Take your MEDS: Digital Signatures from Matrix Code Equivalence

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Abstract. In this paper, we show how to use the Matrix Code Equivalence (MCE) problem as a new basis to construct signature schemes. This extends previous work on using isomorphism problems for signature schemes, a trend that has recently emerged in post-quantum cryptography. Our new formulation leverages a more general problem and allows for smaller data sizes, achieving competitive performance and great flexibility. Using MCE, we construct a zero-knowledge protocol which we turn into a signature scheme named Matrix Equivalence Digital Signature (MEDS). We provide an initial choice of parameters for MEDS, tailored to NIST's Category 1 security level, yielding public keys as small as 2.7 kB and signatures ranging from 18.8 kB to just around 10 kB, along with a reference implementation in C.

Keywords: group action · signature scheme · code-based cryptography · post-quantum cryptography· matrix codes

1 Introduction

Post-Quantum Cryptography (PQC) comprises all the primitives that are believed to be resistant against attackers equipped with a considerable quantum computing power. Several such schemes have been around for a long time [43, 39], some being in fact almost as old as RSA [35]; however, the area itself was not formalized as a whole until the early 2000s, for instance with the first edition of the PQCrypto conference [38]. The area has seen a dramatic increase in importance and volume of research over the past few years, partially thanks to NIST's interest and the launch of the PQC Standardization process in 2017 [37]. After 4 years and 3 rounds of evaluation, the process has crystallized certain mathematical tools as standard building blocks (e.g. lattices, linear codes, multivariate equations, isogenies etc.). Some algorithms [34, 46, 41, 31] have now been selected for standardization, with an additional one or two to be selected among

a restricted set of alternates [4, 3, 1] after another round of evaluation. While having a range of candidates ready for standardization may seem satisfactory, research is still active in designing PQC primitives. In particular, NIST has expressed the desire for a greater diversity among the hardness assumptions behind signature schemes, and announced a partial re-opening of the standardization process for precisely the purpose of collecting non-lattice-based protocols.

Cryptographic group actions are a popular and powerful instrument for constructing secure and efficient cryptographic protocols. The most well-known is, without a doubt, the action of finite groups on the integers modulo a prime, or the set of points on an elliptic curve, which give rise to the *Discrete Logarithm Problem (DLP)*, i.e. the backbone of public-key cryptography. Recently, proposals for post-quantum cryptographic group actions started to emerge, based on the tools identified above: for instance, isogenies [18], linear codes [14], trilinear forms [47] and even lattices [30]. All of these group actions provide very promising solutions for cryptographic schemes, for example signatures [22, 8, 47], ring signatures [12, 9] and many others; at the same time, they are very different in nature, with unique positive and negative aspects.

Our Contribution. In this work, we formalize a new cryptographic group action based on the notion of *Matrix Code Equivalence*. This is similar in nature to the *code equivalence* notion at the basis of LESS, and in fact belongs to a larger class of isomorphism problems that include, for example, the lattice isomorphism problem, and the well-known isomorphism of polynomials [39]. The hardness of the MCE problem was studied in [21, 44], from which it is possible to conclude that this is a suitable problem for post-quantum cryptography. Indeed, we show that it is possible to use MCE to build a zero-knowledge protocol, and hence a signature scheme, which we name *Matrix Equivalence Digital Signature*, or simply MEDS. We provide an initial parameter choice, together with several computational optimizations, resulting in a scheme with great flexibility and very competitive data sizes. To give an idea about our performance potential, we include a reference implementation for the full scheme. Furthermore, we show that this group action allows for the construction of (linkable) ring signatures, with performance results that improve on the existing state of the art [9].

2 Preliminaries

Let $\mathbb{F}_q$ be the finite field of q elements. $\mathrm{GL}_n(q)$ and $\mathrm{AGL}_n(q)$ denote respectively the general linear group and the general affine group of degree n over $\mathbb{F}_q$. We use bold letters to denote vectors $\mathbf{a}, \mathbf{c}, \mathbf{x}, \ldots$, and matrices $\mathbf{A}, \mathbf{B}, \ldots$. The entries of a vector $\mathbf{a}$ are denoted by a_i, and we write $\mathbf{a} = (a_1, \ldots, a_n)$ for a (row) vector of dimension n over some field. Similarly, the entries of a matrix $\mathbf{A}$ are denoted by a_{ij}. Random sampling from a set S is denoted by $a \xleftarrow{\$} S$. For two matrices $\mathbf{A}$ and $\mathbf{B}$, we denote the Kronecker product by $\mathbf{A} \otimes \mathbf{B}$. Finally, we denote the set of all $m \times n$ matrices over $\mathbb{F}_q$ by $\mathcal{M}_{m,n}(\mathbb{F}_q)$.

2.1 Cryptographic Group Actions

Definition 1. Let X be a set and $(G, \cdot)$ be a group. A group action is a mapping

$$\begin{aligned} \star : G \times X &\to X \\ (g, x) &\mapsto g \star x \end{aligned}$$

such that, for all $x \in X$ and $g_1, g_2 \in G$, it holds that $g_2 \star (g_1 \star x) = (g_2 \cdot g_1) \star x$.

A group action can have a number of mathematically desirable properties. For example, we say that a group action is:

- *Commutative*: for any $g_1, g_2 \in G$, we have $g_2 \star (g_1 \star x) = g_1 \star (g_2 \star x)$.
- *Transitive*: given $x_1, x_2 \in X$, there is some $g \in G$ such that $g \star x_1 = x_2$.
- *Free*: if $g \star x = x$, then g is the identity.

In particular, a *cryptographic* group action is a group action with some additional properties that are useful for cryptographic applications. To begin with, there are some desirable properties of computational nature. Namely, the following procedures should be efficient:

- *Evaluation*: given x and g, compute $g \star x$.
- *Sampling*: sample uniformly at random from G.
- *Membership testing*: verify that $x \in X$.

Finally, cryptographic group actions should come with security guarantees; for instance, the *vectorization problem* should be hard:

Problem 1 (Group Action Vectorization).
Given: The pair $x_1, x_2 \in X$.
Goal: Find, if any, $g \in G$ such that $g \star x_1 = x_2$.

Early constructions using this paradigm are based on the action of finite groups of prime order, for which the vectorization problem is the discrete logarithm problem. Lately, multiple isogeny-based constructions have appeared: see, for instance, the work of Couveignes in [20] and later by Rostovtsev and Stolbunov [45]. A general framework based on group actions was explored in more detail by [2], allowing for the design of several primitives. The holy grail are those cryptographic group actions that possess both the mathematical and cryptographic properties listed above. Currently, CSIDH [18] is the only post-quantum commutative cryptographic group action, although there is an ongoing debate about the efficiency and quantum hardness of its vectorization problem [15]. In Section 3, we introduce the group action that is relevant to our work.

2.2 Protocols

We give here an explicit characterization of the protocols we will build. The corresponding security definitions are presented only in an informal manner; formal definitions will be included in the full version of this work.

Definition 2. A *Sigma protocol* is a three-pass interactive protocol between two parties: a prover $\mathsf{P} = (\mathsf{P}_1, \mathsf{P}_2)$ and a verifier $\mathsf{V} = (\mathsf{V}_1, \mathsf{V}_2)$. The protocol is composed of the following procedures:

I. Keygen: on input some public data (including system parameters), output a public key pk (the instance) and the corresponding secret key sk (the witness). Give sk to the prover; pk is distributed publicly and is available to all parties. For simplicity, we assume that the public data is available as input in all the remaining procedures.

II. Commit: on input the public key pk, P_1 outputs a public commitment cmt and sends it to the verifier.

III. Challenge: on input the public key pk and the commitment cmt, V_1 samples uniformly at random a challenge ch from the challenge space C and sends it to the prover.

IV. Response: on input the secret key sk, the public key pk, the commitment cmt and the challenge ch, P_2 outputs a response rsp and sends it to the verifier.

V. Verify: on input a public key pk, the commitment cmt, the challenge ch, and the response rsp, V_2 outputs either 1 (accept) if the *transcript* $(\mathsf{cmt}, \mathsf{ch}, \mathsf{rsp})$ is valid, or 0 (reject) otherwise.

A Sigma protocol is usually required to satisfy the following properties. First, if the statement is true, an honest prover is always able to convince an honest verifier. This property is called *Completeness*. Secondly, a dishonest prover cannot convince an honest verifier other than with a small probability. This is captured by the *Soundness* property, which also bounds such probability, usually known as *soundness error* or, informally, *cheating probability*. Finally, the protocol has to be *Zero-Knowledge*, i.e. anyone observing the transcript (including the verifier) learns nothing other than the fact that the statement is true.

Definition 3. A *Digital Signature scheme* is a protocol between 2 parties: a signer S and a verifier V. The protocol is composed of the following procedures:

I. Keygen: on input the public data (including system parameters), output a secret signing key sk for S and the corresponding public verification key pk.

II. Sign: on input a secret key sk and a message msg, output a signature σ.

III. Verify: on input a public key pk, a message msg and a signature σ, V outputs either 1 (accept) if the signature is valid, or 0 (reject) otherwise.

Correctness means that an honest signer is always able to get verified. The usual desired security notion for signature schemes is *Unforgeability*, which guarantees that it is computationally infeasible to forge a valid signature without knowing the secret signing key.

Definition 4. A *Ring Signature scheme* is a protocol between $r + 1$ parties: potential signers $\mathsf{S}_1, \ldots, \mathsf{S}_r$ (the *ring*) and a verifier V. The protocol is composed of the following procedures:

I. Keygen: on input the public data (including system parameters), output a secret key sk_j for each signer S_j, and the corresponding public key pk_j.
II. Sign: on input a set of public keys $\{\mathsf{pk}_1, \ldots, \mathsf{pk}_r\}$, a secret key sk_{j^*} and a message msg, output a signature σ.
III. Verify: on input a set of public keys $\{\mathsf{pk}_1, \ldots, \mathsf{pk}_r\}$, a message msg and a signature σ, V outputs either 1 (accept) if the signature is valid, or 0 (reject) otherwise.

A crucial feature of a ring signature scheme is that anyone in the ring can produce a valid signature (on behalf on the entire ring), and the verifier can check validity, but cannot learn the identity of the signer. This is achieved by requiring that the scheme satisfies the *Anonymity* property. This captures the idea of protecting the identity of the signer, i.e. it should be impossible for a verifier to tell which secret key was used to sign. Then, *Unforgeability* corresponds to the familiar security notion for signature schemes presented above, extended to include all ring participants. In other words, it should be infeasible to forge a valid signature without knowing at least one of the secret keys in the ring. In addition, *Correctness* is also required, as before.

Linkable ring signature schemes possess additional security features. These protocols are composed of the same three procedures described in Definition 4, plus a fourth one, given below:

IV. Link: on input two signatures σ and σ', output either 1 if the signatures were produced with the same secret key, or 0 otherwise.

A linkable ring signature scheme is required to satisfy the following security properties. *Linkability* asks that it is computationally infeasible for an adversary to produce more than r unlinked valid signatures, even if some or all of the r public keys are malformed. *Linkable Anonymity* guarantees that, even if an adversary is able to obtain multiple signatures from the same signer, they are still unable to determine which secret key was used. *Non-Frameability* protects the user's identity by requiring that it is computationally infeasible for an adversary to produce a valid signature linked to one produced by an honest party.

Remark 1. Note that the linkability property is usually formulated in an alternative way, that is, if more than r valid signatures are produced, then the Link algorithm will output 1 on at least two of them. It is then easy to see, as also pointed out in [12, Remark 2.4], that unforgeability is obtained directly by satisfying linkability and non-frameability.

Finally, we recall the definition of *commitment scheme*, which is a tool necessary for our particular construction of ring signatures. This is a non-interactive function $\mathsf{Com} : \{0,1\}^\lambda \times \{0,1\}^* \to \{0,1\}^{2\lambda}$ mapping a message string of arbitrary length to a commitment string of 2λ bits. The first λ bits of input, chosen uniformly at random, guarantee that the input message is hidden in a very strong sense, as captured in the next definition.

Definition 5. Given an adversary A, we define the two following quantities:

$$\mathsf{Adv}^{\mathsf{Bind}}(\mathsf{A}) = \mathsf{Pr}\Big[\mathsf{Com}(\mathsf{r},\mathbf{x}) = \mathsf{Com}(\mathsf{r}',\mathbf{x}') \mid (\mathsf{r},\mathbf{x},\mathsf{r}',\mathbf{x}') \leftarrow \mathsf{A}(1^\lambda)\Big];$$

$$\mathsf{Adv}^{\mathsf{Hide}}(\mathsf{A},\mathbf{x},\mathbf{x}') = \Big|\underset{\mathsf{r}\,\leftarrow\,\{0,1\}^\lambda}{\mathsf{Pr}}\Big[\mathsf{A}(\mathsf{Com}(\mathsf{r},\mathbf{x})) = 1\Big] - \underset{\mathsf{r}\,\leftarrow\,\{0,1\}^\lambda}{\mathsf{Pr}}\Big[\mathsf{A}(\mathsf{Com}(\mathsf{r},\mathbf{x}')) = 1\Big]\Big|.$$

We say that Com is *computationally binding* if, for all polynomial-time adversaries A, the quantity $\mathsf{Adv}^{\mathsf{Bind}}(\mathsf{A})$ is negligible in λ. We say that Com is *computationally hiding* if, for all polynomial-time adversaries A and every pair $(\mathbf{x},\mathbf{x}')$, the quantity $\mathsf{Adv}^{\mathsf{Hide}}(\mathsf{A})$ is negligible in λ.

3 The Matrix Code Equivalence Problem

A $[m \times n, k]$ *matrix code* is a subspace $\mathcal{C}$ of $\mathcal{M}_{m,n}(\mathbb{F}_q)$. These objects are usually measured with the rank metric, where the *distance* between two matrices $\mathbf{A},\mathbf{B} \in \mathcal{M}_{m,n}(\mathbb{F}_q)$ is defined as $d(\mathbf{A},\mathbf{B}) = \mathrm{Rank}(\mathbf{A}-\mathbf{B})$. We denote the basis of the subspace by $\langle \mathbf{C_1},\ldots,\mathbf{C_k}\rangle$, where the $\mathbf{C_i}$'s are linearly independent elements of $\mathcal{M}_{m,n}(\mathbb{F}_q)$. Due to symmetry, without loss of generality, in the rest of the text we will assume $m \leqslant n$.

For a matrix $\mathbf{A} \in \mathcal{M}_{m,n}(\mathbb{F}_q)$, let vec be a mapping that sends a matrix $\mathbf{A}$ to the vector $\mathsf{vec}(\mathbf{A}) \in \mathbb{F}_q^{mn}$ obtained by 'flattening' $\mathbf{A}$, i.e.:

$$\mathsf{vec}:\ \mathbf{A} = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \ldots & a_{m,n} \end{pmatrix} \mapsto \mathsf{vec}(\mathbf{A}) = (a_{1,1},\ldots,a_{1,n},\ldots,a_{m,1},\ldots,a_{m,n}).$$

The inverse operation is denoted by mat, i.e. $\mathsf{mat}(\mathsf{vec}(\mathbf{A})) = \mathbf{A}$. Using the map vec, an $[m \times n, k]$ matrix code can be thought of as an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{mn}$, and thus we can represent it with a generator matrix $\mathbf{G} \in \mathbb{F}_q^{k \times mn}$, in a manner similar to the common representation for linear codes. Indeed, if $\mathcal{C}$ is an $[m \times n, k]$ matrix code over $\mathbb{F}_q$, we denote by $\mathsf{vec}(\mathcal{C})$ the vectorization of $\mathcal{C}$ i.e.:

$$\mathsf{vec}(\mathcal{C}) := \{\mathsf{vec}(\mathbf{A}) \colon \mathbf{A} \in \mathcal{C}\}.$$

In this case, $\mathsf{vec}(\mathcal{C})$ is a k-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{mn}$.

Definition 6. Let $\mathcal{C}$ and $\mathcal{D}$ be two $[m \times n, k]$ matrix codes over $\mathbb{F}_q$. We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exist two matrices $\mathbf{A} \in \mathrm{GL}_m(q)$ and $\mathbf{B} \in \mathrm{GL}_n(q)$ such that $\mathcal{D} = \mathbf{A}\mathcal{C}\mathbf{B}$, i.e. for all $\mathbf{C} \in \mathcal{C}$, $\mathbf{ACB} \in \mathcal{D}$.

The equivalence between two matrix codes can be expressed using the Kronecker product of $\mathbf{A}^\top$ and $\mathbf{B}$, which we denote by $\mathbf{A}^\top \otimes \mathbf{B}$.

Lemma 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two $[m \times n, k]$ matrix codes over $\mathbb{F}_q$. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are equivalent with $\mathcal{D} = \mathbf{A}\mathcal{C}\mathbf{B}$, with $\mathbf{A} \in \mathrm{GL}_m(q)$ and $\mathbf{B} \in \mathrm{GL}_n(q)$. If $\mathbf{G}$ and $\mathbf{G}'$ are generator matrices for $\mathcal{C}$ and $\mathcal{D}$ respectively, then there exists a $\mathbf{T} \in \mathrm{GL}_k(q)$ such that $\mathbf{G}' = \mathbf{TG}(\mathbf{A}^\top \otimes \mathbf{B})$.*

For efficiency, it is standard to write the generator matrices in systematic form (i.e. reduced-row echelon form); we denote this operation by SF. Following Lemma 1, this gives us that $\mathcal{D} = \mathbf{A}\mathcal{C}\mathbf{B}$ if and only if $\mathsf{SF}(\mathbf{G}') = \mathsf{SF}(\mathbf{G}(\mathbf{A}^\top \otimes \mathbf{B}))$. To simplify notation, we introduce the following operator:

$$\pi_{\mathbf{A},\mathbf{B}}(\mathbf{G}) := \mathbf{G}(\mathbf{A}^\top \otimes \mathbf{B}).$$

We are now ready to describe some hard problems connected to the objects we just introduced. The Matrix Code Equivalence (MCE) problem is formally defined as follows:

Problem 2 (Matrix Code Equivalence).
$\mathsf{MCE}(k, n, m, \mathcal{C}, \mathcal{D})$:
Given: Two k-dimensional matrix codes $\mathcal{C}, \mathcal{D} \subset \mathcal{M}_{m,n}(q)$.
Goal: Determine if there exist $\mathbf{A} \in \mathrm{GL}_m(q), \mathbf{B} \in \mathrm{GL}_n(q)$ such that $\mathcal{D} = \mathbf{A}\mathcal{C}\mathbf{B}$.

The map $(\mathbf{A}, \mathbf{B}) : \mathbf{C} \mapsto \mathbf{ACB}$ is an *isometry* between $\mathcal{C}$ and $\mathcal{D}$, in the sense that it preserves the rank i.e. $\mathrm{Rank}\,\mathbf{C} = \mathrm{Rank}(\mathbf{ACB})$. Note that, although we defined MCE as a decisional problem, our signature construction relies on the computational version of it. The following related problem is at the basis of the ring signature construction.

Problem 3 (Inverse Matrix Code Equivalence).
$\mathsf{IMCE}(k, n, m, \mathcal{C}, \mathcal{D}_1, \mathcal{D}_2)$:
Given: Three k-dimensional matrix codes $\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{M}_{m,n}(q)$.
Goal: Determine if there exist $\mathbf{A} \in \mathrm{GL}_m(q), \mathbf{B} \in \mathrm{GL}_n(q)$ such that $\mathcal{D}_1 = \mathbf{A}\mathcal{C}\mathbf{B}$ and $\mathcal{D}_2 = \mathbf{A}^{-1}\mathcal{C}\mathbf{B}^{-1}$.

Problem 3 is closely connected to MCE, in a manner similar to the well-known connection between discrete logarithm and related problems, i.e.:

$$\mathsf{DLOG} \geq \mathsf{DDH} \geq \mathsf{InverseDDH}.$$

Reductions in the opposite direction are not known, as explained for example in [6]. Although this does not provide any further indication about the complexity of such problems, no explicit weaknesses are known either, and the problems are generally considered hard. A more generic overview about hardness of group action-based problems (of which the discrete logarithm is only a particular instantiation) is given in [26], with similar conclusions. For our specific case, we present a discussion about the concrete hardness of IMCE in Section 6.

Finally, we present a *multiple-instance* version of MCE, which is at the base of one of the optimizations, using *multiple public keys*, which we will describe in Section 5. It is easy to see that this new problem reduces to MCE, as done for instance in [8] for the Hamming case.

Problem 4 (Multiple Matrix Code Equivalence).
$\mathsf{MMCE}(k, n, m, r, \mathcal{C}, \mathcal{D}_1, \ldots, \mathcal{D}_r)$:
Given: $(r+1)$ k-dimensional matrix codes $\mathcal{C}, \mathcal{D}_1, \ldots, \mathcal{D}_r \subset \mathcal{M}_{m,n}(\mathbb{F}_q)$.
Goal: Find – if any – $\mathbf{A} \in \mathrm{GL}_m(q), \mathbf{B} \in \mathrm{GL}_n(q)$ such that $\mathcal{D}_i = \mathbf{A}\mathcal{C}\mathbf{B}$ for some $i \in \{1, \ldots, r\}$.

The MCE problem has been shown to be at least as hard as the Code Equivalence problem in the Hamming metric [21]. Furthermore, under moderate assumptions, MCE is equivalent to the homogeneous version of the Quadratic Maps Linear Equivalence problem (QMLE) [44], which is considered the hardest among polynomial equivalence problems. An extensive security evaluation will be given in Section 6, encompassing an overview of the best attack techniques and concrete security estimates. From this, we infer a choice of parameters in Section 7.2.

To conclude, we now lay out the details of the MCE-based group action, given by the action of isometries on k-dimensional matrix codes. That is, the set X is formed by the k-dimensional matrix codes of size $m \times n$ over some base field $\mathbb{F}_q$, and the group $G = \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$ acts on this set via isometries as follows:

$$\begin{array}{rccc} \star: & G \times X & \to & X \\ & ((\mathbf{A}, \mathbf{B}), \mathcal{C}) & \mapsto & \mathbf{A}\mathcal{C}\mathbf{B} \end{array}$$

We write $\mathcal{G}_{m,n}(q)$ to denote this group of isometries and $\mathcal{M}_{k,m,n}(q)$ for the set of k-dimensional matrix codes; to simplify notation, we drop the indices k, m, n and q when clear from context. Then, for this MCE-based group action the Vectorization Problem is precisely Problem 2. This action is not commutative and in general neither transitive nor free. We can restrict the set $\mathcal{M}$ to a single well-chosen orbit to make the group action both transitive and free. In fact, picking any orbit generated from some starting code $\mathcal{C}$ ensures transitivity, and the group action is free if the chosen code $\mathcal{C}$ has trivial automorphism group $\mathrm{Aut}_{\mathcal{G}}(\mathcal{C}) := \{\varphi \in \mathcal{G} : \varphi(\mathcal{C}) = \mathcal{C}\}$. The lack of commutativity is both positive and negative: although it limits the cryptographical design possibilities, e.g. key exchange becomes non-trivial, it prevents quantum attacks to which commutative cryptographic group actions are vulnerable, such as Kuperberg's algorithm for the dihedral hidden subgroup problem [32].

With regards to efficiency, it is immediate to notice that our group action is very promising, given that the entirety of the operations in the proposed protocols is simple linear algebra; this is in contrast with code-based literature (where complex decoding algorithms are usually required) and other group actions (e.g. isogeny-based) which are burdened by computationally heavy operations. Further details about performance are given in Section 7.

4 Protocols from Matrix Code Equivalence

The efficient non-commutative cryptographic group action provided by MCE, described in the previous section, yields a promising building block for post-quantum cryptographic schemes. In this section, we obtain a digital signature scheme by first designing a Sigma protocol and then applying the Fiat-Shamir transformation [27]. With a similar procedure, we are able to obtain (linkable) ring signatures as well.

Public Data
$q, m, n, k, \lambda \in \mathbb{N}$.
$\mathsf{hash} : \{0,1\}^* \to \{0,1\}^{2\lambda}$.

II. Commit(pk)

1. $\tilde{\mathbf{A}}, \tilde{\mathbf{B}} \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
2. Compute $\tilde{\mathbf{G}} = \mathsf{SF}(\pi_{\tilde{\mathbf{A}},\tilde{\mathbf{B}}}(\mathbf{G}_0))$.
3. Compute $h = \mathsf{hash}(\tilde{\mathbf{G}})$.
4. Set $\mathsf{cmt} = h$.
5. Send cmt to verifier.

IV. Response(sk, pk, cmt, ch)

1. If $\mathsf{ch} = 0$ set $(\mu, \nu) = (\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$.
2. If $\mathsf{ch} = 1$ set $(\mu, \nu) = (\tilde{\mathbf{A}}\mathbf{A}^{-1}, \mathbf{B}^{-1}\tilde{\mathbf{B}})$.
3. Set $\mathsf{rsp} = (\mu, \nu)$.
4. Send rsp to verifier.

I. Keygen()

1. $\mathbf{G}_0 \xleftarrow{\$} \mathbb{F}_q^{k \times mn}$ in standard form
2. $(\mathbf{A}, \mathbf{B}) \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
3. Compute $\mathbf{G}_1 = \mathsf{SF}(\pi_{\mathbf{A},\mathbf{B}}(\mathbf{G}_0))$.
4. Set $\mathsf{sk} = (\mathbf{A}, \mathbf{B})$ and $\mathsf{pk} = (\mathbf{G}_0, \mathbf{G}_1)$.

III. Challenge()

1. $c \xleftarrow{\$} \{0,1\}$.
2. Set $\mathsf{ch} = c$.
3. Send ch to prover.

V. Verify(pk, cmt, ch, rsp)

1. If $\mathsf{ch} = 0$ compute $h' = \mathsf{hash}(\mathsf{SF}(\pi_{\mu,\nu}(\mathbf{G}_0)))$.
2. If $\mathsf{ch} = 1$ compute $h' = \mathsf{hash}(\mathsf{SF}(\pi_{\mu,\nu}(\mathbf{G}_1)))$.
3. Accept if $h' = \mathsf{cmt}$ or reject otherwise.

Fig. 1. MCE Sigma Protocol

4.1 Sigma Protocols and Signatures

The first building block in our work is the Sigma protocol in Figure 1, in which a Prover proves the knowledge of an isometry $(\mathbf{A}, \mathbf{B})$ between two equivalent matrix codes. The security result is given in Theorem 1.

Theorem 1. *The Sigma protocol described above is complete, 2-special sound and honest-verifier zero-knowledge assuming the hardness of the* MCE *problem.*

Proof. We prove the three properties separately.

Completeness. An honest prover will always have his response accepted by the verifier. In fact, for the case $c = 0$, we have

$$h' = \mathsf{hash}(\mathsf{SF}(\pi_{\mu,\nu}(\mathbf{G}_0))) = \mathsf{hash}(\mathsf{SF}(\pi_{\tilde{\mathbf{A}},\tilde{\mathbf{B}}}(\mathbf{G}_0)) = \mathsf{hash}(\tilde{\mathbf{G}}) = h.$$

For the case $c = 1$, instead, observe that

$$\pi_{\tilde{\mathbf{A}}\mathbf{A}^{-1},\mathbf{B}^{-1}\tilde{\mathbf{B}}}(\mathbf{G}_1) = \pi_{\tilde{\mathbf{A}},\tilde{\mathbf{B}}}\left(\pi_{\mathbf{A}^{-1},\mathbf{B}^{-1}}(\mathbf{G}_1)\right).$$

Since $\mathbf{G}_1 = \mathsf{SF}(\pi_{\mathbf{A},\mathbf{B}}(\mathbf{G}_0))$, then $\mathbf{G}_1 = \mathbf{T}\left(\pi_{\mathbf{A},\mathbf{B}}(\mathbf{G}_0)\right)$ for some $\mathbf{T} \in \mathrm{GL}_k(q)$. Hence, we have that

$$\pi_{\tilde{\mathbf{A}}\mathbf{A}^{-1},\mathbf{B}^{-1}\tilde{\mathbf{B}}}(\mathbf{G}_1) = \pi_{\tilde{\mathbf{A}},\tilde{\mathbf{B}}}\left(\pi_{\mathbf{A}^{-1},\mathbf{B}^{-1}}\left(\mathbf{T}\left(\pi_{\mathbf{A},\mathbf{B}}(\mathbf{G}_0)\right)\right)\right) = \mathbf{T}\left(\pi_{\tilde{\mathbf{A}},\tilde{\mathbf{B}}}(\mathbf{G}_0)\right).$$

Now, computing the systematic form and applying hash, we again have $h' = h$.

2-Special Soundness. To get the extractor which produces the witness, we prove that having obtained two valid transcripts with the same commitment and a different challenge, we can extract a solution for the underlying MCE problem. This demonstrates that this Sigma protocol is a Proof of Knowledge with soundness error 1/2.

Let $(h, 0, \mathsf{rsp}_0)$ and $(h, 1, \mathsf{rsp}_1)$ be two valid transcripts, where $\mathsf{rsp}_0 = (\mu_0, \nu_0)$ and $\mathsf{rsp}_1 = (\mu_1, \nu_1)$. Since both pass verification, we have that

$$\mathsf{hash}(\mathsf{SF}(\pi_{\mu_0,\nu_0}(\mathbf{G}_0))) = h = \mathsf{hash}(\mathsf{SF}(\pi_{\mu_1,\nu_1}(\mathbf{G}_1))).$$

Then, since we assume that hash is collision resistant, it must be that

$$\mathsf{SF}(\pi_{\mu_0,\nu_0}(\mathbf{G}_0)) = \mathsf{SF}(\pi_{\mu_1,\nu_1}(\mathbf{G}_1)).$$

From this, it is easy to see that $\mathsf{SF}(\pi_{\mu^*,\nu^*}(\mathbf{G}_0)) = \mathbf{G}_1$, for an isometry $(\mu^*, \nu^*) = = (\mu_1^{-1}\mu_0, \nu_0\nu_1^{-1})$, i.e. (μ^*, ν^*) is a solution to the MCE problem.

Honest-Verifier Zero-Knowledge. To show this, we provide a simulator S which, without the knowledge of the witness, is able to produce a transcript which is indistinguishable from one obtained after an interaction with an honest verifier. When the challenge is $c = 0$, the prover's response is $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$, and does not involve the witness. Hence, it is obvious that S can obtain a flawless simulation by simply performing the same steps as an honest prover would. Thus,

$$\mathsf{Pr}\Big[\mathsf{V}_2(\mathsf{pk}, \mathsf{cmt}, 0, \mathsf{rsp}) = 1 \mid (\mathsf{cmt}, 0, \mathsf{rsp}) \leftarrow \mathsf{S}(\mathsf{pk})\Big] = 1.$$

When $c = 1$, the simulator generates two random matrices $(\mathbf{A}^*, \mathbf{B}^*)$, then sets the commitment to $\mathsf{cmt} = \mathsf{hash}(SF(\pi_{\mathbf{A}^*,\mathbf{B}^*}(\mathbf{G}_1)))$ and the response to $\mathsf{rsp} = (\mathbf{A}^*, \mathbf{B}^*)$. When $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ is uniformly generated, then the matrices $(\tilde{\mathbf{A}}\mathbf{A}^{-1}, \mathbf{B}^{-1}\tilde{\mathbf{B}})$ are uniformly generated too, and hence follow the same distribution as $(\mathbf{A}^*, \mathbf{B}^*)$. Furthermore, the triple $(\mathsf{cmt}, 1, \mathsf{rsp})$ is valid and therefore

$$\mathsf{Pr}\Big[\mathsf{V}_2(\mathsf{pk}, \mathsf{cmt}, 1, \mathsf{rsp}) = 1 \mid (\mathsf{cmt}, 1, \mathsf{rsp}) \leftarrow \mathsf{S}(\mathsf{pk})\Big] = 1.$$

This concludes the proof. □

Applying the Fiat-Shamir transformation, we obtain the signature scheme depicted in Figure 2.

Public key and signature size. We begin by calculating the communication costs for the Sigma protocol of Figure 1. Note that, for the case $c = 0$, the response (μ, ν) consists entirely of randomly-generated objects, and is efficiently represented by a single seed (that can be used to generate both matrices). This yields the following cost per round, in bits:

$$\begin{cases} 3\lambda + 1 & \text{if } c = 0 \\ 2\lambda + 1 + (m^2 + n^2)\lceil \log_2(q) \rceil & \text{if } c = 1 \end{cases}$$

Public Data

$q, m, n, k, t = \lambda \in \mathbb{N}$.
$\mathsf{hash} : \{0,1\}^* \to \{0,1\}^t$.

II. Sign(sk)

1. For all $i = 0, \ldots, t-1$:
   i. $\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
   ii. Compute $\tilde{\mathbf{G}}_i = \mathsf{SF}(\pi_{\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i}(\mathbf{G}_0))$.
2. Compute $h = \mathsf{hash}(\tilde{\mathbf{G}}_0, \ldots, \tilde{\mathbf{G}}_{t-1}, \mathsf{msg})$.
3. Parse $h = [h_0| \ldots |h_{t-1}]$, $h_i \in \{0,1\}$.
4. For all $i = 0, \ldots, t-1$:
   i. Set $(\mu_i, \nu_i) = (\tilde{\mathbf{A}}_i \mathbf{A}_{h_i}^{-1}, \mathbf{B}_{h_i}^{-1} \tilde{\mathbf{B}}_i)$.
5. Set $\sigma = (h, \mu_0, \ldots, \mu_{t-1}, \nu_0, \ldots, \nu_{t-1})$.
6. Send σ to verifier.

I. Keygen()

1. $\mathbf{G}_0 \xleftarrow{\$} \mathbb{F}_q^{k \times mn}$ in standard form
2. Set $\mathbf{A}_0 = \mathbf{I}_m$, $\mathbf{B}_0 = \mathbf{I}_n$.
3. $\mathbf{A}_1, \mathbf{B}_1 \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
4. Compute $\mathbf{G}_1 = \mathsf{SF}(\pi_{\mathbf{A}_1, \mathbf{B}_1}(\mathbf{G}_0))$.
5. Set $\mathsf{sk} = (\mathbf{A}_1, \mathbf{B}_1)$ and $\mathsf{pk} = (\mathbf{G}_0, \mathbf{G}_1)$.

III. Verify(pk, msg, σ)

1. Parse $h = [h_0| \ldots |h_{t-1}]$, $h_i \in \{0,1\}$.
2. For all $i = 0, \ldots, t-1$:
   i. Set $\hat{\mathbf{G}}_i = \mathsf{SF}(\pi_{\mu_i, \nu_i}(\mathbf{G}_{h_i}))$.
3. Compute $h' = \mathsf{hash}(\hat{\mathbf{G}}_0, \ldots, \hat{\mathbf{G}}_{t-1}, \mathsf{msg})$.
4. Accept if $h' = h$ or reject otherwise.

Fig. 2. The basic signature scheme

remembering that seeds are λ bits and hash digests 2λ to avoid collision attacks.

For the signature scheme we calculate the sizes as follows. First, since the matrix $\mathbf{G}_0$ is random, it can also be represented via a short seed, and therefore can be included in the public key at negligible cost (see Algorithm I. of Figure 2). Keeping in mind that the number of rounds t is equal to the value of the desired security level λ, the protocol above yields the following sizes (in bits):

- Public key size: $\lambda + k(mn-k)\lceil \log_2(q) \rceil$
- Average signature size: $t\left(1 + \dfrac{\lambda + (m^2+n^2)\lceil \log_2(q) \rceil}{2}\right)$.

4.2 Ring Signatures

The protocol of Figure 1 can be easily adapted to serve as a building block for a ring signature, by providing the ring of users with individual public keys, corresponding to equivalent matrix codes. Any user can then answer the verifier's challenge via the selected private key. The construction crucially utilizes a dedicated primitive known as *Index-Hiding Merkle Tree (IHMT)*. This primitive was first introduced in [12] as a variation on the traditional construction of Merkle trees. With this variant, in fact, the position of a leaf is not revealed, which is necessary to ensure anonymity. This can be accomplished by specifying a different method to construct the tree, based on an alternative ordering (e.g. lexicographic). For further details, we refer the reader to [12]. A visual representation is given in Figure 3, where we remind the reader that Algorithm I. Keygen is executed *once for each prover*, Algorithm II. Commit is the same regardless of the chosen prover, and Algorithm IV. Response is instead unique for the specific prover identified by $j^* \in \{1, \ldots, r\}$.

It is easy to see that the construction in Figure 3 satisfies the Completeness, 2-Special Soundness and Honest-Verifier Zero-Knowledge properties, similarly to

Public Data

$q, m, n, k, \lambda, r \in \mathbb{N}$.
$\mathsf{Com} : \{0,1\}^\lambda \times \{0,1\}^* \to \{0,1\}^{2\lambda}$.

II. Commit(pk)

1. $\tilde{\mathbf{A}}, \tilde{\mathbf{B}} \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
2. For all $j = 1, \ldots, r$:
   i. Sample $\mathsf{r}_j \xleftarrow{\$} \{0,1\}^\lambda$.
   ii. Compute $\tilde{\mathbf{G}}_j = \mathsf{SF}(\pi_{\tilde{\mathbf{A}},\tilde{\mathbf{B}}}(\mathbf{G}_j))$.
   iii. Compute $h_j = \mathsf{Com}(\mathsf{r}_j, \tilde{\mathbf{G}}_j)$.
3. Build IHMT from $(h_1, \ldots, h_r)$ and get root.
4. Set cmt = root.
5. Send cmt to verifier.

IV. Response(sk_{j^*}, pk, cmt, ch)

1. If ch $= 0$:
   i. Set $(\mu, \nu) = (\tilde{\mathbf{A}}\mathbf{A}_{j^*}, \mathbf{B}_{j^*}\tilde{\mathbf{B}})$.
   ii. Get path corresponding to leaf j^* in IHMT for $(h_1, \ldots, h_r)$.
   iii. Set bits $= \mathsf{r}_{j^*}$.
   iv. Set rsp $= (\mu, \nu, \mathsf{path}, \mathsf{bits})$.
2. If ch $= 1$:
   i. Set $(\mu, \nu) = (\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$.
   ii. Set bits $= (\mathsf{r}_1, \ldots, \mathsf{r}_r)$.
   iii. Set rsp $= (\mu, \nu, \mathsf{bits})$.
3. Send rsp to verifier.

I. Keygen(j)

1. $\mathbf{G}_0 \xleftarrow{\$} \mathbb{F}_q^{k \times mn}$ in standard form.
2. $(\mathbf{A}_j, \mathbf{B}_j) \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
3. Compute $\mathbf{G}_j = \mathsf{SF}(\pi_{\mathbf{A}_j,\mathbf{B}_j}(\mathbf{G}_0))$.
4. Set $\mathsf{sk}_j = (\mathbf{A}_j, \mathbf{B}_j)$ and $\mathsf{pk}_j = \mathbf{G}_j$.
5. Set $\mathsf{pk} = (\mathbf{G}_0, \mathsf{pk}_1, \ldots, \mathsf{pk}_r)$

III. Challenge()

1. $c \xleftarrow{\$} \{0,1\}$.
2. Set ch $= c$.
3. Send ch to prover.

V. Verify(pk, cmt, ch, rsp)

1. If ch $= 0$:
   i. Compute $\hat{\mathbf{G}} = \mathsf{SF}(\pi_{\mu,\nu}(\mathbf{G}_0))$.
   ii. Compute $h' = \mathsf{Com}(\mathsf{bits}, \hat{\mathbf{G}})$.
   iii. Get root$'$ from IHMT using h' and path.
2. If ch $= 1$:
   i. For all $j = 1, \ldots, r$:
      a. Compute $\hat{\mathbf{G}}_j = \mathsf{SF}(\pi_{\mu,\nu}(\mathbf{G}_j))$.
      b. Compute $h'_j = \mathsf{Com}(\mathsf{r}_j, \hat{\mathbf{G}}_j)$.
   ii. Build IHMT from $(h'_1, \ldots, h'_r)$ and get root$'$.
3. Accept if root$'$ = cmt, reject otherwise.

Fig. 3. MCE Ring Sigma Protocol

the protocol of Figure 1. The proof is analogue to the proof of Theorem 1, with only minor differences (such as the reversal in the roles of the challenges), that have no noticeable impact; this is therefore omitted in the interest of space. The protocol can then be turned into a ring signature scheme using Fiat-Shamir as usual, i.e. in the same manner as what was done for the protocol of Figure 1. Furthermore, we can make the ring signature scheme *linkable*, by means of a few modifications which we explain next. To avoid needless repetition, we avoid presenting the ring signature scheme in its extended form (as in Figure 2) and we move on instead to discussing the linkable variant; we will then present the linkable ring signature scheme, in its entirety, to conclude this section.

To begin, recall the group action $\star : G \times X \to X$, formalized in Section 3, with $X = \mathcal{M}$ the set of k-dimensional matrix codes, $G = \mathcal{G}$ the group of isometries for such codes, and the action given by $\star : ((\mathbf{A}, \mathbf{B}), \mathcal{C}) \mapsto \mathbf{A}\mathcal{C}\mathbf{B}$. We now require another group action $* : G \times Y \to Y$, satisfying the following properties.

Definition 7. Let $\star : G \times X \to X$ and $* : G \times Y \to Y$ be two group actions. We define the following properties for the pair $(\star, *)$:

– *Linkability*: Given $(x, y) \in X \times Y$, it is hard to output $g, g' \in G$ with $g \star x = g' \star x$ and $g * y \neq g' * y$.
– *Linkable Anonymity*: Given $(x, y) \in X \times Y$, the pair $(g \star x, g * y)$ is indistinguishable from (x', y'), where g and (x', y') are sampled uniformly at random from G and $X \times Y$, respectively.
– *Non-Frameability*: Given $(x, y) \in X \times Y$, $x' = g \star x$ and $y' = g * y$, for g sampled uniformly at random from G, it is hard to output $g' \in G$ with $g' * y = y'$.

Note how these three properties recall the Linkability, Linkable Anonymity and Non-Frameability properties introduced in Section 2.2. Indeed, showing that the pair of group actions satisfies the above properties is the key to constructing a secure linkable ring signature scheme. Also, as noted before, one could define unforgeability in a manner similar to the last property (by asking to output $g' \in G$ with $g' \star x = x'$) but this is not necessary, since this is a direct consequence of linkability and non-frameability. Informally, then, one can treat elements $y \in Y$ as "tags", that can be checked to establish the link. To this end, it is convenient to introduce an efficiently computable function $\mathsf{Link} : Y \times Y \to \{0, 1\}$, defined by $\mathsf{Link}(y, y') = 1 \Leftrightarrow y = y'$. For our purposes, we require $Y = X = \mathcal{M}$ and define $*$ as $\star^{-1}$. Thus, $* : ((\mathbf{A}, \mathbf{B}), \mathcal{C}) \mapsto \mathbf{A}^{-1}\mathcal{C}\mathbf{B}^{-1}$. We will then show that the required properties hold for any given code $\mathcal{C}$, with the IMCE problem as a basis for its security.

Theorem 2. *The pair of group actions $(\star, *)$ described above is linkable, linkably anonymous and non-frameable assuming the hardness of the* IMCE *problem.*

Proof. We prove the three properties separately.

Linkability. Consider a code $\mathcal{C}$ defined by $\mathbf{G}$, together with two isometries $g = (\mathbf{A}, \mathbf{B})$ and $g' = (\mathbf{A}', \mathbf{B}')$. Suppose that $\mathsf{SF}(\pi_{\mathbf{A},\mathbf{B}}(\mathbf{G})) = \mathsf{SF}(\pi_{\mathbf{A}',\mathbf{B}'}(\mathbf{G}))$ or, equivalently, that $\mathsf{SF}(\mathbf{G}(\mathbf{A}^\top \otimes \mathbf{B})) = \mathsf{SF}(\mathbf{G}((\mathbf{A}')^\top \otimes \mathbf{B}'))$. Then, it must be that $\mathbf{A}^\top \otimes \mathbf{B} = (\mathbf{A}')^\top \otimes \mathbf{B}'$, unless there is an automorphism of $\mathcal{C}$ hidden in the product. As noted in [44], it is plausible to assume that a random code $\mathcal{C}$ admits only the trivial automorphism, for all parameter choices we are interested in. If so, such trivial automorphism are pairs of scalar multiples of the identity for scalars $\alpha \in \mathbb{F}_q$. Hence, either we have $(\mathbf{A}^\top \otimes \mathbf{B})^{-1} = ((\mathbf{A}')^\top \otimes \mathbf{B}')^{-1}$; or, $(\mathbf{A}^\top \otimes \mathbf{B})^{-1} = \alpha^{-1}((\mathbf{A}')^\top \otimes \mathbf{B}')^{-1}$. In both cases, we have that $\mathsf{SF}(\pi_{\mathbf{A}^{-1},\mathbf{B}^{-1}}(\mathbf{G})) = \mathsf{SF}(\pi_{(\mathbf{A}')^{-1},(\mathbf{B}')^{-1}}(\mathbf{G}))$ i.e. $g * y = g' * y$.

Linkable Anonymity. This follows directly from the hardness of the IMCE problem. As discussed in Section 3, the problem is believed to be only marginally easier than MCE. In Section 6, we analyse dedicated attack techniques for IMCE.

Non-Frameability. For this property, an adversary A is given again a code $\mathcal{C}$ defined by $\mathbf{G}$ and codes x' and y' defined by $\mathsf{SF}(\pi_{\mathbf{A},\mathbf{B}}(\mathbf{G}))$ and $\mathsf{SF}(\pi_{\mathbf{A}^{-1},\mathbf{B}^{-1}}(\mathbf{G}))$

respectively, where $g = (\mathbf{A}, \mathbf{B})$ is chosen uniformly at random in $\mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$. The adversary A is asked to find $(\mathbf{A}', \mathbf{B}')$ such that

$$\mathsf{SF}(\pi_{(\mathbf{A}')^{-1},(\mathbf{B}')^{-1}}(\mathbf{G})) = y' = \mathsf{SF}(\pi_{\mathbf{A}^{-1},\mathbf{B}^{-1}}(\mathbf{G})).$$

We can use such an adversary to build a distinguisher D for the IMCE problem, as follows. Suppose $(\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2)$ is the given instance of the problem, and let ϵ be the success probability of A. To begin, D calls A on $(\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2)$, and A will reply with $g' = (\mathbf{A}', \mathbf{B}')$ that satisfies the equation above. From what we have seen in the proof of the Linkability property above, this implies that $((\mathbf{A}')^\top \otimes \mathbf{B}')^{-1} = (\mathbf{A}^\top \otimes \mathbf{B})^{-1}$, modulo trivial automorphisms. Either way, we have that $\mathsf{SF}(\pi_{\mathbf{A},\mathbf{B}}(\mathbf{G})) = \mathsf{SF}(\pi_{\mathbf{A}',\mathbf{B}'}(\mathbf{G}))$. Thus, it is enough for D to compute $\mathsf{SF}(\pi_{\mathbf{A}',\mathbf{B}'}(\mathbf{G}))$ and $\mathsf{SF}(\pi_{(\mathbf{A}')^{-1},(\mathbf{B}')^{-1}}(\mathbf{G})))$, and check whether they define $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. D then answers 1 if this is the case, or 0 otherwise. Since the probability that a randomly-drawn pair $(\mathcal{D}_1, \mathcal{D}_2)$ satisfies this condition is negligible, the probability of success of D is essentially the same as ϵ.

Together, these three properties complete the proof. □

Having clarified the nature of the tools at hand, we are now ready to introduce the linkable version of the ring signature scheme. We first explain compactly how to modify the protocol of Figure 3, then give a full description in Figure 4.

- As part of the public data, choose a collision-resistant hash function $\overline{\mathsf{hash}}$.
- The commit procedure (Algorithm II.) now additionally uses a matrix $\mathbf{T} = \mathsf{SF}(\pi_{\mathbf{A}_{j^*}^{-1},\mathbf{B}_{j^*}^{-1}}(\mathbf{G}_0))$; this is the "tag" for prover j^*, which will be trasmitted alongside the commitment to the verifier. The prover then computes $\tilde{\mathbf{T}} = \mathsf{SF}(\pi_{\tilde{\mathbf{A}}^{-1},\tilde{\mathbf{B}}^{-1}}(\mathbf{T}))$. After obtaining the root of the IHMT, the prover obtains $\bar{h} = \overline{\mathsf{hash}}(\tilde{\mathbf{T}}, \mathsf{root})$ and the commitment is set to $\bar{h}$ instead of root.
- Algorithm IV. is identical, except that, in the case $c = 0$, the response has to include also the matrices $\bar{\mu}, \bar{\nu} = \mathbf{A}_{j^*}\tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{B}_{j^*}$.
- Finally, Algorithm V. is modified to involve a check on the tag, as well. The case $c = 1$ is trivial, with the verifier essentially repeating the commitment procedure (Algorithm II.) as in the original protocol, only this time checking equality with $\bar{h}$ rather than root. For the case $c = 0$, instead, the verifier additionally computes $\hat{\mathbf{T}} = \mathsf{SF}(\pi_{\bar{\mu}^{-1},\bar{\nu}^{-1}}(\mathbf{G}_0))$, then uses this value to obtain $\overline{\mathsf{hash}}(\hat{\mathbf{T}}, \mathsf{root}')$ and verify equality with $\bar{h}$.

It is relatively easy to see that the new protocols satisfy all the necessary security properties. For instance, the ring Sigma protocol, with the above modifications, still satisfies the Completeness, 2-Special Soundness and Honest-Verifier Zero-Knowledge properties. The linkable ring signature scheme of Figure 4 is then just an application of the Fiat-Shamir transform. Once again, such proofs are omitted due to space limitations; the interested reader can consult for example [12], where proofs are given in all generality (see e.g. Theorems 4.3, 4.4 and 4.7). We will present computational costs for the (linkable) ring signature scheme in the next section, after discussing optimizations.

Public Data

$q, m, n, k, t = \lambda, r \in \mathbb{N}$. $\mathsf{Com} : \{0,1\}^\lambda \times \{0,1\}^* \to \{0,1\}^{2\lambda}$. $\mathsf{hash}, \overline{\mathsf{hash}} : \{0,1\}^* \to \{0,1\}^t$.

II. Sign($\mathsf{sk}_{j^*}, \mathsf{pk}$)

1. Compute the tag $\mathbf{T} = \mathsf{SF}(\pi_{\mathbf{A}_{j^*}^{-1}, \mathbf{B}_{j^*}^{-1}}(\mathbf{G}_0))$.
2. For all $i = 0, \ldots, t-1$:
   i. $\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
   ii. Set $\tilde{\mathbf{T}}_i = \mathsf{SF}(\pi_{\tilde{\mathbf{A}}_i^{-1}, \tilde{\mathbf{B}}_i^{-1}}(\mathbf{T}))$.
   iii. For all $j = 1, \ldots, r$:
      a. Sample $\mathsf{r}_{i,j} \xleftarrow{\$} \{0,1\}^\lambda$.
      b. Set $\tilde{\mathbf{G}}_{i,j} = \pi_{\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i}(\mathbf{G}_j)$.
      c. Set $h_{i,j} = \mathsf{Com}(\mathsf{r}_{i,j}, \mathsf{SF}(\tilde{\mathbf{G}}_{i,j}))$.
   iv. Build IHMT from $(h_{i,1}, \ldots, h_{i,r})$ and get root_i.
   v. Compute $\bar{h}_i = \overline{\mathsf{hash}}(\tilde{\mathbf{T}}_i, \mathsf{root}_i)$.
3. Conpute $h = \mathsf{hash}(\bar{h}_0, \ldots, \bar{h}_{t-1}, \mathbf{T}, \mathsf{msg})$.
4. Parse $h = [h_0 | \ldots | h_{t-1}]$, $h_i \in \{0,1\}$.
5. For all $i = 0, \ldots, t-1$:
   i. If $h_i = 0$:
      a. Set $(\mu_i, \nu_i) = (\tilde{\mathbf{A}}_i \mathbf{A}_{j^*}, \mathbf{B}_{j^*} \tilde{\mathbf{B}}_i)$.
      b. Set $(\bar{\mu}_i, \bar{\nu}_i) = (\mathbf{A}_{j^*} \tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i \mathbf{B}_{j^*})$.
      c. Get path_i corresponding to leaf j^* in IHMT for $(h_{i,1}, \ldots, h_{i,r})$.
      d. Set $\mathsf{bits}_i = \mathsf{r}_{i,j^*}$.
      e. Set $\mathsf{rsp}_i = \{\mu_i, \nu_i, \bar{\mu}_i, \bar{\nu}_i, \mathsf{path}_i, \mathsf{bits}_i\}$.
   ii. If $h_i = 1$:
      a. Set $(\mu_i, \nu_i) = (\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i)$.
      b. Set $\mathsf{bits}_i = (\mathsf{r}_{i,1}, \ldots, \mathsf{r}_{i,r})$.
      c. Set $\mathsf{rsp}_i = \{\mu_i, \nu_i, \mathsf{bits}_i\}$.
6. Set $\sigma = (h, \mathsf{rsp}_0, \ldots, \mathsf{rsp}_{t-1}, \mathbf{T})$.
7. Send σ to verifier.

I. Keygen(j)

1. $\mathbf{G}_0 \xleftarrow{\$} \mathbb{F}_q^{k \times mn}$ in standard form.
2. $(\mathbf{A}_j, \mathbf{B}_j) \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
3. Compute $\mathbf{G}_j = \mathsf{SF}(\pi_{\mathbf{A}_j, \mathbf{B}_j}(\mathbf{G}_0))$.
4. Set $\mathsf{sk}_j = (\mathbf{A}_j, \mathbf{B}_j)$ and $\mathsf{pk}_j = \mathbf{G}_j$.
5. Set $\mathsf{pk} = (\mathbf{G}_0, \mathsf{pk}_1, \ldots, \mathsf{pk}_r)$

III. Verify($\mathsf{pk}, \mathsf{msg}, \sigma$)

1. Parse $h = [h_0 | \ldots | h_{t-1}]$, $h_i \in \{0,1\}$.
2. For all $i = 0, \ldots, t-1$:
   i. If $h_i = 0$:
      a. Compute $\hat{\mathbf{G}}_i = \mathsf{SF}(\pi_{\mu_i, \nu_i}(\mathbf{G}_0))$.
      b. Compute $h'_i = \mathsf{Com}(\mathsf{bits}_i, \hat{\mathbf{G}}_i)$.
      c. Get root_i from IHMT using h'_i and path_i.
      d. Compute $\hat{\mathbf{T}}_i = \mathsf{SF}(\pi_{\bar{\mu}_i^{-1}, \bar{\nu}_i^{-1}}(\mathbf{G}_0))$.
   ii. If $h_i = 1$:
      a. For all $j = 1, \ldots, r$:
         † Compute $\hat{\mathbf{G}}_{i,j} = \mathsf{SF}(\pi_{\mu_i, \nu_i}(\mathbf{G}_j))$.
         ‡ Compute $h'_{i,j} = \mathsf{Com}(\mathsf{r}_{i,j}, \hat{\mathbf{G}}_{i,j})$.
      b. Build IHMT from $(h'_{i,1}, \ldots, h'_{i,r})$ and get root'_i.
      c. Compute $\hat{\mathbf{T}}_i = \mathsf{SF}(\pi_{\mu_i^{-1}, \nu_i^{-1}}(\mathbf{T}))$.
   iii. Compute $\bar{h}'_i = \overline{\mathsf{hash}}(\hat{\mathbf{T}}_i, \mathsf{root}'_i)$.
3. Compute $h' = \mathsf{hash}(\bar{h}'_0, \ldots, \bar{h}'_{t-1}, \mathbf{T}, \mathsf{msg})$.
4. Accept if $h' = h$ or reject otherwise.

Fig. 4. MCE linkable ring signature scheme

5 Matrix Equivalence Digital Signature — MEDS

By applying the following optimizations from literature to the basic Fiat-Shamir-based signature scheme described in Section 4, we obtain our Matrix Equivalence Digital Signature (MEDS).

Multiple keys. The first optimization is a popular one in literature [22, 13, 8], and it consists of utilizing multiple public keys, i.e. multiple equivalent codes $\mathbf{G}_0, \ldots, \mathbf{G}_{s-1}$, each defined as $\mathbf{G}_i = \mathsf{SF}(\pi_{\mathbf{A}_i, \mathbf{B}_i}(\mathbf{G}_0))$ for uniformly chosen secret

keys[6] $(\mathbf{A}_i, \mathbf{B}_i)$. This allows to reduce the soundness error from $1/2$ to $1/2^\ell$, where $\ell = \lceil \log_2 s \rceil$. The optimization works by grouping the challenge bits into strings of ℓ bits, which can then be interpreted as binary representations of the indices $\{0, \ldots, s-1\}$, thus dictating which public key will be used in the protocol. Security is preserved since the proof of unforgeability can be easily modified to rely on a multi-instance version of the underlying problem: in our case, MMCE (Problem 4). Note that, although in literature s is chosen to be a power of 2, this does not have to be the case. In this work, we will instead select the value of s based on the best outcome in terms of performance and signature size.

Remark 2. This optimization comes at the cost of an s-fold increase in public-key size. As shown for instance in [22], it would be possible to reduce this impact by using Merkle trees to a hash of the tree commitment of all the public keys. This, however, would add some significant overhead to the signature size, because it would be necessary to include the paths for all openings. Considering the sizes of the objects involved, such an optimization is not advantageous in our case.

Fixed-weight challenges. Another common optimization is the use of fixed-weight challenges. The idea is to generate the challenge string h with a fixed number of 1s and 0s, i.e. Hamming weight, rather than uniformly at random. This is because, when $h_i = 0$, the response (μ_i, ν_i) consists entirely of randomly-generated objects, and so one can just transmit the seed used for generating them. This creates a noticeable imbalance between the two types of responses, and hence it makes sense to minimize the number of 1 values. To this end, one can utilize a so-called *weight-restricted hash function*, that outputs values in $\mathbb{Z}^t_{2,w}$, by which we denote the set of vectors with elements in $\{0,1\}$ of length t and weight w. In this way, although the length of the challenge strings increases, the overall communication cost scales down proportionally to the value of w. In terms of security, this optimization only entails a small modification in the statement of the Forking Lemma, and it is enough to choose parameters such that $\log_2 \binom{t}{w} \geq \lambda$. Note that this optimization can easily be combined with the previous one, by mandating hash digests in $\mathbb{Z}^t_{s,w}$ and choosing parameters such that $\log_2 \left(\binom{t}{w}(s-1)^w\right) \geq \lambda$. In practice, this can be achieved with a hash function $\mathsf{hash} : \{0,1\}^* \to \{0,1\}^\lambda$, by expanding the output to a t-tuple $(h_0, \ldots, h_{t-1})$, $0 \leq h_i < s$ of weight w.

Seed tree. Finally, the signature size can be optimized again using a *seed tree*. This primitive allows to generate the many seeds used throughout the protocol in a recursive way, starting from a master seed mseed and building a binary tree, via repeated PRNG applications, having t seeds as leaves. When the required $t-w$ values need to be retrieved, it is then enough to reveal the appropriate sequence of nodes. This reduces the space required for the seeds from $\lambda(t-w)$ to λN_{seeds}, where N_{seeds} can be upper bounded by $2^{\lceil \log_2(w) \rceil} + w(\lceil \log_2(t) \rceil - \lceil \log_2(w) \rceil - 1)$, as shown in [29]. We refer the reader to Section 2.7 of [12] for more details.

[6] Again, for convenience, we choose $\mathbf{A}_0 = \mathbf{I}_m$, $\mathbf{B}_0 = \mathbf{I}_n$.

Public Data

$q, m, n, k, \lambda, t, s, w \in \mathbb{N}$.
$\mathsf{hash} : \{0,1\}^* \to \{0,1\}^\lambda$.

II. Sign(sk)

1. For all $i = 0, \ldots, t-1$:
   i. $\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
   ii. Compute $\tilde{\mathbf{G}}_i = \mathsf{SF}(\pi_{\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i}(\mathbf{G}_0))$.
2. Compute $h = \mathsf{hash}(\tilde{\mathbf{G}}_0, \ldots, \tilde{\mathbf{G}}_{t-1}, \mathsf{msg})$.
3. Expand h to $(h_0, \ldots, h_{t-1})$, $0 \leq h_i < s$.
4. For all $i = 0, \ldots, t-1$:
   i. Set $(\mu_i, \nu_i) = (\tilde{\mathbf{A}}_i \mathbf{A}_{h_i}^{-1}, \mathbf{B}_{h_i}^{-1} \tilde{\mathbf{B}}_i)$.
5. Set $\sigma = (\mu_0, \ldots, \mu_{t-1}, \nu_0, \ldots, \nu_{t-1}, h)$.
6. Send σ to verifier.

I. Keygen()

1. $\mathbf{G}_0 \xleftarrow{\$} \mathbb{F}_q^{k \times mn}$ in standard form.
2. Set $\mathbf{A}_0 = \mathbf{I}_m$, $\mathbf{B}_0 = \mathbf{I}_n$.
3. For all $j = 1, \ldots, s-1$:
   i. $\mathbf{A}_j, \mathbf{B}_j \xleftarrow{\$} \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$.
   ii. Compute $\mathbf{G}_j = \mathsf{SF}(\pi_{\mathbf{A}_j, \mathbf{B}_j}(\mathbf{G}_0))$.
4. Set $\mathsf{sk} = (\mathbf{A}_1^{-1}, \mathbf{B}_1^{-1}, \ldots, \mathbf{A}_{s-1}^{-1}, \mathbf{B}_{s-1}^{-1})$.
5. Set $\mathsf{pk} = (\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_{s-1})$.

III. Verify(pk, msg, σ)

1. Expand h to $(h_0, \ldots, h_{t-1})$, $0 \leq h_i < s$.
2. For all $i = 0, \ldots, t-1$:
   i. Set $\hat{\mathbf{G}}_i = \mathsf{SF}(\pi_{\mu_i, \nu_i}(\mathbf{G}_{h_i}))$.
3. Compute $h' = \mathsf{hash}(\hat{\mathbf{G}}_0, \ldots, \hat{\mathbf{G}}_{t-1}, \mathsf{msg})$.
4. Accept if $h' = h$ or reject otherwise.

Fig. 5. The MEDS Protocol

To give a complete picture, we present the MEDS protocol in Figure 5, in its final form, including all applicable variants. The various parameters control different optimization: for instance s refers to the number of public keys used, whereas w refers to the fixed weight of the challenge hash string. Parameter choices will be thoroughly discussed in Section 7.2. Note that some of these optimizations can be applied in an identical way to the (linkable) ring signature scheme; however, we leave an explicit description of such a scheme, as well as a full-fledged implementation of it, to a dedicated future work.

Public key and signature size. With these various optimizations, we obtain the following public key and signature size for MEDS:

- MEDS public key size: $\lambda + (s-1)k(mn-k)\lceil \log_2(q) \rceil$
- MEDS signature size:

$$\underbrace{\lambda}_{h} + \underbrace{w(m^2+n^2)\lceil \log_2(q) \rceil}_{\{\mu_i, \nu_i\}_{h_i=1}} + \underbrace{\lambda N_{\text{seeds}}}_{\{\mu_i, \nu_i\}_{h_i=0}}$$

The optimizations described above can also be applied to the ring signature scheme, which gives us the following sizes:

- Ring signature public key size: $\lambda + rk(mn-k)\lceil \log_2(q) \rceil$
- Ring signature size:

$$\underbrace{w\lceil \log_2 t \rceil}_{h} + \underbrace{w((m^2+n^2)\lceil \log_2(q) \rceil + 2\lambda \lceil \log r \rceil + \lambda)}_{\{\mathsf{rsp}_i\}_{h_i=0}} + \underbrace{\lambda N_{\text{seeds}}}_{\{\mathsf{rsp}_i\}_{h_i=1}}$$

For the linkable variant, the cost of the middle term, i.e. $\{\mathsf{rsp}_i\}_{h_i=0}$, is increased to $w(2(m^2+n^2)\lceil \log_2(q) \rceil + 2\lambda \lceil \log r \rceil + \lambda)$, and the signature additionally includes $k(mn-k)\lceil \log_2(q) \rceil$ bits for the tag $\mathbf{T}$.

6 Concrete Security Analysis

Recent works [21, 44] investigate the hardness of MCE by connecting it to other equivalence problems, namely, the Code Equivalence problem in the Hamming metric and the Quadratic Maps Linear Equivalence problem (QMLE). The connection to these problems, as well as the complexity analysis from [21, 44] will serve as a basis for determining the practical hardness of MCE and IMCE, which allows us to choose parameters for MEDS at a desired security level. For better understanding, we include the definition of the related QMLE problem.

Problem 5. $\mathsf{QMLE}(k, N, \mathcal{F}, \mathcal{P})$:
Given: Two k-tuples of multivariate polynomials of degree 2

$$\mathcal{F} = (f_1, f_2, \ldots, f_k),\ \mathcal{P} = (p_1, p_2, \ldots, p_k) \in \mathbb{F}_q[x_1, \ldots, x_N]^k.$$

Goal: Find – if any – matrices $\mathbf{S} \in \mathrm{GL}_N(q), \mathbf{T} \in \mathrm{GL}_k(q)$ such that

$$\mathcal{P}(\mathbf{x}) = (\mathcal{F}(\mathbf{x}\mathbf{S}))\mathbf{T}.$$

We denote by hQMLE, inhQMLE, BMLE and inhBMLE the related problems when the polynomials are homogeneous of degree 2, inhomogeneous, homogeneous bilinear and inhomogeneous bilinear, respectively. Further, we denote by MCRE the variant of MCE when the matrix $\mathbf{A}$ is the identity matrix.

In the rest of the section, we will mostly use the Big O notation $\mathcal{O}$ to express the complexity of algorithms. Where we are not interested in the polynomial factor we will use $\mathcal{O}^*$. We note that despite the notation, the estimates are quite tight and provide a good basis for choosing parameters.

6.1 Attacks on MCE

Recall that the goal of an adversary against MCE is to recover the matrices $\mathbf{A}$ and $\mathbf{B}$, given a description of the matrix codes $\mathcal{C}$ and $\mathcal{D}$. The most naïve attack would be to try every $\mathbf{A} \in \mathrm{GL}_m(q)$ and $\mathbf{B} \in \mathrm{GL}_n(q)$ until we find the correct isometry, amounting to a complexity of $\mathcal{O}(q^{n^2+m^2})$. The naïve attack can be improved by noting that MCRE is easy [21], and so, by guessing either $\mathbf{A}$ or $\mathbf{B}$, we get an MCRE instance. This approach gives a complexity of $\mathcal{O}^*(q^{\min\{m^2,n^2\}})$.

Algebraic attacks. Recently, in [44], it was shown that MCE is equivalent to BMLE. One of the natural attack avenues is thus to model the problem as an algebraic system of polynomial equations over a finite field. This approach was taken in [25], where the general Isomorphism of Polynomials (IP) problem was investigated. Here, we present a detailed modelling and complexity analysis of BMLE not known in the previous literature on IP.

First, fix arbitrary bases $(\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(k)})$ and $(\mathbf{D}^{(1)}, \ldots, \mathbf{D}^{(k)})$ of the codes $\mathcal{C}$ and $\mathcal{D}$ respectively. In terms of the bases, the MCE problem can be rephrased as finding $\mathbf{A} \in \mathrm{GL}_m(q), \mathbf{B} \in \mathrm{GL}_n(q)$ and $\mathbf{T} = (t_{ij}) \in \mathrm{GL}_k(q)$ such that:

$$\sum_{1 \leqslant s \leqslant k} t_{rs}\mathbf{D}^{(s)} = \mathbf{A}\mathbf{C}^{(r)}\mathbf{B}, \quad \forall r, 1 \leqslant r \leqslant k \tag{1}$$

The system (1) consists of knm equations in the $m^2+n^2+k^2$ unknown coefficients of the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{T}$. The quadratic terms of the equations are always of the form $\alpha a_{ij} b_{i'j'}$ for some coefficients a_{ij} and $b_{i'j'}$ of $\mathbf{A}$ and $\mathbf{B}$ respectively which means the system (1) is bilinear. Note that the coefficients of $\mathbf{T}$ appear only linearly. As previously, we can guess the m^2 variables from $\mathbf{A}$, which will lead us to a linear system that can be easily solved. However, we can do better by exploiting the structure of the equations.

For ease of readability of the rest of the paragraph denote by $\mathbf{M}_{i\_}$ and $\mathbf{M}_{\_i}$ the i-th row and i-th column of a matrix $\mathbf{M}$. As a crucial observation, note that, in (1), for $i \neq j$, the unknown coefficients from two rows $\mathbf{A}_{i\_}$ and $\mathbf{A}_{j\_}$ don't appear in the same equation. Symmetrically, the same holds for $\mathbf{B}_{\_i}$ and $\mathbf{B}_{\_j}$, but we will make use of it for the matrix $\mathbf{A}$. Thus, we can consider only part of the system, and control the number of variables from $\mathbf{A}$. The goal is to reduce the number of variables that we need to guess before obtaining an overdetermined linear system, and we want to do this in an optimal way. Consider the first α rows from $\mathbf{A}$. Extracting the equations that correspond to these rows in (1) leads us to the system:

$$\sum_{1 \leqslant s \leqslant k} t_{rs} \mathbf{D}^{(s)}_{i\_} = \mathbf{A}_{i\_} \mathbf{C}^{(r)} \mathbf{B}, \quad \forall r, i, \ 1 \leqslant r \leqslant k, \ 1 \leqslant i \leqslant \alpha. \tag{2}$$

Guessing the αm coefficients from $\mathbf{A}_{i\_}$ leads to a linear system of αkn equations in $n^2 + k^2$ variables. Choosing $\alpha = \lceil \frac{n^2+k^2}{kn} \rceil$, the complexity of the approach becomes $\mathcal{O}(q^{m\lceil \frac{n^2+k^2}{kn} \rceil}(n^2+k^2)^3)$. For the usual choice of $m = n$, and if we assume $k \approx 2.5n - 3.5n$, as is applicable for MEDS, this reduces to at least $\alpha = 3$ and a complexity of $\mathcal{O}(q^{3n}n^6)$.

Note that, one can directly solve the bilinear system (2) using for example XL [19] and the analysis for bilinear systems from [40] (similar results can be obtained from [23]). We have verified, however, that due to the large number of variables compared to the available equations, the complexity greatly surpasses the one of the simple linearization attack presented above.

In order to improve upon this baseline algebraic attack, we will model the problem differently and completely avoid the t_{rs} variables. This modelling is in the spirit of the minors modellings of MinRank as in [24, 7].

As previously, let $\mathbf{G}$ and $\mathbf{G}'$ be the $k \times mn$ generator matrices of the equivalent codes $\mathcal{C}$ and $\mathcal{D}$ respectively. Then from Lemma 1, $\tilde{\mathbf{G}} = \mathbf{G}(\mathbf{A}^\top \otimes \mathbf{B})$ is a generator matrix of $\mathcal{D}$ for some invertible matrices $\mathbf{A}$ and $\mathbf{B}$. We will take the coefficients of $\mathbf{A}$ and $\mathbf{B}$ to be our unknowns. A crucial observation for this attack is that each row $\tilde{\mathbf{G}}_{i\_}$ of $\tilde{\mathbf{G}}$ is in the span of the rows of $\mathbf{G}'$, since $\mathbf{G}'$ and $\tilde{\mathbf{G}}$ define the same code. This means that adding $\tilde{\mathbf{G}}_{i\_}$ to $\mathbf{G}'$ does not change the code, i.e.,

$$^{(i)}\mathbf{G}' = \begin{pmatrix} \mathbf{G}' \\ \tilde{\mathbf{G}}_{i\_} \end{pmatrix}$$

is not of full rank. From here, all maximal minors $\left| \begin{pmatrix} ^{(i)}\mathbf{G}'_{\_j_1} & ^{(i)}\mathbf{G}'_{\_j_2} \dots ^{(i)}\mathbf{G}'_{\_j_{k+1}} \end{pmatrix} \right|$ of $^{(i)}\mathbf{G}'$, for every $\{j_1, j_2, \dots, j_{k+1}\} \subset \{1, 2, \dots, mn\}$, are zero.

Now, as in a minors modeling of MinRank, we can form equations in the unknown coefficients of $\mathbf{A}$ and $\mathbf{B}$ by equating all maximal minors to zero, which amounts to a total of $\binom{mn}{k+1}$ equations. Since the unknown coefficients of $\mathbf{A}$ and $\mathbf{B}$ appear only in the last row of the minors, and only bilinearly, the whole system is also bilinear. Thus we have reduced the problem to solving the bilinear system

$$\left\{ |\left({}^{(i)}\mathbf{G}'_{\_j_1}\ {}^{(i)}\mathbf{G}'_{\_j_2}\ldots{}^{(i)}\mathbf{G}'_{\_j_{k+1}}\right)| = 0, \begin{array}{l}\text{for all } i \in \{1,2,\ldots,k\}\\ \text{and all } \{j_1,j_2,\ldots,j_{k+1}\} \subset \{1,2,\ldots,mn\}\end{array}\right. \tag{3}$$

in the m^2+n^2 unknown coefficients of $\mathbf{A}$ and $\mathbf{B}$.

At first sight, (3) seems to have more than enough equations to fully linearize the system. However, the majority of these equations are linearly dependent. In fact, there are only $(mn-k)k$ linearly independent equations. To see this, fix some i and consider a minor $|\left({}^{(i)}\mathbf{G}'_{\_j_1}\ {}^{(i)}\mathbf{G}'_{\_j_2}\ldots{}^{(i)}\mathbf{G}'_{\_j_{k+1}}\right)|$ of ${}^{(i)}\mathbf{G}'$. Since all rows except the first don't contain any variables, the equation

$$|\left({}^{(i)}\mathbf{G}'_{\_j_1}\ {}^{(i)}\mathbf{G}'_{\_j_2}\ldots{}^{(i)}\mathbf{G}'_{\_j_{k+1}}\right)| = 0$$

basically defines the linear dependence between the columns ${}^{(i)}\mathbf{G}'_{\_j_1},\ldots{}^{(i)}\mathbf{G}'_{\_j_{k+1}}$. But the rank of the matrix is k, so all columns can be expressed through some set of k independent columns. Thus, in total, for a fixed i we have $mn-k$ independent equations and in total $(mn-k)k$ equations for all i.

Alternatively, we can obtain the same amount of equations from $\tilde{\mathbf{G}}$ and the generator matrix $\mathbf{G}'^{\perp}$ of the dual code of $\mathcal{D}$. Since $\tilde{\mathbf{G}}$ should also be a generator matrix of $\mathcal{D}$, we construct the system:

$$\mathbf{G}'^{\perp} \cdot \tilde{\mathbf{G}}^{\top} = \mathbf{0},$$

which is again a system of $(mn-k)k$ bilinear equations in n^2+m^2 variables.

The complexity of solving the obtained system using either of the modellings strongly depends on the dimension of the code – it is the smallest for $k = mn/2$, and grows as k reduces (dually, as k grows towards mn). In Section 7 we give the concrete complexity estimate for solving the system for the chosen parameters using bilinear XL and the analysis from [40].

The attack does not seem to benefit a lot from being run on a quantum computer. Since the costly part comes from solving a huge linear system for which there are no useful quantum algorithms available, the only way is to 'Groverize' an enumeration part of the algorithm. One could enumerate over one set of the variables, either of $\mathbf{A}$ or $\mathbf{B}$, typically the smaller one, and solve a biliner system of less variables. Grover's algorithm could then speed up quadratically this enumeration. However, since in the classical case the best approach is to not use enumeration, this approach only makes sense for quite small values of the field size i.e. only when $q < 4$. In this parameter regime, however, combinatorial attacks perform significantly better, so this approach becomes irrelevant.

Algorithm 1 Collision-search algorithm

```
1: function BuildList(F, P)
2:     L ← ∅
3:     repeat
4:         x ←$ F_q^{m+n}
5:         if P(x) then L ← L ∪ {x}
6:     until |L| = ℓ
7:     return L
8: function CollisionFind(F, P)
9:     L_1 ← BuildList(F, P)
10:    L_2 ← BuildList(P, P)
11:    for all (x, y) ∈ {L_1 × L_2} do
12:        φ ← inhQMLE(x, y)
13:        if φ ≠ ⊥ then
14:            return solution φ
15:    return ⊥
```

Birthday-based collision-search algorithm. It was shown in [44] that MCE is equivalent to hQMLE, so it is possible to apply the graph-theoretic algorithm from [17] to MCE instances.

To do so, consider an hQMLE instance $\mathsf{hQMLE}(k, N, \mathcal{F}, \mathcal{P})$. The algorithm can be generalized as a collision-search algorithm comprised of two steps, as given in Algorithm 1. Step 1 is to build two lists of *distinguished* elements satisfying a particular, predefined distinguishing property $\mathbb{P}$ of the given systems of polynomials $\mathcal{F}$ and $\mathbb{P}$. This first step is depicted by the two calls to the BuildList function in Algorithm 1. The size of the lists is set such that we have a 63% chance of finding a collision, which amounts to roughly a square root of the number of distinguished elements. We denote by d the density of the distinguishing property, i.e. the proportion of elements satisfying $\mathbb{P}$. Step 2 is to check for a collision between the two lists. Given a pair $(\mathbf{x}, \mathbf{y})$, we build an *inhomogeneous* QMLE instance and run an isomorphism-search solver on it. A crucial observation is that a collision allows us to derive linear constraints, and thus transform the hQMLE problem to the inhQMLE problem. This second step corresponds to the main loop in the CollisionFind function in Algorithm 1.

This whole approach works under two assumptions: first, that there exist such distinguishing elements, and secondly, that the solver for inhomogeneous QMLE instances is efficient. Heuristic evidence suggests that solving *random* instances of the inhQMLE problem using an algebraic approach takes $\mathcal{O}(N^9)$ [25], however, the derived inhQMLE instances from the collision-search attack are not random enough. These specific instances have a solver with a complexity of $\mathcal{O}(q^\kappa)$ [16], where κ is a parameter related to $\mathbb{P}$ and thus to d. As κ is typically small, this approach is still efficient in practice. Following the analysis from [44], the concrete complexity of the attack when $k \leqslant 2(n+m)$ is as follows:

$$\mathsf{max}(\sqrt{q^{m+n}/d} \cdot C_{\mathbb{P}}, dq^{m+n} \cdot C_{\mathsf{iQ}}), \tag{4}$$

with success probability of $\approx 63\%$, where $C_{\mathbb{P}}$ denotes the cost of checking whether an element satisfies the distinguishing property, and C_{iQ} denotes the cost of a single query to inhQMLE. To see this, note that the concrete complexity of Algorithm 1 is defined as the complexity of the dominating step. The first step consists of building the lists of distinguished elements. We need a list of size $|L| = \sqrt{q^{m+n}d}$, and finding a single element takes $1/d \cdot C_{\mathbb{P}}$, hence $\sqrt{q^{m+n}/d} \cdot C_{\mathbb{P}}$. The second step, requires a query to the inhQMLE solver for each pair of elements

in $\{L_1 \times L_2\}$, at a running time of $dq^{m+n} \cdot C_{\mathsf{iQ}}$. Finally, the memory complexity is simply the size of the lists.

Asymptotically, the complexity is $\mathcal{O}^*(q^{\frac{2}{3}(n+m)})$ by balancing the steps [44]. For reference, the density is calculated as $d^{-1} = \prod_{i=0}^{\kappa-1} \frac{(q^k-q^i)(q^{m+n}-q^i)}{q^\kappa-q^i} = \mathcal{O}(q^{\kappa^2+\kappa(k-m-n)})$, for a certain choice of an integer value of κ that minimizes Equation (4). It is also pointed out in [44] that when $k \geq 2(m+n)$, we can no longer assume that we have distinguished elements, so we need to consider *all* elements in the collision search. Thus, we have $d = 1$ and the complexity of the algorithm is simply $\mathcal{O}(q^{m+n})$. In that case, we can consider choosing arbitrarily one element $\mathbf{x}$ and checking for a collision with all other elements $\mathbf{y} \in \mathbb{F}_q^{m+n}$. This approach yields the same complexity as the previous one, but is superior because it is deterministic and has negligible memory requirements. Note that this approach was also proposed in [17], but as an inferior (in terms of *time* complexity) deterministic variant, rather than as a solution for the lack of a distinguishing property. As this attack can be applied to any parameter set, it presents an upper-bound on the complexity of a classical collision-search algorithm.

For a quantum version of Algorithm 1, both BuildList and CollisionFind can be seen as unstructured searches of a certain size, hence Grover's algorithm applies to both: we can build the list L using only $\sqrt{\ell \cdot d^{-1}}$ searches, and we can find a collision using only $\sqrt{|L_1 \times L_2|}$ queries to the solver. This requires both $\mathbb{P}$ and $\mathsf{inhQMLE}$ to be performed in superposition. The balance between both sides remains the same. In total, the complexity of the quantum version becomes $\mathcal{O}^*(q^{\frac{1}{3}(n+m)})$.

Collision-search algorithm using non-trivial roots. When viewing an MCE instance as an hQMLE instance, it is possible to use certain bilinear properties to improve Algorithm 1. When $n = m$, such instances have approximately q^{2n-k-1} non-trivial roots, which can be used to improve a subroutine of Algorithm 1, and to make it deterministic instead of probabilistic [44]. In practice, such non-trivial roots exist **i)** almost always when $k < 2n$, **ii)** with probability $1/q$ for $k = 2n$, **iii)** with probability $1/q^{k+1-2n}$ for $k > 2n$. The complexity of this approach is $\mathcal{O}^*(q^n)$, if such non-trivial roots exist. This complexity is proven under the assumption that the complexity of the inhomogenous QMLE solver is no greater than $\mathcal{O}(q^n)$, which holds trivially when $k \geq n$ [44], and heuristically when $k < n$. Finding the non-trivial roots can also be done using a bilinear XL algorithm [40]. We do not consider this approach in our analysis, as it is only interesting for a subset of parameters where the systems are (over)determined, i.e. when k is close to $m + n$.

For a quantum version of this attack, Grover's algorithm can be used to find the non-trivial zeros. The complexity of then is lower bounded by $\mathcal{O}^*(q^{\frac{n}{2}})$.

Leon's algorithm in the rank metric. Leon [33] proposed an algorithm against the code equivalence problem in the Hamming metric that relies on the basic property that isometries preserve the weight of the codewords and that the

weight distribution of two equivalent codes is the same. Thus, finding the set of codewords of smallest weight in both codes reveals enough information to find a permutation that maps one set to the other, which with high probability is the unknown isometry between the codes. This algorithm is quite unbalanced and heavy on the 'codewords finding' side, since it requires finding all codewords of minimal weight. Beullens [11] proposed to relax the procedure and instead perform a collision based algorithm, much in the spirit of Algorithm 1: Build two lists of elements of the codes of particular weight (the distinguishing property from [11] actually also includes the multiset of entries of a codeword) and find a collision between them. As in Leon's algorithm and Algorithm 1, the 'collision finding' part employs an efficient subroutine for reconstructing the isometry.

One can immediately see that this approach easily translates to matrix codes and can be used to solve MCE. Here, finding codewords of a given rank r is equivalent to an instance of MinRank for k matrices of size $m \times n$ over $\mathbb{F}_q$. To ease our analysis and provide a lower bound of the complexity of this approach, we will assume that a MinRank algorithm, given a MinRank instance for rank r outputs for free a list of L codewords of rank r (provided there exist as many). Following the analysis of [7], we take the complexity of this part to be

$$\mathcal{O}(q^{k-k_0}k_0(r+1)\left(\binom{n}{r}\binom{k_0+b-1}{b}\right)^2) \tag{5}$$

where b is the smallest integer for which linearization in the support minors modelling from [7] is possible, and k_0 is a hybridization parameter for which the complexity is minimized.

For the collision part, notice that given two codewords $\mathbf{C}_1$ from $\mathcal{C}$ and $\mathbf{D}_1$ from $\mathcal{D}$, it is not possible to determine the isometry $(\mathbf{A}, \mathbf{B})$, as there are many isometries possible between single codewords. Thus, there is no efficient way of checking that these codewords collide nor finding the correct isometry. On the other hand, a pair of codewords is typically enough. For the pairs $(\mathbf{C}_1, \mathbf{C}_2)$ and $(\mathbf{D}_1, \mathbf{D}_2)$ we can form the system of $2mn$ linear equations

$$\begin{cases} \mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B} \\ \mathbf{A}^{-1}\mathbf{D}_2 = \mathbf{C}_2\mathbf{B} \end{cases} \tag{6}$$

in the m^2+n^2 unknown coefficients of $\mathbf{A}$ and $\mathbf{B}$. When $m = n$, which is a typical choice in practice, the system is overdetermined, and can be solved in $\mathcal{O}(n^6)$. It is then easy to check whether the obtained isometry maps $\mathcal{C}$ to $\mathcal{D}$. We will thus assume, as a lower bound, that we find collisions between pairs of codewords.

Now, let $C(r)$ denote the number of codewords of rank r in a k-dimensional $m \times n$ matrix code. Then, using a birthday argument, two lists of size $\sqrt{2C(r)}$ of rank r codewords of $\mathcal{C}$ and $\mathcal{D}$ are enough to find two collisions. To detect the two collisions, we need to generate and solve systems as in Equation (6) for all possible pairs of elements from the respective lists, so $\binom{\sqrt{2C(r)}}{2}^2$ systems in total. Since $C(r) \approx q^{r(n+m-r)-nm+k}$, the total complexity amounts to

$$\mathcal{O}(q^{2(r(n+m-r)-nm+k)}(m^2+n^2)^\omega).$$

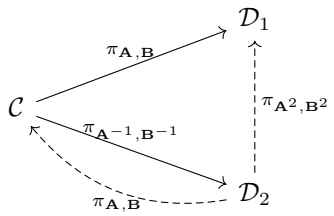
Note that a deterministic variant of this approach has the same asymptotic complexity. Choosing two rank r codewords of $\mathcal{C}$ and checking them for a 2-collision against all pairs of rank r codewords of $\mathcal{D}$ requires solving $\binom{C(r)}{2}$ systems.

Finally, we choose r so that both parts – the MinRank and the collision part are as close to a balance as possible. Section 7 discuses further the complexity of this approach for the chosen parameters of our scheme.

When considering the quantum version of the algorithm, we can apply the same reasoning as in the case of the collision based Algorithm 1, and obtain quadratic speedup in the collision part. Because of the hybridization, the MinRank part also benefits from using Grover, especially for larger fields.

6.2 Attacks on IMCE

The security of the linkable version of the ring signature scheme relies on the hardness of the IMCE problem. Thus, given three codes $\mathcal{C}$, $\mathcal{D}_1$ and $\mathcal{D}_2$, we need to determine whether they form a triangle of the following form, where the full lines represent the exact mappings the problem asks for, whereas the dashed ones are mappings that if found also break the problem. (The diagram uses loose notation in favor of readability.)



Extended algebraic attack. Algebraically, IMCE can be treated in exactly the same manner as the minors modeling of MCE in the previous section. The types of equations are the same, and with the same structure. The only difference is that the algebraic system that we construct has twice as many equations, i.e $2(mn-k)k$, coming from the isometry between $\mathcal{C}$ and D_1 and between $\mathcal{D}_1$ and $\mathcal{D}_2$ but the amount of variables is the same – n^2+m^2, since the isometry is the same. This means that the cost of the attack is reduced for the same choice of parameters, so adding the linkability property requires larger parameters in the regime where the algebraic attack performs the best.

Extended collision-search algorithm. Our extended collision-search attack consists of finding an isometry between *any* of the three codes. Specifically, finding a collision between $\mathcal{C}$ and $\mathcal{D}_1$ allows us to derive $\pi_{\mathbf{A},\mathbf{B}}$, one between $\mathcal{C}$ and $\mathcal{D}_2$ yields $\pi_{\mathbf{A}^{-1},\mathbf{B}^{-1}}$, and one between $\mathcal{D}_1$ and $\mathcal{D}_2$ yields $\pi_{\mathbf{A}^2,\mathbf{B}^2}$. We first reduce the problem to an equivalent QMLE-based problem: solve any of the three QMLE instances $(k, m+n, \mathcal{F}, \mathcal{P}_1)$, $(k, m+n, \mathcal{F}, \mathcal{P}_2)$ or $(k, m+n, \mathcal{P}_2, \mathcal{P}_1)$, with

$$\mathcal{P}_1(\mathbf{x}) = (\mathcal{F}(\mathbf{xS}))\mathbf{T_1},\ \mathcal{P}_2(\mathbf{x}) = (\mathcal{F}(\mathbf{xS}^{-1}))\mathbf{T_2},\ \mathcal{P}_1(\mathbf{x}) = (\mathcal{P}_2(\mathbf{xS}^2))\mathbf{T_2}^{-1}\mathbf{T_1}$$

and $\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^\top \end{bmatrix}$. Then, we apply a modified version of Algorithm 1, where we now build three lists instead of two, and on Line 11, we pick pairs $(\mathbf{x}, \mathbf{y})$ from $\{L_1 \times L_2\} \cup \{L_1 \times L_3\} \cup \{L_2 \times L_3\}$, instead of just $\{L_1 \times L_2\}$.

To derive the complexity of the attack, denote by $N = dq^{m+n}$ the total number of distinguished elements. Then it can be deduced, using a birthday based analysis that the probability of not having a collision when the lists contain ℓ elements each is $P = \prod_{i=1}^{\ell}(1 - \frac{i}{N})(1 - \frac{2i}{N}) \approx e^{-\frac{3\ell^2}{2N}}$. To have a 63% chance of collision, we need to build lists of size $\ell = c_2\sqrt{N}$, with $c_2 = \sqrt{\frac{2}{3}}$. For comparison, performing a similar analysis for the collision-search algorithm for the general MCE problem (where we have only two lists), this constant is $c_1 = \sqrt{2}$.

Remark 3. For Leon's algorithm for IMCE, the arguments are similar to the extended collision search algorithm. Thus it seems that apart from possibly a constant speed-up, the algorithm does not benefit from the IMCE context.

7 Implementation and Performance of MEDS

In this section we give an assessment of the performance of MEDS. We begin with important considerations about the implementation of the signature scheme. Then we provide concrete parameter choices for MEDS and a first preliminary evaluation of its performance based on a C reference implementation.

7.1 Implementation

Besides performance, the security of a cryptographic implementation is of crucial importance. By "implementation security" here we mean the resilience of an implementation against threats such as timing attacks, physical side-channel attacks, and fault injection attacks. While the requirement for side-channel and fault attacks heavily depends on whether in practice physical access is possible for an attacker, it is widely considered best practice to provide protection against timing attacks as baseline for all cryptographic implementations. In order to do this, the implementation must be *constant time*, i.e., timing variations based on secret data must be prevented. This typically means that branching and memory access based on secret data must be avoided. There are generic constructions to achieve timing security for basically any given cryptographic scheme. However, if the design of a cryptographic primitive does not take constant-time requirements into consideration, such generic constructions can be computationally expensive. Therefore, defining a cryptographic scheme such that it supports an efficient protection against timing attacks can make it easier to implement the scheme securely which in turn can make a scheme more secure and efficient in practice.

In the case of MEDS, we need to consider the implementation security of key generation and signing. Verification only operates on public data and hence does not require further protection. The basic operations of MEDS during key generation and signing are:

– field arithmetic,
– matrix multiplication,
– generating random invertible matrices, and
– computing a canonical form of a matrix.

All these operations must be implemented securely.

Field arithmetic. We are using a relatively small prime field in MEDS. Each field element can be stored in a single byte. Hence, for finite field addition and multiplication, we can simply perform integer arithmetic followed by a reduction modulo the prime. On most architectures, constant-time operations for adding and multiplying two bytes are available. The reduction modulo the prime can be implemented using standard approaches in literature (e.g., by Granlund and Montgomery [28], Barrett [10], or Montgomery [36]). Inversion of field elements can efficiently be implemented in constant time using Fermat's little theorem and optimal addition chains.

In the C reference implementation, we simply use the modulo operation for reduction and Fermat's little theorem for inversion to get a first impression on the performance of the scheme. For using MEDS in practice, the timing side-channel security in particular of the modulo operation needs to be verified.

Matrix multiplication. The basic schoolbook algorithm for multiplying matrices is not data depended and hence constant time. More sophisticated approaches like the method by Arlazarov, Dinic, Kronrod, and Faradžev [5] may depend on potentially secret data, but most likely do not significantly improve the performance for the finite field and the matrix sizes used in MEDS. Asymptotically more efficient matrix multiplication algorithms likely are not more efficient for the given matrix dimensions neither. Hence, for the C reference implementation, we are simply using schoolbook multiplication. For optimized implementations, a constant time algorithm must be used.

Random invertible matrix generation. On several occasions throughout the MEDS operations, we need to generate random invertible matrices: For key generation (cf. step 3i. in Figure 5), we need to generate the secret matrices $\mathbf{A}_j, \mathbf{B}_j \in \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$ and their inverses, for signing (cf. step 1i. in Figure 5), we need to generate potentially secret matrices $\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i \in \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$, and for verification, we need to re-generate some matrices $\tilde{\mathbf{A}}_i, \tilde{\mathbf{B}}_i \in \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$ from a seed for cases where $h_i = 0$.

There are several approaches for generating random invertible matrices:

1. *Trial-and-error approach:* Generate a random matrix $\mathbf{M}$ and attempt to compute its inverse. If $\mathbf{M}$ does not have an inverse, try again with a new random matrix. This approach requires constant-time Gaussian elimination if $\mathbf{M}$ needs to be kept secret and it might require several attempts before a random invertible matrix has been found.
2. *Constructive approach:* Construct a matrix $\mathbf{M}$ that is known to be inveritble. One approach for this is described in [42]: Generate a random lower-left

triangular matrix $\mathbf{L}$ with the diagonal all 1 and an upper-right triangular matrix $\mathbf{U}$ with the diagonal all $\neq 0$ as well as a corresponding permutation matrix $\mathbf{P}$ akin to the result of an LUP decomposition. Then compute the random invertible matrix $\mathbf{M}$ as $\mathbf{M} = \mathbf{P}^{-1}\mathbf{L}\mathbf{U}$.
Since generation of and multiplication with $\mathbf{P}$ are expensive to implement in constant time, one can follow the approach of [47], leave out $\mathbf{P}$, and compute $\mathbf{M}$ as $\mathbf{M} = \mathbf{L}\mathbf{U}$ directly. This, however, covers only a subset of $((q-1)/q)^n$ matrices of all invertibe matrices in $\mathbb{F}_q^{n\times n}$. The inverse $\mathbf{M}^{-1}$ of the matrix can then be computed as $\mathbf{M}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$.

The fastest approach in our experiments was the constructive approach following [47]. However, if also the inverse of the matrix is required (as for key generation), the trial-and-error approach turned out to be more efficient. Hence, we are using the trial-and-error approach during key generation and the constructive approach for signing (and for verification as the same matrices as during signing need to be computed from the seeds).

Canonical matrix form. MEDS requires to compute a canonical form of matrices before hashing. However, during signing, this must be computed in constant time. Computing the reduced row-echelon form of a matrix in constant time is expensive. Canonical matrix forms that can be computed in constant time more efficiently include the *systematic form* and the *semi-systematic* form of a matrix, used, e.g., in Classic McEliece [3]. Both the systematic and the semi-systematic form are special cases of the reduced row-echelon form.

For the systematic form, all pivoting elements are required to reside on the left diagonal of the matrix, i.e., a matrix $\mathbf{G} \in \mathbb{F}_q^{k\times mn}$ in systematic form has the shape $(\mathbf{I}_k|\mathbf{G}')$ with $\mathbf{G}' \in \mathbb{F}_q^{k\times(mn-k)}$ and $\mathbf{I}_k$ denoting the $k \times k$ identity matrix. The requirements for the semi-systematic form are more relaxed: Following [3, Sect. 2.2.1], we say that a matrix $\mathbf{G}$ is in (μ, ν)-semi-systematic form if $\mathbf{G}$ has r rows (i.e., no zero rows), the pivot element in row $i \leq r - \mu$ also is in column i and the pivot element in row $i > r - \mu$ is in a column $c \leq i - \mu + \nu$.

However, not all matrices admit a systematic or semi-systematic form. In this case, we need to restart the computation with new random data. The probability that a matrix $\mathbf{G} \in \mathbb{F}_q^{k\times mn}$, $k \leq mn$ is full rank is $\prod_{i=1}^{k}(q^{mn} - q^{i-1})/q^{kmn}$. Therefore, the probability that $\mathbf{G}$ has a systematic form is $\prod_{i=1}^{k}(q^k - q^{i-1})/q^{k^2}$ and the probability that it has a semi-systematic form is $\prod_{i=1}^{k-\mu}(q^k - q^{i-1})/q^{(k-\mu)k} \cdot \prod_{i=1}^{\mu}(q^\nu - q^{i-1})/q^{\mu\nu}$. The probability and the cost of constant-time implementation for the semi-systematic form depend on μ and ν.

This gives us the following three options for avoiding to compute a reduced row-echelon form in constant time for $\tilde{\mathbf{G}}_i$ during signing:

1. *Basis change:* After computing $\tilde{\mathbf{G}}'_i = \pi_{\tilde{\mathbf{A}}_i,\tilde{\mathbf{B}}_i}(\mathbf{G}_0)$, perform a basis change $\tilde{\mathbf{G}}''_i = \mathbf{M}_i\tilde{\mathbf{G}}'_i$ with a secret random invertible matrix $\mathbf{M}_i \in \mathbb{F}_q^{k\times k}$. Then compute a canonical form $\tilde{\mathbf{G}}_i = \mathsf{SF}(\tilde{\mathbf{G}}''_i)$ on a public $\tilde{\mathbf{G}}''_i$ without the need for a constant time implementation. This removes the requirement of a constant time computation of the canonical form but introduces extra cost for the

generation of and multiplication with a random invertible matrix $\mathbf{M}_i$. Instead of an invertible matrix $\mathbf{M}_i$, just a random matrix can be used. With low probability (see above), a random matrix does not have full rank and the computation of the canonical form fails. In that case, the process needs to be restarted with a different $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$.

2. *Semi-systematic form:* Compute the semi-systematic form. This requires a less expensive constant-time implementation than for the reduced-row-echelon form. However, computing the canonical form might fail if no semi-systematic form exists, in which case the computation needs to be restarted.
3. *Systematic form:* Compute the systematic form. This can be implemented even more easily and cheaper in constant time than computing the semi-systematic form. However, systemization failure is more frequent and it is more likely that computations need to be restarted.

We expect the implementation cost for variant 1 to be the higher than the other two. For the specific parameter sets we propose in Section 7.2, the probability that $\tilde{\mathbf{G}}_i$ does not have a systematic form is about 0.0015%. Therefore, even though the failure probability when computing the semi-systematic form can be reduced compare to the systematic form with well-chosen μ and ν, the overall overhead (including cost for constant-time implementation) of computing the semi-systematic form (variant 2) is likely higher than the overall overhead for of computing the systematic form (variant 3). Hence, we decided to use the systematic form in the reference implementation. However, a thorough investigation of the performance of the different variants on common computing platforms is required in order to identify the variant that performs best overall.

7.2 Parameter Choice and Evaluation

Table 1 provides an overview of 128-bit security parameters for MEDS, highlighting different performance and key/signature size trade-offs. Table 2 shows the resulting performance of these parameter sets from our C reference implementation on an AMD Ryzen 7 PRO 5850U CPU. The C reference implementation follows the implementation discussion above but does not apply any platform-specific optimizations. We expect that optimized implementations can significantly increase the performance.

With implementation efficiency in mind, we select $q = 65521$ as the largest 16-bit prime so that one $\mathbb{F}_q$ element fits into two bytes. We choose $n = m = 12$ and $k = 10$, which results in 149 bits of classical security, well above the 128-bit target. Quantumly, the security strength of these parameters is 85 bits. In both cases, Leon's algorithm performs the best, although in the classical case, the algebraic approach is also almost as good with a cost of $\approx 2^{151}$ operations. Note that the chosen ratio between n and k seems to be the goldilocks zone for our scheme. For k larger, the algebraic attack becomes significantly better, and the same is true for Leon's attack when k is smaller.

[8] https://bench.cr.yp.to/supercop.html

Parameter Set	q	n	k	s	t	w	ST	PK	SIG	AT	QAT	FS
MEDS-2696-st	65 521	12	10	2	256	30	✓	2696	18 768	149.05	85.55	−129.74
MEDS-10736-st	65 521	12	10	5	448	16	✓	10 736	10 512	149.05	85.55	−128.28
MEDS-13416-st	65 521	12	10	6	160	20	✓	13 416	12 688	149.05	85.55	−130.01
MEDS-13416	65 521	12	10	6	160	20	–	13 416	13 776	149.05	85.55	−130.01
MEDS-40216-st	65 521	12	10	16	128	16	✓	40 216	10 000	149.05	85.55	−128.85
MEDS-683416-st	65 521	12	10	256	64	11	✓	683 416	6784	149.05	85.55	−127.37

Table 1. Parameters for MEDS, for $\lambda = 128$ bits of classical security. 'ST' for seed tree. 'PK' for 'public key size' and 'SIG' for 'signature size in bytes. 'AT' for 'attack cost' in terms of bit security and 'FS' for 'Fiat-Shamir' probability logarithmic to base 2.

Parameter Set	Key Generation		Signing		Verification	
	(ms)	(mcyc.)	(ms)	(mcyc.)	(ms)	(mcyc.)
MEDS-2696-st	0.25	0.48	59.78	113.64	59.40	112.91
MEDS-10736-st	0.94	1.79	106.65	202.74	106.46	202.38
MEDS-13416-st	1.15	2.18	37.66	71.60	37.75	71.76
MEDS-13416	1.15	2.18	37.57	71.42	37.69	71.65
MEDS-40216-st	3.44	6.54	30.67	58.30	30.17	57.35
MEDS-683416-st	57.92	110.11	15.37	29.23	15.09	28.69

Table 2. Performance of MEDS in time (ms) and mega cycles (mcyc.) at 1900 MHz on an AMD Ryzen 7 PRO 5850U CPU following the SUPERCOP setup[8] computed as median of 1024 randomly seeded runs.

In this setting, we can vary s, t, and w for different trade-offs of public key and signature sizes as well as performance. To improve the efficiency of vectorized implementations using SIMD instructions in the future, we select t as multiple of 16. In general, we are using all optimizations discussed in Section 5. However, we provide one parameter set without using the seed tree (without '-st' in the name of the parameter set).

Remark 4. Given the parameters as presented, we heuristically assume that the automorphism group of the codes is trivial with overwhelming probability. It is computationally infeasible to compute the precise automorphism group of codes of this size, however computational data on smaller sized codes shows that the probability of a random code having trivial automorphism group grows rapidly as q, n, and m increase.

Parameter set MEDS-2696-st provides the smallest public key with slightly over 2.5 kB and a signature of about 18 kB. MEDS-10736-st provides balanced public key and signature sizes, with both around 10 kB, and the smallest sum of signature and public key size; however, this comes with a larger computational cost due to the larger $t = 448$. Both MEDS-13416-st and MEDS-13416 have balanced public key and signature sizes as well, each about 13 kB, at a smaller computational cost due to a smaller t. Removing the seed tree optimization comes with an increase in signature size of about 1 kB. Finally, sets MEDS-40216-st and MEDS-683416-st push the public key size to an extreme in the pursue of reducing signature size and computational cost.

Overall, Table 1 and Table 2 highlight the large degree of flexibility offered by the MEDS scheme. All parameter sets are competitive with respect to existing literature, such as the LESS-FM scheme.

Finally, we discuss performance for the ring signature scheme. With the MEDS-2696-st parameter set, the size of a signature is in fact given by approximately $(\log_2 r + 19.26)$ kB; in the linkable case, this increases to $(\log_2 r + 39.22)$ kB. These numbers are close to the lattice instantiation (Falafl) given in [12]; judging by the timings in Table 2, we also expect it to be much faster than the isogeny instantiation (Calamari). On the other hand, the work of [9] obtains smaller sizes, but does not propose a linkable variant. Note that both sizes and timings can be improved by choosing dedicated parameters and designing an ad-hoc implementation, which we leave as part of a future work.

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A Formal security definitions

Definition 8. For two sets X and W, let R be a relation on $X \times W$. If $(x, w) \in R$, we say that w is the *witness* for the *instance* x. We define a Sigma protocol for the relation R as in Definition 2, with $\mathsf{pk} = x$ and $\mathsf{sk} = w$. We then define the following properties for such a Sigma protocol:

- *Completeness*: when $(x, w) \in R$, a honest prover is accepted with probability 1, i.e.

$$\mathsf{Pr}\left[\mathsf{V}_2(x, \mathsf{cmt}, \mathsf{ch}, \mathsf{rsp}) = 1 \;\middle|\; \begin{array}{c} \mathsf{cmt} \leftarrow \mathsf{P}_1(x) \\ \mathsf{ch} \xleftarrow{\$} C \\ \mathsf{rsp} \leftarrow \mathsf{P}_2(x, w, \mathsf{cmt}, \mathsf{ch}) \end{array}\right] = 1.$$

- 2-*Special Soundness*: there exists an extractor algorithm E such that, for an instance x, any two valid transcripts, $(\mathsf{cmt}, \mathsf{ch}_1, \mathsf{rsp}_1)$ and $(\mathsf{cmt}, \mathsf{ch}_2, \mathsf{rsp}_2)$,

the algorithm output $\mathsf{E}(x, \mathsf{cmt}, \mathsf{ch}_1, \mathsf{rsp}_1, c_2, \mathsf{rsp}_2)$ is a witness for $\mathcal{R}$ with high probability. More formally, the probability

$$\mathsf{Pr}\left[(x,w) \in \mathcal{R} \,|w \leftarrow \mathsf{E}(x, \mathsf{cmt}, \mathsf{ch}_1, \mathsf{rsp}_1, \mathsf{ch}_2, \mathsf{rsp}_2)\right] = 1.$$

– *Honest-Verifier Zero-Knowledge*: for any $(x, w) \in R$, there exists a simulator S, with input only the public key x, which outputs a valid transcript $(\mathsf{cmt}, \mathsf{ch}, \mathsf{rsp})$ in polynomial time, such that

$$\mathsf{Pr}\left[\mathsf{V}_2(x, \mathsf{cmt}, \mathsf{ch}, \mathsf{rsp}) = 1 \,|(\mathsf{cmt}, \mathsf{ch}, \mathsf{rsp}) \leftarrow \mathsf{S}(x)\right] = 1.$$

Moreover, the output distribution of S on input (x, ch) is equal to the distribution of those outputs generated via an honest execution, conditioned on the verifier using ch as the challenge.

Definition 9. A digital signature is expected to satisfy the following properties:

1. *Correctness*: A honest prover is accepted with probability 1, i.e.

$$\mathsf{Pr}\left[\mathsf{Verify}(\mathsf{pk}, \mathsf{msg}, \sigma) = 1 \,|\sigma \leftarrow \mathsf{Sign}(\mathsf{sk}, \mathsf{msg})\right] = 1.$$

2. *Unforgeability*: The signature scheme is said to be unforgeable if any adversary $\mathcal{A}$, with access to any number of signature Σ, is not able to sign a new message except with a negligible probability. In other words, the probability

$$\mathsf{Pr}\left[\mathsf{Verify}(\mathsf{pk}, \mathsf{msg}, \sigma) = 1 \,|(\mathsf{msg}, \sigma) \notin \Sigma, \sigma \leftarrow \mathsf{AdvSign}(\mathsf{pk}, \mathsf{msg})\right]$$

is small.

Definition 10. A ring signature scheme is expected to satisfy the following properties:

1. *Correctness*: A ring signature is correct if for every ring of r signers, for every index $I \in \{1, \ldots, r\}$ and for every message msg, we have

$$\mathsf{Pr}\left[\mathsf{Verify}(K, \mathsf{msg}, \sigma) \,\middle|\, \begin{array}{c}(\mathsf{sk}_j, \mathsf{pk}_j) \leftarrow \mathsf{Keygen},\ \forall\, j \in \{1, \ldots, r\}, \\ K := \{\mathsf{pk}_1, \ldots, \mathsf{pk}_r\}, \\ \sigma \leftarrow \mathsf{Sign}(\mathsf{sk}_{j^*}, \mathsf{msg}, K)\end{array}\right] = 1.$$

2. *Anonymity*: A ring signature is anonymous if for every ring of r signers, any adversary $\mathcal{A}$ has at most a negligible advantage when playing the following game against a challenger:
   (A) The challenger first runs the algorithm Keygen to obtain r secret/public key pairs $(\mathsf{sk}_i, \mathsf{pk}_i)$, $i \in \{1, \ldots, r\}$. He samples a bit $b \xleftarrow{\$} \{0, 1\}$.
   (B) The challenger gives all the pk_j, $i = 1, \ldots, r$ to $\mathcal{A}$.
   (C) $\mathcal{A}$ sends to the challenger a challenge $(K, \mathsf{msg}, j_0, j_1)$. The set of public keys K must contain the public keys pk_{j_0} and pk_{j_1}. The challenger computes the signature $\sigma^* \leftarrow \mathsf{Sign}(\mathsf{sk}_{j_b}, \mathsf{msg}, K)$ and sends it to $\mathcal{A}$.
   (D) $\mathcal{A}$ outputs a bit b^* and wins if $b = b^*$.

3. *Unforgeability*: A ring signature is anonymous if for every ring of r signers, any adversary $\mathcal{A}$ has at most a negligible advantage when playing the following game against a challenger:
   (A) The challenger first runs the algorithm Keygen to obtain r secret/public key pairs $(\mathsf{sk}_i, \mathsf{pk}_i)$, $i \in \{1, \ldots, r\}$. He calls $K' = \{\mathsf{pk}_1, \ldots, \mathsf{pk}_r\}$ the set with the public keys. He finally initialises two empty sets S_1 and S_2.
   (B) The challenger gives the set K' to $\mathcal{A}$.
   (C) $\mathcal{A}$ can create signing and corruption queries a polynomial number of times:
      – $(\mathsf{AdvSign}, j, \mathsf{msg}, K)$: the challenger checks if $\mathsf{pk}_j \in K \subseteq K'$. If that is true, he computes $\sigma \leftarrow \mathsf{Sign}(\mathsf{pk}_j, \mathsf{msg}, K)$. The challenger gives σ to $\mathcal{A}$ and adds (j, msg, K) to S_1.
      – $(\mathsf{AdvCorrupt}, j)$: the challenger adds pk_j to S_2 and returns sk_j to $\mathcal{A}$.
   (D) $\mathcal{A}$ outputs $(K^*, \mathsf{msg}^*, \sigma^*)$. If $K^* \subset K' \backslash S_2, (\cdot, \mathsf{msg}^*, K^*) \notin S_1$ and $\mathsf{Verify}(K^*, \mathsf{msg}^*, \sigma^*) = 1$, then the adversary wins.

Definition 11. A linkable ring signature scheme is expected to satisfy the following properties

1. *Linkability*: A linkable ring signature is called *linkable* if for every ring of r signers, any adversary $\mathcal{A}$ has at most a negligible advantage when playing the following game against a challenger:
   (A) $\mathcal{A}$ runs the algorithm Keygen and outputs the set $K' = \{\mathsf{pk}_1, \ldots, \mathsf{pk}_r\}$ and the set of tuples $\{(\sigma_1, \mathsf{msg}_1, K_1), \ldots, (\sigma_{r+1}, \mathsf{msg}_{r+1}, K_{r+1})\}$.
   (B) $\mathcal{A}$ wins if the following three conditions hold:
      – $\forall j \in \{1, \ldots, r+1\}$, $K_i \subseteq K'$.
      – $\forall j \in \{1, \ldots, r+1\}$, the algorithm $\mathsf{Verify}(K_j, \mathsf{msg}_j, \sigma_i)$ outputs 1.
      – $\forall i, j \in \{1, \ldots, r+1\}$ such that $i \neq j$, $\mathsf{Link}(\sigma_i, \sigma_j) = 0$.
2. *Linkable Anonymity*: A linkable ring signature is linkable anonymous if for every ring of r signers, any adversary $\mathcal{A}$ has at most a negligible advantage when playing the following game against a challenger:
   (A) The challenger first runs the algorithm Keygen to obtain r key pairs $(\mathsf{sk}_j, \mathsf{pk}_j), j \in \{1, \ldots, r\}$. He calls $K' = \{\mathsf{pk}_1, ..., \mathsf{pk}_r\}$ the set with the public keys. He samples a bit $b \xleftarrow{\$} \{0, 1\}$.
   (B) The challenger gives K' to $\mathcal{A}$.
   (C) The adversary chooses and outputs two public keys $(\mathsf{pk}_{i_0}, \mathsf{pk}_{i_1}) \in K'$. The corresponding secret keys are denoted by $(\mathsf{sk}_{i_0}, \mathsf{sk}_{i_1})$.
   (D) The challenger gives to $\mathcal{A}$ the set $\{(\mathsf{pk}_i, \mathsf{sk}_i) \colon 1 \leq i \leq r, i \notin \{i_0, i_1\}\}$.
   (E) A queries for signatures, giving as inputs to the challenger a public key $\mathsf{pk} \in \{\mathsf{pk}_{i_0}, \mathsf{pk}_{i_1}\}$, a message msg and a ring K that contains $\{\mathsf{pk}_{i_0}, \mathsf{pk}_{i_1}\}$:
      – If $\mathsf{pk} = \mathsf{pk}_{i_0}$, the challenger outputs $\sigma \leftarrow \mathsf{Sign}(\mathsf{sk}_{i_b}, \mathsf{msg}, K)$.
      – If $\mathsf{pk} = \mathsf{pk}_{i_1}$, the challenger outputs $\sigma \leftarrow \mathsf{Sign}(\mathsf{sk}_{1-b}, \mathsf{msg}, \mathbb{F}_q)$.
   (F) $\mathcal{A}$ outputs a bit b^*, and he wins the game if $b = b^*$.
3. *Non-Frameability*: A linkable ring signature is non-frameable if for every ring of r signers, any adversary $\mathcal{A}$ has at most a negligible advantage when playing the following game against a challenger:

(A) The challenger first runs the algorithm Keygen to obtain r key pairs $(\mathsf{sk}_j, \mathsf{pk}_j), j \in \{1, \ldots, r\}$. He calls $K' = \{\mathsf{pk}_1, ..., \mathsf{pk}_r\}$ the set with the public keys. He also intialises two empty sets S_1 and S_2.

(B) The challenger gives the set K' to the adversary $\mathcal{A}$.

(C) A can create signing and corruption queries a polynomial number of times:
- $(\mathsf{AdvSign}, j, \mathsf{msg}, K)$: the challenger checks if $\mathsf{pk}_j \in K \subseteq K'$. If that is true, he computes $\sigma \leftarrow \mathsf{Sign}(\mathsf{sk}_j, \mathsf{msg}, K)$. The challenger gives σ to $\mathcal{A}$ and adds (j, msg, K) to S_1.
- $(\mathsf{AdvCorrupt}, j)$: the challenger adds pk_j to S_2 and returns sk_j to $\mathcal{A}$.

(D) $\mathcal{A}$ outputs $(K^*, \mathsf{msg}^*, \sigma^*)$; he wins if the following conditions hold:
- $\mathsf{Verify}(K^*, \mathsf{msg}^*, \sigma^*) = 1$ and $(\cdot, \mathsf{msg}^*, K^*) \notin S_1$;
- $\mathsf{Link}(\sigma^*, \sigma) = 1$ for some signature σ given by the challenger starting from a query of the form $(i, \mathsf{msg}, K) \in S_1$ with $\mathsf{pk}_i \in K' \backslash S_2$.

the adversary wins.