorbitant. So one has to identify an "in-between" specification, and (a) relate it to the abstract specification, and (b) derive an implementation that knows how to satisfy the in-between specification or the original. Neither of these problems is addressed by OSL. The authors mention that they want to synthesize implementations from specifications. Though, for the rich specification language we consider (general transition systems), whether a specification even *has* an implementation is undecidable. Thus, to instantiate the ideology that OSL and we share, one needs human input (to write down T_E, and relate it to the specification and the implementation).

In an under-appreciated work, Fournet et al. [33] develop a compiler, based on CompCert [51], that formally connects the semantics of a higher-level program to the constraint formalism (specifically R1CS constraints). This work is complementary to Distiller—it provides the compiler correctness condition described in Section 2.

Leo [27] also has the goal of formally verified translations to constraints. Leo develops a compiler and uses the ACL2 [43] theorem prover to validate each stage of translation. However, this falls short of a verified compiler, as in Fournet et al. Moreover, the authors of Leo want to validate hand-crafted gadgets. It is not clear how to do this, since ACL2 cannot easily "reverse" R1CS instances to lift them to higher-level semantics. As a consequence, crucial pieces of verification are works in progress [27, §6.4]. By contrast, Distiller incorporates widgets soundly and completely, by treating them at the source code level and using refinement.

Refinement. The idea of developing a program from a specification in a step-wise refinement process goes back to early work by Dijkstra [29, 30] and Wirth [85]. The formal concept of refinement relations and mappings to relate the observable behaviors of transition systems was first explored in the 1980s [47, 48, 54]. It is a cornerstone of modern Formal Methods; applications include reasoning about concurrent and distributed systems, establishing program equivalence, and verifying security properties.

Abadi and Lamport [1] showed that refinement mappings yield a complete proof technique for establishing refinement. Though, in general, the technique requires the transition systems to be augmented with *history* variables (recording information about past states) and *prophecy* variables (predicting information about future states). Other related proof techniques for establishing refinement, for instance, based on (weak) simulation relations [57, 64, 77] are less suited for our purposes as they do not immediately provide a blueprint for computing satisfying assignments.

The notion of refinement considered in Distiller takes a monolithic view of transition systems, which makes it difficult to reason compositionally about subroutines. *Contextual refinement* [32] relates the observable behavior of subroutines subject to all possible client programs that may use them, thereby enabling compositional reasoning. For the relatively simple programming models supported by most existing probabilistic proof front-ends (no concurrency, object-oriented features, or higher-order functions), considering contextual refinement instead of *global refinement* does not add substantial complexity to the verification effort.

Distiller uses *mechanized proofs* (§6). Specifically, it uses a lightweight encoding of refinement proofs in the language of the deductive program verifier Viper [60]. The proofs are partially automated using SMT solvers. However, this is a choice. Nothing in our approach precludes or necessitates particular approaches to mechanization. In particular, there is a large body of work on refinement calculi that mechanize the correct step-wise refinement of programs and system models [2, 8, 56, 58, 59]. More recently, the many applications of proofs concerning products and couplings of two or more programs have fueled the development of relational program logics that provide frameworks for proof mechanization [9–11, 19, 34, 73, 87]. Several of these formal reasoning systems have been implemented in tools, including for instance TLA+ [49], Rodin [3], EasyCrypt [12], and ReLoC [34].

8 Discussion and conclusion

Distiller improves a key metric in implementations of probabilistic proofs, namely the number of arithmetic constraints required to encode the validity of the execution of a computation. The improvements typically range from small integer factors to orders of magnitude, depending on the computation.

Nevertheless, Distiller has limitations. First, we use Viper not on the actual code that the Pequin toolchain works over but rather on a representation of that code in Viper; this leaves a potential safety gap. Similarly, although we are encouraged by certified compiler work in this research area [33], we ourselves use Pequin, which is not verified; this is not fundamental but does mean that the built system has another potential gap. Speaking of our built system, it stops short of being automated end-to-end; coupling involves some manual code shuffling (§6). However, this step is mechanical, and we plan to automate it.

Despite these limitations, Distiller has taken a crucial step forward. It introduces, for the first time, a framework for widgets that are correct by construction. This framework radically expands the space of potential widgets, thereby allowing probabilistic proofs to do much more, by paying much less.

Acknowledgments

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A Details of refinement mapping: merging

This section provides details on showing that the **assume** statements of $\hat{T}_{IE}$ (Figure 4) of the merge computation express invariants of T_{IE}. This guarantees that the **assume** statements will always evaluate to true in T_I (Figure 1), and completes the refinement $T_{IE} \leq \hat{T}_{IE}$. See Section 4.2 for the full discussion.

We justify the addition of the **assume** statements by providing supporting invariants at relevant portions of T_{IE}. Figure 11 shows $\hat{T}_{IE}$ with the desired invariants explicitly added. There are four **assume** statements added to T_{IE} that require justification, on Lines 33, 35, 37, and 38. We omit the proof of the **assume** statement on Line 11 as it is easy to show.

The first **assume** (Line 33) states that the most recently added element `B[i]` is properly ordered with respect to its preceding element. To prove this, we establish the invariants on Lines 21 to 23 at the inner loop, and the invariant on Line 15 at the outer loop. These invariants imply that, at each iteration of the outer loop, `B[i]` is set to a value greater than `B[i-1]`, provided `found` is `true` after the inner loop terminates. To show the latter, first observe that `k` $=$ `L` holds after the inner loop terminates. Now, if `found` were `false` at this point, then Line 20 would imply

$$\sum_{n=0}^{L-1} \texttt{curr[}n\texttt{]} \geq \sum_{n=0}^{L-1} \texttt{A}_\texttt{n}\texttt{.len} \,.$$

However, the invariants on Line 14 together with the loop condition `i < B.len` of the outer loop and the equation on Line 11 imply

$$\sum_{n=0}^{L-1} \texttt{curr[}n\texttt{]} < \texttt{B.len} = \sum_{n=0}^{L-1} \texttt{A}_\texttt{n}\texttt{.len} \,.$$

This yields a contradiction. Hence, `found` must be `true` after the inner loop terminates.

To prove the second **assume** (Line 35), we can again use the invariant at Line 21 and the fact that `found` must hold after termination of the inner loop.

Similarly, the third `assume` (Line 37) follows from the invariants at Lines 21 and 22 at the inner loop, and 13 at the outer loop.

The final **assume** (Line 38) follows from the identity `running_min == A`$_{\texttt{kstar}}$`[curr[kstar]]`. This is sufficient because `k_i` is set to `kstar`, `j_i` is set to `curr[kstar]`, and `B[i]` is set to `running_min`.

The remaining invariants that are not used directly to prove the **assume** statements are needed to ensure that the invariants are inductive. In particular, Line 24 at the inner loop is needed to show that Line 15 is maintained by each iteration of the outer loop.

The invariants, thus, justify the addition of the **assume** statements in $\hat{T}_{IE}$, proving the refinement $T_{IE} \leq \hat{T}_{IE}$.

B Details of MSC Example

In this section, we expand on the discussion of the maximum strongly connected component problem in Section 5.4.

Given a directed graph $G = (V, E)$ with nodes V and edges $E \subseteq V \times V$, the problem is to partition V into the maximal strongly connected components $C_0, \ldots, C_{k-1}$ of G. We represent the C_i implicitly using an array `rank` that maps every node $v \in V$ to the index of its maximal strongly connected component. That is, we define for all $i \in [0, k)$

$$C_i = \{v \in V \mid \texttt{rank}[v] = i\} \quad .$$

Given this, the formal problem statement is to find k and `rank` such that the following three conditions hold:

1. For all $v \in V$, $0 \leq \texttt{rank}[v] < k$.
2. For all $i \in [0, k)$, there exists a cycle c_i in G that visits exactly the nodes in C_i.
3. For all $i \in [0, k)$ and all cycles c in G, if c visits some node in C_i then c visits only nodes in C_i.

Weakest specification. T_S encodes the above specification by nondeterministically choosing k, `rank`$[v]$ for each node $v \in V$, and the cycles c_i for each component $i \in [0, k)$. Condition 3, which enforces that the components C_i are maximal, is encoded by a

```
1   void merge (L,A0,..,AL-1,B) {
2   ℓ0 : int[L] curr = {0};
3       int len, running_min, kstar, found, k_i, j_i;
4       len = 0;
5   ℓ1 : for (int k = 0; k < L; k++)
6         invariant len == ∑_{n=0}^{k-1} An.len
7       {
8        len += Ak.len;
9       }
10      B.len = len;
11      assume B.len == ∑_{k=0}^{L-1} Ak.len;
12  ℓ2 : for (int i = 0; i < B.len; i++)
13        invariant ∀ j :: 0 ≤ j < L ⇒ 0 ≤ curr[j]
14        invariant ∑_{n=0}^{L-1} curr[n] == i
15        invariant i == 0 ∨ ∀ j :: 0 ≤ j < L ∧ curr[j] < Aj.len ⇒ B[i-1] < Aj[curr[j]]
16      {
17        found = false;
18  ℓ3 :   for (int k = 0; k < L; k++)
19          invariant !found ⇒ ∀ j :: 0 ≤ j < k ⇒ curr[j] ≥ Aj.len
20          invariant !found ⇒ ∑_{n=0}^{k-1} curr[n] ≥ ∑_{n=0}^{k-1} An.len
21          invariant found ⇒ 0 ≤ kstar < L
22          invariant found ⇒ curr[kstar] < Akstar.len
23          invariant found ⇒ running_min == Akstar[curr[kstar]]
24          invariant found ⇒ ∀ j :: 0 ≤ j < k ∧ j ≠ kstar ∧ curr[j] < Aj.len ⇒ running_min < Aj[curr[j]]
25        {
26          if (curr[k] < Ak.len && (!found || Ak[curr[k]] < running_min)) {
27            running_min = Ak[curr[k]];
28            kstar = k;
29            found = true;
30          }
31        }
32        B[i] = running_min;
33        assume i == 0 || B[i-1] < B[i];
34        k_i = kstar;
35        assume 0 <= k_i && k_i < L;
36        j_i = curr[kstar];
37        assume 0 <= j_i && j_i < Ak_i.len
38        assume B[i] == Ak_i[j_i];
39        curr[kstar]++;
40      }
41  ℓ4: return;
42  }
```

Figure 11: Pseudocode for the transition system $\hat{T}_{IE}$ with invariants.

nested loop, where the inner loop enumerates all cycles in G that visit exactly one node twice. As there are $O(|V|!)$ possible cycles that need to be considered, the size of the constraints generated from T_S grows exponentially in $|V|$.

Implementation We use Dijkstra's MSC algorithm [30, Chapter 25] as our T_I. The algorithm is similar to Tarjan's algorithm [74] but lends itself more directly to an efficient translation into constraints.

Dijkstra observed that a set of strongly connected components $C_0, \ldots, C_{k-1}$ is maximal iff it can be ordered so that all edges leaving a C_i target only nodes in components preceding C_i. That is, consider the graph $G_C = (V_C, E_C)$ such that $V_C = \{C_0, \ldots, C_{k-1}\}$ and $(C_i, C_j) \in E_C$ iff $i \neq j$ and $(v,w) \in E$ for some $v \in C_i$ and $w \in C_j$. Then the above ordering of the C_i is a (reverse) topological sort of G_C. The existence of this topological sort implies that G_C is acyclic and therefore the C_i cannot be joined into larger components. Given Dijkstra's observation, one can replace Condition 3 in T_S with the following:

3*. For all $(v,w) \in E$, $\mathsf{rank}[w] \leq \mathsf{rank}[v]$.

The algorithm computes a `rank` that satisfies this revised condition. For pedagogy we discuss the abstract version of the algorithm shown in Figure 12, which assumes mathematical sets as an built-in type. Dijkstra's concrete version of the algorithm implements the relevant set operations using auxiliary variables to achieve a running time that is linear in the size of G.

The code highlighted in blue can be ignored for now. The algorithm iterates over V and E using two working sets `VW` and `EW` to keep track of the nodes and edges that still need to be processed. The stack `cc` is used to keep track of a chain of disjoint strongly connected components that are currently being traversed.

The algorithm maintains the invariant that each component is connected via an edge in G to its next component higher up on the stack. Moreover, the algorithm maintains that the set `VC` is the union of all the components in `cc`. A node leaves `VC` (and, hence, `cc`) once its maximal strongly connected component has been identified and its rank has been assigned. The variable `k` keeps track of the number of maximal components that have been identified so far.

Whenever the stack `cc` becomes empty, the outermost loop nondeterministically chooses an unprocessed node `v` from `VW` and adds it as a new singleton component to `cc` (lines 14–17). While the stack is nonempty, the algorithm pops the topmost component `C` from `cc`. Then it attempts to nondeterministically choose an unprocessed edge `(u,w)` in `EW` such that its source node `u` is in `C` (lines 21–22). If no such edge exists, then `C` must be maximal and its nodes are removed from `VC` (lines 40–43). Otherwise, if the target node `w` of the chosen edge is in $\mathtt{VC} \setminus \mathtt{C}$, then by the invariant of `cc` (that every component in VC is pointing to the next component higher up on the stack), there exists a cycle connecting all the components in `cc` starting with the component that contains `w`, all the way to `C`. The loop on lines 26–32 thus compacts these components to a new component `C`. If on the other hand the target node `w` is an unprocessed node, then the current component `C` is added back to the stack and then `C` is updated to the new topmost component consisting of `w` (lines 33–38). In all other cases, `w` is either already in `C` or is already in an identified

```
1  datatype comp =
2    Node(node)
3  | Rotate(node, comp, node)
4  | Concat(comp, comp)
5
6  int msc(V, E, rank) {
7    set<edge> EW = E;
8    set<node> VW = V;
9    set<node> VC = ∅
10   stack<set<node> * comp * node * node> cc = empty;
11   comp[] cycle;
12   int k = 0;
13   while (VW != ∅) {
14     int v = choose(VW);
15     VC = {v};
16     VW = VW \ {v};
17     push(cc, ⟨{v}, Node(v), v, v⟩);
18     do {
19       set<node> C, comp c, node i, _ = pop(cc);
20       while ({(u,w) ∈ EW | u ∈ C} != ∅) {
21         edge (u,w) = choose({(u,w) ∈ EW | u ∈ C});
22         EW = EW \ {(u,w)};
23         if (w ∈ VC \ C) {
24           // compact the chain
25           c = Rotate(i, c, u);
26           do {
27             set<node> C1, comp c1, node i1, node o1 =
28               pop(cc);
29             C = C ∪ C1;
30             i = w ∈ C1 ? w : i1;
31             c = Concat(Rotate(i, c1, o1), c);
32           } while (w ∉ C)
33         } else if (w ∈ VW) {
34           VW = VW \ {w};
35           VC = VC ∪ {w};
36           push(cc, ⟨C,i,c,u⟩);
37           C = {w}; c = Node(w); i = w;
38         }
39       }
40       for (w ∈ C) rank[w] = k;
41       cycle[k] = c;
42       k = k + 1;
43       VC = VC \ C;
44     } while (VC != ∅);
45   }
46   assert isMSC(V, E, k, rank, cycle);
47   return k;
48 }
```

Figure 12: Pseudocode for implementation T_I of MSC. The code in blue is the augmentation needed for proving $T_I \leq T_E$.

```
1  int msc_efficient(V, E, rank) {
2    comp[] cycle;
3    int k;
4    havoc k;
5    for (v ∈ V) havoc rank[v];
6    for (i ∈ [0,k)) havoc cycle[k];
7    assume isMSC(V, E, k, rank, cycle);
8    return k;
9  }
```

Figure 13: Pseudocode for T_E of MSC.

maximal strongly connected component. In these cases, the edge can be discarded because it is guaranteed to satisfy Condition 3* based on the partial reverse topological sort that has already been computed.

To reduce the size of the generated constraints for T_I, we flatten the nested loops into a single loop that is executed $|V|+|E|$ times. In each iteration, T_I performs up to 13 `LOAD` and 8 `STORE` operations. These operations dominate the size of the generated constraints. In particular, the `STORE` operations are embedded under 4 layers of conditionals making them 16× more expensive than a `LOAD`.

Efficient specification. The idea behind T_E is to directly check the three conditions 1, 2, and 3*. Its pseudocode is shown in Figure 13. T_E nondeterministically assigns `k`, `rank`, and an auxiliary array `cycle`. The latter stores for each alleged connected component C_i, a cycle c_i that visits the nodes in C_i. The three conditions are assumed on Line 7 using the function `isMSC`. Before we discuss `isMSC` in detail, we argue $T_I \leq T_E$, which boils down to justifying the `assume` at Line 7 of Figure 13 by showing that the `assert` at Line 46 of Fig. 12 always holds. The crux of the proof is to establish conditions 1, 2, and 3* as an invariant before the `assert`.

It is easy to see that T_I ensures Condition 1. To see that it ensures 3*, first note that when a component `C` is assigned its rank it is the topmost component in the chain `cc`. Moreover, its assigned rank `k` is larger than the rank of all previously assigned components. Hence, we must show that all edges leaving `C` point only to these *older* components. To this end, observe that the algorithm maintains the invariant that for the topmost component `C` in the chain, all edges starting in `C` that have already been processed either remain in `C` or target an older component. Since `C` is assigned its `rank` only after all its edges have been processed, it follows that Condition 3* is satisfied.

For showing Condition 2, T_E nondeterministically chooses for each alleged connected component C_i, a cycle c_i that visits the nodes in C_i. Hence, to show $T_I \leq T_E$, we need to augment T_I with auxiliary code to compute the cycles stored in `cycle`. This is the code highlighted in blue in Figure 12.

The augmentation maintains for each connected component C in `cc`, an associated cycle c that visits all nodes in C. When C is assigned its rank, c is copied to `cycle` (Line 41). To maintain the invariant that c visits all nodes in C, the code must also construct a new cycle each time the algorithm computes a new component by compacting a cyclic chain in `cc`. Therefore, the augmentation additionally maintains for each component C, the target node i in C for the incoming edge from its predecessor in `cc`, and the source node o in C for the outgoing edge to its successor in `cc` (unless C is the topmost component, in which case o can be chosen arbitrarily).

Instead of representing each cycle c explicitly as a path in G, we represent it symbolically using an auxiliary tree-like datatype `comp`. To motivate this symbolic representation, consider the construction of an explicit cycle for a new compacted component obtained by the loop on lines 26–32. To this end, we need to construct a path `c` from `w` to `u` that visits all nodes in the new component. Since there exists an edge back from `u` to `w`, we obtain the desired cycle. The loop must therefore maintain the invariant that `c` is a path that visits all nodes in the components that have already been compacted into `C`. Moreover, this path must end in `u` and start in `i`. The latter is needed so that we can concatenate this path with another path that visits all nodes in the component `C1` that precedes `C` in `cc` and that will be compacted next. The component `C1` has an associated cycle `c1` and is connected to its neighbors via `i1` and `o1`. First, we *rotate* `c1` to obtain a path that starts and ends in the node `i1`, thus visiting all nodes in `C1` at least once. Next, we extend this path with the segment of `c1` that goes from `i1` to `o1`. Finally, we concatenate the resulting path with `c` using the edge $(\texttt{o1},\texttt{i})$ that must exist by the invariant of `cc`. The final path goes from `i1` to `u` and visits all nodes in $\texttt{C1}\cup\texttt{C}$ (at which point `i1` becomes the new `i` for $\texttt{C1}\cup\texttt{C}$), reestablishing the loop invariant.

The issue with this construction is that the size of the obtained cycle `c` can be exponential in the size of the compacted components. This blow-up would affect both the running time of the augmented T_I as well as the size of $\mathcal{C}(T_E)$. By representing the cycles using the type `comp`, we avoid this blow-up. Intuitively, a `comp` value `c` is a program that provides instructions for constructing a cycle that visits all the nodes appearing in `c`. The size of this program is linear in the number of visited nodes. Moreover, checking whether a given `comp` program `c` indeed constructs a cycle that contains all the nodes appearing in `c` can be done in linear time. This checker is our missing ingredient for T_E to guarantee Condition 2.

The `comp` programs have a tree-like structure where each vertex p is labeled with an instruction to construct a path from the paths computed by p's children. There are three kinds of instructions, corresponding to the three different operations involved in the construction of cycles during compaction. The instruction `Node(`n`)` for a node n constructs a (trivially cyclic) path that consists only of the node n. The instruction `Rotate(`l,c,r`)` constructs a new path from l to r that visits all nodes in c, provided c constructs a cycle that contains l and r. Finally, `Concat(`c_1,c_2`)` concatenates the two paths constructed by c_1 and c_2, provided there exists an edge in G from the end point of c_1 to the start point of c_2.

The implementation of `isMSC` is shown in Figure 14. It uses the function `isCycle` to check for each $i \in [0,k)$ whether `cycle[`i`]` constructs a cycle in G that visits all nodes appearing in `cycle[`i`]`. Moreover, it ensures that the `rank` of each of these nodes is indeed i. The auxiliary array `seen` is used to ensure that each node in V occurs in some `cycle[`i`]`.

```
1  bool isMSC(V, E, k, rank, cycle) {
2    bool b = true;
3    bool[V] seen = {false};
4
5    for (i ∈ [0,k))
6      b = b && isCycle(V, E, rank, cycle[i], seen, i);
7    for (v ∈ V) b = b && seen[v];
8    for ((u,v) ∈ E) b = b && rank[v] <= rank[u];
9
10   return b;
11 }
12
13 bool isCycle(V, E, i, rank, c, seen) {
14   match c {
15     case Node(n) =>
16       seen[n] = true;
17       return n ∈ V && rank[n] == i
18     case Rotate(l,c1,r) =>
19       return !seen[l] && !seen[r]
20         && isCycle(V, E, i, rank, c1, seen)
21         && seen[l] && seen[r]
22     case Concat(c1, c2) =>
23       return isPath(V, E, i, rank, c1, seen)
24         && isPath(V, E, i, rank, c2, seen)
25         && (right(c1),left(c2)) ∈ E
26         && (right(c2),left(c1)) ∈ E
27   }
28 }
29
30 bool isPath(V, E, i, rank, c, seen) {
31   match c {
32     case Node(n) =>
33       seen[n] = true;
34       return n ∈ V && rank[n] == i
35     case Rotate(l,c1,r) =>
36       return !seen[l] && !seen[r]
37         && isCycle(V, E, i, rank, c1, seen)
38         && seen[l] && seen[r]
39     case Concat(c1, c2) =>
40       return isPath(V, E, i, rank, c1, seen)
41         && isPath(V, E, i, rank, c2, seen)
42         && (right(c1),left(c2)) ∈ E
43   }
44 }
45
46 node left(c) {
47   match c {
48     case Node(n) => return n
49     case Rotate(l, _, _) => return l
50     case Concat(c1, _) => return left(c1)
51   }
52 }
53
54 node right(c) {
55   match c {
56     case Node(n) => return n
57     case Rotate(_, _, r) => return r
58     case Concat(_, c1) => return right(c1)
59   }
60 }
```

Figure 14: Pseudocode for checker used by T_E of MSC.

We briefly discuss the $T_E \leq T_S$ refinement proof. The proof boils down to establishing conditions 1, 2, and 3 as an invariant after the `assume` at Line 7 of Figure 13. First, note that if `k` is negative, then the loop on Line 5 of Figure 14 is skipped. Hence, the next loop on Line 7 will set `b` to `false` because at least one node will not be seen, assuming V is nonempty. Therefore, the checker implies Condition 1. Next, to prove that $\texttt{isCycle}(V, E, \texttt{rank}, \texttt{cycle}[i], \texttt{seen}, i)$ implies Condition 2 for component i, we construct an *interpreter* for `comp` programs. Given a `comp` program `c` that satisfies `isCycle`, the interpreter executes `c` to construct the cyclic path in G encoded by `c`. Finally, the loop on Line 8 directly checks Condition 3*. As we have argued earlier, this implies Condition 3.

The combined size of all `comp` programs is linear in $|V|$ and so is checking their correctness (lines 5 and 6 in Figure 14). As a result, for dense graphs ($|E| \approx |V|^2$), the cost to enforce Condition 2 is insignificant. While checking Condition 3* is still linear in $|E|$, it involves only two `LOAD`s per edge and no `STORE`. A detailed cost analysis yields an expected reduction in total constraint size for dense graphs by a factor of 17.5 for sufficiently large $|E|$. For shallow graphs ($|E| \approx |V|$), we still obtain a reduction by a factor of two.

C Details of MST Memory

Here we delve into the details of our technique in Section 5.5 as well as giving background on Pequin's implementation of Buffet's [79] RAM technique.

The complexity of our amortized T_I is dominated by $261 \cdot |E| \cdot \alpha(|V|)$ disjoint set related RAM operations while our T_E is dominated by $2 \cdot |E| \cdot \alpha(|V|)$ specialized disjoint set memory operations, each of which is $2.5\times$ as expensive as a Buffet RAM operation.

In Buffet's RAM model, operations are 4-tuples `(time, op, val, addr)`, referring to the time, operation, value, and address respectively. The consistency of a transcript of RAM operations is checked by sorting these tuples by `addr` with ties decided by `pc`, and then checking the pairwise consistency of `LOAD` and `STORE` operations at the same address. Sorting tuples is done via a permutation network that contains $O(s \cdot n \log n)$ constraints in general where n is the number of inputs and s is the size of the tuple.

For our refined approach, we combine the `LOAD`s and `STORE`s into a single `UPDATE` operation, and we define a `FIND-SET` operation that completely eliminates the nested while loops while maintaining amortized complexity. These operations are 5-tuples, `(time, op, old parent new parent, node)`, and the transcript is divided into external and internal data-structure operations. The external operations are those called by the main algorithm and are generated when the program is unrolled during constraint compilation. To achieve amortized complexity with minimal overhead, we introduce a fixed size buffer of fully nondeterministic internal `UPDATE` operations which ensure the consistency of the external `FIND-SET` operations. Every `FIND-SET` call is certified by a series of `UPDATE` operations which result in a fixed-point where `old parent`, `new parent`, and `node` are the same (the root).

The ordering of these operations on the initial transcript does not matter because the transcript is sorted before any consistency checks are performed. To check memory consistency, we follow the techniques for standard RAM, and we also run a second sort over the transcript by `time` to check the fixed-point property of `FIND-SET`. Our 5-tuples are sorted first by `time`, with ties decided by placing `FIND-SET` first and the requisite number of `UPDATE` operations in their proper order. The overhead of doing a second reordering of the transcript ($2\times$) combined with the overhead of handling 5-tuples instead of 4-tuples ($1.25\times$) results in $2.5\times$ more expensive memory operations.

We make one additional major change to memory by replacing the `time` with `weight` since there is a direct correspondence between the two. Then, instead of sorting the edges by weight, we sort them into two groups, those in the MST and those not contained in the MST. This alternative ordering allows us to split the main loop into two loops, and we can remove the nested conditional from one of the loops. Instead of the expensive operations in the conditional block being unrolled into constraints $|E|$ times, they only appear $|V|-1$ times (from the unrolling of the first loop). Calling any memory operations out of time order in the circuit (in both Buffet's RAM and our disjoint set memory model) does not matter because they are later reordered for consistency checks.

Replacing time with weight and reordering iterations allows us to remove the overhead of nested conditionals in loops, and introducing a buffer for internal operations allows for cheaper amortized memory operations. Together these improvements result in a substantial $52.2\times$ reduction in the cost of memory operations on large input sizes with additional smaller order improvements, particularly when $|E| \gg |V|$.

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