

# Efficient Threshold FHE with Application to Real-Time Systems

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## Abstract

Threshold Fully Homomorphic Encryption (ThFHE) enables arbitrary computation over encrypted data while keeping the decryption key to be distributed across multiple parties at all time. ThFHE is a key enabler for threshold cryptography and, more generally, secure distributed computing. Existing ThFHE schemes inherently require highly inefficient parameters and are unsuitable for practical deployment. In this paper, we take the first step towards to make ThFHE practically usable by (i) proposing a novel ThFHE scheme with a new analysis resulting in significantly improved parameters; (ii) and providing the first ThFHE implementation benchmark based on Torus FHE.

- We propose the *first* ThFHE scheme with a *polynomial modulus-to-noise ratio* that supports practically efficient parameters while retaining provable security based on standard quantum-safe assumptions. We achieve this via a novel Rényi divergence-based security analysis of our proposed threshold decryption mechanism.
- We present a highly optimized software implementation of our proposed ThFHE scheme that builds upon the existing Torus FHE library and supports (distributed) decryption on highly resource-constrained ARM-based handheld devices. To the best of our knowledge, this is the first practically efficient implementation of *any* ThFHE scheme. Along the way, we implement several extensions to the Torus FHE library, including a Torus-based linear integer secret sharing subroutine to support ThFHE key sharing and distributed decryption for any threshold access structure.

We illustrate the efficacy of our proposal via an end-to-end use case involving encrypted computations over a real medical database, and distributed decryptions of the computed result on resource-constrained handheld devices.

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# 1 Introduction

**Outsourced Computation.** The recent advent of cloud computing technologies [Hay08, WVLY<sup>+</sup>10] enables individuals and organizations to outsource heavy computations over large databases to potentially untrusted third-party servers. However, this poses new challenges for the security and privacy of the data, particularly when the data contains sensitive information such as individual medical records, etc. For compliance, regulation, and other essential privacy requirements, the data must be kept secure at rest, in transit, and during computation.

**Fully Homomorphic Encryption (FHE).** While traditional encryption procedures are useful for securing data at rest and in transit, they often fail to achieve any security during computation. Fully Homomorphic Encryption (FHE) [Gen09, BGV14, BGG<sup>+</sup>18] resolves this problem by enabling computation on encrypted data. This motivates a significant body of research work [SS10, CNT12, DM15, CGGI20, CGBH<sup>+</sup>18, FSK<sup>+</sup>21] to focus onto building practically efficient fully homomorphic encryption systems.

**Threshold Cryptography.** While FHE resolves the crucial problem of computation on encrypted data, one must carefully store the decryption key securely to get any real benefit out of it. Typical enterprise solutions of key management involve using secure hardware solutions such as HSMs, SGXs etc. While they provide reasonable security in practice, they often suffer from a lack of programmability, cumbersome setup procedures, scalability, high cost, side-channel attacks etc [KHF<sup>+</sup>19, LSG<sup>+</sup>18]. An alternative approach, that uses threshold cryptography [Sha79, DF90, DDFY94] is offered by enterprises like Hashicorp Vault<sup>1</sup>. In that approach, the key is shared among multiple servers (say  $T$ ) to avoid a “single point of failure” and a threshold number of them (say  $t$ ) can collaborate to recompute the decryption key. However, this defies the purpose as a single compromise at the decryption server, during a key-reconstruction, would reveal the key entirely. An ideal solution must have the decryption key distributed *at all time*. In particular, this is achieved by a ThFHE (Threshold-FHE) scheme [AJL<sup>+</sup>12, MW16, BGG<sup>+</sup>18], where the decryption is performed jointly by any threshold number of parties without reconstructing the key at any one place. In particular, parties compute partial decryption with their shares of the key and send them over to the decryptor, who, once obtains  $t$  such partial decryptions in total (may include her partial decryption), combines them to get the message.

**Practical ThFHE.** While there are several ThFHE schemes in the literature [AJL<sup>+</sup>12, MW16, BGG<sup>+</sup>18, MS<sup>+</sup>11, JRS17], the state-of-art is far from being practical. This is in contrast to the literature in FHE, in that many practical proposals and prototypes exist<sup>2</sup>. Perhaps the most crucial bottleneck of the existing schemes comes from the security requirement imposed by the threshold decryption procedure, which might involve up to  $t - 1$  corrupted servers (we only consider passive/semi-malicious corruption here). In slightly more detail, the modulus to noise ratio used in the existing threshold schemes must be set *super-polynomial* (in the security parameter) compared to the non-threshold FHE schemes that require only a polynomial modulus to noise ratio. The use of super-polynomial modulus-

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<sup>1</sup><https://www.vaultproject.io/>

<sup>2</sup><https://homenc.github.io/HElib/>,

<https://www.microsoft.com/en-us/research/project/microsoft-seal/>,

<https://tfhe.github.io/tfhe/>,

<https://palisade-crypto.org/>

noise ratio stems from a technique called *smudging* (alternatively *noise flooding*), which is used to achieve security when the parties are corrupt during the distributed decryption. In this work, we propose the first ThFHE scheme, which uses polynomial modulus to noise ratio – we achieve this by adapting a Rényi divergence based technique for distinguishing problems with public sampleability property as discussed in [BLRL<sup>+</sup>18, TT15]. This dramatically improves the system’s efficiency, as shown by our prototype implementation in software – this is the first benchmark for a ThFHE scheme.

## 1.1 Our Contribution

In this work we significantly improve the state-of-art for practical ThFHE scheme by both *new theoretical analysis* and *first prototype implementations*. Finally, we complement this by providing a use case for a real-world, end-to-end system that securely outsources medical data while avoiding the single point of failure by distributing the key among different lightweight devices that medical personnel hold.

**The First Practical Threshold FHE Scheme.** Our construction is based on the prior constructions [AJL<sup>+</sup>12, MW16, CM15]. In particular, we plug-in the threshold decryption technique from Asharov et al. [AJL<sup>+</sup>12] into the FHE scheme by Gentry, Sahai and Waters [GSW13] (GSW) – as a result, we get a *single-key* ThFHE version of the scheme by Mukherjee and Wichs [MW16] with two crucial differences: (i) the smudging noise is sampled from a Gaussian distribution; (ii) a polynomial modulus is used. In our analysis, which is inspired by the works such as [BLRL<sup>+</sup>18, TT15, ASY22], we use Rényi Divergence instead of statistical distance, which essentially made those changes possible keeping the security intact. As a result, we obtain the *first ThFHE scheme with polynomial modulus to noise ratio*.<sup>1</sup>

**First Software Prototype for Threshold FHE.** We provide the first prototype implementation of a ThFHE system with a benchmark in software. We expand further below.

- In our software implementation, we provide an extension of the existing library for Torus-FHE<sup>2</sup>. We also provide the first software implementation of a linear integer secret sharing scheme extended from [DT06] to support Torus Ring-LWE secret key sharing, which may be of independent interest. Our extended Torus-FHE library supports arbitrary  $t$  out of  $T$  threshold decryption while maintaining polynomial modulus-to-noise ratio.
- To emulate our intended use-case of decryption in handheld devices, we develop a portable implementation of the threshold decryption routines. We provide the results from its experimentation on a Raspberry Pi 3b board that uses a 64 bit ARM CPU.

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<sup>1</sup>Remarkably, polynomial modulus to noise ratio not only improves the efficiency significantly, but also makes the scheme potentially more secure – this is because such a ratio for the underlying Learning with Errors problem [Reg09] implies reduction to the corresponding worst-case lattice problem with polynomial approximation factor, which are believed to be significantly harder than the same problem with super-polynomial approximation factor, which is obtained if a super-polynomial ratio is used. For more details, we refer to, for example, [BV14].

<sup>2</sup><https://tfhe.github.io/tfhe/>

**A Practical Use-case.** Finally, as a use-case, we provide a detailed description of an end-to-end secure computation system over outsourced encrypted medical data. The goal is to have encrypted medical data stored in the cloud, such that any heavy computation may be performed on that encrypted data. At the same time, the decryption key must be stored in an easily accessible but secure way. In particular, a medical personnel who owns many lightweight devices should be able to access the result of the computation by using  $t$  devices, but if any  $t - 1$  device are compromised,<sup>1</sup> then the decryption key must not be revealed, even if the compromised device participates in several decryption sessions. For example, in a (5,8)-threshold decryption system, any five devices should be able to perform the distributed decryption, and the decryption key should remain secure until the number of compromised parties is less than five. Furthermore, the system should be such that the encryption or the computation on the encrypted data should be oblivious to the values of  $t$  or  $T$ . In particular, one may think about changing those values later. Our system satisfies all of the above mentioned aspects.

## 2 Related Work

**Threshold FHE.** The concept of ThFHE, introduced by Asharov et al. [AJL<sup>+</sup>12], has been majorly studied in two related but slightly different contexts: (i) to build low-round multiparty computation protocols [AJL<sup>+</sup>12, MW16, GLS15, BJMS20]; (ii) and as a key enabler for threshold cryptography [BGG<sup>+</sup>18, JRS17]. At a technical level these two categories of schemes follow slightly different definitions because of different application requirements. The MPC-motivated works (category (i) above) consider mainly  $(T, T)$ -threshold settings (Badrinarayan et al. [BJMS20] is an exception), whereas the later works are focused towards achieving  $(t, T)$  ( $t \leq T$ ) setting (which is standard in the threshold cryptography literature). Furthermore, the former works (necessarily, due to requirement of MPC) considered distributed key-generation for single-key schemes,<sup>2</sup> unless, of course, a specialized public-key infrastructure was assumed. The only distributed step considered by the threshold-inspired works (category-(ii) above) was distributed decryption, in that every party has a common ciphertext and their own share of secret decryption key; and then each party broadcasts a partially decrypted ciphertext generated locally, which are then combined together to obtain the decrypted value – this is similar to threshold public-key encryption [BBH06, Fra90, DF90, SG02]. The distributed decryption step is modular and essentially agnostic of how the ciphertext is generated. In particular, such decryption protocol can be plugged-in to schemes with appropriate distributed key-generation protocol or can be used in a multi-key scheme a la [MW16, BJMS20] (or even with a symmetric-key scheme).<sup>3</sup> Therefore, distributed decryption step appear in both categories of the above

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<sup>1</sup>In this work we consider a semi-malicious model of corruption a la [AJL<sup>+</sup>12, MW16] which assumes that corrupt parties behave as per the protocol description except they can choose arbitrary values for randomness – this is stronger than the passive security model where parties choose good randomness but weaker than fully malicious setting where parties behave in completely arbitrarily manner.

<sup>2</sup>The multi-key schemes are the exceptions. For multi-key schemes such as Mukherjee and Wichs [MW16] the key-generation step was naturally dispensed with, which was the key-step to achieve round-optimal MPC in the common random string model.

<sup>3</sup>In a  $(T, T)$  setting the distributed key-generation is trivial [AJL<sup>+</sup>12]. In the  $(t, T)$  setting the key-shares must be consistent with the secret  $(t - 1)$  degree polynomial, and hence a non-trivial protocol is required. One may just think about using a generic MPC protocol for this a la [BJMS20]. More efficient protocols are considered recently [GHL22]. This is not the focus of our work.

work. Our focus here is more aligned with the threshold cryptography literature, and hence we follow the second approach.

One common aspect of all of the above distributed decryption constructions is the use of the so-called noise smudging technique to achieve a simulation-based security guarantee when up to  $(t - 1)$  parties are (semi-maliciously) corrupt. The main idea is to sample noise from a uniform distribution and then use it to “smudge” (alternatively “flood”) the “sensitive LWE noise” contained in the partially decrypted ciphertext. The analysis (based on simple statistical distance measurements) crucially relies on the smudging noise being super-polynomially larger than the LWE noise; then to ensure correctness one must use a super-polynomial modulus-to-noise ratio – this results in impractical parameters. In this paper, we instead use a novel Rényi divergence-based analysis inspired by [BLRL<sup>+</sup>18, TT15] – this allows us to use a polynomially large smudging noise and subsequently a polynomial modulus-to-noise ratio, thereby putting ThFHE in the practical regime.

**Comparison with Very Recent Works in Threshold FHE.** In a very recent work, Boudgoust et al. [BS23] also proposed a poly-modulus threshold FHE scheme based on LWE assumption. They follow a two-step proof strategy: (a) argue one-way CPA security based on Rényi divergence, and (b) use an additional transformation to achieve full-fledged IND-CPA security, which either requires a random oracle heuristic or incurs additional overheads on top of the original threshold FHE scheme. On the other hand, we present a strategy to *directly* prove full-fledged IND-CPA security of the original threshold FHE scheme based on Rényi divergence by using a novel *public sampleability argument*. Our approach yields direct and potentially more efficient constructions of threshold FHE with provable security in the standard model; in particular, we avoid the random oracle heuristic as well as the additional overheads that are seemingly inherent to the proof techniques in [BS23]. We also present an end-to-end prototype implementation of our proposed system and evaluate its performance for a practical use-case; to the best of our knowledge, such a practical evaluation is not presented in [BS23].

In another recent work, Lee et al. [LMK<sup>+</sup>22] showed improved bootstrapping methods for FHEW/ TFHE and used their techniques to realize threshold FHE. Our goal in this paper is orthogonal since we focus purely on the threshold decryption procedure with the aim of achieving a polynomial modulus-to-noise ratio. Our techniques are therefore agnostic of the bootstrapping procedure used during homomorphic evaluations.

**Multiparty Homomorphic Encryption.** Mouchet et al. [MTBH21] recently considered a new notion of multiparty homomorphic encryption scheme (MPHE), which is very similar to the Asharov’s et al. [AJL<sup>+</sup>12]’s threshold FHE notion, that has both distributed key-generation plus distributed decryption, albeit for a  $(T, T)$  access structure. They also included an implementation benchmark. However, a major shortcoming of their definition is the absence of a simulation-based definition for their partial decryption protocol – so it does not capture a realistic threat model where adversary can corrupt parties while participating in the decryption procedure. Therefore, they did not need to use any noise smudging. Therefore, their implementation can not be counted as a predecessor of ours. In another work by Ananth et al. [AJJM20] defines a another primitive, which they also call multiparty homomorphic encryption – this is a slightly weaker variant of multi-key FHE, in that the decryption computation complexity grows with the circuit being evaluated. Another work by Padron and Vargas [PV21] defines an even weaker primitive (where the evaluator holds part of the secret-key) and calls it multiparty homomorphic encryption. Our notion

of ThFHE is different from all these notions.

**Multi-Key FHE.** In a multi-key FHE scheme [LATV12, CM15, MW16, BP16, PS16, CZW17, CO17, CCS19, AJJM20] parties encrypt their input with individual keys (generated locally) and then broadcast them; subsequently an extended ciphertext is constructed using all the encryptions from the involved parties and any arbitrary homomorphic operation can be performed on the extended ciphertext. As mentioned earlier, our distributed/threshold decryption can be adapted to this setting. For example, our implementation may possibly be combined with the multi-key implementation of Chen et al. [CCS19] to obtain the first practical multi-key FHE scheme with distributed decryption. We note that, Chen et al.’s implementation does not focus on distributed decryption and hence does not encounter smudging noise issues. Instead we focus only on distributed/threshold decryption (with the necessary notion of simulatability) in the  $(t, T)$  setting.

**Software Frameworks.** Recent works have accelerated FHE (non-threshold) implementations via GPU based parallelizations. Based on [CGGI20], a Python library **NuFHE**<sup>1</sup> has been developed. In [CDS15], the **Cingulata** (formerly, Armadillo) C++ toolchain and run-time environment were introduced for running programs over FHE ciphertexts, which now supports Torus FHE. **Lattigo**<sup>2</sup> [MBTPH20] on the other hand is a Go based module that builds secure protocols based on Multiparty-Homomorphic-Encryption and Ring-Learning-With-Errors-based Homomorphic Encryption Primitives. Some recent extensions proposed in [MBH22, MTBH21] do support threshold decryption; however, all of these implementations fundamentally require a superpolynomial modulus-to-noise ratio. Additionally, they only support *leveled* homomorphic versions (i.e., without bootstrapping) of the BGV [BGV14], BFV [Bra12, FV12] and CKKS [CKKS17] FHE schemes. Our ThFHE implementation builds upon and extends the Torus FHE library in a natural way (including the bootstrapping procedure), and is cross-compatible with all of these computation frameworks.

### 3 Preliminaries and Background

In this section, we introduce the notations used throughout this paper. We also present some preliminary background material on cryptographic primitives used in this paper.

#### 3.1 Notations and Mathematical Background

**Notations.** We use  $\mathbb{T}$  to denote the Torus (i.e., the set of all real numbers modulo 1). We write  $x \leftarrow \chi$  to represent that an element  $x$  is sampled uniformly at random from a set/distribution  $\mathcal{X}$ . For  $a, b \in \mathbb{Z}$  such that  $a, b \geq 0$ , we denote by  $[a]$  and  $[a, b]$  the set of integers lying between 1 and  $a$  (both inclusive), and the set of integers lying between  $a$  and  $b$  (both inclusive). We refer to  $\lambda \in \mathbb{N}$  as the security parameter, and denote by  $\text{poly}(\lambda)$  and  $\text{negl}(\lambda)$  any generic (unspecified) polynomial function and negligible function in  $\lambda$ , respectively.<sup>3</sup>

<sup>1</sup><https://nufhe.readthedocs.io/en/latest/>

<sup>2</sup><https://github.com/tuneinsight/lattigo>

<sup>3</sup>Note that a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is said to be negligible in  $\lambda$  if for every positive polynomial  $p$ ,  $f(\lambda) < 1/p(\lambda)$  when  $\lambda$  is sufficiently large.

**Threshold Access Structure.** For any  $T, t \in \mathbb{N}$  such that  $t \leq T$ , a  $(t, T)$ -threshold access structure over any set  $\mathcal{P} = \{P_1, \dots, P_T\}$  is defined as a collection of qualified subsets of the form

$$\mathbb{A}_{(t,T)} = \{\overline{\mathcal{P}} \subseteq \mathcal{P} : |\overline{\mathcal{P}}| \geq t\},$$

which (informally) states that any subset with  $t$  or more parties is a qualified subset. If  $\mathbb{A}_{(t,T)}$  is a *minimal*  $(t, T)$ -threshold access structure, then it only consists of subsets of size exactly  $t$ ; in other words, we have  $|\mathbb{A}_{(t,T)}| = \binom{T}{t}$ . Observe that this access structure can be represented efficiently [BGG<sup>+</sup>18]. In particular, there exists a polynomial-size circuit that takes as input  $T$ -length vectors and outputs a bit, such that for every valid subset  $S \in \mathbb{A}_{(t,T)}$ , on input the  $T$ -sized binary vector  $V = \{V_i\}_{i \in [T]}$  with  $V_i = 1$  if and only if  $P_i \in S$ , the circuit outputs 1 (see [BGG<sup>+</sup>18] for details).

**Rényi Divergence.** Let  $\text{Supp}(P)$  and  $\text{Supp}(Q)$  denote the supports of distributions  $P$  and  $Q$  respectively, such that  $\text{Supp}(P) \subseteq \text{Supp}(Q)$ . For  $a \in (1, +\infty)$ , the Rényi divergence of order  $a$  is

$$R_a(P||Q) = \left( \sum_{x \in \text{Supp}(P)} \frac{P(x)^a}{Q(x)^{a-1}} \right)^{\frac{1}{a-1}}.$$

The above definition extends naturally to continuous distributions (see [BLRL<sup>+</sup>18] for details).

## 3.2 Fully Homomorphic Encryption (FHE)

Fully Homomorphic Encryption (FHE) is a form of encryption that permits computations directly over encrypted data without decrypting it first. The result of this computation is also encrypted. Below, we recall the definition of fully homomorphic encryption (FHE) [Gen09, GHS12] for any message space  $\mathcal{M}$ .

**Definition 1** (Fully Homomorphic Encryption). *A fully homomorphic encryption (FHE) scheme is a tuple of four algorithms (Gen, Enc, Dec, Eval) with respect to a class of Boolean functions  $\mathcal{F} = \{\mathcal{F}_\ell\}_{\ell \in \mathbb{N}}$  (represented as Boolean circuits with  $\ell$ -bit inputs) such that the tuple (Gen, Enc, Dec) is an IND-CPA-secure public-key encryption (PKE) scheme as defined below, and the evaluation algorithm Eval satisfies the homomorphism and compactness properties as defined below:*

**IND-CPA security:** *For any  $(\text{pk}, \text{sk}) \leftarrow \text{Gen}(1^\lambda)$ , for any messages  $\text{m}_0, \text{m}_1 \in \mathcal{M}$ , and for any probabilistic polynomial-time (PPT) adversary  $\mathcal{A}$ , letting  $\text{ct}_0 \leftarrow \text{Enc}(\text{pk}, \text{m}_0)$  and  $\text{ct}_1 \leftarrow \text{Enc}(\text{pk}, \text{m}_1)$ ,*

$$\begin{aligned} |\Pr[\mathcal{A}(\text{pk}, \text{m}_0, \text{m}_1, \text{ct}_0) = 1] - \Pr[\mathcal{A}(\text{pk}, \text{m}_0, \text{m}_1, \text{ct}_1)]| &= 1 \\ &\leq \text{negl}(\lambda). \end{aligned}$$

**Homomorphism:** *For any (Boolean) function  $f : \{0, 1\}^\ell \rightarrow \{0, 1\} \in \mathcal{F}$  and any sequence of  $\ell$  messages  $\text{m}_1, \dots, \text{m}_\ell$ , letting  $(\text{pk}, \text{sk}) \leftarrow \text{Gen}(1^\lambda)$ , and  $\text{ct}_i \leftarrow \text{Enc}(\text{pk}, \text{m}_i)$  for each  $i \in [\ell]$ , we have the following:*

$$\text{Dec}(\text{sk}, \text{Eval}(\text{pk}, f, \text{ct}_1, \dots, \text{ct}_\ell)) = f(\text{m}_1, \dots, \text{m}_\ell).$$



**Compactness:** There exists a polynomial  $p(\lambda)$  such that, for any (Boolean) function  $f : \{0, 1\}^\ell \rightarrow \{0, 1\} \in \mathcal{F}$  and any sequence of  $\ell$  messages  $m_1, \dots, m_\ell$ , letting  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ , and  $ct_i \leftarrow \text{Enc}(pk, m_i)$  for each  $i \in [\ell]$ , we have

$$|ct^* \leftarrow \text{Eval}(pk, f, ct_1, \dots, ct_\ell)| \leq p(\lambda),$$

where  $p(\lambda)$  is independent of size of  $f$  and the number  $\ell$  of inputs.

In the definition mentioned above, we assumed that the evaluation key is included as part of the public key.

### 3.3 Threshold FHE

In this section, we define Threshold FHE (or ThFHE in short).

**Definition 2** (Threshold Fully Homomorphic Encryption (ThFHE)). Let  $\mathbb{S}$  be a class of efficient access structures on a set of parties  $\mathcal{P} = \{P_1, \dots, P_T\}$ . A ThFHE scheme for  $\mathbb{S}$  over a message space  $\mathcal{M}$  is a tuple of probabilistic polynomial-time algorithms

$$\text{ThFHE} = (\text{ThFHE.Gen}, \text{ThFHE.Enc}, \text{ThFHE.Eval}, \\ \text{ThFHE.PartialDec}, \text{ThFHE.Combine}),$$

defined as follows [BGG<sup>+</sup>18]:

- $\text{ThFHE.Gen}(1^\lambda, 1^d, \mathbb{A})$ : On input the security parameter  $\lambda$ , a depth bound  $d$ , and an access structure  $\mathbb{A} \in \mathbb{S}$ , the setup algorithm outputs an encryption (public) key  $pk$ , a decryption (secret) key  $sk$ , and a set of secret key shares  $sk_1, \dots, sk_T$ .
- $\text{ThFHE.Enc}(pk, \mu)$ : On input  $pk$  and a plaintext  $\mu$ , the encryption algorithm outputs a ciphertext  $ct$ .
- $\text{ThFHE.Eval}(pk, C, ct_1, \dots, ct_\ell)$ : On input a public key  $pk$ , a (Boolean) circuit  $C$  of depth at most  $d$ , and a set of ciphertexts  $ct_1, \dots, ct_\ell$ , the evaluation algorithm outputs a ciphertext  $ct^*$ .
- $\text{ThFHE.PartialDec}(sk_i, ct)$ : On input a secret key share  $sk_i$  and a ciphertext  $ct$ , the partial decryption algorithm outputs a partial decryption  $p_i$ .
- $\text{ThFHE.Combine}(\{p_i\}_{i \in \mathcal{S}})$ : On input a set of partial decryptions  $\{p_i\}_{i \in \mathcal{S}}$  for some subset  $\mathcal{S} \subseteq \{P_1, \dots, P_T\}$ , the combination algorithm either outputs a plaintext  $\mu$  or the symbol  $\perp$ .

**Correctness and Compactness.** We require that a ThFHE scheme satisfies compactness and correctness. These properties are very similar in flavor to those for traditional FHE, with the only difference being that correctness of decryption now requires that recombining partial decryptions of a ciphertext output by the evaluation algorithm returns the correct evaluation of the corresponding circuit on the underlying plaintexts. See [BGG<sup>+</sup>18] for formal definitions.

**Security Definition.** In the rest of this section, we focus on a security definition for ThFHE that we introduce in this paper. At a high level, our definition can be viewed as a sequential “amalgamation” of the notions of simulation and semantic security for ThFHE discussed in prior works [MW16, BGG<sup>+</sup>18]. Looking ahead, this definition crucially enables us to present a Rényi divergence-based analysis of security for our proposed ThFHE scheme with polynomial modulus-to-noise ratio.

Consider a ThFHE scheme over a message space  $\mathcal{M}$  for a threshold access structure  $\mathbb{A}_{t,T}$  for  $T$  parties  $\mathcal{P} = \{P_1, \dots, P_T\}$ . Let  $\lambda$  be the security parameter and  $d$  be the depth bound for the ThFHE scheme. We define below a game  $G_{\text{ThFHE}, \mathcal{A}, \mathbb{A}_{t,T}}(1^\lambda, 1^d)$  between a PPT challenger  $\mathcal{C}$  and a PPT adversary  $\mathcal{A}$ .

$G_{\text{ThFHE}, \mathcal{A}, \mathbb{A}_{t,T}}(1^\lambda, 1^d)$ :

**Initialization phase.**

1. The challenger  $\mathcal{C}$  runs  $\text{ThFHE.Gen}(1^\lambda, 1^d, \mathbb{A}_{t,T})$  to obtain  $(\text{pk}, \text{sk}, \text{sk}_1, \dots, \text{sk}_T)$ , and provides the public key  $\text{pk}$  to the adversary  $\mathcal{A}$ .
2. The adversary  $\mathcal{A}$  outputs a set  $S \subset \{P_1, \dots, P_T\}$  such that  $S \notin \mathbb{A}_{t,T}$ , and receives the set of secret key shares  $\{\text{sk}_i\}_{P_i \in S}$  from  $\mathcal{C}$ .

**Honest encryption query phase.**

1.  $\mathcal{A}$  produces a set of  $\ell$  plaintext queries  $\{\mu_i\}_{i \in [\ell]}$ .
2. In response,  $\mathcal{C}$  provides to  $\mathcal{A}$  the set of “honestly encrypted” ciphertexts  $\{\text{ct}_i\}_{i \in [\ell]}$ , where  $\text{ct}_i = \text{ThFHE.Enc}(\text{pk}, \mu_i)$ .

**Partial decryption query phase.**

1. The adversary  $\mathcal{A}$  issues  $Q$  (polynomially many) circuits  $\{C_i : \mathcal{M}^\ell \rightarrow \mathcal{M}\}_{i \in [Q]}$ .
2.  $\mathcal{A}$  receives  $(\hat{\text{ct}}_i, \{p_{i,j}\})$  for each  $i \in [Q]$  and  $P_j \notin S$ , where

$$\begin{aligned} \hat{\text{ct}}_i &= \text{ThFHE.Eval}(\text{pk}, C_i, \text{ct}_1, \dots, \text{ct}_\ell), \\ p_{i,j} &= \text{ThFHE.PartialDec}(\text{sk}_j, \hat{\text{ct}}_i). \end{aligned}$$

**Challenge phase.**

1. The adversary  $\mathcal{A}$  outputs two messages  $m_0, m_1 \in \mathcal{M}$ .
2. The challenger  $\mathcal{C}$  randomly samples a bit  $b \leftarrow \{0, 1\}$  and provides  $\mathcal{A}$  with  $\text{ct}^* = \text{ThFHE.Enc}(\text{pk}, m_b)$ .

**Output phase.**

1. The adversary  $\mathcal{A}$  eventually outputs a bit  $b' \in \{0, 1\}$ .
2. if  $b' = b$ , the game outputs 1, otherwise it outputs 0.

Note that the above phases are sequential; in particular, we do not allow partial decryption queries after the adversary sees the challenge ciphertext  $\text{ct}^*$ ; otherwise, the adversary can trivially win by crafting partial decryption queries using  $\text{ct}^*$  that leak information about the bit  $b$ .

We say that a threshold FHE scheme  $\text{ThFHE}$  is secure if, for any security parameter  $\lambda \in \mathbb{N}$ , for any depth  $d = \text{poly}(\lambda)$ , for any threshold access structure  $A_{t,T}$ , and for any PPT adversary  $\mathcal{A}$ , letting  $\gamma_\beta = \Pr[\mathbf{G}_{\text{ThFHE}, \mathcal{A}, A_{t,T}}(1^\lambda, 1^d) = \beta]$ , for  $\beta \in \{0, 1\}$  (where the probability is over the random coins used by  $\text{ThFHE.Gen}$ ,  $\text{ThFHE.Enc}$ ,  $\text{ThFHE.Eval}$  and the adversary  $\mathcal{A}$ ), we have  $|\gamma_0 - \gamma_1| \leq \text{negl}(\lambda)$ .

**Discussion on Our Security Definition.** As mentioned earlier, our definition of security effectively combines in sequence the definitions of simulation and semantic security for  $\text{ThFHE}$  from prior works [MW16, BGG<sup>+</sup>18]. Informally, a  $\text{ThFHE}$  scheme is said to provide semantic security if a PPT adversary cannot efficiently distinguish between encryptions of arbitrarily chosen plaintext messages  $m_0$  and  $m_1$  [BGG<sup>+</sup>18]. Additionally, a  $\text{ThFHE}$  scheme is said to provide simulation security if there exists an efficient algorithm to *simulate* partial decryptions on ciphertexts produced by evaluating one or more circuits on (honestly generated) ciphertexts, given only the output of evaluating the circuit(s) on the underlying plaintexts [MW16, BGG<sup>+</sup>18]. Our definition can be viewed as an indistinguishability-based security notion where we essentially require these two notions of security to hold *simultaneously* against a PPT adversary that is given the secret key shares for an *invalid access structure set*  $\mathcal{S}$ .

Observe that, intuitively, if a  $\text{ThFHE}$  scheme satisfies both semantic and simulation security (as is desirable), it also satisfies our security definition. We briefly outline a simple hybrid argument that validates this observation. Consider a hybrid game in which we use the simulator as used by the proof of simulation security to simulate the partial decryptions *without knowing the master secret key*, while generating the challenge ciphertext using the public key. This hybrid is indistinguishable from the original game in our security proof by the simulation security of the  $\text{ThFHE}$  scheme. At this point, our security game essentially reduces to the semantic security game, since the secret key is no longer used when responding to the adversary’s queries. Hence, our definition captures a natural unification of the two notions of security.

Notably, our definition allows us to port existing technical machinery for Rényi divergence-based analysis of other lattice-based cryptosystems [BLRL<sup>+</sup>18] to the context of threshold FHE (unfortunately, the original simulation security definition from prior works [MW16, BGG<sup>+</sup>18] is not amenable to such techniques). In particular, we can use techniques from [BLRL<sup>+</sup>18] to argue that for any  $\text{ThFHE}$  scheme, as long as the adversary’s views of the real and simulated partial decryptions in our security game have a bounded Rényi divergence, it cannot distinguish between encryptions of  $m_0$  and  $m_1$  with non-negligible probability without breaking the original semantic security guarantees of  $\text{ThFHE}$ . Looking ahead, for our proposed  $\text{ThFHE}$  scheme, we can achieve the desired bounds on the Rényi divergence while only using a polynomial modulus-to-noise ratio, which is the technical crux of our contribution. On the contrary, it is seemingly hard to achieve the original notion of simulation security proposed in [MW16, BGG<sup>+</sup>18] without a superpolynomial modulus-to-noise ratio.

### 3.4 Linear Integer Secret Sharing Scheme (LISSS)

In this work, we base our constructions and software implementation of Threshold FHE on a special class of secret sharing schemes called Linear Integer Secret Sharing Scheme (LISSS) defined below.

**Definition 3** (LISSS). *Let  $\mathcal{P} = \{P_1, \dots, P_T\}$  be a set of parties, and let  $\mathbb{S}$  be a class of efficient access structures on  $\mathcal{P}$ . A secret sharing scheme  $\text{SS}$  with secret space  $\mathcal{K} = \mathbb{Z}_p$  for some prime  $p$  is called a linear integer secret sharing scheme (LISSS) if there exist the following algorithms:*

- **SS.Share**( $k \in \mathcal{K}, \mathbb{A}$ ): *There exists a matrix  $\mathbf{M} \in \mathbb{Z}_p^{d \times e}$  with dimensions determined by the access structure  $\mathbb{A} \in \mathbb{S}$  called the distribution matrix, and each party  $P_i$  is associated with a partition  $T_i \subseteq [d]$ . To create the shares on a secret  $k \in \mathcal{K}$ , the sharing algorithm uniformly samples  $\rho_2, \dots, \rho_e \leftarrow \mathbb{Z}_p$ , defines a vector  $\mathbf{s} = (s_1, \dots, s_d)^\mathbf{T} = \mathbf{M} \cdot (k, \rho_2, \dots, \rho_e)^\mathbf{T}$ , and outputs to each party  $P_i$  the corresponding set of shares  $\text{share}_i = \{s_j\}_{j \in T_i}$ .*
- **SS.Combine**( $\{\text{share}_i\}_{P_i \in \overline{\mathcal{P}}}$ ): *For any qualified subset of parties  $\overline{\mathcal{P}} \in \mathbb{A}$ , there exists a set of efficiently computable “recovery coefficients”  $\{c_j\}_{j \in \cup_{P_i \in \overline{\mathcal{P}}} T_i}$ , such that*

$$\sum_{j \in \cup_{P_i \in \overline{\mathcal{P}}} T_i} c_j \cdot \mathbf{M}[j] = (1, 0, \dots, 0),$$

where  $\mathbf{M}[j]$  denotes the  $j$ -th row of the matrix  $\mathbf{M}$  described earlier. Then, the final secret  $k$  can be re-computed using these recovery coefficients as

$$k = \sum_{j \in \cup_{P_i \in \overline{\mathcal{P}}} T_i} c_j \cdot s_j.$$

**Definition 4** ( $\{-1, 0, 1\}$ -LISSS). *Let  $\mathcal{P} = \{P_1, \dots, P_T\}$  be a set of parties, and let  $\mathbb{S}$  be a class of efficient access structures on  $\mathcal{P}$ . Any LISSS scheme  $\text{SS} = (\text{SS.Share}, \text{SS.Combine})$  as defined above is a  $\{-1, 0, 1\}$ -LISSS if it is guaranteed that for any set of “recovery coefficients”  $\{c_j\}$  generated by **SS.Combine** (on input the set of shares corresponding to a qualified subset of parties  $\overline{\mathcal{P}} \in \mathbb{A}$  for an access structure  $\mathbb{A} \in \mathbb{S}$ ), we must have  $c_j \in \{-1, 0, 1\}$ .*

In this paper, we use a special instance of  $\{-1, 0, 1\}$ -LISSS, called the Benaloh-Leichter LISSS [DT06]. We expand more on Benaloh-Leichter LISSS in Section 4.3.

## 4 Our Proposal: Torus-FHE with Threshold Decryption

In this section, we present our construction of first practical threshold FHE. We introduce two protocols - threshold secret sharing of the decryption key and threshold decryption, to realize our final ThFHE. Along the way, we describe our two main theoretical contributions - an extension of the standard LISSS secret sharing scheme due to Benaloh and Leichter [DT06] to support the secret key structure which consists of binary polynomials, and the usage of Rényi Divergence based analysis to achieve only a small polynomial blowup in the noise level for our proposed threshold ThFHE built upon Torus-FHE scheme. We first

describe the generic decryption algorithm of any Ring-LWE based FHE scheme and then build its thresholdized construction. Our security proofs rely on the hardness of the LWE problem in the ring setting with binary secrets (the Binary RLWE problem in short; see Appendix A for the formal definition).

**Remark.** We remark here that our threshold decryption technique can, in fact, be generalized to any lattice-based encryption scheme where the decryption procedure involves computing a linear function of the secret key (in particular, Regev-style decryption based on computing an inner-product of the ciphertext vector and the secret key vector). However, since our concrete goal is to realize a threshold version of the Torus-FHE scheme from [CGGI20], we keep our theoretical discussion aligned with the Torus-FHE scheme (and the Torus-FHE library) for ease of exposition.

#### 4.1 Decryption in Torus-FHE

For ease of exposition, we start with describing the generic decryption algorithm of a Ring-LWE based Torus-FHE scheme over a message space  $\mathcal{M} = \mathbb{T}[X]/(X^N + 1)$ . We assume TRLWE to be an instantiation of such a scheme, represented by a tuple of PPT algorithms as follows,

$$\text{TRLWE} = (\text{TRLWE.Gen}, \text{TRLWE.Enc}, \text{TRLWE.Eval}, \text{TRLWE.Dec}).$$

The scheme has two fixed parameters  $N$  and  $k$  to denote size of polynomials and number of polynomials respectively. The secret key (say,  $\mathbf{SK}$ ) in TRLWE has the following structure.

$$\mathbf{SK} = \left( \sum_{j=1}^N SK_{1,j}x^{j-1}, \dots, \sum_{j=1}^N SK_{k,j}x^{j-1} \right),$$

with  $SK_{i,j} \in \{0, 1\} \forall 1 \leq i \leq k, \forall 1 \leq j \leq N$ .

The ciphertext in TRLWE can be written as  $\text{CT} = (A, B)$ , where  $B = \sum_{i=1}^k A[i] \cdot \mathbf{SK}[i] + \mathbf{m} + e$ . Here  $\mathbf{m}$  is the underlying plaintext and  $A$  can be represented as

$$A = \left( \sum_{j=1}^N A_{1,j}x^{j-1}, \dots, \sum_{j=1}^N A_{k,j}x^{j-1} \right),$$

with each  $A_{i,j} \in \mathbb{T}$ . Also,  $A[i] \cdot \mathbf{SK}[i]$  is the polynomial multiplication between  $i^{\text{th}}$  polynomial of  $A$  and  $i^{\text{th}}$  polynomial of  $\mathbf{SK}$  modulo  $(x^N + 1)$ . In order to avoid notational complexity, we will henceforth use  $A \cdot \mathbf{SK}$  to denote  $\sum_{i=1}^k A[i] \cdot \mathbf{SK}[i]$  in the paper. And,  $e = \sum_{j=1}^N e_j x^{j-1}$  is RLWE noise polynomial with each  $e_j \leftarrow \mathcal{G}$ , where  $\mathcal{G}$  is a Gaussian distribution.

As our contribution focuses on distributed decryption of a Ring-LWE ciphertext and we rely on a public key adaptation [Rot11] of underlying FHE scheme to perform the encryption and evaluation operations, we do not discuss these algorithms (TRLWE.Enc, TRLWE.Eval) here. But TRLWE.Dec needs to be modified in order to support threshold decryption. We discuss original decryption algorithm here.

**TRLWE.Dec( $\mathbf{SK}$ , CT):** Given the secret key  $\mathbf{SK}$  and a ciphertext  $\text{CT} = (A, B)$ , the decryption algorithm proceeds in two steps as follows:

- $\text{TRLWE.Decode}_0(\mathbf{SK}, \text{CT})$ : On input ciphertext  $\text{CT}$  and secret key  $\mathbf{SK}$ , this step of the decryption calculates  $\Phi = B - A \cdot \mathbf{SK}$ , which is equal to  $\mathbf{m} + e$ . Here,  $\mathbf{m}$  is the underlying plaintext and  $e$  is a Torus ring-LWE noise polynomial.
- $\text{TRLWE.Decode}_1(\Phi)$ : This final step rounds up each of the  $N$  coefficients of  $\Phi$  to return the coefficients of the plaintext message  $\mathbf{m}$ .

The security of TRLWE follows from the hardness of the Binary Ring Learning with Errors (BRLWE) problem (see Appendix A for the formal definition). Note that although a reduction from binary LWE to LWE exists [Mic18], a reduction for its ring-variant is not yet known; nonetheless, binary RLWE is widely [BBPS19, BD20] believed to be computationally hard.

Our main contribution is a proposal for thresholdizing the decryption of the aforementioned TRLWE scheme. We focus on the specific case of  $(T, T)$ -threshold decryption and its security analysis based on Rényi Divergence in subsequent sections. The generalized treatment of  $(t, T)$ -threshold decryption appears in Appendix B.

## 4.2 Achieving $(T, T)$ -Distributed Decryption

Let us assume  $\mathcal{P} = \{P_1, \dots, P_T\}$  is the set of  $T$  parties and they are willing to perform  $\text{TRLWE.Dec}$  on a Torus Ring-LWE ciphertext  $\text{CT} = (A, B)$  in a distributed way. We are in the *dealer-based model*, i.e., we assume that a trusted dealer uses some secret sharing algorithm to distribute the Torus Ring-LWE secret  $\mathbf{SK}$  to each  $P_i$  as  $SH_i$ , such that  $\mathbf{SK} = \sum_{i=1}^T SH_i$ . In this context, each  $P_i \in \mathcal{P}$  individually performs the following steps:

- $\text{TRLWE.PartialDec}(SH_i, \text{CT})$ : On input of the secret share  $SH_i$  and the ciphertext  $\text{CT} = (A, B)$ , this algorithm generates partially decrypted ciphertext  $\text{part\_decrypt}_i = A \cdot SH_i + e_{sm}^i$ . Here,  $e_{sm}^i$  is the smudging noise polynomial added by  $P_i$ , where each coefficient of  $e_{sm}^i$  is sampled from the Gaussian smudging noise distribution  $\mathcal{G}_{sm}$  (we expand on the smudging noise subsequently in Section 4.4). The partial decryption  $\text{part\_decrypt}_i$  is then broadcast to the rest of the  $(T - 1)$  parties.
- $\text{TRLWE.Combine}(\{\text{part\_decrypt}_i\}_{i \in [T]}, \text{CT})$ : This algorithm takes as input of all the partially decrypted ciphertexts  $\text{part\_decrypt}_i$  (where  $i \in [T]$ ) and the ciphertext  $\text{CT} = (A, B)$ , and combines them as  $\Phi = B - \sum_{i=1}^T \text{part\_decrypt}_i$ . Note that  $\Phi$  is essentially  $(\mathbf{m} + e - \sum_{i=1}^T e_{sm}^i)$ .
- $\text{TRLWE.Decode}_1(\Phi)$ : On input of the phase  $\Phi$ , each of its  $N$  coefficients are rounded up to retrieve  $N$  coefficients of the message  $\mathbf{m}$ .

Clearly, this  $(T, T)$  distributed decryption is very specific, as participation of each party is mandatory to perform a distributed decryption. Next, we generalize this to  $(t, T)$  threshold decryption for any  $0 < t < T$ . Our proposal relies on a  $(t, T)$  threshold secret sharing, which is an extended version of the original Benaloh-Leichter LISSS [DT06] and is elaborated in Section 4.3. Due to page limitation we describe our proposed  $(t, T)$  threshold decryption algorithm in Appendix B.

### 4.3 Extending Benaloh-Leichter LISSS

We next aim to generalize the aforementioned threshold decryption protocol to support  $(t, T)$ -threshold decryption for any  $t \leq T$ , which requires an appropriate LISSS (see Section 3.4) to support  $(t, T)$ -threshold secret sharing. For this purpose, we resort to using Benaloh-Leichter LISSS [DT06], as it supports efficient final combination of partial decryptions, in contrast to multiplying partial decryptions with large Lagrange coefficients while using Shamir's secret sharing [Sha79], that leads to noise-blowup in the ciphertext and incorrect decryption. Original scheme shares a scalar secret, but the secret of TRLWE is composed of  $k$  number of  $N$ -sized binary polynomials as described in Section 4.1. Hence, we describe an extended version of Benaloh-Leichter  $(t, T)$ -threshold secret sharing scheme to support Torus-RLWE secret key sharing. Let us assume  $\mathbf{SK}$  is the Torus-RLWE secret key, which is to be shared among  $T$  parties belonging to the set  $\mathcal{P} = \{P_1, \dots, P_T\}$ . We first describe some pre-processing steps required for  $(t, T)$ -threshold secret sharing.

**Formation of Distribution Matrix  $M$ .** Formation of distribution matrix  $M$  depends upon the monotone Boolean formula (MBF<sup>1</sup>), representing a  $(t, T)$ -threshold access structure. Also, any MBF, being a combination of AND and OR of Boolean variables, we are able to construct distribution matrix of any monotone Boolean formula by taking care of the following three cases:

**A single Boolean variable.** Distribution matrix of each Boolean variable  $x_i$  is represented by  $I_k$ , the identity matrix of dimension  $k$ .

**AND-ing of two MBFs.** Let us suppose, matrix  $M_{f_a}$  and  $M_{f_b}$  are the distribution matrices for MBFs  $f_a$  and  $f_b$  respectively and have dimension  $d_a \times e_a$  and  $d_b \times e_b$  respectively. Then we form  $M_{f_a \wedge f_b}$  to represent  $f_a \wedge f_b$  as follows:

$$\begin{array}{|c|c|c|c|} \hline c_a^k & c_a^k & C_a & 0 \\ \hline 0 & c_b^k & 0 & C_b \\ \hline \end{array}$$

Here,  $c_a^k$  and  $c_b^k$  denote first  $k$  columns and  $C_a$  and  $C_b$  denote the rest of the columns of  $M_{f_a}$  and  $M_{f_b}$  respectively. Resulting  $M_{f_a \wedge f_b}$  has dimension  $(d_a + d_b) \times (e_a + e_b)$ .

**OR-ing of two MBFs.** Assuming matrices  $M_{f_a}$  and  $M_{f_b}$  of dimension  $d_a \times e_a$  and  $d_b \times e_b$  respectively to be the distribution matrices for Boolean formula  $f_a$  and  $f_b$  respectively. Then we form  $M_{f_a \vee f_b}$  of dimension  $(d_a + d_b) \times (e_a + e_b - k)$  to represent  $f_a \vee f_b$  as following:

$$\begin{array}{|c|c|c|} \hline c_a^k & C_a & 0 \\ \hline c_b^k & 0 & C_b \\ \hline \end{array}$$

Here,  $c_a^k$  and  $c_b^k$  denote first  $k$  columns of  $M_{f_a}$  and  $M_{f_b}$  respectively.  $C_a$  and  $C_b$  denote the rest of the columns of  $M_{f_a}$  and  $M_{f_b}$  respectively.

It can be easily verified that, the distribution matrix  $M$  for  $(t, T)$ -threshold secret sharing has dimension  $d \times e$ , where  $d = \binom{T}{t} kt$  and  $e = \left(\binom{T}{t} kt - \left(\binom{T}{t} - 1\right)k\right)$ .

**Formation of Share Matrix  $\rho$ .** Though  $\rho$  is a vector in the original scheme [DT06], in our extended version,  $\rho$  is a matrix with dimension  $e \times N$ . Its first  $k$  rows are populated

<sup>1</sup>By MBF, we refer to Boolean formulae having a single output and consisting of only AND and OR combination of variables.

from the coefficients of  $k$  binary polynomials in  $\mathbf{SK}$ . The rest of the rows of the matrix are filled uniformly randomly from  $\{0, 1\}$ .

**Sharing.** The number of  $t$ -sized subsets of  $\mathcal{P}$  is  $\binom{T}{t}$ . We enumerate over all these subsets and tag each of them with corresponding enumerating serial number and call it the `group_id`. Once the sharing process is complete, each party  $P_i$  gets  $\binom{T-1}{t-1}$  number of key shares to store, for each possible  $t$ -sized group, that  $P_i$  can belong to.

To differentiate among these key shares, we tag each key share with following two attributes:

- `party_id`: refers to which party the key share belongs to.
- `group_id`: refers to the  $t$ -sized group for the key share.

We provide the secret sharing algorithm in Algorithm 1.

At the end of this sharing procedure, total  $d = \binom{T}{t}kt$  rows of *shares* matrix, produces  $\binom{T}{t}t$  number of key shares, each tagged with specific `group_id` and `party_id`. The `findParties(gid, t, T)` procedure, mentioned in Algorithm 1, returns a list of party ids present in  $gid^{\text{th}}$   $t$ -sized group(subset) of  $\mathcal{P}$ .

**Reconstruction.** Any  $t$ -sized group of parties should be able to reconstruct  $\mathbf{SK}$ , with the help of the key shares, they have. Given a  $t$ -sized group  $\mathcal{P}' = \{P'_1, P'_2, \dots, P'_t\} \subset \mathcal{P}$ , each of the  $t$  parties will have one key share with `group_id` corresponding to  $\mathcal{P}'$ . Let us denote these  $t$  key shares as  $\{SH_1, SH_2, \dots, SH_t\}$ . We observe (Appendix D) that exactly one share among them will have non-binary coefficients in its  $k$  polynomials. We call the party, having non-binary key share, the `group_leader` of the  $t$ -sized group. In any  $t$ -sized group, the party with minimum value of `party_id` is the `group_leader`.

Now, without loss of generality, let us assume  $P'_1$  is the `group_leader` of  $\mathcal{P}'$  and its non-binary key share is  $SH_1$ . Then the secret  $S$  can be reconstructed as:  $\mathbf{SK} = SH_1 - \sum_{i=2}^t SH_i$ . Hence, recovery coefficient  $c_1$  is 1 for the `group_leader` and  $c_i$  is  $-1$  for each of other  $(t-1)$  parties. We exploit this reconstruction property in final combination stage of  $(t, T)$ -threshold decryption technique.

**Size of Secret Shares.** After applying  $(t, T)$ -threshold secret sharing on  $\mathbf{SK}$ , each party gets  $\binom{T-1}{t-1}$  key shares to store. For any  $t$ -sized group, the `group_leader`'s share size (in number of bits) is upper bounded by  $\lceil \log_2 t \rceil \cdot N \cdot k$ , and each of the other  $(t-1)$  parties has share of size exactly  $N \cdot k$  bits. This can be proved by close observation of the secret shares.

Using the aforementioned extension of the Benaloh-Leichter LISSS scheme, we generalize the distributed decryption protocol for Torus-FHE described in Section 4.2 to work for any general  $(t, T)$  for  $t \leq T$ . Due to lack of space in the body, the detailed description is deferred to Appendix B.

#### 4.4 Polynomial Modulus-to-Noise Ratio via Rényi Divergence

We now elaborate on our main theoretical contribution, namely, achieving a polynomial modulus-to-noise ratio (i.e. a polynomial ratio between the modulus  $q$  and the Ring LWE noise  $e$ ) for our proposed threshold version of Torus-FHE (abbreviated as TRLWE henceforth) via: (a) a novel usage of Gaussian smudging noise during partial decryption (as described



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**Algorithm 1**  $t$ -out-of- $T$  Secret Sharing

---

```
1: function SHARESECRET( $t, T$ )
2:    $shares \leftarrow M \cdot \rho$ 
3:    $row \leftarrow 1$ 
4:   while  $row \leq d$  do
5:      $gid \leftarrow \lceil row/kt \rceil$ 
6:      $pt \leftarrow \text{FINDPARTIES}(gid, t, T)$ 
7:     for  $i = 1$  to  $t$  do
8:        $rowcount \leftarrow row + (i - 1)k$ 
9:        $curr\_share \leftarrow \text{TRLWEKEY}()$  ▷ New TRLWE Key
10:      for  $j = 0$  to  $k - 1$  do
11:         $curr\_share[j] \leftarrow shares[rowcount + j]$ 
12:         $cur\_share.party\_id \leftarrow pt[i - 1]$ 
13:         $cur\_share.group\_id \leftarrow gid$ 
14:       $row \leftarrow row + kt$ 
```

---

earlier in Section 4.2 and Appendix B), and (b) application of Rényi Divergence for distinguishing problems with public sampleability property to prove the security of our proposed Threshold FHE scheme TRLWE (under our proposed security definition in Section 3.3) as well as to get efficient choice of parameters for the scheme.

**Our Approach: Rényi Divergence-based Analysis of Smudging Noise.** In this paper, due to our novel approach of using Gaussian smudging noise and then using a Rényi Divergence based analysis akin to that of [BLRL<sup>+</sup>18, TT15] as opposed to the statistical distance based analysis used in prior works [MW16, BGG<sup>+</sup>18], it suffices to sample the smudging noise from a Gaussian distribution with standard deviation only polynomially larger than the standard deviation of the Gaussian distribution pertaining to the RLWE noise. As a result, from a theoretical point of view, we obtain the *first ThFHE scheme with polynomial modulus to noise ratio*. From an implementation point of view, it leads to a massive improvement in practical performance of our prototype implementation in software (presented in Section 5). We expand on our approach below.

**Analyzing  $(T, T)$ -Distributed Decryption.** For the ease of exposition, we now describe the Rényi Divergence-based analysis of our proposed distributed decryption protocol for TRLWE for the special case of  $(T, T)$ -distributed decryption (described originally in Section 4.2). Due to lack of space in the body, we defer the analysis of the more general case of  $(t, T)$ -threshold decryption to Appendix C, but the main technical ideas are already captured here.

**The Adversarial Model.** Recall from Section 4.2 that for the case of  $(T, T)$ -distributed decryption, the Torus Ring-LWE secret  $\mathbf{SK}$  is linearly secret-shared across  $\{P_1, \dots, P_T\}$  as  $\mathbf{SK} = \sum_{i=1}^T SH_i$ , where party  $P_i$  holds the secret key share  $SH_i$ . Now consider a scenario where, as per our security definition in Section 3.3, an adversary  $\mathcal{A}$  corrupts all but one party (say party  $P_1$  without loss of generality), and gains access to the secret key shares of all of the corrupted parties (i.e.,  $SH_2, \dots, SH_T$ ). As per our security definition,  $\mathcal{A}$  receives honest encryptions of  $\ell$ -many plaintexts of its choice. Now suppose that  $\mathcal{A}$  issues  $Q$ -many partial decryption queries. In each query, it provides a new circuit  $C$  of bounded

depth to a challenger. The challenger computes a resultant ciphertext  $\widehat{\text{CT}} = (A, B)$  by homomorphically evaluating  $C$  on the  $\ell$  honest ciphertexts. The adversary  $\mathcal{A}$  is then allowed to see the partial decryption of  $\widehat{\text{CT}}$  by the honest party  $P_1$ , computed (in the “real” security game) as  $\text{part\_decrypt}_1 = A \cdot SH_1 + e_{sm}$ , where  $e_{sm}$  is the smudging noise polynomial added by party  $P_1$  (each coefficient is sampled from a Gaussian distribution  $\mathcal{G}_{sm}$  with standard deviation  $\sigma$ ).

**“Simulating” an Honest Partial Decryption.** We now construct a simulator  $\mathcal{S}$  that “simulates” a partial decryption on behalf of the honest party  $P_1$  *without* the knowledge of the partial decryption key  $SH_1$ , but simply from the knowledge of the underlying plaintext  $m$  and the knowledge of the corrupted partial decryption keys  $\{SH_j\}_{j \in [2, T]}$ . Before delving into the description of the simulator  $\mathcal{S}$ , we briefly motivate the construction of such a simulator  $\mathcal{S}$ . Observe that  $\mathcal{S}$  has no additional information beyond what  $\mathcal{A}$  already knows. So,  $\mathcal{A}$  is not able to distinguish  $\text{CT}_0$  from  $\text{CT}_1$ , i.e., the encryption of two plaintexts  $m_0$  and  $m_1$  of its choice, due to hardness of Binary Ring-LWE assumption, on which the original Torus-FHE scheme relies.

We now construct the simulator  $\mathcal{S}$  as follows. Given the ciphertext  $\widehat{\text{CT}} = (A, B)$ , the underlying plaintext message  $m$ , and the corrupted partial decryption keys  $\{SH_j\}_{j \in [2, T]}$ , the simulator  $\mathcal{S}$  outputs a “simulated” partial decryption

$$\text{part\_decrypt}_1^{\text{Sim}} = B - m - \sum_{i=2}^T A \cdot SH_i + e_{sm},$$

where  $e_{sm}$  is the smudging noise polynomial (again, each coefficient of this polynomial is sampled from a Gaussian distribution  $\mathcal{G}_{sm}$  with standard deviation  $\sigma$ ). Now, observe that, letting  $\gamma = B - m - \sum_{i=2}^T A \cdot SH_i$ , we have

$$\text{part\_decrypt}_1 = \gamma - e + e_{sm}, \quad \text{part\_decrypt}_1^{\text{Sim}} = \gamma + e_{sm},$$

where  $e$  is the RLWE noise polynomial embedded in  $\widehat{\text{CT}}$ .

**Rényi Divergence-based Analysis.** Let  $\eta$  be the set of fixed parameters instantiating the security game described in Section 3.3 as follows.

$$\eta = (\mathbf{PK}, \mathbf{SK}, \{\mathbf{SK}_i\}_{i \in [T]}, \{\mu_i\}_{i \in [\ell]}, \{C_i\}_{i \in [Q]}, m_0, m_1).$$

Let a distribution

$$D_b^\eta(r) = (\mathbf{PK}, \{\mathbf{SK}_i\}_{i \in [2, T]}, \{\text{CT}_i\}_{i \in [\ell]}, \{\widehat{\text{CT}}_i\}_{i \in [Q]}, \{p_i\}_{i \in [Q]}, m_0, m_1, \text{CT}_b)$$

represent the view of the adversary, when the challenger  $\mathcal{C}$  samples bit  $b$  in the challenge phase. The set of noise values  $r = \{r_i\}$  is used in computation of honest party  $P_1$ ’s partial decryption  $p_i = \gamma + r_i$  and each  $r_i$  is sampled either from distribution of  $(e_{sm} - e)$  or from distribution of  $e_{sm}$ , depending on whether  $\mathcal{C}$  provides real or simulated partial decryptions to  $\mathcal{A}$ .

Let  $\delta$  and  $\delta'$  denote the advantages with which  $\mathcal{A}$  distinguishes  $D_0^\eta(r)$  from  $D_1^\eta(r)$  in presence of real or simulated partial decryptions respectively. Assuming that the aforementioned distinguishing problems are “publicly sampleable” [BLRL<sup>+</sup>18], the relation below follows from known results in [BLRL<sup>+</sup>18]:  $\delta' \geq \frac{\delta}{4R_\alpha(\Psi|\Psi')} \cdot \left(\frac{\delta}{2}\right)^{\frac{\alpha}{\alpha-1}}$ , where  $\Psi$  and  $\Psi'$  denote the

distribution of  $(e_{sm} - e)$  and the distribution of  $e_{sm}$  respectively and  $R_a(\Psi||\Psi')$  is the Rényi divergence of order  $a$  between the distributions  $\Psi$  and  $\Psi'$  order  $a$ .

**Arguing Public Sampleability.** In order to invoke the aforementioned relation, we first need to argue that the aforementioned distinguishing problems satisfy the notion of public sampleability as defined in [BLRL<sup>+</sup>18]. Given a bit  $b' \in \{0, 1\}$  and a sample  $x$  from  $D_b^\eta(r)$ , we can publicly sample a fresh element  $x'$  of  $D_{b'}^\eta(r)$  by replacing the last component  $\text{CT}_b$  of  $x$  with  $\text{CT}_{b'} = \text{TRLWE.Enc}(\mathbf{PK}, m_{b'})$ . With fixed  $\eta$  and fixed set of noise values in  $r$ , rest of the components in  $x$  remain same in the output sample  $x'$ . Thus the distribution  $D_b^\eta(r)$  is indeed publicly sampleable.

**Completing the Proof.** We can now invoke known results from [TT15] and the *multiplicative property* of Rényi Divergence to argue that for any  $a \in (1, \infty)$ , we have,

$$R_a(\Psi||\Psi') \leq \exp\left(\frac{a \cdot \pi \cdot N \cdot \|e\|_\infty^2}{\sigma^2}\right),$$

where  $\|e\|_\infty$  denotes the infinity norm of the degree  $(N - 1)$ -RLWE noise polynomial  $e$ . Assuming that  $\|e\|_\infty \leq c\alpha$ , where  $c$  is some constant and  $\alpha$  is the standard deviation of RLWE noise distribution  $\mathcal{G}$ , we have,  $R_a(\Psi||\Psi') \leq \exp\left(\frac{a \cdot \pi \cdot N \cdot c^2 \cdot \alpha^2}{\sigma^2}\right)$ .

Finally, for the scenario where the adversary  $\mathcal{A}$  sees a maximum of  $Q = \text{poly}(\lambda)$  such partial decryption samples, we invoke the multiplicative properties of Rényi Divergence to state the following:

$$R_a(\Psi||\Psi') \leq \exp\left(\frac{a \cdot \pi \cdot Q \cdot N \cdot c^2 \cdot \alpha^2}{\sigma^2}\right).$$

**Parameter Choices (Lower Bounds).** At this point, we are ready to propose the asymptotic parameter choices for our ThFHE scheme TRLWE supporting  $(T, T)$ -threshold decryption. Assume that the adversary  $\mathcal{A}$  sees at most  $Q = \text{poly}(\lambda)$  partial decryption samples, let  $\sigma$  and  $\alpha$  be the standard deviation parameters for the Gaussian distributions pertaining to the smudging noise and RLWE noise, respectively, and let  $c$  be a constant such that  $|e| \leq c\alpha$  ( $e$  being the RLWE noise polynomial). It suffices for us to choose  $\sigma$  such that

$$\sigma \geq c \cdot \alpha \cdot \sqrt{Q \cdot N},$$

since this yields  $R_a(\Psi||\Psi') \leq \exp(a \cdot \pi)$ , and hence

$$\delta' \geq \frac{\delta}{4} \cdot \left(\frac{\delta}{2}\right)^{\frac{a}{a-1}} \cdot \exp(-a \cdot \pi).$$

Taking any value of  $a > 1$  yields the desired condition on  $\delta$  and  $\delta'$ , i.e., non-negligible  $\delta$  would result in non-negligible  $\delta'$ . Note that it suffices for  $\sigma$  to be *only polynomially larger* than  $\alpha$ . Hence, our scheme is secure whenever  $\sigma \geq c \cdot \alpha \cdot \sqrt{Q \cdot N}$ .

**Parameter Choices (Upper Bounds).** It remains to answer the question of *upper bounding* the amount of smudging noise that each party can add, and here we allow the maximum possible smudging noise that does not affect the correctness of the distributed decryption protocol. Formally, let  $q = 2^{\lambda_1}$  be the TRLWE modulus (or equivalently, suppose that the Torus-FHE scheme supports a maximum precision of  $\lambda_1$  bits) and let  $p = 2^{\lambda_2}$  be the size of

the space of message-polynomial coefficients (or equivalently, suppose that the Torus-FHE scheme supports message-polynomial coefficients with a precision of  $\lambda_2$  bits) such that  $p \leq q$ . At a high level, to ensure the correctness of  $(T, T)$ -distributed decryption, we need the *total* noise to be upper bounded by  $\Delta/2$ , where  $\Delta = q/p = 2^{\lambda_1 - \lambda_2}$ . More formally, for correctness of  $(T, T)$ -distributed decryption to hold, we must have  $\|e\|_\infty + T \cdot \|e_{sm}\|_\infty < \Delta/2$ , where  $\|\cdot\|_\infty$  denotes the infinity norm of some polynomial. Since  $\|e_{sm}\|_\infty > \|e\|_\infty$  (by the lower bound argument presented above), we choose

$$\|e_{sm}\|_\infty < \Delta/2(T + 1).$$

Combined with the lower bounds imposed by the Rényi divergence-based security analysis presented earlier, we thus avoid the super-polynomial modulus-to-noise ratio (ratio between modulus  $q$  and any coefficient of RLWE noise polynomial  $e$ ) incurred by all prior works on ThFHE, thereby yielding the *first ThFHE scheme with polynomial modulus-to-noise ratio*.

The above Rényi Divergence-based analysis immediately generalizes to  $(t, T)$ -threshold decryption for any  $t \leq T$ . The detailed analysis for the general case appears in Appendix C.

## 5 Software Implementation and Experimental Evaluation

We now describe a prototype implementations of our  $(t, T)$ -threshold decryption scheme over Torus-FHE on two extreme varieties of computing platforms - a high-end x86-based server, and a low-end resource-constrained ARM-based platform<sup>1</sup>. We report the implementation and performance of TRLWE, a public-key adaptation of the ThFHE scheme built upon Torus-FHE library equipped with our proposed threshold decryption mechanism. Although Torus-FHE library implements a symmetric-key version of the underlying FHE scheme, we use the idea of [Rot11] to implement its public-key version and extend it to support threshold decryption in order to get the desired implementation of TRLWE. We stress that this is, to the best of our knowledge, the first practical implementation of any ThFHE scheme. In particular, while a very recent work [BS23] also *theoretically* proposes a threshold FHE scheme with polynomial modulus-to-noise ratio, they do not report any implementation or performance results to the best of our knowledge.

In our setting, the threshold secret sharing is done by a trusted cloud server with sufficient computational resources. Subsequently, homomorphic evaluations also happen on encrypted data stored at the cloud server. The key focus of our implementation is in realizing the proposed threshold decryption algorithm on resource-constrained handheld devices; hence our experiments and evaluation focus purely on the performance of our threshold decryption implementation.

For the sake of completeness, we implement our threshold decryption algorithm on two kinds of platforms, lying at two extreme ends of the spectrum of computational capabilities: (a) a high-end workstation with an Intel(R) Xeon(R) CPU E5-2690 v4 CPU (2.60GHz clock-frequency), 28 physical cores, and 128GB RAM, and (b) a low-end Raspberry Pi 3b board with a Quad Core 1.2GHz Broadcom BCM2837 64bit CPU and 1 GB RAM running Raspberry Pi OS Lite (Linux kernel version: 5.10.63-v7+).

<sup>1</sup>Our implementation code and additional (low-level) implementation details are available at: [https://anonymous.4open.science/r/ThFHE\\_artifacts-2FD3](https://anonymous.4open.science/r/ThFHE_artifacts-2FD3)

Our first implementation is optimized for high performance and, as verified by our experiments, yields extremely fast threshold decryption times. Our second implementation is optimized for extracting maximum performance out of a low-end resource-constrained platform, and yields reasonably practical threshold decryption times. Before describing our evaluation, we present some more details of our implementation.

## 5.1 Implementation Details

We implement the natural public-key analogue of the original `Torus-FHE` library (which implemented a secret-key version of the scheme), while leaving implementation of the homomorphic evaluation process unchanged. This makes our implementation cross-compatible with other libraries (e.g., `NuFHE`) that build directly upon `Torus-FHE`. Since our core contribution lies in thresholdizing the decryption process, which only requires the secret key, we keep our discussion limited to generating and sharing the secret key.

We extend the `Torus-FHE` library to support threshold key generation and threshold decryption. We use the `Torus-RLWE` secret key generation routine to generate the secret key with a set of parameters that is chosen by relying on our proposed Rényi divergence-based security argument (see Appendix C.2 for the detailed analysis). In particular, this analysis enables a polynomial modulus-to-noise ratio, which crucially allows our implementation to be practically deployable on a resource-constrained platform.

Once the key has been generated, we build the distribution matrix  $M$  and share matrix  $\rho$  (see Section 4.3). The steps to create the distribution matrix as described in Section 4.3, when implemented directly in software, results in a recursive implementation, which potentially results in high memory access overheads, and is unsuitable for resource-constrained platforms. However we can avoid these excess function call overheads and generate the distribution matrix iteratively in one go by exploiting a regular pattern, which is there in the distribution matrix inductively for any  $(t, T)$ -threshold access structure.

For the partial and threshold decryption functions, we have two implementations. The first implementation targets a high-end processor, and directly leverages `Torus-FHE` APIs for fast polynomial multiplication using Fast Fourier Transform (FFT), as is required in the partial decryption phase. The other is a portable implementation suited for low-end resource-constrained handheld devices. In particular, the latter replaces the FFT polynomial multiplication, which depends on x86 AVX instructions for efficiency, with a naïve school-book multiplication. This is done to keep the implementation as architecture-agnostic and lightweight as possible. In the porting process, we have removed multiple dynamic memory allocation steps to achieve better memory efficiency. Also, our observation that each of the participating parties except one receives a binary key share through  $(t, T)$ -threshold secret sharing significantly contributes to reduce the cycle counts in polynomial multiplication in both implementations.

## 5.2 Experimental Evaluation

In this section we validate our proposed threshold decryption technique by an implementation over `Torus-FHE` library and the steps involved are summarized in Algorithm 2. The output of `BootstrappedOR` in step 3 here is a `Torus-LWE` ciphertext and we convert it to

Table 1: Parameters used in experimental setting

Parameter	Value
$k$ (Number of polynomials in Torus-LWE ciphertext)	1
$n$ (Torus-LWE dimension)	1024
$N$ (Degree of Torus-RLWE polynomial is $(N - 1)$ )	1024
$\alpha$ (Standard deviation of Torus-RLWE noise)	$2^{-25}$
$\sigma$ (Standard deviation of smudging noise)	$2^{-6}$

a Torus-RLWE ciphertext in step 4, in order to support packing of multiple plaintext bits together.

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**Algorithm 2** Software implementation of cryptosystem with  $(t, T)$ -threshold decryption

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**Input:**  $inp_1 \in \mathbb{N}, inp_2 \in \mathbb{N}, t \in \mathbb{N}, T \in \mathbb{N}, \mathcal{P} \subset [1, T]$  s.t  $|\mathcal{P}| = t$

**Output:**  $outp \leftarrow inp_1 \vee inp_2$

- 1:  $(LweSK, LwePK) \leftarrow \text{LWEKEYGEN}$
  - 2:  $ciph_1 \leftarrow \text{ENCRYPT}(LwePK, inp_1), ciph_2 \leftarrow \text{ENCRYPT}(LwePK, inp_2)$
  - 3:  $eval\_res \leftarrow \text{BOOTSTRAPPEDOR}(ciph_1, ciph_2, LwePK)$
  - 4:  $(ring\_cipher, RLweSK) \leftarrow \text{CONVERTLWETORLWE}(eval\_res, LweSK)$
  - 5:  $\text{SHARESECRET}(RLweSK, t, T) \quad \triangleright$  Now all parties get their key shares. Each party  $i \in \mathcal{P}$  calculates  $outp$  on its own.
  - 6:  $outp \leftarrow \text{THRESHOLDDECRYPT}(ring\_cipher, \mathcal{P}, t, T, i)$
  - 7: **return**  $outp$
- 

In accordance with our intended use-case, we experimentally evaluate steps 1 through 5 of Algorithm 2 on a high-end server, and step 6 on both the high-end server and a low-end resource-constrained handheld device. In particular, in our experiments we have measured the time taken by steps 5 and 6. Note that step 6 includes both partial decryption and final combination. The concrete parameters used in our experiments are listed in Table 1.

The choices for  $n$ ,  $N$  and standard deviation of Torus-RLWE noise are compatible with the Torus-FHE library. For smooth conversion in step 4 of Algorithm 2, we fix  $k = 1$ . We choose the last parameter based on a lower-bound given by our Rényi divergence based analysis and an upper-bound imposed by correctness.

Figures 1 and 2 show the secret key sharing time, the partial decryption time, the final decryption time, and the plain decryption time on a high-end workstation in terms of milliseconds and clock cycles respectively, while Figures 3 and 4 show the partial and final decryption times in milliseconds and clock cycle counts respectively on the low-end Raspberry Pi 3b platform. The partial decryption time in all the figures follow a constant trend as in our use case, it is done parallelly in individual devices and the vector or polynomial sizes do not change with the number of parties. We emphasize that, as a direct consequence of the efficient parameter choices for threshold FHE enabled by our Rényi Divergence-based analysis, the threshold decryption timing is practical even on a highly resource-constrained ARM-based platform.

Finally, Figure 5 shows that the time required for threshold decryption<sup>1</sup> is only slightly

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<sup>1</sup>We report the *end-to-end* threshold decryption time, wherein we add up the time for a single partial

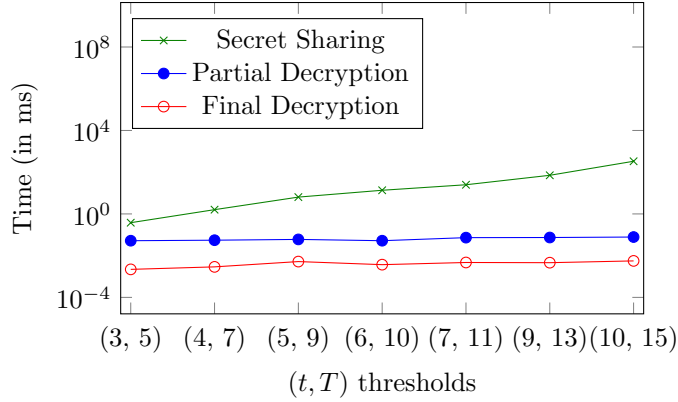


Figure 1: Secret Sharing and Threshold Decryption Time in High-End Server. Note that the y-axis is in logarithmic scale.

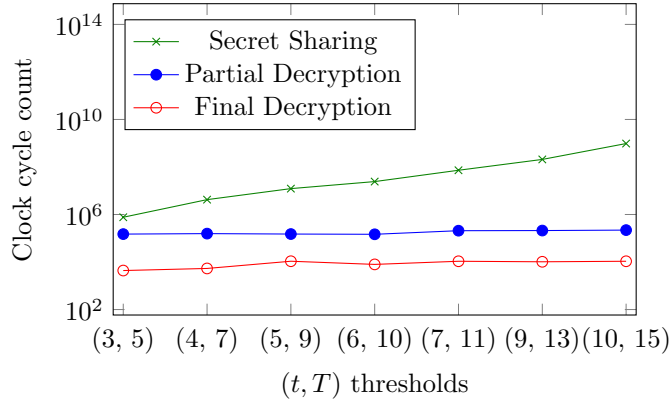


Figure 2: Secret Sharing and Threshold Decryption Clock Cycles in High-End Server

higher than that of plain decryption using the Torus-FHE library, for both the high-end workstation and the resource constrained device. In other words, our proposed threshold decryption procedure incurs only minimal overhead over the plain decryption algorithm specified in the original Torus-FHE library. To the best of our knowledge, this is the first prototype of threshold FHE with the capability of executing the threshold decryption algorithm practically on resource-constrained platforms.

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decryption (since the partial decryption phase is meant to be done in parallel by each participating party) and the time for the final re-combination of partial decryptions. Also, the overall threshold decryption time is dominated by the former component, which is independent of  $(t, T)$ , and hence, the overall threshold decryption time in Figure 5 grows only minimally with increasing  $(t, T)$  values.

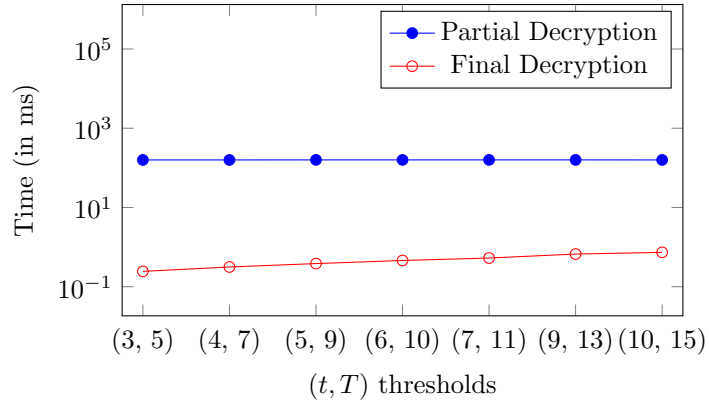


Figure 3: Threshold Decryption Time in Handheld device

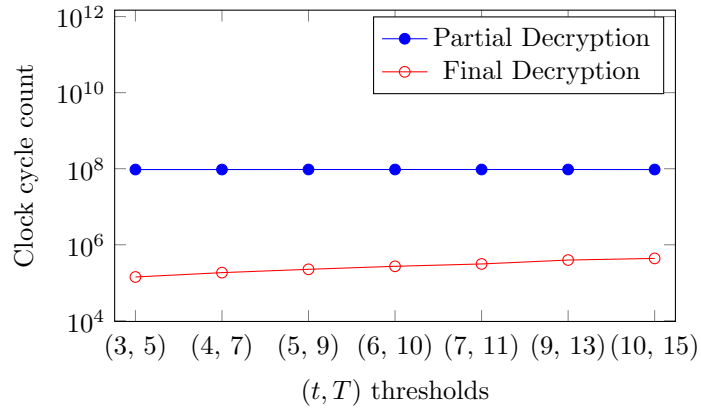


Figure 4: Threshold Decryption Clock Cycles in Handheld device

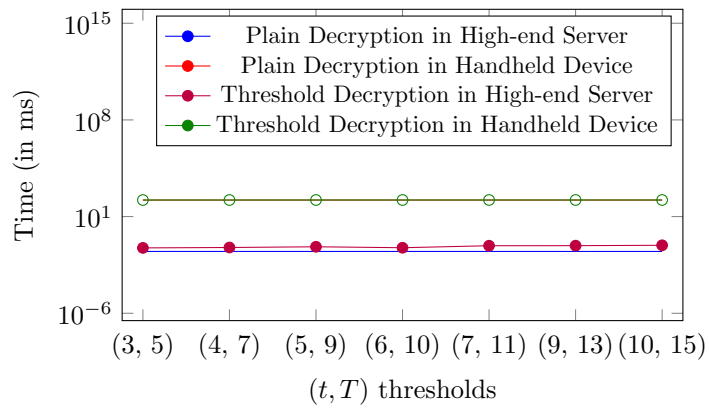


Figure 5: Time comparison between Plain Decryption and Threshold Decryption



## 6 Case-Study: Computing over Encrypted Medical Data

In this section, we use our proposed ThFHE scheme over the Torus to realize an end-to-end usecase of outsourced computations over encrypted medical datasets, where the final outcome is computed in a distributed manner by multiple entities (e.g. doctors, research laboratories, or other medical practitioners). Concretely, we illustrate the efficacy of our proposal via experiments evaluating encrypted computations over a real medical database, as well as distributed decryptions of the computed result on resource-constrained handheld devices, where both the encryption and distributed decryption operations are performed using our proposed ThFHE scheme. The encrypted computation that we perform is a  $K$ -Nearest Neighbours (KNN) classification [SCK14] that outputs an encrypted prediction bit indicating the possibility of cardiovascular disease (where the classification is done based on a patient’s encrypted medical records and pre-computed encrypted training data).

### 6.1 Encrypted KNN Computation

The encrypted KNN algorithm (given in Algorithm 3) takes as input: (a) an encrypted set of test data (which is to be predicted), (b) an encrypted set of training data (to train the KNN algorithm), (c) the bootstrapping key  $bk$ , which can be considered as a part of public key  $k$  in the algorithm and (d) KNN parameter  $K$  to output an encrypted single prediction bit. Following the approach outlined in [RC19], we sub-divide the encrypted KNN computation algorithm into three parts as described below.

**Encrypted Manhattan Distance Computation.** First, the encrypted Manhattan distances between the testing data and all the training data are (homomorphically) computed and stored in the *distant* variable. The Manhattan distance is preferred over other distances to avoid the “curse of dimensionality” problem in machine learning [AHK01]. To compute the difference between two ciphertexts ( $\text{ThFHE}_{DIFF}$  between  $\text{Encrypt}(k, \text{Plain}_1)$  and  $\text{Encrypt}(k, \text{Plain}_2)$ ), we use the 2’s complement form representation.

**Sorting over Encrypted Data.** In this step, the neighbours are sorted in ascending order based on the calculated distances. The encrypted-bubble-sort implementation directly uses encrypted-comparison and sorting techniques from prior-works, including [RC19, CSS20, CS20, ÇDSS15]. Our encrypted bubble sort implementation intakes the bootstrapping key, the patient’s encrypted data and their corresponding encrypted Manhattan distances, and outputs the sorted patient data based on these encrypted distances.

**Prediction over Encrypted Data.** The (encrypted) decisions output by the KNN computation are added to get  $\text{Encrypt}(k, \text{count})$  (line 7, Algorithm 3), which is then compared homomorphically with the threshold value ( $\text{Encrypt}(k, \lfloor K/2 \rfloor)$ ) to arrive at the (encrypted) decision. The final plaintext decision is recovered via threshold decryption.

### 6.2 Experimental results

We now present experimental results illustrating the practical performance of the encrypted KNN algorithm and the associated threshold decryption computation over a real-world

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**Algorithm 3** KNN over encrypted medical data

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**Input:**  $test\_data = \text{ENCRYPT}(k, test\_patient)$ ,  
 $train\_data = \{\text{ENCRYPT}(k, patient_1), \dots, \text{ENCRYPT}(k, patient_n)\}$ ,  
 $bk$  = bootstrapping key,  $K$  = KNN parameter.  
**Output:**  $decisional\_bit = \text{ENCRYPT}(k, predicted\_bit)$

- 1: Initialize a Torus-LWE ciphertext array  $distant$  of size  $n$
- 2: **for**  $i = 1$  **to**  $n$  **do**
- 3:    $distant[i] \leftarrow \text{MANHATTAN}(test\_data, train\_data[i], bk)$
- 4:  $sorted\_train\_data \leftarrow \text{BUBBLESORT}(distant, train\_data, bk)$
- 5: Initialize a counter ciphertext  $count = \text{ENCRYPT}(k, zero)$  to count the decision of K-Nearest Neighbours
- 6: **for**  $i = 1$  **to**  $K$  **do**
- 7:    $count \leftarrow count + \text{DECISION}(sorted\_train\_data[i])$
- 8: Initialize a Torus-LWE variable  $threshold = \text{ENCRYPT}(k, \lfloor K/2 \rfloor)$
- 9:  $decisional\_bit \leftarrow \text{DIFFERENCE}(threshold, count, bk)$
- 10: **return**  $decisional\_bit$

---

Table 2: Encrypted KNN execution time

$K$	Neighbour size	Prediction time without OpenMP (in minutes)	Prediction time with OpenMP (in minutes)	Prediction time with OpenMP (cycle count in $10^{12}$ )
5	10	154.88	15.96	2.49
	20	369.90	29.71	4.64
	30	558.36	49.96	7.79
	40	850.70	60.11	9.38
	50	1062.86	72.90	11.37
7	10	194.81	18.50	2.89
	20	431.41	33.66	5.25
	30	726.13	50.08	7.81
	40	975.01	63.83	9.96
	50	1282.08	79.78	12.45
9	10	235.30	24.23	3.78
	20	527.98	46.66	7.28
	30	844.36	60.55	9.45
	40	1146.26	80.81	12.61
	50	1498.43	99.31	15.49

medical dataset. The experiments were performed on an Intel(R) Xeon(R) E5-2690 v4 processor with 2.60GHz clock, 132GB of RAM and 56 cores.

**Dataset Used.** For our experiments, we use a publicly available cardiovascular disease

related dataset<sup>1</sup>. The dataset contains 70000 data instances and 12 features. For our experiments, we fix one of these features as target feature to be predicted based upon the data of the remaining 11 features.

**Performance.** Table 2 shows the execution time of KNN algorithm in two variants, with (using OpenMP) and without any parallel processing techniques. The OpenMP version has big advantage of parallelizing multiple loops to facilitate smaller execution time as shown in Table 2 for  $K = 5, 7$ , and 9. The number of OpenMP threads used during each execution is equal to the neighbour size listed in Table 2.

## 7 Conclusion and Future Work

We presented the design, analysis and practical implementation for a novel threshold FHE scheme from the hardness of Binary Ring-LWE with polynomial modulus-to-noise ratio. We showed, for the first time, that threshold FHE can actually be deployed in a fast, scalable and reasonably resource-efficient manner for real-time applications via benchmarking experiments on two extreme varieties of computing platforms - a high-end x86-based server and a low-end resource-constrained ARM-based platform. We showcased an end-to-end implementation of our proposed system and used it for fast, scalable yet secure  $k$ -nearest-neighbor computations over encrypted medical data outsourced to a cloud service provider.

Our work gives rise to many interesting directions of future research. In particular, we leave it as an open question to extend our Rényi divergence-based security analysis techniques to the setting of multi-key FHE with threshold decryption, for which all known realizations still require super-polynomial modulus-to-noise ratio.

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<sup>1</sup><https://www.kaggle.com/sulianova/cardiovascular-disease-dataset>

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## A The LWE Assumption and its Variants

In this section we recall the Learning with Errors (LWE) assumption and some of its variants, including the Ring LWE (RLWE) assumption and the Binary RLWE assumption.

**The LWE Assumption.** Let  $\lambda \in \mathbb{N}$  be a security parameter, and let  $q, n, m = \text{poly}(\lambda)$ . For each  $i \in [m]$ , let

$$\mathbf{a}_i \leftarrow \mathbb{Z}_q^n, \quad b_i = \mathbf{a}_i \cdot \mathbf{s} + e_i, \quad u_i \leftarrow \mathbb{Z}_q,$$

where  $\mathbf{s} \leftarrow \mathbb{Z}_q^n$  is a uniformly sampled secret vector,  $e_i \leftarrow \psi$  where  $\psi$  is a Gaussian noise distribution over  $\mathbb{Z}_q$ , and  $\mathbf{a}_i \cdot \mathbf{s}$  denotes the vector dot-product between the vectors  $\mathbf{a}_i$  and  $\mathbf{s}$ . The LWE hardness assumption states that, for any probabilistic polynomial-time (PPT) adversary  $\mathcal{A}$ , the following holds:

$$|\Pr[\mathcal{A}(\{\mathbf{a}_i, b_i\}_{i \in [m]}) = 0] - \Pr[\mathcal{A}(\{\mathbf{a}_i, u_i\}_{i \in [m]}) = 0]| < \text{negl}(\lambda).$$

**The RLWE Assumption.** Let  $\lambda \in \mathbb{N}$  be a security parameter, and let  $q, N, m = \text{poly}(\lambda)$ . For each  $i \in [m]$ , let

$$A_i(X) \leftarrow \mathbb{Z}_q[X]/(X^N + 1), \quad B_i(X) = A_i(X) \cdot S(X) + E_i(X),$$

$$U_i(X) \leftarrow \mathbb{Z}_q[X]/(X^N + 1),$$

where  $S(X) \leftarrow \mathbb{Z}_q[X]/(X^N + 1)$  is a uniformly sampled secret polynomial,  $E_i(X) \leftarrow \psi[X]/(X^N + 1)$  where  $\psi$  is a Gaussian noise distribution over  $\mathbb{Z}_q$ , and  $A_i(X) \cdot S(X)$  denotes the polynomial multiplication modulo  $(X^N + 1)$  between  $A_i(X)$  and  $S(X)$ . The Ring LWE (RLWE) hardness assumption states that, for any PPT adversary  $\mathcal{A}$ , we have:

$$\left| \Pr[\mathcal{A}(\{A_i(X), B_i(X)\}_{i \in [m]}) = 0] - \Pr[\mathcal{A}(\{A_i(X), U_i(X)\}_{i \in [m]}) = 0] \right| < \text{negl}(\lambda).$$

**Binary RLWE.** The Binary RLWE (BRLWE) hardness assumption is a variant of the RLWE hardness assumption described above where, the secret key polynomial  $S(X)$  is sampled from  $\mathbb{B}[X]/(X^N + 1)$  as opposed to  $\mathbb{Z}_q[X]/(X^N + 1)$ , where  $\mathbb{B} = \{0, 1\}$ . Note that although an equivalence between the LWE with binary secrets assumption and the standard LWE assumption is known [Mic18], a similar result for BRLWE and RLWE is not known to the best of our knowledge. However, the BRLWE hardness assumption is widely believed to hold [BBPS19, BD20].

## B Generalized Threshold Decryption Protocol

In this section, we describe the generalized  $(t, T)$ -threshold decryption algorithm for our proposed threshold Torus-FHE. Here, we use the extended version of Benaloh-Leichter LISSS, proposed in Section 4.3 to share the secret key across the various parties (as opposed to a simple additive sharing in the  $(T, T)$ -case in Section 4.2). Consequently, we need to modify the `TRLWE.PartialDec` and `TRLWE.Combine` algorithms to enable correct and efficient decryption by any  $t'$ -sized subset of the  $T$  parties for  $t' \geq t$ .

Let  $\mathcal{P} = \{P_1, \dots, P_T\}$  be any set of  $T$  parties and let  $\mathcal{P}' = \{P_{id_1}, \dots, P_{id_t}\} \subset \mathcal{P}$  be a  $t$ -sized subset of  $\mathcal{P}$  with `group_id`  $j$ , authorized to threshold-decrypt a ciphertext  $\text{CT} = (A, B)$ . Also, without loss of generality, let us assume  $id_1 < \dots < id_t$ , so that  $P_{id_1}$  is the `group_leader` of  $\mathcal{P}'$ . We begin by assuming that all of the  $T$  parties in  $\mathcal{P}$  have already received their key shares after successful execution of the  $(t, T)$ -threshold secret sharing scheme on  $\mathbf{SK}$  (Section 4.3). Hence, each  $P_{id_i} \in \mathcal{P}'$  has exactly one key share corresponding to `group_id`  $j$ . We denote these  $t$  key shares as  $\{SH_{id_1, j}, \dots, SH_{id_t, j}\}$ . Let us recall from Section 4.3 that,

$$\mathbf{SK} = SH_{id_1, j} - \sum_{i=2}^t SH_{id_i, j}.$$

The threshold decryption of CT consists of the following steps, performed by each  $P_{id_i} \in \mathcal{P}'$  individually:

- `TRLWE.PartialDec`( $SH_{id_i, j}, \text{CT}$ ): On input Torus Ring-LWE ciphertext CT and a key share  $SH_{id_i, j}$ ,  $P_{id_i}$  calculates the following:

$$part\_decrypt_{id_i} = A \cdot SH_{id_i, j} + e_{sm}^{id_i},$$

where  $e_{sm}^{id_i}$  is a smudging noise polynomial and each coefficient of  $e_{sm}^{id_i}$  is sampled from a Gaussian smudging noise distribution  $\mathcal{G}_{sm}$ . Then,  $P_{id_i}$  broadcast  $part\_decrypt_{id_i}$  to rest of the  $(t - 1)$  parties.

- `TRLWE.Combine`( $\{part\_decrypt_{id_i}\}_{i \in [t]}, \text{CT}$ ): On input all  $t$  partial decryptions, each party calculates the phase

$$\phi = B - (part\_decrypt_{id_1} - \sum_{i=2}^t part\_decrypt_{id_i}),$$

where  $\phi$  equals  $\mathbf{m} + e - e_{sm}^{id_1} + \sum_{i=2}^t e_{sm}^{id_i}$ .

- `TRLWE.Decode1`( $\phi$ ): Each of the  $N$  coefficients of  $\phi$  is rounded up to extract the coefficients of the message  $\mathbf{m}$ .

**Properties.** The correctness and compactness of the proposed  $(t, T)$ -threshold decryption scheme directly follows from the proofs of threshold-FHE scheme mentioned in Section 3.3. Its proof of security is presented in Appendix C.

## C Proof of security and parameter choices of generalized $(t, T)$ -threshold FHE

In this section, we prove that our proposed  $(t, T)$ -threshold FHE scheme satisfies the notion of security, as described in Section 3.3 using a novel Rényi divergence-based technique. More specifically, we show that distinguishing advantage of an adversary in the security game in real world is related to its distinguishing advantage in a simulated world game. Subsequently, we also provide a detailed discussion on the asymptotic choice of noise parameters for our proposed scheme.

## C.1 Proof of Security

Let us recall the instantiation of our proposed  $(t, T)$ -threshold FHE TRLWE from Section 4.1 and its threshold decryption technique from Appendix B. Also recall the notion of security in the form of a game between adversary  $\mathcal{A}$  and challenger  $\mathcal{C}$  from Section 3.3 with respect to our proposed threshold FHE scheme TRLWE.

**A Close Look at Partial Decryptions.** First we take a close look at the partial decryption component returned by the challenger  $\mathcal{C}$  in the query phase 2 of the security game. We allow the corrupted subset  $S$  in the security game to be of maximal size, i.e.,  $|S| = (t - 1)$ . Let us assume for some  $i \in [Q]$ ,  $\widehat{\text{CT}}_i$  is the evaluated ciphertext from  $i^{\text{th}}$  circuit  $C_i$  on input set of ciphertexts  $\text{CT}_{i \in [\ell]}$ . A partial decryption of  $\widehat{\text{CT}}_i = (\hat{A}_i, \hat{B}_i)$  by some honest party  $P_j \notin S$  corresponds to a  $t$ -sized group  $P_j \cup S$  with `group_id`  $g$  and we denote it with  $p_{i,j}$ . Challenger  $\mathcal{C}$  computes  $p_{i,j}$  as follows,

$$p_{i,j} = \hat{A}_i \cdot SH_{j,g} + e_{sm},$$

where  $SH_{j,g}$  is  $P_j$ 's secret share corresponding to `group_id`  $g$  and  $e_{sm}$  is a smudging noise polynomial with each coefficient sampled from  $\mathcal{G}_{sm}$ . However using the linear reconstruction property of Benaloh-Leichter  $\{0, 1\}$ -LISSS (Section 4.3), we can alternatively express  $p_{i,j}$  as following,

$$p_{i,j} = \hat{B}_i - m_i - e - \sum_{P_k \in S} \hat{A}_i \cdot SH_{k,g} + e_{sm},$$

where  $e$  is RLWE noise polynomial with each coefficient sampled from  $\mathcal{G}$ . Letting  $\gamma_i = \hat{B}_i - m_i - \sum_{P_k \in S} \hat{A}_i \cdot SH_{k,g}$ , we get

$$p_{i,j} = \gamma_i + r_{i,j},$$

such that  $r_{i,j}$  is sampled from distribution of  $(e_{sm} - e)$ .

We can publicly simulate the partial decryption by some honest party  $P_j \notin S$  as follows,

$$p'_{i,j} = \gamma_i + r'_{i,j},$$

where  $r'_{i,j}$  is sampled from distribution of  $e_{sm}$ .

**Defining Some Distributions.** Recall the security game between the challenger  $\mathcal{C}$  and the adversary  $\mathcal{A}$  in Section 3.3 with respect to our proposed scheme TRLWE. Let  $\eta$  is the set of some fixed parameters in the game as follows,

$$\eta = (\mathbf{PK}, \mathbf{SK}, \{\mathbf{SK}_i\}_{i \in [T]}, S, \{\mu_i\}_{i \in [\ell]}, \{C_i\}_{i \in [Q]}, m_0, m_1).$$

Let  $r = \{r_{i,j}\}_{i \in [Q], P_j \notin S}$  be a set of noise parameters. We define the distribution  $D_b^\eta(r)$  parameterized by  $\eta$  as follows,

$$D_b^\eta(r) = (\mathbf{PK}, \{\mathbf{SK}_i\}_{P_i \in S}, \{\text{CT}_i\}_{i \in [\ell]}, \{\widehat{\text{CT}}_i\}_{i \in [Q]}, \{p_{i,j}\}_{i \in [Q], P_j \notin S}, m_0, m_1, \text{CT}_b).$$

Here each component is analogous to the components described in  $\mathbf{G}_{\text{ThFHE}, \mathcal{A}, \mathbb{A}_t, T}(1^\lambda, 1^d)$ . Recall that the partial decryption components  $\{p_{i,j}\}_{i \in [Q], P_j \notin S}$  of some evaluated ciphertext  $\widehat{\text{CT}}_i = (\hat{A}_i, \hat{B}_i)$  for  $i \in [Q]$  corresponding to an honest party  $P_j$  is computed as follows:

$$p_{i,j} = \gamma_i + r_{i,j}.$$

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**Algorithm 4** Public Sampling Algorithm PS
 

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**Input:**

$$x = (\mathbf{PK}, \{\mathbf{SK}_i\}_{P_i \in S}, \{\mathbf{CT}_i\}_{i \in [\ell]}, \{\widehat{\mathbf{CT}}_i\}_{i \in [Q]}, \{p_{i,j}\}_{i \in [Q], P_j \notin S}, m_0, m_1, \mathbf{CT}_b),$$

$$b' \in \{0, 1\}.$$

**Output:**  $x' \in D_b^\eta(r)$ .

- 1: Generate a ciphertext  $\mathbf{CT}_{b'} = \text{TRLWE.Enc}(\mathbf{PK}, m_{b'})$ , which is a fresh encryption of  $m_{b'}$ . The computation requires  $\mathbf{PK}$ , which is publicly available.
- 2: Return a fresh sample  $x'$  as,

$$x' = (\mathbf{PK}, \{\mathbf{SK}_i\}_{P_i \in S}, \{\mathbf{CT}_i\}_{i \in [\ell]}, \{\widehat{\mathbf{CT}}_i\}_{i \in [Q]}, \{p_{i,j}\}_{i \in [Q], P_j \notin S}, m_0, m_1, \mathbf{CT}_{b'}).$$


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Also the component  $\mathbf{CT}_b$  of  $D_b^\eta(r)$  denotes that the challenge ciphertext  $\mathbf{CT}^*$  in the game is actually an encryption of  $m_b$ .

**Public Sampleability of  $D_b^\eta(r)$ .** We argue the public sampleability [BLRL<sup>+</sup>18] of  $D_b^\eta(r)$  by providing a public sampling algorithm PS in Algorithm 4. Given any sample

$$x = (\mathbf{PK}, \{\mathbf{SK}_i\}_{P_i \in S}, \{\mathbf{CT}_i\}_{i \in [\ell]}, \{\widehat{\mathbf{CT}}_i\}_{i \in [Q]}, \{p_{i,j}\}_{i \in [Q], P_j \notin S}, m_0, m_1, \mathbf{CT}_b),$$

from  $D_b^\eta(r)$  with unknown bit  $b$  and a bit  $b'$ , it generates fresh sample

$$x' = (\mathbf{PK}, \{\mathbf{SK}_i\}_{P_i \in S}, \{\mathbf{CT}_i\}_{i \in [\ell]}, \{\widehat{\mathbf{CT}}_i\}_{i \in [Q]}, \{p_{i,j}\}_{i \in [Q], P_j \notin S}, m_0, m_1, \mathbf{CT}_{b'})$$

of  $D_{b'}^\eta(r)$  efficiently. With the fixed set of noise parameters  $r = \{r_{i,j}\}_{i \in [Q], P_j \notin S}$ , the partial decryption components  $\{p_{i,j}\}_{i \in [Q], P_j \notin S}$  remain exactly same in a fresh sample of  $D_{b'}^\eta(r)$ . Also,  $\mathbf{CT}_{b'}$  being a fresh encryption of  $m_{b'}$  is a valid component of  $D_{b'}^\eta(r)$ . Thus correctness of the algorithm is trivial.

**Proof Outline.** Notice that  $D_b^\eta(r)$  captures the view of the adversary in the security game when the challenger  $\mathcal{C}$  samples the bit  $b$  in the challenge phase and  $r = \{r_{i,j}\}$  is the set of noise values sampled from the distribution of  $e_{sm} - e$  and used in computing partial decryptions  $\{p_{i,j}\}$  for all  $i \in [Q]$  and  $P_j \notin S$ . However in simulated world, we can simulate partial decryptions  $p'_{i,j} = \gamma_i + r'_{i,j}$ , by just sampling each  $r'_{i,j}$  from the distribution of  $e_{sm}$ . Let us denote a problem  $\mathbf{P}$  as distinguishing a sample of  $D_0^\eta(r)$  from a sample of  $D_1^\eta(r)$  and the problem  $\mathbf{P}'$  as distinguishing a sample of  $D_0^\eta(r')$  from a sample of  $D_1^\eta(r')$ . Using a novel Rényi divergence based analysis, we show that non-negligible distinguishing advantage of problem  $\mathbf{P}$  leads to non-negligible distinguishing advantage of problem  $\mathbf{P}'$ . But due to hardness of binary ring-LWE problem, no PPT adversary can distinguish  $\mathbf{CT}_0$  from  $\mathbf{CT}_1$  in the simulated world, as it gains no effective information about the actual secret shares of honest parties by seeing the simulated partial decryptions. Thus distinguishing advantage of problem  $\mathbf{P}'$  is already known to be negligible due to binary ring-LWE assumption. Now by contradiction we can say that the distinguishing advantage in  $\mathbf{P}$  is negligible, making our TRLWE a secure ThFHE scheme.

**Rényi Divergence based Analysis.** Recall from Theorem 4.2 of [BLRL<sup>+</sup>18], due to public sampleability property of  $D_b^\eta(r)$ , if there exists a  $\tau$ -time distinguisher  $\mathcal{D}$  for problem

P with distinguishing probability  $\delta$ , then there must exist a distinguisher  $\mathcal{D}'$  for Problem P' with distinguishing probability  $\delta'$  with run-time  $\tau'$ , such that,

$$\delta' \geq \frac{\delta}{4R_a(\Psi||\Psi')} \cdot \left(\frac{\delta}{2}\right)^{\frac{a}{a-1}},$$

$$\tau' \leq \frac{64}{\delta^2} \log\left(\frac{8R_a(\Psi||\Psi')}{\delta^{a/(a-1)+1}}\right)(\tau_S + \tau).$$

Here,  $\tau_S$  is the run-time of public sampling algorithm for  $D_b^\eta(r)$ . Also  $\Psi$  is the distribution of  $(e_{sm} - e)$  and  $\Psi'$  is the distribution of  $e_{sm}$ . Now, with the results from Lemma 5 in [TT15] and *multiplicative property* of Rényi Divergence we argue that for any  $a \in (1, \infty)$ :

$$R_a(\Psi||\Psi') \leq \exp\left(\frac{a \cdot \pi \cdot N \cdot \|e\|_\infty^2}{\sigma^2}\right),$$

where  $\|e\|_\infty$  denotes the infinity norm of the degree  $(N - 1)$ -RLWE noise polynomial  $e$ . Assuming that  $\|e\|_\infty \leq c\alpha$ , where  $c$  is some constant and  $\alpha$  is the standard deviation of RLWE noise distribution  $\mathcal{G}$ , we have

$$R_a(\Psi||\Psi') \leq \exp\left(\frac{a \cdot \pi \cdot N \cdot c^2 \cdot \alpha^2}{\sigma^2}\right).$$

Finally, considering the scenario that the adversary  $\mathcal{A}$  does  $Q = \text{poly}(\lambda)$  number of queries, and thus is able to see a total of  $Q \cdot (T - t + 1)$  number of partial decryptions corresponding to  $(T - t + 1)$  number of honest parties, we invoke the multiplicative properties of Rényi Divergence from [TT15] to state the following:

$$R_a(\Psi||\Psi') \leq \exp\left(\frac{a \cdot \pi \cdot Q \cdot (T - t + 1) \cdot N \cdot c^2 \cdot \alpha^2}{\sigma^2}\right).$$

Observe that it suffices for us to choose  $\sigma$  such that

$$\sigma \geq c \cdot \alpha \cdot \sqrt{Q \cdot (T - t + 1) \cdot N},$$

since this yields  $R_a(\Psi||\Psi') \leq \exp(a \cdot \pi)$ , and hence:

$$\delta' \geq \frac{\delta}{4} \cdot \left(\frac{\delta}{2}\right)^{\frac{a}{a-1}} \cdot \exp(-a \cdot \pi) = \frac{1}{2} \cdot \left(\frac{\delta}{2}\right)^{\frac{2a-1}{a-1}} \cdot \exp(-a \cdot \pi).$$

and for the run-time we have,

$$\tau' \leq \frac{64}{\delta^2} \log\left(\frac{8 \cdot \exp(a \cdot \pi)}{\delta^{a/(a-1)+1}}\right)(\tau_S + \tau).$$

Hence the condition  $\sigma \geq c \cdot \alpha \cdot \sqrt{Q \cdot (T - t + 1) \cdot N}$  (i.e., smudging noise is only polynomially larger than RLWE noise) implies that, for any  $a > 1$ , non-negligible  $\delta$  would result in non-negligible  $\delta'$ . This completes the proof of security for our proposed TRLWE scheme supporting  $(t, T)$ -threshold decryption.

## C.2 Correctness of $(t, T)$ -Threshold Decryption

In this section, we discuss the upper bounds for the different noise parameters in order to ensure correctness of our proposed  $(t, T)$ -threshold decryption procedure.

**Some Notations.** Let us assume  $q = 2^{\lambda_1}$  to be the modulus in TRLWE and  $|\mathcal{M}| = p = 2^{\lambda_2}$  to be size of the space of coefficients of message-polynomial such that  $p \leq q$ . Now, let  $\Delta = \frac{q}{p} = 2^{\lambda_1 - \lambda_2}$  denote the distance between two consecutive value of a message coefficient in  $\mathcal{M}$ . Note that we assume  $\Delta = 1$  throughout the paper, as in Torus-FHE library  $\lambda_1 = \lambda_2 = 32$  have been considered.

**TRLWE noise.** When applying  $\text{TRLWE.Decode}_0$  on a TRLWE ciphertext  $\text{CT} = (A, B)$ , we effectively compute  $\Phi = B - A \cdot \mathbf{SK}$ , which essentially equals  $\Delta \cdot \mathbf{m} + e$ . Now,  $\Delta$  being a constant we can rewrite  $\Phi$  as  $\sum_{i=0}^{N-1} (\Delta \cdot \mathbf{m}_i + e_i)x^i$ . Next, we round up and approximate each coefficient of  $\Phi$  during  $\text{TRLWE.Decode}_1$  as  $\Delta \cdot \mathbf{m}_i + e_i \xrightarrow{\text{round}} \Delta \cdot \mathbf{m}_i \xrightarrow{\text{approximate}} \mathbf{m}_i$ . For correctness, we need  $|e_i| < \Delta/2$ .

**Smudging noise.** The parties eventually combine their own partial decryptions in order to compute an unmasking component  $\text{part\_decrypt}_f$ . Without loss of generality, for party  $P_1$ ,  $\text{part\_decrypt}_f$  is computed as:

$$\left( A \cdot SH_1 - \sum_{j=2}^t A \cdot SH_j + e_{sm}^1 - \sum_{j=2}^t e_{sm}^j \right).$$

The final message recovery step proceeds as:

$$B - \text{part\_decrypt}_f = \Delta \cdot \mathbf{m} + e - \left( e_{sm}^1 - \sum_{l=2}^t e_{sm}^l \right).$$

Here,  $e = e_0 + e_1x + \dots + e_{N-1}x^{N-1}$  is the RLWE noise polynomial and  $e_{sm}^i = e_{sm,0}^i + e_{sm,1}^i x + \dots + e_{sm,N-1}^i x^{N-1}$  is the smudging noise polynomial added by party  $P_i$ . Hence, for correct decryption the following condition should hold for each  $z \in [0, N-1]$ :

$$|e_z - e_{sm,z}^1 + \sum_{l=2}^t e_{sm,z}^l| < \frac{\Delta}{2}.$$

Let  $\|e\|_\infty$  and  $\|e_{sm}\|_\infty$  denote the infinity norms of the RLWE noise polynomial  $e$  and the smudging noise polynomial  $e_{sm}$ , respectively. Then, we must have:

$$\|e\|_\infty + t \cdot \|e_{sm}\|_\infty < \Delta/2.$$

Since  $\|e_{sm}\|_\infty > \|e\|_\infty$  (by the lower bound argument presented above), it suffices to choose  $\|e_{sm}\|_\infty < \Delta/2(t+1)$ .

## D Observing the Pattern of Secret Shares

We state our observation on the pattern of the secret shares, generated by the  $(t, T)$ -threshold secret sharing using Benaloh-Leichter LISSS (Section 4.3), in the form of a theorem and provide the corresponding proof here.

**Theorem 1.**  $\mathcal{P}' = \{P_{id_1}, P_{id_2}, \dots, P_{id_t}\} \subset \mathcal{P} = \{P_1, P_2, \dots, P_T\}$  is a  $t$ -sized group with `group_id` value of `gid`, where  $id_1 < id_2 < \dots < id_t$ .  $\forall 1 \leq i \leq t$ ,  $P_{id_i}$  has a key share  $SH_i$ , tagged with `group_id` value of `gid`. Then all key shares except  $SH_1$ , have only binary coefficients in their  $k$  polynomials, while  $SH_1$  will have coefficient value upper-bounded by  $t$  in its  $k$  polynomials.

In order to prove Theorem 1, we will first state two lemmas related to the structure of the distribution matrix  $M$  for  $(t, T)$  threshold secret sharing of a TRLWE secret key  $S$ . We consider the number of polynomials in  $S$  is  $k$  and  $I_k$  denotes the identity matrix of dimension  $k$ .

The first lemma is about the pattern of the distribution matrix for Boolean formula of the form  $x_1 \wedge x_2 \wedge \dots \wedge x_t$  for any  $t$ .

**Lemma 1.** We consider  $\mathbf{0}$  to be a notation of zero matrix of dimension  $k \times k$ . Then, distribution matrix  $M_f$  for Boolean formula  $f = x_1 \wedge x_2 \wedge \dots \wedge x_t$  follows the following structure.

$$\begin{bmatrix} I_k & I_k & I_k & \dots & I_k & I_k \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_k \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_k & \mathbf{0} \\ \vdots & & & & & \\ \mathbf{0} & \mathbf{0} & I_k & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & I_k & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}_{kt \times kt}$$

*Proof of Lemma 1.* We prove the lemma by induction on the value of  $t$ .

For  $t = 1$ ,  $f = x_1$  and  $M_f = I_k$ . Hence, the stated matrix structure is satisfied by default. For  $t = 2$ ,  $f = x_1 \wedge x_2$ . We follow the ANDing procedure (see Section 4.3) of  $M_{x_1} = I_k$  and  $M_{x_2} = I_k$  and get  $M_{x_1 \wedge x_2} = \begin{bmatrix} I_k & I_k \\ \mathbf{0} & I_k \end{bmatrix}$ , which clearly satisfies the claimed structure.

Let us assume that the claimed structure of the distribution matrix holds for  $t = i$ , i.e., for  $f = x_1 \wedge x_2 \wedge \dots \wedge x_i$ ,  $M_f$  is as shown below. Also,  $x_{i+1}$  being a Boolean variable,  $M_{x_{i+1}} = I_k$ . ANDing  $M_f$  and  $M_{x_{i+1}}$  produces  $M_{f_1} = M_{f \wedge x_{i+1}}$  as shown below.  $M_f$  has a dimension of  $ki \times ki$  and  $M_{f_1}$  has a dimension of  $k(i+1) \times k(i+1)$ .

$$M_f = \begin{bmatrix} I_k & I_k & I_k & \dots & I_k & I_k \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_k \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_k & \mathbf{0} \\ \vdots & & & & & \\ \mathbf{0} & \mathbf{0} & I_k & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & I_k & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

$$M_{f_1} = \begin{bmatrix} I_k & I_k & I_k & I_k & \dots & I_k & I_k \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_k \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_k & \mathbf{0} \\ \vdots & & & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_k & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_k & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & I_k & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

Clearly, the structure is maintained for  $t = i + 1$ . Hence, by induction, the lemma is true for any  $t \geq 1$ .  $\square$

And the second lemma is about the pattern of distribution matrix for Boolean formula consisting of disjunction of  $l$  number of such  $t$ -sized conjunctive terms, i.e.,  $(x_{1,1} \wedge x_{1,2} \wedge \dots \wedge x_{1,t}) \vee \dots \vee (x_{l,1} \wedge x_{l,2} \wedge \dots \wedge x_{l,t})$ .

**Lemma 2.** *Let us assume that  $f' = (x_{1,1} \wedge x_{1,2} \wedge \dots \wedge x_{1,t}) \vee \dots \vee (x_{l,1} \wedge x_{l,2} \wedge \dots \wedge x_{l,t})$  is a Boolean formula, where  $\forall 1 \leq i \leq l, 1 \leq j \leq t$ ,  $x_{i,j}$  is a binary variable and each of the  $(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,t})$  terms is represented by distribution matrix  $M_f$ , as stated in Lemma 1. We denote first  $k$  columns of  $M_f$  by  $F$  of dimension  $kt \times k$  and the rest of the columns of  $M_f$  by  $R$  of dimension  $kt \times k(t-1)$ .  $\mathbf{0}$  denotes zero matrix of dimension  $kt \times k(t-1)$ . Then distribution matrix  $M_{f'}$  has the following structure:*

$$\begin{bmatrix} F & R & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ F & \mathbf{0} & R & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & & & \\ F & \mathbf{0} & \dots & \mathbf{0} & R & \mathbf{0} \\ F & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & R \end{bmatrix}_{lkt \times (lkt - (l-1)k)}$$

*Proof of Lemma 2.* We prove the lemma by induction on the value of  $l$ .

For  $l = 1$ ,  $f' = f = (x_{1,1} \wedge x_{1,2} \wedge \dots \wedge x_{1,t})$  and  $M_{f'} = M_f = [F \ R]$ , which satisfies the claimed structure by default.

For,  $l = 2$ ,  $f' = (x_{1,1} \wedge x_{1,2} \wedge \dots \wedge x_{1,t}) \vee (x_{2,1} \wedge x_{2,2} \wedge \dots \wedge x_{2,t})$ . We perform ORing on  $M_{x_{1,1} \wedge x_{1,2} \wedge \dots \wedge x_{1,t}} = M_f$  and  $M_{x_{2,1} \wedge x_{2,2} \wedge \dots \wedge x_{2,t}} = M_f$  (see Section 4.3) and get

$$M_{f'} = \begin{bmatrix} F & R & \mathbf{0} \\ F & \mathbf{0} & R \end{bmatrix}_{2kt \times (2kt - k)}$$

This structure follows the lemma.

Let us assume that the structure is maintained  $\forall l \leq j$ . So, with  $f' = (x_{1,1} \wedge x_{1,2} \wedge \dots \wedge x_{1,t}) \vee \dots \vee (x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,t})$  and  $f'' = (x_{j+1,1} \wedge x_{j+1,2} \wedge \dots \wedge x_{j+1,t})$ ,  $M_{f'}$  has a dimension of  $jkt \times jkt - (j-1)k$  and  $M_{f''}$  has a dimension of  $kt \times kt$ .  $M_{f'}$  follows the structure as shown below.  $M_{f''} = [F \ R]$ . Now, ORing  $M_{f'}$  and  $M_{f''}$  produces  $M_{f_2} = M_{f' \vee f''}$  with dimension  $(j+1)kt \times ((j+1)kt - jk)$  as shown below.

$$M_{f'} = \begin{bmatrix} F & R & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ F & \mathbf{0} & R & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & & & \\ F & \mathbf{0} & \dots & \mathbf{0} & R & \mathbf{0} \\ F & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & R \end{bmatrix}$$



$$M_{f_2} = \begin{bmatrix} F & R & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ F & \mathbf{0} & R & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & & & & \\ F & \mathbf{0} & \dots & \mathbf{0} & R & \mathbf{0} & \mathbf{0} \\ F & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & R & \mathbf{0} \\ F & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & R \end{bmatrix}$$

So, the lemma is true for  $l = (j + 1)$ .

Hence, by induction the lemma is true for any  $l \geq 1$ .  $\square$

Now we use Lemma 1 and Lemma 2 to provide here the proof of Theorem 1.

*Proof of Theorem 1.* Let us recall from Section 4.3 that the monotone Boolean formula for  $(t, T)$ -threshold secret sharing can be written as  $f = (x_{1,1} \wedge x_{1,2} \wedge \dots \wedge x_{1,t}) \vee \dots \vee (x_{l,1} \wedge x_{l,2} \wedge \dots \wedge x_{l,t})$ , where  $l = \binom{T}{t}$ . If  $\mathbf{0}$  denotes zero matrix of dimension  $kt \times (kt - k)$ , from Lemma 1 and Lemma 2, we know that structure of the corresponding distribution matrix  $M$  with dimension  $\binom{T}{t}kt \times (\binom{T}{t}kt - (\binom{T}{t} - 1)k)$  is as follows:

$$M = \begin{bmatrix} F & R & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ F & \mathbf{0} & R & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & & & \\ F & \mathbf{0} & \dots & \mathbf{0} & R & \mathbf{0} \\ F & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & R \end{bmatrix}$$

$$F = \begin{bmatrix} I_k \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \quad R = \begin{bmatrix} I_k & I_k & I_k & \dots & I_k \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_k \\ \vdots & & & & \\ \mathbf{0} & I_k & \mathbf{0} & \dots & \mathbf{0} \\ I_k & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

A detailed look into the above matrix  $M$  reveals that  $F$  has a structure of dimension  $kt \times k$  and  $R$  has a structure with dimension  $kt \times (kt - k)$  as shown in above matrix structure. In  $F$  and  $R$ ,  $\mathbf{0}$  denotes a zero matrix of dimension  $k \times k$ . It is obvious from the structure of  $M$  that each of its  $\binom{T}{t}$  horizontal sections contain exactly one  $F$  and one  $R$  along with  $(\binom{T}{t} - 1)$  zero matrices  $\mathbf{0}_{kt \times (kt - k)}$ . Now, the structure of  $F$  shows that each of its first  $k$  rows contains one '1' entry. No other row below has any '1' in it and the structure of  $R$  reveals that each of its first  $k$  rows contains exactly  $(t - 1)$  number of '1' in it. Each of the other rows below contains exactly one '1' in it. Hence, each of the first  $k$  rows of any one horizontal section (out of total  $\binom{T}{t}$  sections) of  $M$  has exactly  $t$  number of '1' in it. Each of the rest of the rows below in that section contains exactly one '1' in it.

Let us recall that, each section of  $M$  corresponds to one section of *shares* ( $shares = M \cdot \rho$ ) from Section 4.3 in the paper, i.e, the key shares of any  $t$ -sized subset of collaborating parties.

$\rho$  is a binary matrix. During matrix multiplication, dot product between one row of  $M$  and one column of  $\rho$  produces an entry in *shares*. Dot product between two binary vectors is always upper bounded by the number of '1' in any of the two vectors. As, each of first  $k$  rows of any section of  $M$  contains exactly  $t$  number of '1', the entries of first  $k$  rows of any

section in *shares* are always upper bounded by  $t$ . First  $k$  rows of any section of *shares* form one key-share. Clearly, that key share will have non-binary entries in it. Similarly, each of the other  $(kt - k)$  rows below in any section of  $M$  contains exactly one '1', so the entries of the  $(kt - k)$  number of rows below in any section of *shares* are upper bounded by 1. In other words, those entries can be either 0 or 1. Hence, rest of the  $(t - 1)$  key shares of any  $t$ -sized subset of parties, have only binary entries in it.

Hence we conclude that, in our proposed  $(t, T)$  threshold LISSS for a  $t$ -sized subset of parties  $PT' = \{P_{id_1}, P_{id_2}, \dots, P_{id_t}\}$ , where  $id_1 < id_2 < \dots < id_t$ , all the parties except  $P_{id_1}$  will have binary key shares.  $\square$