

Accountable Threshold Signatures with Proactive Refresh

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Abstract. An *accountable* threshold signature (ATS) is a threshold signature scheme where every signature identifies the quorum of signers who generated that signature. They are widely used in financial settings where signers need to be held accountable for threshold signatures they generate. In this paper we initiate the study of *proactive refresh* for accountable threshold signatures. Proactive refresh is a protocol that lets the group of signers refresh their shares of the secret key, without changing the public key or the threshold. We give several definitions for this notion achieving different levels of security. We observe that certain natural constructions for an ATS cannot be proactively refreshed because the secret key generated at setup is needed for accountability. We then construct three types of ATS schemes with proactive refresh. The first is a generic construction that is efficient when the number of signers is small. The second is a hybrid construction that performs well for a large number of signers and satisfies a strong security definition. The third is a collection of very practical constructions derived from ATS versions of the Schnorr and BLS signature schemes; however these practical constructions only satisfy our weaker notion of security.

1 Introduction

A threshold signature scheme [22] protects the secret signing key by splitting it into n shares so that any t shares can sign. An *accountable* threshold signature scheme, also called an ATS, is a type of threshold scheme where the signature identifies the quorum set that generated the signature. In particular, there is a *tracing* algorithm that takes as input the public key \mathbf{pk} , a message m , and a valid signature on m , and outputs a quorum $\mathcal{J} \subseteq [n]$ of size at least t that must have participated in generating the signature. More precisely, a set of signers \mathcal{J} should be unable to cause the tracing algorithm to blame a signer outside of \mathcal{J} for a signature generated by \mathcal{J} (see Section 3 for the complete definition). Since every signature must encode the quorum that generated it, signature length must be at least $\lceil \log_2 \binom{n}{t} \rceil$ bits.

Accountable threshold signatures (ATS) come up often in real-world settings: if a rogue transaction is signed by a threshold of trustees, the signature should identify the trustees responsible. For this reason, they are widely used in blockchain applications (e.g., [2]).

The trivial t -out-of- n ATS scheme is one where every signer locally generates a public-private key pair for a standard (non-threshold) signature scheme. The complete public key is the concatenation of all n local public keys. When t parties need to sign a message m , they each sign the message using their local secret key, and the final signature is the concatenation of all t signatures. The verifier accepts such an ATS signature if it contains t valid signatures. The tracing algorithm can trivially determine which parties participated in generating a given valid signature. This trivial ATS is used widely, for example in Bitcoin multisig transactions [2]. While the scheme has many benefits, signature size and verification time are at least linear in $t\lambda$, where λ is the security parameter. Several ATS constructions achieve much lower signature size and verification time [37, 6, 39, 11, 14, 13].

Proactive refresh. Consider an adversary that is able to corrupt one signing party every week and learn its key share. After t weeks the adversary will learn enough key shares to forge a signature on

any message of its choice. To thwart such a dynamic adversary, Ostrovsky and Yung [40] introduced the concept of *proactive refresh*. Every epoch, say once a day, the n parties will engage in a protocol that refreshes their secret key shares without changing the public key. The requirement is that an adversary that corrupts fewer than t parties in every epoch will not be able to forge signatures, even though in aggregate the adversary may corrupt all n parties. Several subsequent works designed proactive refresh protocols for specific threshold systems [30, 29, 26, 25, 42, 4, 27, 18, 1, 32, 20, 34].

Our results. In this paper we initiate the study of proactive refresh for accountable threshold signatures (ATS). An ATS with proactive refresh, or ATS-PR, is the same as an ATS with the addition of a share update protocol. At the beginning of every epoch all n parties participate in this update protocol to refresh their key shares, without changing the public key. The scheme must be unforgeable and accountable against an adversary that can corrupt a different set of parties at every epoch. We will define this more precisely in a minute.

At first, refreshing the secret keys of parties in an ATS may seem impossible. Each party’s secret key is used to hold that party accountable for a rogue signature. If we refresh the party’s secret key, the tracing algorithm will no longer be able to trace a rogue signature to that party. For example, in the trivial ATS described above it is not possible to refresh the secret keys without changing the public key: once an adversary learns the secret key of one party, it will always be able to forge signature shares on behalf of that party. Nevertheless, we show that by encoding a little more information in the signature, it is possible for the tracing algorithm to work correctly despite the fact that all the secret keys change at the beginning of every epoch.

In Section 3 we define a number of security models for an ATS-PR that capture multi-epoch unforgeability and accountability properties. We give two natural definitions of unforgeability, denoted *uf-0* and *uf-1*, that are an adaption of the threshold unforgeability definitions of Bellare et al. [5] to the settings of proactive refresh. We next give two definitions of accountability, denoted *acc-0* and *acc-1*. In *acc-1* the adversary can corrupt an arbitrary number of parties at every epoch, and can issue arbitrary signature queries to the parties at every epoch. Eventually, the adversary produces a message-signature pair (m^*, σ^*) that will trace to a signing set $\mathcal{J} \subseteq [n]$. If in some epoch e' the adversary obtained enough key shares and signature shares to sign m^* on behalf of the set \mathcal{J} , then the adversary did not break accountability — the set \mathcal{J} effectively signed m^* at epoch e' . Therefore, to break accountability we require the adversary to satisfy the complementary condition: in every epoch e we require that there is some $i_e \in [n]$ such that the adversary did not corrupt party i_e in epoch e and did not ask party i_e to sign m^* in epoch e , and yet $i_e \in \mathcal{J}$. This means that in every epoch, some party is incorrectly blamed for signing m^* . The definition requires that no efficient adversary can satisfy this condition. We discuss this further in Section 3. Definition *acc-0* is weaker and requires that for some $i \in \mathcal{J}$, the adversary never corrupted party i nor did it ever ask it to sign m^* , across all epochs. In summary, we obtain four notions of security denoted $(\text{uf-}b \wedge \text{acc-}b')$ for $b, b' \in \{0, 1\}$.

Constructions. Next, we present five constructions. We begin with a generic combinatorial construction that performs well when $\binom{n}{t}$ is polynomial size. The scheme is built from any generic n -out-of- n threshold signature scheme (not necessarily accountable) that supports a proactive refresh. There are many examples built from RSA [30, 29, 26, 25, 42], Schnorr [29, 33, 36, 34], and BLS [16, 11]. It satisfies *uf-1* \wedge *acc-1* security, our strongest notion of security.

In Section 5 we describe a generic construction that satisfies *uf-1* \wedge *acc-1* security even when $\binom{n}{t}$ is large. The scheme is built by combining two schemes:

- a refreshable n -out-of- n threshold scheme \mathcal{S}_1 that is not accountable, and

- a t -out-of- n accountable threshold scheme \mathcal{S}_2 that is not-refreshable.

We build a two-level ATS-PR scheme where the scheme \mathcal{S}_1 is used to sign \mathcal{S}_2 public keys, and \mathcal{S}_2 is used to sign messages. At the beginning of epoch number i the parties do: (i) refresh their \mathcal{S}_1 secret keys, (ii) run a distributed key generation (DKG) protocol to generate fresh \mathcal{S}_2 secret keys and a public key \mathbf{pk}_i ; and (iii) sign the newly generated ATS public key \mathbf{pk}_i using the scheme \mathcal{S}_1 . A signature on a message m is a triple $(\mathbf{pk}_i, \sigma_1, \sigma_2)$, where σ_1 is the \mathcal{S}_1 signature on \mathbf{pk}_i , and σ_2 is the \mathcal{S}_2 signature on m . To make this construction practical we need an ATS scheme \mathcal{S}_2 that has short public keys (so that our overall signature is short) and has an efficient DKG. In Section 6 we construct an ATS that has both properties: a constant size public key (i.e., its size is independent of t and n) and a simple DKG. Our construction makes use of techniques used to construct cryptographic accumulators.

We then turn to constructing a refreshable ATS from standard signature schemes such as Schnorr [43] and BLS [16]. In Section 7 we give very practical systems that enables a proactive refresh for a Schnorr-ATS and a BLS-ATS. This leads to practical short ATS schemes that support proactive refresh. However, we describe an attack that shows that the schemes do not provide acc-1 security. Instead, we prove that they provide acc-0 and uf-0 security (we discuss the requirements for proving uf-1 in Section 7).

Future work. We briefly describe two directions for future work. First, our security definitions in Section 3 require that the n parties collaborate and honestly follow the update protocol. Future work could consider a more complex setting where some of the parties participating in the update protocol are corrupt. We expect that techniques from maliciously secure multiparty computation can be used to preserve the security of the update protocol in this case. We discuss this further in Section 8.

Second, our security definitions capture adaptive threshold adversaries (as in [35] and the references therein), where the adversary can choose to corrupt arbitrary parties before and after issuing signature queries in every epoch. Our constructions in Sections 4 and 5 inherit adaptive security from the underlying threshold schemes from which they are built. However, in Section 7, we analyze the practical Schnorr and BLS constructions in a semi-adaptive setting where the adversary must commit to its key queries at the beginning of every epoch, before issuing its adaptive signature queries. The reason is that achieving security against a fully adaptive adversary often leads to complex threshold signature schemes (as discussed in [35]). Future work can consider practical constructions for ATS-PR in the fully adaptive settings.

Additional related work. The notion of an accountable threshold signature (ATS) is closely related to the concept of a multisignature defined in [37] and further developed in [6, 14, 39, 31, 3, 13]. However, there are a number of differences. First, multisignatures are often viewed as a compression mechanism, compressing multiple signatures into one, not a threshold mechanism. The threshold is often left implicit. An ATS imposes an explicit threshold used by the verifier to decide if a signature is valid. Second, the syntax of an ATS allows for centralized key generation or an interactive distributed key generation protocol (DKG). Multisignatures often only allow for local key generation where every signer generates its key share by itself (however, there are some exceptions [13]).

Traditionally, threshold signatures come in two flavors: *fully private* (called PTS) where a signature reveals nothing about the threshold or the signing quorum, or *fully accountable* (called ATS), as in this paper. A recent proposal called TAPS [15] provides both properties: it is fully private to the public, but fully accountable to an authority that holds a secret tracing key.

2 Preliminaries

In this section we present the basic notions and cryptographic primitives that are used in this work. For an integer $n \in \mathbb{N}$ we denote by $[n]$ the set $\{1, \dots, n\}$. For a distribution X we denote by $x \leftarrow \$ X$ the process of sampling a value x from the distribution X . Similarly, for a set \mathcal{X} we denote by $x \leftarrow \$ \mathcal{X}$ the process of sampling a value x from the uniform distribution over \mathcal{X} .

2.1 Threshold Signatures

For simplicity of presentation, we begin by considering non-interactive threshold signature schemes, in which each signer can locally produce their signature shares without interacting with the other signers. In Appendix A we will formally consider interactive schemes as well.

Syntax. A non-interactive threshold signature scheme (see [17, 23, 16, 11, 33, 5] and the references therein) is defined with respect to some public parameters \mathbf{pp} . Looking ahead, in our constructions \mathbf{pp} will include the description of some cryptographic group. This description is typically generated as a function of the security parameter $\lambda \in \mathbb{N}$ by a group-generation algorithm, but in this work we will abstract this process away and fix the group as part of \mathbf{pp} . To avoid over-cluttering in notation, we assume that \mathbf{pp} are known to all algorithms without always providing it as an explicit input.

A threshold signature (TS) scheme is a 4-tuple $\text{TS} = (\text{KGen}, \text{Sign}, \text{Combine}, \text{Vf})$, where:

- **KGen** is the randomized key-generation algorithm. It takes in as input the public parameters \mathbf{pp} , a number n of signers, and a threshold t , and outputs a public key \mathbf{pk} , a public signature combination key \mathbf{pkc} , and an n -vector of secret signing keys $(\mathbf{sk}_1, \dots, \mathbf{sk}_n)$.
- **Sign** is the randomized signing algorithm. It takes in as input a secret key \mathbf{sk}_i and a message m , and outputs a signature share s_i .
- **Combine** is a deterministic signature combination algorithm. It takes in as input the combination key \mathbf{pkc} and signature shares $s_{i_1}, \dots, s_{i_\ell}$ and outputs either a signature σ or \perp . To simplify the notation, we will require throughout that every signature share encodes the signer from which it originated.
- **Vf** is the deterministic verification algorithm. It takes in the public key \mathbf{pk} , a message m , and a signature σ , and outputs either 1 (implying acceptance) or 0 (implying rejection).

We postpone defining the correctness and security of TS schemes to Section 3, where we will present new generalizations of these definitions.

Accountable threshold signatures. An accountable threshold signature (ATS) scheme [38, 16, 3, 9, 13, 39] is a TS scheme equipped with an additional **Trace** algorithm. This algorithm takes as input the public key \mathbf{pk} , a message m , and a signature σ . It outputs either a subset $\mathcal{J} \subseteq [n]$ of signers such that $|\mathcal{J}| \geq t$ (where t is the threshold provided to **KGen**) or \perp . Roughly speaking, the trace correctness property states that if a signature σ on m was produced by a subset $\mathcal{I} \subseteq [n]$ of the signers, then the subset \mathcal{J} output by $\text{Trace}(\mathbf{pk}, m, \sigma)$ is contained in \mathcal{I} . The accountability property asserts that no subset \mathcal{I} of the signers can sign “on behalf” of a subset \mathcal{J} which is not contained in \mathcal{I} (i.e., it cannot make **Trace** output such a \mathcal{J}). We formally define these two properties in Section 3.

2.2 The Forking Lemma

In subsequent sections, we make use of the “forking lemma” of Bellare and Neven [6] (following Pointcheval and Stern [41]). Let $q \geq 1$ be an integer, let \mathcal{H} , \mathcal{X} and \mathcal{Y} be sets such that $|\mathcal{H}| \geq 2$.

Let \mathcal{A} be a randomized algorithm that on input $(x, \vec{h}) \in \mathcal{X} \times \mathcal{H}^q$ returns either a pair $(i, y) \in [q] \times \mathcal{Y}$ or \perp . Let $\mathcal{F}_{\mathcal{A}}$ be an algorithm that takes an input $x \in \mathcal{X}$, and returns either an output $(y, y') \in \mathcal{Y}^2$ or \perp , and is defined as follows:

1. Sample random coins ρ for \mathcal{A} .
2. Sample $h_1, \dots, h_q \leftarrow_{\$} \mathcal{H}$ and invoke $\text{out}_1 \leftarrow \mathcal{A}(x, h_1, \dots, h_q; \rho)$.
3. If $\text{out}_1 = \perp$, return \perp . Otherwise, let $\text{out}_1 = (i, y)$.
4. Sample $h'_1, \dots, h'_q \leftarrow_{\$} \mathcal{H}$ and invoke $\text{out}_2 \leftarrow \mathcal{A}(x, h_1, \dots, h_{i-1}, h'_i, \dots, h_q; \rho)$.
5. If $\text{out}_2 = \perp$, return \perp . Otherwise, let $\text{out}_2 = (i', y')$.
6. If $i' = i$ and $h_i \neq h'_i$ then return (y, y') . Otherwise, return \perp .

The forking lemma (Lemma 1 below) relates the probability that $\mathcal{F}_{\mathcal{A}}$ successfully provides an output (other than \perp) to the probability that \mathcal{A} provides an output.

Lemma 1 ([6]). *For all distributions D over \mathcal{X} , it holds that*

$$\Pr[\mathcal{F}_{\mathcal{A}}(x) \neq \perp : x \leftarrow_{\$} D] \geq \epsilon_D \cdot \left(\frac{\epsilon_D}{q} - \frac{1}{h} \right),$$

where ϵ_D is defined as

$$\epsilon_D := \Pr \left[\mathcal{A}(x, \vec{q}) \neq \perp : \begin{array}{l} x \leftarrow_{\$} D \\ \vec{h} \leftarrow_{\$} \mathcal{H}^q \end{array} \right].$$

3 Accountable Threshold Signatures with Proactive Refresh

In this section we present our definitions for ATS schemes with proactive refresh (ATS-PR). We start by providing the syntactic additions for such schemes (when compared to standard ATS schemes), then define the correctness properties that should be satisfied by them, and finally, we present new security notions. Recall that to simplify the presentation, we first focus on non-interactive schemes, and then formally consider interactive schemes in Appendix A.

3.1 Syntax and Correctness

The key-update procedure. An ATS-PR scheme is an ATS scheme that is additionally equipped with a key-update procedure, whose role is to refresh the signers' secret keys without modifying the public key in any way. We can envision the key-update procedure as dividing time into epochs. An epoch starts once one execution of the key-update procedure ends (or, for the first epoch right after the invocation key generation algorithm), and ends when the next execution of the key-update procedure ends.

Formally, the key-update procedure is a pair $\text{Update} = (\text{Update}_0, \text{Update}_1)$ of algorithms:

- Update_0 is a randomized algorithm that takes in a secret key sk_i^e of signer i in epoch e and the public key pk , and outputs a vector $(\delta_{i,1}^e, \dots, \delta_{i,n}^e)$ of update messages. Each signer i sends $\delta_{i,j}^e$ to the j th signer, for all $j \neq i$.
- Update_1 is a deterministic algorithm that takes in a secret key sk_i^e and n update messages $\delta_{1,i}^e, \dots, \delta_{n,i}^e$. It outputs an updated secret key sk_i^{e+1} for epoch $e+1$ for signer i .

For succinctness, we may write $(\text{sk}_1^{e+1}, \dots, \text{sk}_n^{e+1}) \leftarrow \$ \text{Update}(\text{pk}, \text{sk}_1^e, \dots, \text{sk}_n^e)$ as a shorthand for the random process of first invoking $\text{Update}_0(\text{sk}_i^e, \text{pk})$ for every $i \in [n]$ to randomly sample n^2 update messages $\{\delta_{i,j}\}_{i,j \in [n]}$; and then running $\text{Update}_1(\text{sk}_i^e, (\delta_{1,i}^e, \dots, \delta_{n,i}^e))$ to obtain sk_i^{e+1} for every $i \in [n]$.

Correctness. Informally, the basic correctness requirement for ATS-PR schemes is that Vf should accept honestly-generated signatures in all epochs.

Definition 1 (correctness). *We say that an ATS-PR scheme $\text{PRATS} = (\text{KGen}, \text{Sign}, \text{Combine}, \text{Vf}, \text{Trace}, \text{Update})$ is **correct** if for all public parameters pp , all messages m in the associated message space \mathcal{M}_{pp} , all positive integers n, t and e such that $t \leq n$, and all subsets $\mathcal{J} \subseteq [n]$ of size at least t , it holds that*

$$\Pr[\text{Vf}(\text{pk}, m, \text{Combine}(\text{pkc}, \{\text{Sign}(\text{sk}_j^e, m)\}_{j \in \mathcal{J}})) = 1] = 1,$$

where the probability is over the random variables $(\text{pk}, \text{pkc}, \text{sk}_1^1, \dots, \text{sk}_n^1) \leftarrow \$ \text{KGen}(\text{pp}, n, t)$, and $(\text{sk}_1^{i+1}, \dots, \text{sk}_n^{i+1}) \leftarrow \$ \text{Update}(\text{pk}, \text{sk}_1^i, \dots, \text{sk}_n^i)$ for $i = 2, \dots, e - 1$, and the random coins of Sign .

In addition to the traditional correctness property, an ATS-PR scheme should also provide *trace correctness*. That is, on input a public-key pk , a message m , and a signature σ , the tracing algorithm Trace should output a subset of the set of keys used to generate σ . This should hold irrespective of the epoch in which σ was generated.

Definition 2 (trace correctness). *We say that an ATS-PR scheme $\text{PRATS} = (\text{KGen}, \text{Sign}, \text{Combine}, \text{Vf}, \text{Trace}, \text{Update})$ satisfies **trace correctness** if for all public parameters pp , all messages m in the associated message space \mathcal{M}_{pp} , all positive integers n, t and e such that $t \leq n$, and all subsets $\mathcal{J} \subseteq [n]$ of size at least t , it holds that*

$$\Pr[\text{Trace}(\text{pk}, m, \text{Combine}(\text{pkc}, \{\text{Sign}(\text{sk}_j^e, m, \mathcal{J})\}_{j \in \mathcal{J}})) \subseteq \mathcal{J}] = 1,$$

where the probability is over the random variables $(\text{pk}, \text{pkc}, \text{sk}_1^1, \dots, \text{sk}_n^1) \leftarrow \$ \text{KGen}(\text{pp}, n, t)$, and $(\text{sk}_1^{i+1}, \dots, \text{sk}_n^{i+1}) \leftarrow \$ \text{Update}(\text{pk}, \text{sk}_1^i, \dots, \text{sk}_n^i)$ for $i = 1, \dots, e - 1$, and the random coins of Sign .

3.2 Security Notions for ATS-PR Schemes

We now turn to present our notions of security for ATS-PR schemes. We start with a brief overview of the security notions and then provide formal definitions.

An ATS-PR scheme should satisfy two security requirements, unforgeability and accountability, which extend the traditional security notions of ATS schemes to the setting of proactive refreshes.

Unforgeability. The traditional unforgeability requirement for threshold signatures asserts that an adversary cannot produce a valid signature on a message m without observing either the secret key or a signature share on m of at least t different signers. In the proactive refresh setting, we require that this holds *per epoch*. That is, the adversary should not be able to produce a signature on a message m , unless *there is a specific epoch* in which it observed the secret keys or signature shares on m of at least t signers. Note that this means that the adversary is allowed to observe t or more secret keys or signature shares on m across epochs, and potentially even observe the secret keys of all signers at different points in time. However, in each specific epoch, the total number of secret keys and signature shares on m that the adversary observes should be strictly less than t . Following [5], we consider two flavors of this unforgeability definition, denoted uf-0 and uf-1, depending on

whether or not the adversary is allowed to observe signature shares on the message m^* for which it forges a signature. We present constructions satisfying both notions with different trade-offs.

Accountability. The accountability property of ATS schemes states that an adversary should not be able to produce a valid signature on a message m on behalf of a subset \mathcal{J} of signers without observing the secret key or signature share on m of all the signers in \mathcal{J} . In the proactive refresh setting we present a strong accountability definition that requires that this restriction on the adversary should only hold in each epoch (thus allowing them to observe secret keys/signature shares on m of all signers in \mathcal{J} across different epochs). We also consider a milder accountability definition, which requires that for some signer $j \in \mathcal{J}$, the adversary never observes j 's secret key or a signature share of j on m . Looking ahead, we will present different constructions of ATS-PR schemes satisfying the two notions.

Note that even under our strong accountability definition, if the adversary learns all the secret keys (or signature shares on m) of \mathcal{J} in *the same epoch*, say epoch 1, then they can forever produce signatures on m , even in future epochs. This is inherent, since we want the public verification key to remain the same over time, rendering the verification algorithm oblivious to the epoch in which the message was signed.

Game-based security definitions. We use security games to define the above security notions for ATS-PR schemes, following the framework of Bellare and Rogaway [8]. A game \mathbf{G} consists of an adversary \mathcal{A} interacting with the challenger. The game is specified by a main procedure and possibly additional oracle procedures, which describe the manner in which the challenger replies to oracle queries issued by the adversary. We denote by $\mathbf{G}(\mathcal{A})$ the output of \mathbf{G} when executed with an adversary \mathcal{A} . This $\mathbf{G}(\mathcal{A})$ is a random variable defined over the randomness of both \mathcal{A} and the random choices of the game's main procedure and oracles.

For an ATS-PR scheme PRATS and public parameters pp , the above-described requirements are captured by the security games defined in Figure 1. All games are defined similarly, and the only difference between them is the winning condition for the adversary (i.e., the condition that results in the game outputting 1). In all games, the adversary first specifies the number n of overall signers, the threshold t , and the number E of epochs. This is followed by challenger sampling keys for all E epochs using the **KGen** and **Update** procedures of PRATS. The adversary then interacts with the challenger using two types of queries: Secret-key queries and signature queries. A secret-key query (e, i) reveals to the adversary the secret key of signer i in epoch e . A signature query (m, e, i) provides the adversary with an honestly-generated signature share on m with respect to signer i 's secret key in epoch e . Finally, in all games the adversary should produce a valid forgery; that is, a message m^* and a signature σ^* that passes verification. Each game has additional restrictions on the adversary in light of our informal discussion above. Games $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-0}}$ and $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}}$ capture two notions of unforgeability, whereas games $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{acc-0}}$ and $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{acc-1}}$ capture two notions of accountability. For $b, b' \in \{0, 1\}$, we also define the game $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b \wedge \text{acc-}b'}$, that captures schemes that satisfy both unforgeability and accountability. This will help us state and prove our theorem statements more succinctly.

Two remarks regarding the games in Figure 1 are in order:

- By convention, we assume that $\perp \notin \mathcal{Q}$ for any set \mathcal{Q} . Hence, if the adversary successfully outputs a valid signature σ^* on a message m^* such that $\text{Trace}(\text{pk}, m^*, \sigma^*) = \perp$, then the adversary breaks even our weak accountability notion (that is, wins the acc-0 security game).

Games $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b}$, $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{acc-}b'}$, $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b \wedge \text{acc-}b'}$	
1 :	$\text{flag}_{\text{uf-}0}, \text{flag}_{\text{uf-}1}, \text{flag}_{\text{acc-}0}, \text{flag}_{\text{acc-}1} \leftarrow 0$
2 :	$(\text{st}, n, t, E) \leftarrow \mathcal{A}(\text{pp})$
3 :	$(\text{pk}, \text{pkc}, \text{sk}_1^1, \dots, \text{sk}_n^1) \leftarrow \text{KGen}(\text{pp}, n, t)$
4 :	for $e = \{2, \dots, E\}$ do
5 :	$(\text{sk}_1^e, \dots, \text{sk}_n^e) \leftarrow \text{Update}(\text{pk}, \text{sk}_1^{e-1}, \dots, \text{sk}_n^{e-1})$
6 :	$(m^*, \sigma^*) \leftarrow \mathcal{A}^{\text{skO}(\cdot, \cdot), \text{SignO}(\cdot, \cdot)}(\text{st}, \text{pk}, \text{pkc})$
7 :	if $\text{Vf}(\text{pk}, m^*, \sigma^*) = 0$ then
8 :	return 0
9 :	if $\forall e \in [E], \mathcal{Q}_e^{\text{sk}} < t \wedge \mathcal{Q}_e^{\text{sig}}(m^*) = 0$ then $\text{flag}_{\text{uf-}0} \leftarrow 1$
10 :	if $\forall e \in [E], \mathcal{Q}_e^{\text{sk}} \cup \mathcal{Q}_e^{\text{sig}}(m^*) < t$ then $\text{flag}_{\text{uf-}1} \leftarrow 1$
11 :	if $\text{Trace}(\text{pk}, m^*, \sigma^*) \not\subseteq \cup_{e \in [E]} (\mathcal{Q}_e^{\text{sk}} \cup \mathcal{Q}_e^{\text{sig}}(m^*))$ then $\text{flag}_{\text{acc-}0} \leftarrow 1$
12 :	if $\forall e \in [E], \text{Trace}(\text{pk}, m^*, \sigma^*) \not\subseteq \mathcal{Q}_e^{\text{sk}} \cup \mathcal{Q}_e^{\text{sig}}(m^*)$ then $\text{flag}_{\text{acc-}1} \leftarrow 1$
13 :	Only game $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b}$: return $\text{flag}_{\text{uf-}b}$
14 :	Only game $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{acc-}b}$: return $\text{flag}_{\text{acc-}b}$
15 :	Only game $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b \wedge \text{acc-}b'}$: return $\text{flag}_{\text{uf-}b} \vee \text{flag}_{\text{acc-}b'}$

Oracle $\text{skO}(e, i)$	Oracle $\text{SignO}(m, e, i)$
1 : $\mathcal{Q}_e^{\text{sk}} \leftarrow \mathcal{Q}_e^{\text{sk}} \cup \{i\}$	1 : $\sigma_i \leftarrow \text{Sign}(\text{sk}_i^e, m)$
2 : return sk_i^e	2 : $\mathcal{Q}_e^{\text{sig}}(m) \leftarrow \mathcal{Q}_e^{\text{sig}}(m) \cup \{i\}$
	3 : return σ_i

Fig. 1. The security games $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b}$, $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{acc-}b'}$, $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b \wedge \text{acc-}b'}$ for $b, b' \in \{0, 1\}$ for an ATS-PR scheme $\text{PRATS} = (\text{KGen}, \text{Sign}, \text{Combine}, \text{Vf}, \text{Trace}, \text{Update})$ with public parameters pp . For a set \mathcal{X} and an element x , we let $\mathcal{X} \leftarrow \mathcal{X} \cup \{x\}$ be a shorthand for the following operation: If \mathcal{X} was previously defined, then set $\mathcal{X} \leftarrow \mathcal{X} \cup \{x\}$; if \mathcal{X} is still undefined, then set $\mathcal{X} = \{x\}$.

- If we add the syntactic requirement that `Trace` never outputs \perp on a signature that passes verification, then `acc-1` security implies `uf-1` security. This is because under this requirement, `Trace` always outputs a subset \mathcal{J} of size at least t on valid signatures. If in each epoch the adversary corrupted at most $t - 1$ signers, then in each epoch there must be at least one signer in \mathcal{J} which is uncorrupted by the adversary.

Definition 3 below defines the advantage of an adversary \mathcal{A} in the eight games defined in Figure 1 as the probability that the games output 1 when executed with \mathcal{A} .

Definition 3. Let $\text{PRATS} = (\text{KGen}, \text{Sign}, \text{Combine}, \text{Vf}, \text{Trace}, \text{Update})$ be an ATS-PR scheme with public parameters pp and let $\text{prop} \in \{\text{uf-}b, \text{acc-}b, \text{uf-}b \wedge \text{acc-}b'\}_{b,b' \in \{0,1\}}$. The advantage of an adversary \mathcal{A} in $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{prop}}$ is defined as

$$\text{Adv}_{\text{PRATS}[\text{pp}]}^{\text{prop}}(\mathcal{A}) \stackrel{\text{def}}{=} \Pr \left[\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{prop}}(\mathcal{A}) = 1 \right].$$

Threshold signatures without accountability or proactive refresh. In subsequent sections we will consider TS schemes with proactive refresh but without accountability (i.e., without a `Trace` algorithm). These can be treated as a specific case of ATS-PR schemes in which the `Trace` algorithm is trivial (returns \perp on all inputs). As such, games $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}0}$ and $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}1}$ and Definition 3 readily captures the unforgeability property of such schemes. In addition, we will consider ATS schemes without proactive refresh (i.e., no `Update` procedure). These can also be treated as a special case of ATS-PR schemes in which the number of epochs is fixed at 1. The games defined in Figure 1 together with Definition 3 define the unforgeability and accountability of such schemes by having the game fix the number of epochs E to 1 (instead of receiving it from \mathcal{A}).

Semi-adaptive adversaries. The security games as defined in Figure 1 allow for fully-adaptive adversaries, in the sense that they do not pose any restrictions on the order in which the adversary decides on its oracle queries. Proving security against such adversaries is known to be a challenging task, already for non-accountable threshold signature schemes [35]. Since the problem of fully-adaptive adversaries is not at the focus of this work, we also consider semi-adaptive adversaries. For every epoch e , such adversaries are restricted to issuing all secret-key queries for that epoch before issuing their signature queries for this epoch. This is captured by modifying the security games as follows. The game will maintain a set \mathcal{E} which will include all epochs for which a signature query has been issued by the adversary. On input (e, i) , the oracle `skO` will first check if e is in \mathcal{E} . If so, it will ignore the query, returning \perp . Otherwise, it will continue as defined in Figure 1. This ensures that at every epoch e the adversary must issue all of its key queries for epoch e before issuing a signature query in epoch e .

For a ATS-PR scheme `PRATS` with public parameters pp , and for a security property $\text{prop} \in \{\text{uf-}b, \text{acc-}b, \text{uf-}b \wedge \text{acc-}b'\}_{b,b' \in \{0,1\}}$, denote by $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{sa-prop}}$ the semi-adaptive security game obtained from $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{prop}}$ as described above. The advantage of an adversary in these games is defined similarly to the adversarial advantage in the fully-adaptive security games.

Definition 4. Let $\text{PRATS} = (\text{KGen}, \text{Sign}, \text{Combine}, \text{Vf}, \text{Trace}, \text{Update})$ be an ATS-PR scheme with public parameters pp and let $\text{prop} \in \{\text{uf-}b, \text{acc-}b, \text{uf-}b \wedge \text{acc-}b'\}_{b,b' \in \{0,1\}}$. The advantage of an adversary \mathcal{A} in $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{sa-prop}}$ is defined as

$$\text{Adv}_{\text{PRATS}[\text{pp}]}^{\text{sa-prop}}(\mathcal{A}) \stackrel{\text{def}}{=} \Pr \left[\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{sa-prop}}(\mathcal{A}) = 1 \right].$$

Our constructions in Sections 4 and 5 will assume a (non-accountable) TS scheme with proactive refresh as a building block. In terms of the adaptiveness of the adversary, our constructions will inherit the security guarantees of the assumed TS scheme. Hence, in these sections we will not address the question of adaptivity directly. In Section 7, we will present direct constructions of ATS-PR schemes, and prove them secure against semi-adaptive adversaries (we will remind the reader of this fact in these sections). We leave the task of extending these constructions to handle fully-adaptive adversaries as an interesting open question for future work.

Extending the definitions to the random oracle model. All of the syntactic and security definitions above extend to the random oracle model by granting all algorithms, including the adversary \mathcal{A} , oracle access to a function H chosen uniformly at random from a family \mathcal{H} of functions. In the correctness and security definitions (Definition 3), all probabilities are then also taken over the choice of H .

4 A Combinatorial Construction for Few Signers

In this section we present a simple combinatorial construction of an ATS-PR scheme from any TS-PR scheme (that is, from any threshold signature scheme with proactive refresh but no accountability assurances). The ATS-PR scheme incurs an overhead of $\binom{n}{t}$ in the length of the public key and the secret keys, as well as in the running time of the key generation algorithm, where n is the total number of potential signers and t is the threshold. Hence, this scheme is of interest in the setting where n is relatively small.

Our combinatorial ATS-PR scheme, which we denote by CATS, uses as a building block a TS-PR scheme $\text{PRTS} = (\text{PRTS.KGen}, \text{PRTS.Sign}, \text{PRTS.Combine}, \text{PRTS.Vf}, \text{PRTS.Update} = (\text{PRTS.Update}_0, \text{PRTS.Update}_1))$. For ease of presentation, we assume that PRTS is a non-interactive scheme (recall Sections 2.1 and 3), which will result in our CATS being non-interactive as well. However, the construction and security proof readily extend to interactive schemes as well.¹

In the presentation of CATS , we rely on the following notation: Let $\mathcal{S}_{t,n}$ denote the collection of all t -sized subsets of $[n]$. For $i \in [n]$, let $\mathcal{S}_{t,n}(i)$ denote the collection of all such subsets that include index i .

CATS: a combinatorial ATS-PR scheme (built from a TS-PR scheme PRTS)

CATS.KGen(pp, n, t):

1. Set $\text{pk} \leftarrow \varepsilon$ and $\text{sk}_i \leftarrow \varepsilon$ for each $i \in [n]$.
2. For each $\mathcal{J} \in \mathcal{S}_{t,n}$:
 - (a) Sample $(\text{pk}^{\mathcal{J}}, \text{pkc}^{\mathcal{J}}, (\text{sk}_1^{\mathcal{J}}, \dots, \text{sk}_t^{\mathcal{J}})) \leftarrow \text{PRTS.KGen}(\text{pp}, t, t)$.
 - (b) Update $\text{pk} \leftarrow \text{pk} \parallel \text{pk}^{\mathcal{J}}$ and $\text{pkc} \leftarrow \text{pkc} \parallel \text{pkc}^{\mathcal{J}}$.
 - (c) For each $j \in \mathcal{J}$, update $\text{sk}_j \leftarrow \text{sk}_j \parallel \text{sk}_j^{\mathcal{J}}$.
3. Output $(\text{pk} = (n, t, \text{pk}), \text{pkc} = (n, t, \text{pkc}), (\text{sk}_1, \dots, \text{sk}_n))$.

CATS.Sign(sk_i, m):

1. Parse sk_i as $(\text{sk}_i^{\mathcal{J}})_{\mathcal{J} \in \mathcal{S}_{t,n}(i)}$.
2. For all $\mathcal{J} \in \mathcal{S}_{t,n}(i)$, compute $\sigma_i^{\mathcal{J}} \leftarrow \text{PRTS.Sign}(\text{sk}_i^{\mathcal{J}}, m)$.
3. Output $\sigma_i = (\sigma_i^{\mathcal{J}})_{\mathcal{J} \in \mathcal{S}_{t,n}(i)}$.

¹ We actually only need PRTS to be an n -out-of- n threshold signature scheme, which may be simpler to construct. To avoid introducing additional definitions, we present the construction assuming PRTS is a general TS-PR.

CATS.Combine(pk_c, σ_{i₁}, ..., σ_{i_ℓ}):

1. Parse pk_c as (pk^ℳ)_{ℳ ∈ S_{t,n}} and σ_{i_j} as (σ^ℳ_{i_j})_{ℳ ∈ S_{t,n}(i_j)} for each j ∈ [ℓ].
2. Let ℳ* = {j₁, ..., j_t} be the lexicographically first t-sized subset of {i₁, ..., i_ℓ}.
3. Compute σ ← PRTS.Combine(pk^{ℳ*}, σ^{ℳ*}_{j₁}, ..., σ^{ℳ*}_{j_t}).
4. Output σ' = (ℳ*, σ).

CATS.Vf(pk, m, σ'):

1. Parse pk as (n, t, pk), pk as (pk^ℳ)_{ℳ ∈ S_{t,N}}, and σ' as (ℳ*, σ).
2. If ℳ* < t, output 0.
3. Output 1 if PRTS.Vf(pk^{ℳ*}, m, σ) = 1.
Otherwise, output 0.

CATS.Trace(pk, m, σ):

1. Parse σ' as (ℳ*, σ).
2. Output ℳ*.

CATS.Update:

- CATS.Update₀(pk, sk_i):
 1. Parse pk as (pk^ℳ)_{ℳ ∈ S_{t,n}} and sk_i as (sk^ℳ_i)_{ℳ ∈ S_{t,n}(i)}.
 2. For each ℳ = (j₁, ..., j_t) ∈ S_{t,n}(i):
Sample (δ^ℳ_{j₁}, ..., δ^ℳ_{j_t}) ←_{\$} PRTS.Update₀(pk^ℳ, sk^ℳ_i).
 3. For each j ∈ [n], set δ_{i,j} ← (δ^ℳ_j)_{ℳ ∈ S_{t,n}(i) ∩ S_{t,n}(j)}.
 4. Output (δ_{i,1}, ..., δ_{i,n}}).}
- CATS.Update₁(sk_i, δ_{1,i}, ..., δ_{n,i}):
 1. Parse sk_i as (sk^ℳ_i)_{ℳ ∈ S_{t,n}(i)} and parse δ_{j,i} as (δ^ℳ_i)_{ℳ ∈ S_{t,n}(i) ∩ S_{t,n}(j)} for each j ∈ [N].
 2. For each ℳ = {j₁, ..., j_t} ∈ S_{t,n}(i):
Set $\widetilde{\text{sk}}_i^\mathcal{M} \leftarrow_{\$} \text{PRTS.Update}_1(\text{sk}_i^\mathcal{M}, \delta_{j_1}^\mathcal{M}, \dots, \delta_{j_t}^\mathcal{M})$.
 3. Output $\widetilde{\text{sk}}_i = (\widetilde{\text{sk}}_i^\mathcal{M})_{\mathcal{M} \in S_{t,n}(i)}$.

Theorem 1 below, which is proven in Appendix B.1, reduces the security of CATS to that of the underlying scheme PRTS.

Theorem 1. *Let $b \in \{0, 1\}$. For any adversary \mathcal{A} there exists an adversary \mathcal{B} such that for every public parameters pp it holds that*

$$\text{Adv}_{\text{CATS}[\text{pp}]}^{\text{uf-b} \wedge \text{acc-1}}(\mathcal{A}) \leq \text{bin}_{\max} \cdot \text{Adv}_{\text{PRTS}[\text{pp}]}^{\text{uf-b}}(\mathcal{B}),$$

where bin_{\max} is a bound on the binomial coefficient $\binom{n}{t}$ for n, t chosen by \mathcal{A} in $\mathbf{G}_{\text{CATS}[\text{pp}]}^{\text{uf-b} \wedge \text{acc-1}}$.

On the tightness of the reduction. As evident in the statement of Theorem 1, the reduction used in its proof incurs a loss of factor bin_{\max} . This may be reasonable when the total number n of potential signers is small, which is precisely the scenario in which one might use CATS. Nevertheless, we stress that this loss is a matter of definitional choices. In more detail, CATS uses $\binom{n}{t}$ independent instances PRTS. However, we chose to define our security games in Figure 1 in a manner that only captures single-instance security.² This allows us to more cleanly present the main contributions of the paper. As a result, the proof of Theorem 1 encompasses a multi-instance to single-instance

² Though we stress that our definition for interactive protocols (Appendix A) does capture concurrent executions of the signing protocol.

reduction for threshold signatures, resulting in a loss which is linear in the number of instances. If we were to define multi-instance security of TS-PR schemes (by a straightforward generalization of the games in Figure 1), then we would have a straightforward and tight reduction from the security of CATS to the multi-instance security of PRTS. In particular, if PRTS is instantiated using a scheme which has better-than-trivial multi-instance security, then the loss in Theorem 1 shrinks accordingly.

5 An Efficient Construction with Strong Security Guarantees

In this section we present a generic approach for obtaining ATS-PR schemes. The benefit of this approach over the one presented in Section 4 is that it has efficient instantiations even when $\binom{n}{t}$ is large. The ATS-PR scheme makes use of two basic schemes: An n -out-of- n TS-PR scheme (without accountability) and a t -out-of- n ATS scheme (without proactive refresh). The idea is to set the TS-PR public key as the public key of the new scheme. To refresh the secret keys of the new scheme, we refresh the secret keys of the TS-PR scheme and generate fresh epoch-specific keys for the ATS schemes. The epoch-specific ATS keys are then used to sign messages. To enforce consistency, in each update the new ATS public key is signed using the n -out-of- n TS-PR scheme. The epoch-specific ATS public key and the signature on it are then appended to signatures issued using the epoch-specific ATS signing keys.

We now formally present our generic ATS-PR scheme. The construction relies on the following two building blocks:

1. A threshold signature scheme with proactive refresh $\text{PRTS} = (\text{PRTS.KGen}, \text{PRTS.Sign}, \text{PRTS.Vf}, \text{PRTS.Update})$.³
2. An ATS scheme $\text{ATS} = (\text{ATS.KGen}, \text{ATS.Sign}, \text{ATS.Vf}, \text{ATS.Trace})$. We assume that ATS is equipped with a distributed key generation protocol $\Pi_{\text{ATS.KGen}}$ enabling signers to generate the keys for the scheme in a distributed manner.

When presenting our ATS scheme with proactive refresh we use the following notation. We write $\sigma \leftarrow \$ \text{PRTS.Sign}((\text{sk}_1, \dots, \text{sk}_n), \text{pkc}, m)$ to denote the process of simulating the execution of the (potentially interactive) signing protocol PRTS.Sign , where the i -th signer runs on local input (sk_i, m) and σ is the result of applying PRTS.Combine onto the local outputs of the protocol with key pkc . When presenting the signing procedure, we do so in a general language that also captures interactive protocols. In particular, we also provide the (potentially interactive) Sign algorithm with the subset \mathcal{J} of signers as input (we refer the reader to Section A for a formal definition of interactive ATS-PR schemes).

Our ATS-PR scheme $\text{PRATS} = (\text{PRATS.KGen}, \text{PRATS.Sign}, \text{PRATS.Vf}, \text{PRATS.Update}, \text{PRATS.Trace})$ is then defined as follows.⁴

PRATS: A generic ATS scheme with proactive refresh (built from PRTS and ATS)

$\text{PRATS.KGen}(\text{pp}, n, t)$:

³ As in Section 4, we only need PRTS to support n -out-of- n signing, which may be easier to instantiate than general threshold signature schemes.

⁴ Though the underlying PRTS and ATS scheme might be defined relative to a random oracle, we abstract this fact away for simplicity of presentation. The proof of security would remain essentially unchanged without this simplification.

1. Sample $(\text{PRTS.pk}, \text{PRTS.pkc}, (\text{PRTS.sk}_1, \dots, \text{PRTS.sk}_n)) \leftarrow \text{PRTS.KGen}(\text{pp}, n, n)$.
2. Sample $(\text{ATS.pk}, \text{ATS.pkc}, (\text{ATS.sk}_1, \dots, \text{ATS.sk}_n)) \leftarrow \text{ATS.KGen}(\text{pp}, n, t)$.
3. Compute $\sigma_{\text{pk}} \leftarrow \text{PRTS.Sign}((\text{PRTS.sk}_1, \dots, \text{PRTS.sk}_n), \text{PRTS.pkc}, \text{ATS.pk})$.
4. For $i = 1, \dots, n$ set $\text{sk}_i \leftarrow (\text{TS.sk}_i, \text{ATS.sk}_i, \text{ATS.pk}, \sigma_{\text{pk}})$.
5. Output $(\text{pk} = (n, t, \text{TS.pk}), \text{pkc} = \text{ATS.pkc}, (\text{sk}_1, \dots, \text{sk}_n))$.

PRATS.Sign($\text{sk}_i, m, \mathcal{J}$):

1. Parse sk_i as $(\text{PRTS.sk}_i, \text{ATS.sk}_i, \text{ATS.pk}, \sigma_{\text{pk}})$.
2. Invoke $\text{ATS.Sign}(\text{ATS.sk}_i, m, \mathcal{J})$ and let s_m denote the output of the protocol.
3. Output $s_i = (\mathcal{J}, \text{ATS.pk}, \sigma_{\text{pk}}, s_m)$.

PRATS.Combine($\text{pkc}, (s_{i_1}, \dots, s_{i_\ell})$):

1. Parse each s_i as $(\mathcal{J}_i, \text{ATS.pk}_i, \sigma_{\text{pk}, i}, s_{m, i})$.
2. Let $(\mathcal{J}, \text{ATS.pk}, \sigma_{\text{pk}}) \leftarrow (\mathcal{J}_{i_1}, \text{ATS.pk}_{i_1}, \sigma_{\text{pk}, i_1})$.
If for some $j \in [\ell]$ it holds that $(\mathcal{J}, \text{ATS.pk}, \sigma_{\text{pk}}) \neq (\mathcal{J}_{i_j}, \text{ATS.pk}_{i_j}, \sigma_{\text{pk}, i_j})$, output \perp .
3. Invoke $\sigma_m \leftarrow \text{ATS.Combine}(\text{pkc}, s_{m, i_1}, \dots, s_{m, i_\ell})$.
4. Output $\sigma = (\text{ATS.pk}, \sigma_{\text{pk}}, \sigma_m)$.

PRATS.Vf(pk, m, σ):

1. Parse pk as $(n, t, \text{PRTS.pk})$ and σ as $(\mathcal{J}, \text{ATS.pk}, \sigma_{\text{pk}}, \sigma_m)$.
2. Output 1 if $\text{PRTS.Vf}(\text{PRTS.pk}, \text{ATS.pk}, \sigma_{\text{pk}})$ and $\text{ATS.Vf}(\text{ATS.pk}, m, \sigma_m)$.
Otherwise, output 0.

PRATS.Trace(pk, m, σ):

1. Parse σ as $(\text{ATS.pk}, \sigma_{\text{pk}}, \sigma_m)$.
2. Verify that $\text{PRTS.Vf}(\text{PRTS.pk}, \text{ATS.pk}, \sigma_{\text{pk}}) = 1$ and otherwise, output \perp .
3. Output $\mathcal{J} = \text{ATS.Trace}(\text{ATS.pk}, m, \sigma_m)$.

PRATS.Update(sk_i, pk):

1. Parse sk_i as $(\text{PRTS.sk}_i, \text{ATS.sk}_i, \text{ATS.pk}, \sigma_{\text{pk}})$ and pk as $(n, t, \text{TS.pk})$.
2. Run $\text{PRTS.Update}(\text{PRTS.sk}_i, \text{PRTS.pk})$ and let $\text{PRTS.sk}'_i$ be the output of signer i .
3. Invoke $\text{ATS.KGen}(n, t)$ and let $(\text{ATS.pk}, \text{ATS.sk}'_i)$ denote the output of signer i .
4. Invoke $\text{PRTS.Sign}(\text{PRTS.pk}, \text{PRTS.sk}'_i, \text{ATS.pk})$ and let σ'_{pk} denote the output of the protocol.
5. Output $\text{sk}'_i = (\text{PRTS.sk}'_i, \text{ATS.sk}'_i, \text{ATS.pk}', \sigma'_{\text{pk}})$.

On the efficiency of the scheme. The public key of PRATS consists solely of the public key of the non-accountable scheme PRTS (in addition to n and t), for which we instantiations with short public keys are known (the reader is referred to Section 1 and the references therein). Two main efficiency measures that depend on ATS are:

- Length of signatures, which consist of an encoding of the set \mathcal{J} , a public key of ATS, an ATS signature, and a PRTS signature. Known PRTS do enjoy short signatures.
- The complexity of updates, which is dominated by (1) The key refresh of PRTS; (2) the execution of the distributed key generation protocol ATS.KGen to produce new ATS keys; and (3) invoking the signing algorithm of PRTS to collectively sign the new ATS keys. For items (1) and (3), existing PRTS schemes have efficient key updates and signing protocols.

In light of the above discussion, what is missing is an ATS scheme that simultaneously enjoy (1) a short public key; and (2) an efficient distributed key generation protocol. Although ATS schemes with short public keys are known (see for example [7]), their key generation procedures rely on a trapdoor, and hence do not admit efficient distributed analogs. To fill in this gap, in Section 6

we present a new construction of an ATS with short public keys and an efficient key distribution protocol.

Security. Theorem 2 below reduces the security of PRATS to that of PRTS and ATS. For concreteness, in the statement of the theorem and its proof we assume that PRTS and ATS satisfy uf-1 security. If either of them only satisfies uf-0 then essentially the same proof shows that PRATS satisfies uf-0 security.

Theorem 2. *For any adversary \mathcal{A} there exist adversaries \mathcal{B}_1 and \mathcal{B}_2 such that for every set pp of public parameters it holds that*

$$\text{Adv}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}(\mathcal{A}) \leq \frac{1}{E} \cdot \text{Adv}_{\text{ATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}(\mathcal{B}_1) + \text{Adv}_{\text{PRTS}[\text{pp}]}^{\text{uf-1}}(\mathcal{B}_2),$$

where E is a bound on the number of epochs requested by \mathcal{A} in $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}$.

The proof of Theorem 2 is presented in Appendix B.2.

6 An ATS with Short Public Key and Efficient DKG

In this section we present a new ATS construction with a short public key and an efficient distributed key generation protocol. In conjunction with our generic construction from Section 5, this yields a concretely-efficient ATS-PR construction.

Our scheme relies on the strong RSA assumption. It is instructive to present our scheme vis-a-vis the ATS of Bellare and Neven [7], also relying on RSA. Loosely, their scheme fixes a “global” exponent e . The secret key of signer i is then $\mathbf{H}(\text{nonce}, i)^{1/e}$ and signing with respect to a subset \mathcal{J} of signers is done via proving knowledge of the e -th roots of all elements $\{\mathbf{H}(\text{nonce}, j)\}_{j \in \mathcal{J}}$. As observed by Bellare and Neven, this can be done efficiently using a multi-prover variant of the GQ protocol [28]. The problem in our setting, however, is that deriving the secret keys in their scheme involves computing roots of random group element, thus requiring a trapdoor.⁵

Our construction. To address the aforesaid issue, we draw inspiration from cryptographic accumulators (see [19] as well as [10, 12] and the many references therein). Instead of associating different secret keys with roots of different group elements, we associate different keys with *different roots of the same group element*. That is, the public key consists of one group element Y , and the secret key of signer i is Y^{1/e_i} for some exponent e_i which is deterministically derived from the index i . As we will show, a careful generalization of the GQ protocol allows a subset \mathcal{J} of signers to collectively prove knowledge of the $(\prod_{j \in \mathcal{J}} e_j)$ -th root of Y , yielding an efficient ATS scheme. Moreover, we present an efficient 1-round (i.e., non-interactive) distributed key generation protocol for this scheme.

In detail, our construction is parameterized by a group \mathbb{G} in which the strong RSA problem is conjectured to be hard. Possible instantiations include the group \mathbb{Z}_N^* relative to a bi-prime modulus N , and class groups of imaginary quadratic fields. For each security parameter λ (implied by the description of the group \mathbb{G}) and integer $n = \text{poly}(\lambda)$, we assume an efficiently-computable injective mapping from $[n]$ to primes greater than 2^λ , and we denote by e_i the prime corresponding to $i \in \mathbb{N}$. The scheme makes use of two hash functions, treated as random oracles in the security proof. The first, \mathbf{H}_{com} maps pairs of subsets of signers and group elements to λ -bit strings. The second, \mathbf{H}_{chal} , maps 4-tuples consisting of a message, a group element, a subset of signers, and an additional group element, to exponents.

⁵ Alternatively, one can settle on a long public key $Y_1, \dots, Y_n \in \mathbb{G}^n$, and letting the secret key of signer i be the e -th root of Y_i . Such keys can be generated without a trapdoor, but the long public key is problematic in our setting.

RSAATS[\mathbb{G}]: An RSA-based ATS scheme

KGen(\mathbb{G}, n, t):

1. For $i = 1, \dots, n$: Sample $X_i \leftarrow \mathbb{G}$.
2. Compute $X \leftarrow \prod_{i=1}^n X_i$ and $Y \leftarrow X^{\prod_{i=1}^n e_i}$.
3. For each $i \in [n]$, set $\text{sk}_i \leftarrow X^{\prod_{j \in [n] \setminus \{i\}} e_j}$ (so that $\text{sk}_i^{e_i} = Y$).
4. Output $(\text{pk} = (n, t, Y), \text{pkc} = \perp, (\text{sk}_1, \dots, \text{sk}_n))$.

Sign($\text{sk}_i, \text{pk}, \mathcal{J}, m$):

1. **First Round:**
 - (a) Sample $Z_i \leftarrow \mathbb{G}$, and compute $R_i \leftarrow Z_i^{\prod_{j \in \mathcal{J}} e_j}$.
 - (b) Compute $c_i \leftarrow \text{H}_{\text{com}}(\mathcal{J}, R_i)$.
 - (c) Send c_i to each signer $j \in \mathcal{J} \setminus \{i\}$.
2. **Second Round:**
 - (a) Upon receiving a message c_j from each $j \in \mathcal{J} \setminus \{i\}$, send R_i to all $j \in \mathcal{J} \setminus \{i\}$.
 - (b) For each $j \in \mathcal{J} \setminus \{i\}$: Upon receiving R_j from signer j , verify that $c_j = \text{H}_{\text{com}}(\mathcal{J}, R_j)$. If not, abort the execution of the protocol.
 - (c) Set $R \leftarrow \prod_{j \in \mathcal{J}} R_j$.
3. **Third Round:**
 - (a) Set $h \leftarrow \text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$.
 - (b) Compute $S_i \leftarrow \text{sk}_i^h \cdot Z_i$.
 - (c) Output (h, S_i) .

Combine($\text{pkc}, (\sigma_{i_1}, \dots, \sigma_{i_k})$):

1. Parse each σ_{i_j} as (h^j, S^j) and set $h \leftarrow h^1$. If $h^j \neq h$ for a $j \in [k]$, output \perp .
2. Let $\mathcal{J} = \{i_1, \dots, i_k\}$, and compute $S \leftarrow \prod_{j \in [\mathcal{J}]} S_j$.
3. Output $\sigma = (\mathcal{J}, h, S)$.

Vf(pk, m, σ):

1. Parse pk as (n, t, Y) and σ as (\mathcal{J}, h, S) .
2. Compute $R \leftarrow S^{\prod_{i \in \mathcal{J}} e_i} / Y^{h \cdot \sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j}$.
3. Compute $h' \leftarrow \text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$.
4. Output 1 if $h = h'$ and $|\mathcal{J}| \geq t$. Otherwise, output 0.

Trace(pk, m, σ):

1. Parse σ as (\mathcal{J}, h, S) .
2. Output \mathcal{J} .

Correctness. Observe that for an honestly generated signature (\mathcal{J}, h, S) it holds that

$$\begin{aligned}
 S^{\prod_{i \in \mathcal{J}} e_i} &= \prod_{j \in \mathcal{J}} \left(S_j^{\prod_{i \in \mathcal{J}} e_i} \right) \\
 &= \prod_{j \in \mathcal{J}} \left(\left(\text{sk}_j^h \cdot Z_j \right)^{\prod_{i \in \mathcal{J}} e_i} \right) \\
 &= \prod_{j \in \mathcal{J}} \left(\text{sk}_j^{h \cdot \prod_{i \in \mathcal{J}} e_i} \cdot R_j \right) \\
 &= \prod_{j \in \mathcal{J}} \left(Y^{h \cdot \prod_{i \in \mathcal{J} \setminus \{j\}} e_i} \right) \cdot R.
 \end{aligned}$$

Rearranging, this implies that

$$R = \frac{S^{\prod_{i \in \mathcal{J}} e_i}}{Y^{h \cdot \sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j}},$$

and the verification goes through.

Distributed key generation. The public and secret keys of RSAATS can be generated via a simple distributed protocol. Recall that this is needed to instantiate **ATS** in the generic construction from Section 5. Concretely, this is done by the following steps:

1. Each signer $i \in [n]$ samples a uniformly random group element $X_i \leftarrow \$ \mathbb{G}$, computes $Y_i \leftarrow X_i^{e_i}$, and sends Y_i to all other signers.
2. Upon receiving Y_1, \dots, Y_n from the other signers, each signer sets the public key as $\text{pk} \leftarrow \prod_{i \in [n]} Y_i^{\prod_{j \in [n] \setminus \{i\}} e_j}$, and its secret key as

$$\text{sk}_i \leftarrow X_i^{\prod_{j \in [n] \setminus \{i\}} e_j} \cdot \prod_{k \in [n] \setminus \{i\}} Y_k^{\prod_{j \in [n] \setminus \{k, i\}} e_j} \in \mathbb{G}.$$

Observe that if we denote $X = \prod_{j \in [n]} X_j$, then $\text{pk} = X^{\prod_{j \in [n]} e_j}$ and $\text{sk}_i = X^{\prod_{j \in [n] \setminus \{i\}} e_j}$. Hence, sk_i is indeed the e_i -th root of pk for each i . Looking ahead, our security reduction for RSAATS will internally simulate precisely this key generation process, while planting a strong RSA challenge Y^* as one of the Y_i 's and simulating the role of all other signers. Hence, it will readily prove the security of the scheme when the key generation algorithm **KGen** is replaced by an honest execution of the above distributed key generation protocol.

Security. The security of RSAATS is proven based on the hardness of the strong RSA problem, defined in Definition 5.

Definition 5. Let \mathbb{G} be a group and let \mathcal{A} be an algorithm. We define the advantage of \mathcal{A} in solving the strong RSA problem in \mathbb{G} as

$$\text{Adv}_{\mathbb{G}}^{\text{srsa}}(\mathcal{A}) \stackrel{\text{def}}{=} \left[X^e = Y \wedge e \notin \{-1, 1\} : \begin{array}{l} Y \leftarrow \$ \mathbb{G} \\ (X, e) \leftarrow \$ \mathcal{A}(\mathbb{G}, Y) \end{array} \right]$$

Theorem 3 below, whose proof can be found in Appendix B.3, reduces the security of RSAATS to the hardness of the strong RSA problem.

Theorem 3. For any adversary \mathcal{A} there exists an algorithm \mathcal{B} such that

$$\text{Adv}_{\mathbb{G}}^{\text{srsa}}(\mathcal{B}) \geq \frac{\left(\text{Adv}_{\text{RSSATS}[\mathbb{G}]}^{\text{uf-1} \wedge \text{acc-1}}(\mathcal{A}) \right)^2}{n_{\max}^2 \cdot (q_{\text{sign}} + q_{\text{chal}})} - \frac{q_{\text{sign}}^2 + q_{\text{sign}} \cdot q_{\text{com}} + q_{\text{sign}} \cdot q_{\text{chal}} + q_{\text{sign}}}{|\mathbb{G}|} - \frac{2q_{\text{com}}^2 + 3q_{\text{sign}} \cdot q_{\text{com}} + q_{\text{sign}}^2}{2^\lambda},$$

where n_{\max} is a bound on the number of signers, and q_{sign} , q_{chal} , and q_{com} are bounds on the number of queries issued by \mathcal{A} to its signing oracle, to H_{chal} , and to H_{com} , respectively.

On the necessity of strong RSA. We stress that though the security statement for our ATS scheme relies on the strong RSA assumption, our proof actually reduces the security of the scheme

to a somewhat milder assumption. Concretely, the adversary that we construct in the reduction is restricted to computing the e th root of a randomly sampled group element Y , where e has to be chosen from a small set of pre-determined exponents $\{e_1, \dots, e_n\}$ (where n is the number of signers). This should be contrasted with the strong RSA problem, in which the adversary is free to choose e however it pleases.

7 Shorter Signatures From BLS and Schnorr

In this Section we present very practical ATS-PR schemes in cyclic groups and in bilinear groups, building on Schnorr [43] and BLS [16] signatures, respectively. We start by providing an overview of the main ideas behind these constructions.

Proactive secret sharing. Ostrovsky and Yung [40] and Herzberg et al. [30] described how to proactively refresh Shamir’s t -out-of- n secret sharing scheme [44]. Recall that in Shamir’s scheme, to share a secret s in some finite field \mathbb{F} , the dealer samples a random polynomial f of degree $t - 1$ over \mathbb{F} such that $f(0) = s$. The share x_i of the i th party is then $f(i)$. Any subset of parties of size t can reconstruct the secret using Lagrange interpolation, while any $t - 1$ shares are statistically independent of the secret s . To refresh, Ostrovsky and Yung and Herzberg et al. suggested the following procedure; the parties jointly sample a new polynomial f' of degree $t - 1$ such that $f'(0) = 0$. Then, each party updates its share by $x'_i \leftarrow x_i + f'(i)$. Observe that regardless of the number of share refreshes, the i th party’s share is always of the form $x_i + f''(i)$, where f'' is a polynomial of degree $t - 1$ satisfying $f''(0) = 0$. Hence, by the linearity of Shamir’s secret sharing scheme, the same reconstruction procedure still yields the correct secret s . At the same time, for any number E of epochs, and any $t - 1$ shares $(x_{i_1^e}, \dots, x_{i_{t-1}^e})$ in each epoch e , the distribution over $\{(x_{i_1^e}, \dots, x_{i_{t-1}^e})\}_{e \in [E]}$ is uniformly random in $\mathbb{F}^{(t-1) \times E}$. Hence, an adversary observing $t - 1$ shares of its choice in each epoch learns nothing about the secret s .

As observed in a follow-up work by Herzberg et al. [29], this proactive refresh naturally carries over to discrete-log-based (non-accountable) threshold signature schemes in which the signers’ secret keys form a t -out-of- n Shamir secret sharing of some “global” secret key x .

From secret sharing to ATS. At first, these techniques do not seem to extend to the ATS constructions based on Schnorr and BLS signatures (e.g., [14, 6, 11, 13, 39]). In such constructions, the signers’ secret keys x_1, \dots, x_n are sampled independently in \mathbb{Z}_p , where p is the order of the underlying groups. Then, each subset \mathcal{J} of signers is naturally associated with a distinct corresponding signing key $x_{\mathcal{J}} = \sum_{j \in \mathcal{J}} x_j \in \mathbb{Z}_p$. The corresponding public key $g^{x_{\mathcal{J}}}$ can be computed from the individual public keys g^{x_1}, \dots, g^{x_n} by $g^{\mathcal{J}} = \prod_{j \in \mathcal{J}} g^{x_j}$, which enables accountability. However, if we now try to refresh the keys by adding a secret sharing of 0, the sum $\sum_{j \in \mathcal{J}} x'_j$ of the fresh keys will no longer correspond to the public key $g^{x_{\mathcal{J}}}$, and the signing procedure will produce invalid signatures.

Instead, our approach is to associate to each subset \mathcal{J} the secret key

$$x_{\mathcal{J}} := \sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot x_j \in \mathbb{Z}_p,$$

where $\lambda_j^{\mathcal{J}}$ is the j th Lagrange coefficient used by the subset \mathcal{J} to reconstruct the secret in Shamir’s secret sharing scheme. Now, suppose that at the beginning of an epoch we refresh the secret keys $\{x_i\}_{i \in [n]}$ by setting $x'_i \leftarrow x_i + \delta_i \in \mathbb{Z}_p$, where $\{\delta_i\}_{i \in [n]}$ is a random t -out-of- n secret sharing of 0.

Then, for every subset \mathcal{J} of size t , it holds that

$$\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} x'_j = \sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} (x_j + \delta_j) = \sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} x_j = x_{\mathcal{J}}.$$

Hence, the “collective” secret key $x_{\mathcal{J}}$ of the subset \mathcal{J} remains unchanged from epoch to epoch, despite the proactive refresh. We will see how to put this to use in a minute.

An attack on accountability. The schemes resulting from this idea only satisfy the weaker acc-0 accountability property defined in Section 3. This seems inherent to this linear key update mechanism. To see why, we describe an attack that shows that the schemes do not satisfy acc-1. Consider the simple case of $t = 3$. In the first epoch, the adversary corrupts signers 1, 2 and 4, to learn the keys x_1, x_2, x_4 from which it computes $\lambda_1^{\{1,2,4\}} x_1 + \lambda_2^{\{1,2,4\}} x_2 + \lambda_4^{\{1,2,4\}} x_4$ (recall that $\lambda_i^{\mathcal{J}}$ is the coefficient of x_i in the Shamir secret sharing reconstruction for subset \mathcal{J}). In the second epoch, the adversary corrupts signers 1, 3 and 4, and obtains the refreshed keys x'_1, x'_3, x'_4 , from which it computes $\lambda_1^{\{1,3,4\}} x'_1 + \lambda_3^{\{1,3,4\}} x'_3 + \lambda_4^{\{1,3,4\}} x'_4$. This linear combination of refreshed keys is equal to the linear combination $\lambda_1^{\{1,3,4\}} x_1 + \lambda_3^{\{1,3,4\}} x_3 + \lambda_4^{\{1,3,4\}} x_4$ of the keys x_1, x_3, x_4 in the first epoch because this linear combination causes the refresh randomness $\delta_1, \delta_3, \delta_4$ to cancel (recall that $\delta_1, \delta_3, \delta_4$ are shares in a 3-out-of- n secret sharing of zero). The adversary now has two linear combinations of x_1, x_2, x_3, x_4 from which it can obtain a linear combination of x_1, x_2 and x_3 . In epoch 3 the adversary corrupts signers $\{1, 2, 5\}$ and in epoch 4 it corrupts signers $\{1, 3, 5\}$. This gives another linear combination of x_1, x_2, x_3 . Finally, in epochs 5 and 6 the adversary corrupts signers $\{1, 2, 6\}$ and then $\{1, 3, 6\}$ and this gives a third linear combination of x_1, x_2, x_3 . Now, the adversary has a full rank linear system for x_1, x_2, x_3 which lets it find all three keys. Thus, after six epochs the adversary can sign any message of its choice on behalf of the set $\{1, 2, 3\}$, even though there was never an epoch in which the adversary simultaneously corrupted all three of these signers. This shows that the resulting schemes do not satisfy acc-1. We will show that they satisfy acc-0.

Although the schemes in this section satisfy this weaker notion of accountability, they introduce non-trivial efficiency gains. On the face of it, acc-0 can be achieved (in conjunction with unforgeability) by signing a message twice: Once with a (non-accountable) TS-PR scheme and once with a (non-refreshable) ATS scheme. However, the downside of this approach is that it essentially doubles the length of signatures and the time to sign. In contrast, the schemes that we present in this section preserve the length and performance of their underlying schemes: Our BLS based scheme produces standard BLS signatures (consisting of just 1 group element), and our Schnorr-based produce standard Schnorr signatures (not counting the encoding of the subset \mathcal{J} of signers, which, as discussed in the introduction, cannot be avoided).

Unforgeability: uf-0 vs. uf-1. The unforgeability notion we will prove for all of our constructions in this section is uf-0, rather than uf-1. Restricting ourselves to uf-0 allows us to reduce the security of our constructions to the security of their underlying basic signature scheme (BLS or Schnorr). In contrast, as observed by Bellare et al. [5], proving uf-1 requires stronger “one-more”-type assumptions (e.g., one-more discrete-log). Moreover, focusing on uf-0 security already captures our main ideas, and extending the proofs to handle uf-1 can be done using the ideas from [5].

7.1 A BLS-Based ATS-PR Scheme

Our BLS-Based ATS with proactive refresh is parameterized by a bilinear group \mathcal{G} which is a tuple $(\mathbb{G}_0, \mathbb{G}_1, \mathbb{G}_T, e, g_0, g_1, p)$, where $\mathbb{G}_0, \mathbb{G}_1$ are cyclic groups of order p generated by g_0 and g_1

respectively, and e is a non-degenerate bilinear map, mapping pairs in $\mathbb{G}_0 \times \mathbb{G}_1$ to \mathbb{G}_T . It relies on the existence of a hash function $H : \mathcal{M} \rightarrow \mathbb{G}_0$, where \mathcal{M} is the message space. We will implicitly assume that the number n of signers is upper bounded by the order p of the group and that all arithmetic is over \mathbb{Z}_p .

BLSPR[\mathcal{G}]: A BLS-based ATS-PR scheme
<p><u>KGen(\mathcal{G}, n, t):</u></p> <ol style="list-style-type: none"> For $i = 1, \dots, n$: Sample $x_i \leftarrow \mathbb{Z}_p$, set $\text{sk}_i \leftarrow x_i$, and $X_i \leftarrow g_1^{x_i} \in \mathbb{G}_1$. Output $(\text{pk} = (t, X_1, \dots, X_n), \text{pkc} = \perp, (\text{sk}_1, \dots, \text{sk}_n))$. <p><u>Sign($\text{sk}_i, m$):</u></p> <ol style="list-style-type: none"> Compute $\sigma_i \leftarrow H(m)^{\text{sk}_i} \in \mathbb{G}_0$. Output σ_i. <p><u>Combine($\text{pkc}, (\sigma_{i_1}, \dots, \sigma_{i_{ \mathcal{J} }}$):</u></p> <ol style="list-style-type: none"> For $j \in \mathcal{J}$: Compute $\lambda_j^{\mathcal{J}} \leftarrow \prod_{i \in \mathcal{J} \setminus \{j\}} \frac{i}{i-j} \in \mathbb{Z}_p$. Compute $Y \leftarrow \prod_{j \in \mathcal{J}} \sigma_j^{\lambda_j^{\mathcal{J}}} \in \mathbb{G}_0$. <i>// observe that $Y = H(m)^{x_{\mathcal{J}}}$ where $x_{\mathcal{J}} = \sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} x_j$ is the “collective” secret key of subset \mathcal{J}.</i> Output $\sigma = (\mathcal{J}, Y)$. <p><u>Vf($\text{pk}, m, \sigma$):</u></p> <ol style="list-style-type: none"> Parse pk as (t, X_1, \dots, X_n) and σ as (\mathcal{J}, Y). For $j \in \mathcal{J}$: Compute $\lambda_j^{\mathcal{J}} \leftarrow \prod_{i \in \mathcal{J} \setminus \{j\}} \frac{i}{i-j} \in \mathbb{Z}_p$. Compute $X_{\mathcal{J}} \leftarrow \prod_{j \in \mathcal{J}} X_j^{\lambda_j^{\mathcal{J}}} \in \mathbb{G}_1$. Output 1 if $e(Y, g_1) = e(H(m), X_{\mathcal{J}})$ and $\mathcal{J} \geq t$. Else, output 0. <p><u>Trace(pk, m, σ):</u></p> <ol style="list-style-type: none"> Parse σ as (\mathcal{J}, Y). Output \mathcal{J}. <p><u>Update₀(sk_i, pk):</u></p> <p><i>// Signer i samples a random polynomial f_i of degree $t-1$ and such that $f_i(0) = 0$ and sends $f_i(j)$ to signer j.</i></p> <ol style="list-style-type: none"> Parse pk as (t, X_1, \dots, X_n). Sample $a_1, \dots, a_{t-1} \leftarrow \mathbb{Z}_p$. For $j = 1, \dots, n$: Compute $\delta_{i,j} \leftarrow \sum_{\ell=1}^{t-1} a_{\ell} \cdot j^{\ell} \in \mathbb{Z}_p$. Output $(\delta_{i,1}, \dots, \delta_{i,n})$. <p><u>Update₁($\text{sk}_j, \delta_{1,j}, \dots, \delta_{n,j}$):</u></p> <p><i>// Signer i adds $f(i)$ to its secret key, where $f = f_1 + \dots + f_n$ is a random polynomial of degree $t-1$ satisfying $f(0) = 0$.</i></p> <ol style="list-style-type: none"> Compute $\text{sk}'_j \leftarrow \text{sk}_j + \sum_{i=1}^n \delta_{i,j} \in \mathbb{Z}_p$. Output sk'_j.

Correctness. The correctness of the scheme follows from the fact that in each epoch k , the secret key of i is of the form $\text{sk}_i = x_i + \delta_i^{(k)}$, where the values $\{\delta_1^{(k)}, \dots, \delta_n^{(k)}\}$ form a Shamir t -out-of- n

secret sharing of 0. Hence, it holds that

$$\begin{aligned}
e(Y, g_1) &= e\left(\prod_{j \in \mathcal{J}} \sigma_j^{\lambda_j^{\mathcal{J}}}, g_1\right) = e(\mathbf{H}(m), g_1)^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot \text{sk}_j} \\
&= e(\mathbf{H}(m), g_1)^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot (x_j + \delta_j^{(k)})} = e(\mathbf{H}(m), g_1)^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot x_j} \\
&= \prod_{j \in \mathcal{J}} e(\mathbf{H}(m), X_j)^{\lambda_j^{\mathcal{J}}} \\
&= e(\mathbf{H}(m), \prod_{j \in \mathcal{J}} X_j^{\lambda_j^{\mathcal{J}}}).
\end{aligned}$$

This means that at every epoch, a signature generated by \mathcal{J} , where $|\mathcal{J}| \geq t$, will be accepted.

Security. We now prove the unforgeability and accountability of our BLS-based scheme. Recall that we consider a semi-adaptive variant of our unforgeability definition, in which for each epoch the adversary is forced to issue all of its secret key queries prior to its signing queries. Theorem 4 below reduces the security of our scheme to the unforgeability of standard (single-signer) BLS signatures. Unforgeability for single-signer signature schemes is captured as a special case of our security definitions, by fixing the number n of signers (and the threshold t) to be 1, and the number E of epochs to be 1. For an adversary \mathcal{B} , we denote by $\text{Adv}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}(\mathcal{B})$ its advantage in breaking the unforgeability of BLS. We refer to [17, 16] for a definition of unforgeability of signature schemes, and a description of the BLS signature scheme.

Theorem 4. *Let \mathcal{G} be a bilinear group. Then, for every adversary \mathcal{A} there exists an adversary \mathcal{B} such that*

$$\text{Adv}_{\text{BLSPR}[\mathcal{G}]}^{\text{sa-uf-0} \wedge \text{acc-0}}(\mathcal{A}) \leq 2n_{\max} \cdot \text{Adv}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}(\mathcal{B}),$$

where n_{\max} is an upper bound on the number of signers in the game $\mathbf{G}_{\text{BLSPR}[\mathcal{G}]}^{\text{uf-0} \wedge \text{acc-0}}$.

Roadmap. We prove Theorem 4 by proving unforgeability and accountability separately. Let BLSPR-1 denote a 1-epoch variant of our BLS-based ATS-PR scheme (obtained by fixing the number E of epochs to be 1). Unforgeability is then proven in two steps. First, Lemma 2 reduces the unforgeability of BLSPR-1 to that of BLS. Then, in Lemma 3 we reduce the unforgeability of our (multi-epoch) scheme to the 1-epoch variant.

Lemma 2. *For every adversary \mathcal{A} there exists an adversary \mathcal{B}_1 such that*

$$\text{Adv}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}(\mathcal{A}) \leq n_{\max} \cdot \text{Adv}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}(\mathcal{B}_1).$$

Lemma 3. *For every adversary \mathcal{A} there exists an adversary \mathcal{B}_2 such that*

$$\text{Adv}_{\text{BLSPR}[\mathcal{G}]}^{\text{sa-uf-0}}(\mathcal{A}) \leq \text{Adv}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}(\mathcal{B}_2).$$

The proof of Lemma 2 is similar to the proof of unforgeability for aggregated BLS signatures (see [16, 14, 17]) and is deferred to Appendix B.4. The proof of Lemma 3 can be found in Appendix B.5. *Accountability* of BLSPR is stated in Lemma 4 below, whose proof is deferred to Appendix B.6.

Lemma 4. *For every adversary \mathcal{A} there exists an adversary \mathcal{B} such that*

$$\text{Adv}_{\text{BLSPR}[\mathcal{G}]}^{\text{sa-acc-0}}(\mathcal{A}) \leq n_{\max} \cdot \text{Adv}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}(\mathcal{B}).$$

By definition of the $\mathbf{G}_{\text{BLSPR}[\mathcal{G}]}^{\text{sa-uf-0} \wedge \text{acc-0}}$ security game, Theorem 4 immediately follows from lemmas 2, 3, and 4.

7.2 A 3-Round Schnorr-Based ATS-PR Scheme

Our three-round Schnorr-Based ATS with proactive refresh builds on the recent (non-accountable) threshold Schnorr scheme of Lindell [36]. We adapt the scheme to an ATS (making it accountable) and show how to add a proactive refresh. We first simplify Lindell’s signing protocol: In his protocol, when each signer sends a group element R , it also provides alongside it a non-interactive zero-knowledge proof of knowledge for the discrete log of R . This is needed to prove simulation-based security, but (as we prove) is unnecessary to satisfy our game-based definitions. Then, we augment the protocol with a proactive refresh via the approach discussed at the beginning of this section. At the end of the section we explain how a similar approach can add a proactive refresh to the two-round Schnorr ATS schemes [39, 33, 21, 5].

Our ATS with proactive refresh is parameterized by a cyclic group $\mathbb{G} = \langle g \rangle$ of order p , and a pair of hash functions: H_{com} mapping pairs of subsets of signers and group elements into λ -bit strings, and H_{chal} mapping 4-tuples of the form (message, public key, signers subset, group element) into elements in \mathbb{Z}_p . We will implicitly assume that the number n of signers is upper bounded by the order p of the group, and that all finite field elements are in \mathbb{Z}_p .

Schnorr3-PR[\mathbb{G}] A three-round Schnorr-based ATS-PR scheme

KGen(\mathbb{G}, n, t):

1. For $i = 1, \dots, n$: Sample $x_i \leftarrow \mathbb{Z}_p$, set $\text{sk}_i \leftarrow x_i$, and $X_i \leftarrow g^{x_i} \in \mathbb{G}$.
2. Output $(\text{pk} = (t, X_1, \dots, X_n), \text{pkc} = \perp, (\text{sk}_1, \dots, \text{sk}_n))$.

Sign($\text{sk}_i, \text{pk}, \mathcal{J}, m$):

- **First Round:**
 1. Sample $r_i \leftarrow \mathbb{Z}_p$, and compute $R_i \leftarrow g^{r_i}$.
 2. Compute $c_i \leftarrow H_{\text{com}}(R_i)$.
 3. Send $\text{msg}_{i,1} \leftarrow c_i$ to each signer $j \in \mathcal{J} \setminus \{i\}$.
- **Second Round:**
 1. Upon receiving a message c_j from each $j \in \mathcal{J} \setminus \{i\}$, send $\text{msg}_{i,2} \leftarrow R_i$ to all $j \in \mathcal{J} \setminus \{i\}$.
- **Third Round:**
 1. For each $j \in \mathcal{J} \setminus \{i\}$: Upon receiving R_j from signer j , verify that $c_j = H_{\text{com}}(R_j)$. If not, abort.
 2. Set $R \leftarrow \prod_{j \in \mathcal{J}} R_j$.
 3. Set $h \leftarrow H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \in \mathbb{Z}_p$.
 4. Compute $\lambda_i^{\mathcal{J}} \leftarrow \prod_{j \in \mathcal{J} \setminus \{i\}} \frac{j}{j-i} \in \mathbb{Z}_p$ and $s_i \leftarrow \lambda_i^{\mathcal{J}} \cdot h \cdot \text{sk}_i + r_i \in \mathbb{Z}_p$.
 5. Output s_i .

Combine($\text{pkc}, \{s_j\}_{j \in \mathcal{J}}, \mathcal{J}, R$):

1. Upon receiving a message $s_j \in \mathbb{Z}_p$ from each $j \in \mathcal{J} \setminus \{i\}$, compute $s \leftarrow \sum_{j \in \mathcal{J}} s_j \in \mathbb{Z}_p$.
2. Output $\sigma = (\mathcal{J}, R, s)$.
// observe that $s = h \cdot x_{\mathcal{J}} + r$ where $x_{\mathcal{J}} = \sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} x_j$ is the “collective” secret key of subset \mathcal{J} and $r = \sum_{j \in \mathcal{J}} r_j$.

Vf(pk, m, σ):

1. Parse pk as (t, X_1, \dots, X_n) and σ as (\mathcal{J}, R, s) .
2. Compute $h \leftarrow H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \in \mathbb{Z}_p$.
3. For $j \in \mathcal{J}$: Compute $\lambda_j^{\mathcal{J}} \leftarrow \prod_{i \in \mathcal{J} \setminus \{j\}} \frac{i}{i-j} \in \mathbb{Z}_p$.
4. Output 1 if $\left(\prod_{j \in \mathcal{J}} X_j^{\lambda_j^{\mathcal{J}} \cdot h} \right) \cdot R = g^s$ and $|\mathcal{J}| \geq t$. Otherwise, output 0.

Trace(pk, m, σ):

1. Parse σ as (J, R, s).
2. Output J.

Update₀(sk_i, pk):

// Signer *i* samples a random polynomial f_i of degree $t - 1$ and such that $f_i(0) = 0$ and sends $f_i(j)$ to signer *j*.

1. Parse pk as (t, X_1, \dots, X_n) .
2. Sample $a_1, \dots, a_{t-1} \leftarrow_{\$} \mathbb{Z}_p$.
3. For $j = 1, \dots, n$: Compute $\delta_{i,j} \leftarrow \sum_{\ell=1}^{t-1} a_\ell \cdot j^\ell$.
4. Output $(\delta_{i,1}, \dots, \delta_{i,n})$,

Update₁(sk_j, δ_{1,j}, ..., δ_{n,j}):

// Signer *i* adds $f(i)$ to its secret key, where $f = f_1 + \dots + f_n$ is a random polynomial of degree $t - 1$ satisfying $f(0) = 0$.

1. Compute $\text{sk}'_j = \text{sk}_j + \sum_{i=1}^n \delta_{i,j}$.
2. Output sk'_j .

Correctness. The correctness of the scheme follows from the fact that, in each epoch k , the secret key of signer i is of the form $\text{sk}_i^{(k)} = x_i + \delta_i^{(k)}$, where the values $\{\delta_1^{(k)}, \dots, \delta_n^{(k)}\}$ form a Shamir t -out-of- n secret sharing of 0. Hence, using $h \leftarrow \text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$, for any epoch k , it holds that,

$$\begin{aligned}
g^s &= g^{\sum_{j \in \mathcal{J}} s_j} = g^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot h \cdot \text{sk}_j^{(k)} + r_j} = g^{\sum_{j \in \mathcal{J}} r_j} \cdot g^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot h \cdot (x_j + \delta_j^{(k)})} \\
&= \prod_{j \in \mathcal{J}} R_j \cdot g^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot h \cdot x_j} \cdot (g^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot \delta_j^{(k)}})^h \\
&= R \cdot \prod_{j \in \mathcal{J}} (g^{x_j})^{\lambda_j^{\mathcal{J}} \cdot h} \cdot (g^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot \delta_j^{(k)}})^h \\
&= R \cdot \prod_{j \in \mathcal{J}} X_j^{\lambda_j^{\mathcal{J}} \cdot h} \cdot (g^{\sum_{j \in \mathcal{J}} \lambda_j^{\mathcal{J}} \cdot \delta_j^{(k)}})^h
\end{aligned}$$

Since $\{\delta_1^{(k)}, \dots, \delta_n^{(k)}\}$ form a Shamir t -out-of- n secret sharing of 0, we can use Lagrange interpolation over all $j \in \mathcal{J}$ (where $|\mathcal{J}| = t$) to get:

$$\sum_{j \in \mathcal{J}} \lambda_j \cdot \delta_j^{(k)} = 0.$$

Combining this with the previous equation, we get,

$$g^s = R \cdot \prod_{j \in \mathcal{J}} X_j^{\lambda_j^{\mathcal{J}} \cdot h} \cdot (g^{\sum_{j \in \mathcal{J}} \lambda_j \cdot \delta_j^{(k)}})^h = R \cdot \prod_{j \in \mathcal{J}} X_j^{\lambda_j^{\mathcal{J}} \cdot h} \cdot (g^0)^h = R \cdot \prod_{j \in \mathcal{J}} X_j^{\lambda_j^{\mathcal{J}} \cdot h}$$

This proves that signatures produced in all epochs are correct.

Security. We now prove the unforgeability and accountability of this scheme for semi-adaptive adversaries. Theorem 5 reduces the security of our scheme to the unforgeability of standard (single-signer) Schnorr signatures.

Theorem 5. *Let \mathbb{G} be a cyclic group of order p . Then, for any adversary \mathcal{A} , there exists an adversary \mathcal{B} such that*

$$\text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-uf-0} \wedge \text{acc-0}}(\mathcal{A}) \leq 3n_{\max} \cdot \left(\text{Adv}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}(\mathcal{B}) + \frac{q_S^2 + q_S q_H}{p} + \frac{2q_C^2}{2^\lambda} + \frac{q_S}{|\mathcal{M}|} \right)$$

where n_{\max} is a bound on the number of signers, \mathcal{M} is the message space, and q_S , q_H , and q_C are bounds on the number of signature queries, H_{chal} queries, and H_{com} queries issued by \mathcal{A} .

The proof for Theorem 5 follows a similar road map to the security proof of our BLS-based construction, and can be found in Appendix B.7.

From 3 to 2 rounds Schnorr. We believe that our 3-rounds Schnorr-based ATS-PR scheme can be compressed into a 2-rounds scheme using ideas from the recently-proposed multisignature scheme MuSig2 [39] (see also [33]). In detail, MuSig2 gets rid of the first commitment round (in which signer i commits to R_i using a random oracle) by introducing additional randomness into the protocol. In their scheme, each signers sends 4 random $R_{i,1} \leftarrow g^{r_{i,1}}, \dots, R_{i,4} \leftarrow g^{r_{i,4}}$ group elements instead of one. The “collective” R is then computed as $R \leftarrow R_1 \cdot R_2^\alpha \cdot R_3^{\alpha^2} \cdot R_4^{\alpha^3}$, where $R_i \leftarrow \prod_{j \in \mathcal{J}} R_{j,i}$ and $\alpha \leftarrow H(\text{pk}, R_1, \dots, R_4, m)$ for a random oracle H . In the second round, each signer i sends the sum $s_i \leftarrow x_i \cdot h + \sum_{j=1}^4 \alpha^{j-1} \cdot r_{i,j}$. The aggregate signature is then (R, s) for $s = \sum_{i \in \mathcal{J}} s_i$; this can be verified as standard Schnorr signatures.

We believe that our 3-round scheme depicted above can be transformed into a 2-round scheme by relying on the same ideas. We leave the task of formally defining the resulting scheme and proving its security as an interesting question for future work.

8 Discussion and Extensions

In this section we explore several extensions and directions for future work.

Corruptions during key updates. In our security games, the adversary is oblivious to the key update process. It is reasonable to consider strengthenings in which the adversary is exposed, and potentially controls, some key-related information during key updates. One such strengthening is to provide the adversary with the entire transcript of the key update protocol (that is, all messages $\delta_{i,j}$ produced by the parties using Update_0). Observe that by our discussion in Section 6, our generic scheme from Section 5 remains secure under this strengthening. The situation is a bit more involved when it comes to our constructions in Section 7. There, exposing the adversary to the $\delta_{i,j}$ values by which the secret keys are shifted will render the key update useless. However, this is easily remedied by having the parties privately send their update messages (i.e., $\delta_{i,j}$ send from party i to party j is encrypted under j th public key). Then, security is maintained even if the adversary learns the update messages sent to at most $t - 1$ parties in each key update.

An even stronger security notion might allow the adversary to adversarially choose the update messages of the (up to $t - 1$) currently corrupted parties. On the face of it, it seems that our constructions from Section 7 readily extend to this setting by replacing the secret sharing step during the key updates with a *verifiable* t -out-of- N linear secret sharing scheme (e.g., using Feldman’s protocol [24, 30, 29]). As for the scheme obtained by combining our constructions from Sections 5 and 6, it seems that adopting a commit-and-open approach might be sufficient. In any case, we leave the task of extending our constructions to this setting as an interesting open question for future work.

Distributed and local key generation. Our definitions and constructions are in a setting where key generation is carried out by a central key generation algorithm. When the set of potential signers is a-priori known, a possible extension which has been considered in closely-related scenarios, is to replace this algorithm with a distributed key-generation protocol. In our constructions from Section 7, the (honestly-generated) secret keys are completely independent. Hence, for these constructions, one might even consider a scenario in which the set of potential signers need not be a-priori fixed, and signers can join and locally generate their own key material over time.

An interesting direction for future research is generalizing our security notions to accommodate both of the above extensions, and adjusting our constructions to these settings. Note that this will require in particular employing mechanisms to fence of rouge key attacks, in which the adversary maliciously chooses the keys of corrupted parties based on what it knows about the keys of honest users. In the simpler case where the identity of potential signers is known in advance, such mechanisms can be incorporated already in the distributed key generation protocol. In the setting where keys are locally and non-interactively generated by the parties, the signing protocols need to be adjusted to disallow such attacks. For more information on rogue key attacks and how to protect against them, see [38, 14, 11, 6, 13, 39] and the references therein.

Confirmation vs. tracing. In real-world settings it may be sufficient to use a *confirmation* algorithm instead of a *tracing* algorithm. A confirmation algorithm takes as input a rogue signature and a suspect quorum and outputs “yes” if the suspect quorum is the one that generated the given signature. This can lead to shorter accountable signatures because now it suffices to embed a commitment to the signing quorum in the signature, rather than explicitly encode the signing quorum in the signature.

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A Interactive ATS-PR

In this section, we present our definitions for interactive ATS-PR schemes. We focus on schemes in which the signing protocol is made up of three rounds of communication among the signers. These definitions capture our (non-refreshable) ATS construction from Section 6 and our ATS-PR construction from Section 7.2.

Three-round threshold signature schemes. In addition to threshold signature schemes with non-interactive signing procedures (with or without a proactive refresh), we also consider schemes in which signing is done via an interactive protocol of three rounds or less. The syntax for such schemes closely follows that of non-interactive schemes, with the following exception. In interactive schemes, the signature algorithm Sign is now an interactive protocol, made up of three sub-algorithms $(\text{Sign}_1, \text{Sign}_2, \text{Sign}_3)$, where:

- Sign_1 is a randomized algorithm which takes as input a secret key sk_i , a message m , and a subset \mathcal{J} of indices. It outputs a state $\text{st}_{i,1}$ and a first message $\text{msg}_{i,1}$ to be sent to all in round 1 of the protocol to all signers in $\mathcal{J} \setminus \{i\}$.
- Sign_2 is a deterministic algorithm which takes as input a state $\text{st}_{i,1}$ and incoming messages $\{\text{msg}_{j,1}\}_{j \in \mathcal{J} \setminus \{i\}}$. It outputs a state $\text{st}_{i,2}$ and a second message $\text{msg}_{i,2}$ to be sent to all in round 2 of the protocol to all signers in $\mathcal{J} \setminus \{i\}$.
- Sign_3 is a deterministic algorithm which takes as input a state $\text{st}_{i,2}$ and incoming messages $\{\text{msg}_{j,2}\}_{j \in \mathcal{J} \setminus \{i\}}$. It outputs a signature share s_i .

The syntax above requires knowledge of m and \mathcal{J} at the beginning of the protocol (these are given as inputs to Sign_1). This captures the standard scenario in which a subset \mathcal{J} of signers initiates the signing protocol in a coordinated manner in order to sign a particular message m . Defining the syntax in this manner also has the advantage of being general enough to capture protocols that require knowledge of m and \mathcal{J} already in the onset of the protocol. It should be noted, however, that signers in our interactive protocols (Sections 6 and 7) do not require knowledge of m until the

last message is computed, and require knowledge of \mathcal{J} during the all-but-last rounds only in order to know to whom to send their all-but-last messages.

The correctness definitions for an interactive threshold signature scheme are naturally extended from the non-interactive case, by replacing the non-interactive signing algorithm with an honest execution of the interactive signing protocol.

Security. Accountability and unforgeability of three-round threshold signatures are defined via a natural generalization of the analogous definitions for non-interactive schemes, presented in Figure 2. Importantly, the signing oracle is now replaced with three separate oracles, one for each of the sub-routines making up the signing protocol. The adversary may query any of these oracles on inputs and ordering of its choice. In particular, this allows the adversary to interact with honest signers in many concurrent sessions of the signing protocol, arbitrarily interleaving between them.

For a three-round threshold signatures scheme with proactive refresh $\text{PRTS} = (\text{KGen}, \text{Sign} = (\text{Sign}_1, \text{Sign}_2, \text{Sign}_3), \text{Vf}, \text{Trace}, \text{Update} = (\text{Update}_0, \text{Update}_1))$, the security games capturing its unforgeability and accountability are defined below. As in the non-interactive case, the scheme and corresponding games are parameterized by public parameters pp . To avoid over-cluttering in notation, we assume that pp are included in the public key outputted by KGen .

B Deferred Proofs

B.1 Proof of Theorem 1

Proof. Let $b \in \{0, 1\}$ and let \mathcal{A} be an adversary participating in the $\mathbf{G}_{\text{CATS}[\text{pp}]}^{\text{uf-b}\wedge\text{acc-1}}$ security game. Consider the following algorithm \mathcal{B} that takes part in the $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-b}}$ security game. \mathcal{B} takes as input the public parameters pp and simulates the game $\mathbf{G}_{\text{CATS}[\text{pp}]}^{\text{uf-b}\wedge\text{acc-1}}$ to \mathcal{A} as follows:

1. Invoke $\mathcal{A}(\text{pp})$ to obtain (n, t, E) . Send (t, t, E) to the challenger in $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-b}}$, and receive from the challenger $(\text{pk}^*, \text{pkc}^*)$.
2. Sample $\mathcal{J}^* \leftarrow \$_{\mathcal{S}_{t,n}}$ and set $\text{pk}^{\mathcal{J}^*} \leftarrow \text{pk}^*$ and $\text{pkc}^{\mathcal{J}^*} \leftarrow \text{pkc}^*$.
3. For each $\mathcal{J} = \{j_1, \dots, j_t\} \in \mathcal{S}_{t,n} \setminus \{\mathcal{J}^*\}$:
 - (a) Sample $(\text{pk}^{\mathcal{J}}, \text{pkc}^{\mathcal{J}}, (\text{sk}_{j_1}^{\mathcal{J},1}, \dots, \text{sk}_{j_t}^{\mathcal{J},1})) \leftarrow \text{PRTS.KGen}(\text{pp}, t, t)$.
 - (b) For $e = 1, \dots, E-1$, compute $(\text{sk}_{j_1}^{\mathcal{J},e+1}, \dots, \text{sk}_{j_t}^{\mathcal{J},e+1}) \leftarrow \$_{\text{PRTS.Update}(\text{pk}^{\mathcal{J}}, \text{sk}_{j_1}^{\mathcal{J},e}, \dots, \text{sk}_{j_t}^{\mathcal{J},e})}$.
4. Set $\text{pk} \leftarrow (\text{pk}^{\mathcal{J}})_{\mathcal{J} \in \mathcal{S}_{t,n}}$ and $\text{pkc} \leftarrow (\text{pkc}^{\mathcal{J}})_{\mathcal{J} \in \mathcal{S}_{t,n}}$.
5. Pass pk and pkc to \mathcal{A} and reply to its oracle queries as follows:
 - When \mathcal{A} issues a secret-key query (e, i) to skO : If $i \in \mathcal{J}^*$, then forward this query to the secret-key oracle in the game $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-b}}$. Denote the response by $\text{sk}_i^{\mathcal{J}^*,e}$. Reply to \mathcal{A} with $\text{sk}_i^e = (\text{sk}_i^{\mathcal{J},e})_{\mathcal{J} \in \mathcal{S}_{t,n}(i)}$.
 - When \mathcal{A} issues a signature share query (m, e, i) : If $i \in \mathcal{J}^*$, then forward the query to the signature oracle in $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-b}}$ and get back a partial signature s_i ; denote it as $s_i^{\mathcal{J}^*}$. For $\mathcal{J} \in \mathcal{S}_{t,n}(i) \setminus \{\mathcal{J}^*\}$, compute $s_i^{\mathcal{J}} \leftarrow \$_{\text{PRTS.Sign}(\text{sk}_i^{\mathcal{J},e}, m)}$. Reply to \mathcal{A} with $s_i = (s_i^{\mathcal{J}})_{\mathcal{J} \in \mathcal{S}_{t,n}(i)}$.

When \mathcal{A} outputs a message-signature pair (m^*, σ^*) , \mathcal{B} outputs the same. Observe that \mathcal{B} perfectly simulates the game $\mathbf{G}_{\text{CATS}[\text{pp}]}^{\text{uf-b}\wedge\text{acc-1}}$ to \mathcal{A} . Moreover, by the definition of CATS , whenever $\mathbf{G}_{\text{CATS}[\text{pp}]}^{\text{uf-b}\wedge\text{acc-1}}(\mathcal{A}) = 1$, this implies that $\sigma^* = (\mathcal{J}, \sigma)$ is a valid signature on m^* with respect to PRTS , and that in each epoch e there is at least one user $j_e \in \mathcal{J}$ for which \mathcal{A} did not query for $\text{sk}_{j_e}^e$, nor for the signature share of j_e on m^* . Hence, in the corresponding epoch, \mathcal{B} also did not query for $\text{sk}_{j_e}^{\mathcal{J}^*,e}$ nor for a

Games $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b}$, $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{acc-}b'}$, $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b \wedge \text{acc-}b'}$									
<pre> 1 : $\text{flag}_{\text{uf-}0}, \text{flag}_{\text{uf-}1}, \text{flag}_{\text{acc-}0}, \text{flag}_{\text{acc-}1} \leftarrow 0$ 2 : $(\text{st}, n, t, E) \leftarrow \mathcal{A}(\text{pp})$ 3 : $(\text{pk}, \text{pkc}, \text{sk}_1^1, \dots, \text{sk}_n^1) \leftarrow \text{KGen}(\text{pp}, n, t)$ 4 : for $e = \{2, \dots, E\}$ do 5 : $(\text{sk}_1^e, \dots, \text{sk}_n^e) \leftarrow \text{Update}(\text{pk}, \text{sk}_1^{e-1}, \dots, \text{sk}_n^{e-1})$ 6 : for $i = \{1, \dots, n\}$ do 7 : $\text{sid}_i \leftarrow 0, \mathcal{S}_{i,1} \leftarrow \emptyset, \mathcal{S}_{i,2} \leftarrow \emptyset$ 8 : $(m^*, \sigma^*) \leftarrow \mathcal{A}^{\text{skO}(\cdot, \cdot), \text{Sign}(\cdot)}(\text{st}, \text{pk}, \text{pkc})$ 9 : if $\text{Vf}(\text{pk}, m^*, \sigma^*) = 0$ then 10 : return 0 11 : if $\forall e \in [E], \mathcal{Q}_e^{\text{sk}} < t \wedge \mathcal{Q}_e^{\text{sig}}(m^*) = 0$ then 12 : $\text{flag}_{\text{uf-}0} \leftarrow 1$ 13 : if $\forall e \in [E], \mathcal{Q}_e^{\text{sk}} \cup \mathcal{Q}_e^{\text{sig}}(m^*) < t$ then 14 : $\text{flag}_{\text{uf-}1} \leftarrow 1$ 15 : if $\text{Trace}(\text{pk}, m^*, \sigma^*) \not\subseteq \cup_{e \in [E]} (\mathcal{Q}_e^{\text{sk}} \cup \mathcal{Q}_e^{\text{sig}}(m^*))$ then 16 : $\text{flag}_{\text{acc-}0} \leftarrow 1$ 17 : if $\forall e \in [E], \text{Trace}(\text{pk}, m^*, \sigma^*) \not\subseteq \mathcal{Q}_e^{\text{sk}} \cup \mathcal{Q}_e^{\text{sig}}(m^*)$ then 18 : $\text{flag}_{\text{acc-}1} \leftarrow 1$ 19 : Only game $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b}$: return $\text{flag}_{\text{uf-}b}$ 20 : Only game $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{acc-}b}$: return $\text{flag}_{\text{acc-}b}$ 21 : Only game $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b \wedge \text{acc-}b'}$: return $\text{flag}_{\text{uf-}b} \vee \text{flag}_{\text{acc-}b'}$ </pre>	<table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="text-align: left; padding: 5px;">Oracle $\text{Sign}_1\text{O}(e, i, \mathcal{J}, m)$:</th> <th style="text-align: left; padding: 5px;">Oracle $\text{Sign}_3\text{O}(e, i, \text{sid}, (\widetilde{\text{msg}}_{i_1,2}, \dots, \widetilde{\text{msg}}_{i_\ell,2}))$:</th> </tr> </thead> <tbody> <tr> <td style="width: 50%; padding: 5px; vertical-align: top;"> <pre> 1 : $\text{sid}_i \leftarrow \text{sid}_i + 1$ 2 : $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \cup \{\text{sid}_i\}$ 3 : $(\text{st}_{i,1}^{\text{sid}}, \text{msg}_{i,1}^{\text{sid}}) \leftarrow \text{Sign}_1(\text{sk}_i^e, m, \mathcal{J})$ 4 : return $\text{msg}_{i,1}^{\text{sid}}$ </pre> </td> <td style="width: 50%; padding: 5px; vertical-align: top;"> <pre> 1 : if $\text{sid} \notin \mathcal{S}_{i,2}$ then 2 : return \perp 3 : fi 4 : $\mathcal{Q}_e^{\text{sig}}(m) \leftarrow \mathcal{Q}_e^{\text{sig}}(m) \cup \{i\}$ 5 : $s_i^{\text{sid}} \leftarrow \text{Sign}_3(\text{st}_{i,2}^{\text{sid}}, \widetilde{\text{msg}}_{i_1,2}, \dots, \widetilde{\text{msg}}_{i_\ell,2})$ 6 : $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \cup \{i\}$ 7 : return s_i^{sid} </pre> </td> </tr> <tr> <th style="text-align: left; padding: 5px;">Oracle $\text{Sign}_2\text{O}(e, i, \text{sid}, (\widetilde{\text{msg}}_{i_1,1}, \dots, \widetilde{\text{msg}}_{i_\ell,1}))$:</th> <th style="text-align: left; padding: 5px;">Oracle $\text{skO}(e, i)$:</th> </tr> <tr> <td style="width: 50%; padding: 5px; vertical-align: top;"> <pre> 1 : if $\text{sid} \notin \mathcal{S}_{i,1}$ then 2 : return \perp 3 : fi 4 : $(\text{st}_{i,2}^{\text{sid}}, \text{msg}_{i,2}^{\text{sid}}) \leftarrow \text{Sign}_2(\text{st}_{i,1}^{\text{sid}}, \widetilde{\text{msg}}_{i_1,1}, \dots, \widetilde{\text{msg}}_{i_\ell,1})$ 5 : $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \cup \{\text{sid}\}$ 6 : $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \cup \{\text{sid}\}$ 7 : return $\text{msg}_{i,2}^{\text{sid}}$ </pre> </td> <td style="width: 50%; padding: 5px; vertical-align: top;"> <pre> 1 : $\mathcal{Q}_e^{\text{sk}} \leftarrow \mathcal{Q}_e^{\text{sk}} \cup \{i\}$ 2 : return sk_i^e </pre> </td> </tr> </tbody> </table>	Oracle $\text{Sign}_1\text{O}(e, i, \mathcal{J}, m)$:	Oracle $\text{Sign}_3\text{O}(e, i, \text{sid}, (\widetilde{\text{msg}}_{i_1,2}, \dots, \widetilde{\text{msg}}_{i_\ell,2}))$:	<pre> 1 : $\text{sid}_i \leftarrow \text{sid}_i + 1$ 2 : $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \cup \{\text{sid}_i\}$ 3 : $(\text{st}_{i,1}^{\text{sid}}, \text{msg}_{i,1}^{\text{sid}}) \leftarrow \text{Sign}_1(\text{sk}_i^e, m, \mathcal{J})$ 4 : return $\text{msg}_{i,1}^{\text{sid}}$ </pre>	<pre> 1 : if $\text{sid} \notin \mathcal{S}_{i,2}$ then 2 : return \perp 3 : fi 4 : $\mathcal{Q}_e^{\text{sig}}(m) \leftarrow \mathcal{Q}_e^{\text{sig}}(m) \cup \{i\}$ 5 : $s_i^{\text{sid}} \leftarrow \text{Sign}_3(\text{st}_{i,2}^{\text{sid}}, \widetilde{\text{msg}}_{i_1,2}, \dots, \widetilde{\text{msg}}_{i_\ell,2})$ 6 : $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \cup \{i\}$ 7 : return s_i^{sid} </pre>	Oracle $\text{Sign}_2\text{O}(e, i, \text{sid}, (\widetilde{\text{msg}}_{i_1,1}, \dots, \widetilde{\text{msg}}_{i_\ell,1}))$:	Oracle $\text{skO}(e, i)$:	<pre> 1 : if $\text{sid} \notin \mathcal{S}_{i,1}$ then 2 : return \perp 3 : fi 4 : $(\text{st}_{i,2}^{\text{sid}}, \text{msg}_{i,2}^{\text{sid}}) \leftarrow \text{Sign}_2(\text{st}_{i,1}^{\text{sid}}, \widetilde{\text{msg}}_{i_1,1}, \dots, \widetilde{\text{msg}}_{i_\ell,1})$ 5 : $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \cup \{\text{sid}\}$ 6 : $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \cup \{\text{sid}\}$ 7 : return $\text{msg}_{i,2}^{\text{sid}}$ </pre>	<pre> 1 : $\mathcal{Q}_e^{\text{sk}} \leftarrow \mathcal{Q}_e^{\text{sk}} \cup \{i\}$ 2 : return sk_i^e </pre>
Oracle $\text{Sign}_1\text{O}(e, i, \mathcal{J}, m)$:	Oracle $\text{Sign}_3\text{O}(e, i, \text{sid}, (\widetilde{\text{msg}}_{i_1,2}, \dots, \widetilde{\text{msg}}_{i_\ell,2}))$:								
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Oracle $\text{Sign}_2\text{O}(e, i, \text{sid}, (\widetilde{\text{msg}}_{i_1,1}, \dots, \widetilde{\text{msg}}_{i_\ell,1}))$:	Oracle $\text{skO}(e, i)$:								
<pre> 1 : if $\text{sid} \notin \mathcal{S}_{i,1}$ then 2 : return \perp 3 : fi 4 : $(\text{st}_{i,2}^{\text{sid}}, \text{msg}_{i,2}^{\text{sid}}) \leftarrow \text{Sign}_2(\text{st}_{i,1}^{\text{sid}}, \widetilde{\text{msg}}_{i_1,1}, \dots, \widetilde{\text{msg}}_{i_\ell,1})$ 5 : $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \cup \{\text{sid}\}$ 6 : $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \cup \{\text{sid}\}$ 7 : return $\text{msg}_{i,2}^{\text{sid}}$ </pre>	<pre> 1 : $\mathcal{Q}_e^{\text{sk}} \leftarrow \mathcal{Q}_e^{\text{sk}} \cup \{i\}$ 2 : return sk_i^e </pre>								

Fig. 2. The security games $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b}$, $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{acc-}b'}$, $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-}b \wedge \text{acc-}b'}$ for $b, b' \in \{0, 1\}$ for a three-round ATS-PR scheme $\text{PRATS} = (\text{KGen}, \text{Sign}, \text{Combine}, \text{Vf}, \text{Trace}, \text{Update})$ with public parameters pp . In line 8, we write $\text{Sign}(\cdot)$ as a short hand for denoting that \mathcal{A} has oracle access to the three oracles $\text{Sign}_1\text{O}(\cdot, \cdot, \cdot, \cdot, \cdot)$, $\text{Sign}_2\text{O}(\cdot, \cdot, \cdot, \cdot)$ and $\text{Sign}_3\text{O}(\cdot, \cdot, \cdot, \cdot)$. As in Figure 1, for a set \mathcal{X} and an element x , we let $\mathcal{X} \leftarrow \mathcal{X} \cup \{x\}$ be a shorthand for the following operation: If \mathcal{X} was previously defined, then set $\mathcal{X} \leftarrow \mathcal{X} \cup \{x\}$; if \mathcal{X} is still undefined, then set $\mathcal{X} = \{x\}$.

signature share of j_e on m^* with respect to this key. Therefore, if it additionally holds that $\mathcal{J} = \mathcal{J}^*$, then $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-b}}(\mathcal{B}) = 1$ as well. Since the view of \mathcal{A} is independent of the choice of \mathcal{J}^* , this implies that

$$\text{Adv}_{\text{PRTS}[\text{pp}]}^{\text{uf-b}}(\mathcal{B}) \geq \frac{1}{\text{bin}_{\max}} \cdot \text{Adv}_{\text{CATS}[\text{pp}]}^{\text{uf-b} \wedge \text{acc-1}}(\mathcal{A}),$$

concluding the proof of Theorem 1.

B.2 Proof of Theorem 2

Proof. Let \mathcal{A} be an adversary playing game $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}$ and let (M^*, S^*) denote the message and signature \mathcal{A} outputs at the end of the game. Assume without loss of generality that with probability 1, the signature S^* is a 3-tuple containing a public key for ATS, a PRTS signature, and an ATS signature. Let $S^* = (PK^*, S_0^*, S_1^*)$, and let S_{pk} denote the set of ATS public keys sampled by the challenger in $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}$, either by calling ATS.KGen as a subroutine of PRATS.KGen or by simulating $\Pi_{\text{ATS.KGen}}$ as part of PRATS.Update .

By total probability,

$$\begin{aligned} \text{Adv}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}(\mathcal{A}) &= \Pr \left[\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}(\mathcal{A}) = 1 \wedge PK^* \in S_{\text{pk}} \right] \\ &\quad + \Pr \left[\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}(\mathcal{A}) = 1 \wedge PK^* \notin S_{\text{pk}} \right]. \end{aligned}$$

Theorem 2 then follows from Lemma 6 and Lemma 5 below.

Lemma 5. *There exist an adversary \mathcal{B}_1 such that for all pp*

$$\Pr \left[\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}(\mathcal{A}) = 1 \wedge PK^* \in S_{\text{pk}} \right] \leq \frac{1}{E} \cdot \text{Adv}_{\text{ATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}(\mathcal{B}_1)$$

Proof (of Lemma 5). Consider the following adversary \mathcal{B}_1 playing in $\mathbf{G}_{\text{ATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}$. The adversary \mathcal{B}_1 invokes $\mathcal{A}(\text{pp})$ and simulates $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}$ as follows:

1. Receive (n, t, E) from \mathcal{A} . Forward (n, t) to the challenger in $\mathbf{G}_{\text{ATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}$.
2. Receive ATS.pk^* from the challenger in $\mathbf{G}_{\text{ATS}[\text{pp}]}^{\text{uf-1} \wedge \text{acc-1}}$.
Sample $(\text{PRTS.pk}, (\text{PRTS.sk}_1, \dots, \text{PRTS.sk}_n)) \leftarrow_{\$} \text{PRTS.KGen}(n, n)$ and send the public key $\text{PRATS.pk} = (n, t, \text{PRTS.pk})$ to \mathcal{A} .
3. Sample $e^* \leftarrow_{\$} [E]$ and answer the signing and secret key queries of \mathcal{A} as in the following manner:
 - For $e \in [E] \setminus \{e^*\}$: Sample $(\text{ATS.pk}^{(e)}, (\text{ATS.sk}_1^{(e)}, \dots, \text{ATS.sk}_n^{(e)})) \leftarrow_{\$} \text{ATS.KGen}(n, t)$ and compute

$$\sigma_{\text{pk}}^{(e)} \leftarrow_{\$} \text{PRTS.Sign}(\text{PRTS.pk}, (\text{PRTS.sk}_1, \dots, \text{PRTS.sk}_n), \text{ATS.pk}^{(e)}).$$

For each secret key query of the form (e, i) , send $(\text{PRTS.sk}_i, \text{ATS.sk}_i^{(e)}, \text{ATS.pk}_i^{(e)}, \sigma_{\text{pk}}^{(e)})$ to \mathcal{A} . For each query to one of the signing oracles, simulate the response of the oracle using the knowledge of the secret keys $\text{PRTS.sk}_1, \dots, \text{PRTS.sk}_n$ and $\text{ATS.sk}_1^{(e)}, \dots, \text{ATS.sk}_n^{(e)}$.

- For $e = e^*$: Compute

$$\sigma_{\text{pk}}^{(e^*)} \leftarrow_{\S} \text{PRTS.Sign}(\text{PRTS.pk}, (\text{PRTS.sk}_1, \dots, \text{PRTS.sk}_n), \text{ATS.pk}^*).$$

For each secret key query of the form (e, i) , send the index i as the input to a secret key query in the game $\mathbf{G}_{\text{ATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ and receive a key ATS.sk_i^* in response. Send $(\text{PRTS.sk}_i, \text{ATS.sk}_i^*, \text{ATS.pk}^*, \sigma_{\text{pk}}^{(e^*)})$ to \mathcal{A} as the response to the secret key query.

For each query to a signing oracle, forward the query to the corresponding oracles in $\mathbf{G}_{\text{ATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ (omitting the epoch index e^* from the input). If the query is to Sign_1O or Sign_2O , relay the response to \mathcal{A} . If it is to Sign_3O , then the oracle in $\mathbf{G}_{\text{ATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ responds with a

signature σ_m on a message m on behalf of a subset \mathcal{J} ; reply to \mathcal{A} with $(\text{ATS.pk}^*, \sigma_{\text{pk}}^{(e^*)}, \sigma_m)$.

4. Receive a message m^* and a signature $\sigma^* = (\text{pk}^*, \sigma_0^*, \sigma_1^*)$. If the output of the simulated $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ is 1 and $\text{pk}^* = \text{ATS.pk}^*$, then output (m^*, σ_1^*) . Otherwise, output \perp .

Observe that whenever the output of the simulated $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ game is 1, the forgery outputted by \mathcal{A} is a valid one, and in particular σ_1^* is a valid ATS signature on m^* with respect to pk^* . This also means that either it holds that $\text{PRATS.Trace}(\text{PRATS.pk}, m^*, \sigma^*) = \perp$, or in epoch e^* there is an index $j^* \in \mathcal{J}^*$ for which \mathcal{A} did not query for its secret key or its signature share on m^* . In the former case, this implies that $\text{ATS.Trace}(\text{pk}^*, m^*, \sigma_1^*) = \perp$, and in the latter, this implies that \mathcal{B}_1 did not issue the corresponding queries as well.

Finally, whenever $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}(\mathcal{A}) = 1 \wedge PK^* \in S_{\text{pk}}$ occurs and \mathcal{B}_1 guesses correctly e^* as the epoch relative to which \mathcal{A} outputs the forgery (note that this is well defined when $PK^* \in S_{\text{pk}}$), it is also the case that $\mathbf{G}_{\text{ATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}(\mathcal{B}_1)$. Conditioned on $PK^* \in S_{\text{pk}}$, the probability that $\text{pk}^* = \text{ATS.pk}^*$ is $1/E$, since the view of \mathcal{A} is independent of the epoch e^* which is chosen uniformly at random. This implies the lemma.

Lemma 6. *There exist an adversary \mathcal{B}_2 such that for all pp*

$$\Pr \left[\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}(\mathcal{A}) = 1 \wedge PK^* \notin S_{\text{pk}} \right] \leq \text{Adv}_{\text{PRTS}[\text{pp}]}^{\text{uf-b}\wedge\text{uf-1}}(\mathcal{B}_2)$$

Proof (of Lemma 6). Consider the following adversary \mathcal{B}_2 playing in $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-1}}$. The adversary \mathcal{B}_2 invokes $\mathcal{A}(\text{pp})$ and simulates $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ as follows:

1. Receive (n, t, E) from \mathcal{A} . Forward (n, n, E) to the challenger in $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-1}}$.
2. Receive PRTS.pk from the challenger in $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-1}}$ and forward PRTS.pk to \mathcal{A} as the public key in $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$.
3. Answer \mathcal{A} 's oracle queries as follows:
 - For $i = 1, \dots, E$: Sample

$$(\text{ATS.pk}^{(i)}, (\text{ATS.sk}_1^{(i)}, \dots, \text{ATS.sk}_n^{(i)})) \leftarrow_{\S} \text{ATS.KGen}(n, t).$$

and compute an n -out-of- n signature with respect to epoch i on $\text{ATS.pk}^{(i)}$, via access to the signing oracle in $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-1}}$. Denote the resulting signature by $\sigma_{\text{pk}}^{(i)}$ for each $i \in [E]$.

- In response to a secret-key query for index $\ell \in [n]$ and epoch $e \in [E]$: Forward the query to the secret-key oracle of $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-1}}$ and obtain a secret key $\text{PRTS.sk}_\ell^{(e)}$. Respond to \mathcal{A} with $\text{PRATS.sk}_\ell^{(e)} = (\text{PRTS.sk}_\ell^{(e)}, \text{ATS.sk}_\ell^{(e)}, \text{ATS.pk}^{(e)}, \sigma_{\text{pk}}^{(e)})$.

- In response to a signature queries for epoch $e \in [E]$, simulate the signing oracles of $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ using knowledge of $\text{ATS.sk}_1^{(e)}, \dots, \text{ATS.sk}_n^{(e)}$ and the values $\text{ATS.pk}^{(e)}$ and $\sigma_{\text{pk}}^{(e)}$.
- 4. Receive a message m^* and a signature $\sigma^* = (\text{pk}^*, \sigma_0^*, \sigma_1^*)$. If the output of the simulated $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ is 1 and $\text{pk}^* \notin \{\text{ATS.pk}^{(1)}, \dots, \text{ATS.pk}^{(E)}\}$, then output $(\text{pk}^*, \sigma_0^*)$. Otherwise, output \perp .

When the output of the simulated $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}$ game is 1, it holds the forgery outputted by \mathcal{A} is a valid one, and in particular σ_0^* is a valid PRTS signature on pk^* with respect to PRTS.pk . This also means that in each epoch \mathcal{A} issued at most $t - 1$ secret key queries, and so the same is true for \mathcal{B}_2 . Finally, conditioned on $PK^* \notin S_{\text{pk}}$, \mathcal{B}_2 never queried its signing oracle on pk^* . Hence, the event $\mathbf{G}_{\text{PRTS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}(\mathcal{B}_2)$ is contained in the conjunction $\mathbf{G}_{\text{PRATS}[\text{pp}]}^{\text{uf-1}\wedge\text{acc-1}}(\mathcal{A}) = 1 \wedge PK^* \in S_{\text{pk}}$, implying the lemma.

B.3 Proof of Theorem 3

Proof. Let \mathcal{A} be an adversary participating in the game $\mathbf{G}_{\text{RSAATS}[\mathbb{G}]}^{\text{uf-1}\wedge\text{acc-1}}$ and $q_{\text{chal}}, q_{\text{sign}}$ be bounds on the number of H_{chal} queries and signature queries issued by \mathcal{A} in $\mathbf{G}_{\text{RSAATS}[\mathbb{G}]}^{\text{uf-1}\wedge\text{acc-1}}$, respectively, and let $q = q_{\text{chal}} + q_{\text{sign}}$. Assume without loss of generality that \mathcal{A} does not issue the same query to H_{com} or to H_{chal} more than once. Consider the following algorithm \mathcal{B}_0 . On inputs (\mathbb{G}, Z) and (h_1, \dots, h_q) , the algorithm \mathcal{B}_0 simulates $\mathbf{G}_{\text{RSAATS}[\mathbb{G}]}^{\text{uf-1}\wedge\text{acc-1}}$ to \mathcal{A} as follows:

1. Invoke $\mathcal{A}(\mathbb{G})$ and receive (n, t) from \mathcal{A} .
2. Sample $i^* \leftarrow_{\$} [n]$. For $j \in [n] \setminus \{i^*\}$ sample $X_j \leftarrow_{\$} \mathbb{G}$ and compute $\text{sk}_\ell \leftarrow Z^{\prod_{j \in [n] \setminus \{i^*, \ell\}} e_\ell} \cdot \left(\prod_{j \in [n] \setminus \{i^*\}} X_j \right)^{\prod_{k \in [n] \setminus \{\ell\}} e_k}$.
3. Compute $Y \leftarrow Z^{\prod_{j \in [n] \setminus \{i^*\}} e_j} \cdot \left(\prod_{j \in [n] \setminus \{i^*\}} X_j \right)^{\prod_{\ell \in [n]} e_\ell}$ and pass $\text{pk} = (n, t, Y)$ to \mathcal{A} .
4. Answer oracle queries as follows:
 - Initialize a dictionary \mathcal{L}_{com} of input-output pairs for H_{com} . When \mathcal{A} issues a query $q = Z$ to H_{com} , if $\mathcal{L}_{\text{com}}[q]$ is defined, then let $c = \mathcal{L}[q]$. If not, let $c \leftarrow \{0, 1\}^\lambda$. If there exists a query q' in \mathcal{L}_{com} such that $\mathcal{L}_{\text{com}}[q'] = c$, then output \perp and terminate. Otherwise, record $\mathcal{L}_{\text{com}}[q] = c$. Reply with c .
 - Initialize a counter $t = 0$ and a dictionary $\mathcal{L}_{\text{chal}}$ of input-output pairs for H_{chal} . When \mathcal{A} issues a query $q = (m, \text{pk}, \mathcal{J}, R)$ to H_{chal} , if $\mathcal{L}_{\text{chal}}[q]$ is defined, then let $h = \mathcal{L}_{\text{chal}}[q]$. Otherwise let $t = t + 1$ and $h = h_t$ and record $\mathcal{L}_{\text{chal}}[q] = h$. Reply with h .
 - In response to a secret-key query for index $\ell \in [n]$ reply as follows. If $\ell = i^*$, abort the simulation and output \perp . Otherwise, reply with sk .
 - In response to signing queries: If the session identifier provided by \mathcal{A} is inconsistent with previous queries, return \perp to \mathcal{A} . Otherwise, simulate users $j \neq i^*$ honestly using the knowledge of sk_j , simulating the random oracles and recording their responses.
For $j = i^*$:
 - (a) *First round:* Sample $c_{i^*} \leftarrow \{0, 1\}^\lambda$ and send c_{i^*} as the first message of user i^* in the protocol.
 - (b) *Second round:* Let $t = t + 1$ and $h = h_t$, sample $S_{i^*} \leftarrow \mathbb{G}$, and compute $R_{i^*} = S_{i^*}^{\prod_{j \in \mathcal{J}} e_j} \cdot Y^{-h \cdot \prod_{j \in \mathcal{J} \setminus \{i^*\}} e_j}$. If $\mathcal{L}_{\text{com}}[R_{i^*}]$ has already been defined, then output \perp and terminate. Otherwise, record $\mathcal{L}_{\text{com}}[R_{i^*}] = c_{i^*}$.

Given $\{c_j\}_{j \in \mathcal{J} \setminus \{i^*\}}$, for each j find the group element R_j such that $\mathcal{L}_{\text{com}}[R_j] = c_j$. If for some j there is no such value, then skip to Step 4c. If for some j there exist two elements R_j and R'_j such that $\mathcal{L}_{\text{com}}[R_j] = c_j = \mathcal{L}_{\text{com}}[R'_j]$, then output \perp and terminate the simulation. Let $R = \prod_{j \in \mathcal{J}} R_j$. If $\mathcal{L}_{\text{chal}}[(m, \text{pk}, \mathcal{J}, R)]$ is already defined then output \perp and terminate. Otherwise, record $\mathcal{L}_{\text{chal}}[(m, \text{pk}, \mathcal{J}, R)] = h$ and send R_{i^*} as the second message in the protocol.

- (c) *Third round:* Receive all openings $\{R_j\}_{j \in \mathcal{J} \setminus \{i^*\}}$ provided as the second-round messages. For all $j \in \mathcal{J} \setminus \{i^*\}$: If $\mathcal{L}_{\text{com}}[R_j]$ is undefined, then sample $c \leftarrow \{0, 1\}^\lambda$ and record $\mathcal{L}_{\text{com}}[R_j] = c$. Verify that for all $j \in \mathcal{J} \setminus \{i^*\}$ it holds that $\mathcal{L}_{\text{com}}[R_j]$ is consistent with the commitment c_j from the first round. If for some j this is not the case, reply \perp to \mathcal{A} . Otherwise, reply with S_{i^*} sampled in Step 4b.

At the end of the simulation, the forger \mathcal{A} outputs a message m^* and a signature $\sigma^* = (\mathcal{J}^*, R^*, S^*)$. If $i^* \notin \mathcal{J}^*$, if \mathcal{A} never queried H_{chal} with $q = (m^*, \text{pk}, \mathcal{J}^*, R^*)$ or if the output of the simulated game $\mathbf{G}_{\text{RSAATS}[\mathbb{G}]}^{\text{uf-1} \wedge \text{acc-1}}$ is 0, then \mathcal{B}_0 outputs \perp and terminate. Otherwise, let ℓ^* denote the index of the H_{chal} -query in which \mathcal{A} queried for $\text{H}_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*)$. Then, \mathcal{B}_0 outputs $(\ell^*, (h_{\ell^*}, \sigma^*))$.

We say that \mathcal{B}_0 *succeeds* if its output is different than \perp , and we turn to bound the probability that \mathcal{B}_0 succeeds. Recall that \mathcal{B}_0 outputs \perp if one of the following events occurs:

- The simulated $\mathbf{G}_{\text{RSAATS}[\mathbb{G}]}^{\text{uf-1} \wedge \text{acc-1}}$ game comes to a conclusion and its output is 0. Denote this event by E_1 .
- \mathcal{A} queries the secret key of i^* or outputs a forgery with respect to a subset that does not include i^* . Denote this event by E_2 .
- In a q query to H_{com} that was not previously defined, \mathcal{B}_0 samples a value c which a previous query was mapped to. Denote this event by E_3 . Since c is sampled uniformly at random from $\{0, 1\}^\lambda$, and \mathcal{L}_{com} are recorded either during signature or H_{com} queries, the probability that E_3 occurs is bounded by $q_{\text{com}} \cdot (q_{\text{com}} + q_{\text{sign}}) \cdot 2^{-\lambda}$.
- In some signing query, $\mathcal{L}_{\text{com}}[R_{i^*}]$ was already defined before R_{i^*} was computed in Step 4b. Denote this event by E_4 . Observe that the distribution of R_{i^*} is uniform in \mathbb{G} . Hence, the probability that E_3 occurs is bounded by $q_{\text{sign}} \cdot (q_{\text{com}} + q_{\text{sign}}) / |\mathbb{G}|$.
- In some signing query, \mathcal{A} provided a commitment c_j which is a collision between (at least) two values in \mathcal{L}_{com} . Denote this event by E_5 . Since all c_j values stored in \mathcal{L}_{com} are chosen uniformly at random, the birthday bound implies that the probability for E_5 is at most $(q_{\text{com}} + q_{\text{sign}})^2 \cdot 2^{-\lambda}$.
- In some signing query, $\mathcal{L}_{\text{chal}}[(m, \text{pk}, \mathcal{J}, R)]$ was already defined for R computed in Step 4b. Denote this event by E_6 . Since R is distributed uniformly in \mathbb{G} , and $\mathcal{L}_{\text{chal}}$ values are recorded in either H_{chal} queries or signature queries, it holds that the probability of E_6 is bounded by $q_{\text{sign}} \cdot (q_{\text{chal}} + q_{\text{sign}}) / |\mathbb{G}|$.

Taking everything together, we obtain that the probability that \mathcal{B}_0 succeeds is bounded by

$$\begin{aligned}
\Pr[\mathcal{B}_0 \text{ succeeds}] &\geq \Pr[\overline{\text{E}}_1 \wedge \overline{\text{E}}_2 \wedge \overline{\text{E}}_3 \wedge \overline{\text{E}}_4 \wedge \overline{\text{E}}_5 \wedge \overline{\text{E}}_6] \\
&\geq \Pr[\overline{\text{E}}_1 \wedge \overline{\text{E}}_2] - \Pr[\text{E}_3] - \Pr[\text{E}_4] - \Pr[\text{E}_5] - \Pr[\text{E}_6] \\
&\geq \Pr[\overline{\text{E}}_1 \wedge \overline{\text{E}}_2] - \frac{q_{\text{sign}}^2 + q_{\text{sign}} \cdot q_{\text{com}} + q_{\text{sign}} \cdot q_{\text{chal}} + q_{\text{sign}}}{|\mathbb{G}|} \\
&\quad - \frac{2q_{\text{com}}^2 + 3q_{\text{sign}} \cdot q_{\text{com}} + q_{\text{sign}}^2}{2^\lambda}.
\end{aligned}$$

Observe that the event $\overline{E_1} \wedge \overline{E_2}$ occurs when the output of the simulated $\mathbf{G}_{\text{RSAATS}[\mathbb{G}]}^{\text{uf-1}\wedge\text{acc-1}}$ game is 1, \mathcal{A} never queries for the secret key of i^* , and the forgery outputted by \mathcal{A} is with respect a subset \mathcal{J}^* that contains i^* . Note that for the output of the game to be 1, there has to be at least one index $j^* \in \mathcal{J}^*$ such that \mathcal{A} never queries for the secret key of j^* . Hence, and since i^* is chosen uniformly at random from $[n]$, we get that

$$\Pr[\overline{E_1} \wedge \overline{E_2}] \geq \frac{1}{n_{\max}} \cdot \Pr[\mathbf{G}_{\text{RSAATS}[\mathbb{G}]}^{\text{uf-1}\wedge\text{acc-1}} = 1] = \frac{1}{n_{\max}} \cdot \text{Adv}_{\text{RSSATS}[\mathbb{G}]}^{\text{uf-1}\wedge\text{acc-1}}(\mathcal{A}).$$

Putting things together, it holds that

$$\begin{aligned} & \Pr[\mathcal{B}_0 \text{ succeeds}] \\ & \geq \frac{\text{Adv}_{\text{RSSATS}[\mathbb{G}]}^{\text{uf-1}\wedge\text{acc-1}}(\mathcal{A})}{n_{\max}} - \frac{q_{\text{sign}}^2 + q_{\text{sign}} \cdot q_{\text{com}} + q_{\text{sign}} \cdot q_{\text{chal}} + q_{\text{sign}}}{|\mathbb{G}|} \\ & \quad - \frac{2q_{\text{com}}^2 + 3q_{\text{sign}} \cdot q_{\text{com}} + q_{\text{sign}}^2}{2^\lambda}. \end{aligned} \tag{1}$$

Now consider the algorithm \mathcal{B} attempting to solve the strong RSA problem. On input (\mathbb{G}, Y) , \mathcal{B} runs the forking algorithm $\mathcal{F}_{\mathcal{B}_0}$ for \mathcal{B}_0 guaranteed by Lemma 1. If the output of $\mathcal{F}_{\mathcal{B}_0}$ is \perp , then \mathcal{B} outputs \perp and terminate. Otherwise, \mathcal{B} obtains a pair $((h, \sigma = (\mathcal{J}, R, S)), (h', \sigma' = (\mathcal{J}', R', S')))$ such that $h \neq h'$. Moreover, by the definition of \mathcal{B}_0 and that the following equations hold:

$$S \prod_{i \in \mathcal{J}} e_i = R \cdot Y^{h \cdot \sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j} \tag{2}$$

$$S' \prod_{i \in \mathcal{J}'} e_i = R' \cdot Y^{h' \cdot \sum_{i \in \mathcal{J}'} \prod_{j \in \mathcal{J}' \setminus \{i\}} e_j} \tag{3}$$

Additionally, by the definition of \mathcal{B}_0 , we also know that $R = R'$ and $J = J'$, that $i^* \in \mathcal{J} \cap \mathcal{J}'$. Assume without loss of generality that $h > h'$ (otherwise, the proof is symmetric). Hence, dividing Eq. (2) by (3) and rearranging, we obtain that

$$(S/S') \prod_{i \in \mathcal{J}} e_i = Y^{(h-h') \cdot (\sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j)} \tag{4}$$

We argue that $\prod_{i \in \mathcal{J}} e_i$ and $(h-h') \cdot (\sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j)$ are coprime.

Claim 6 $\text{GCD}(\prod_{i \in \mathcal{J}} e_i, (h-h') \cdot (\sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j)) = 1$.

We postpone the proof of the claim to the end of the section. We now explain how \mathcal{B} decides on its output. First, by Claim 6 and Bezout's theorem, there exist integers a and b , efficiently computable using Euclid's extended algorithm, such that

$$a \cdot \prod_{i \in \mathcal{J}} e_i + b \cdot (h-h') \cdot \left(\sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j \right) = 1$$

This implies that

$$\begin{aligned} Y &= Y^{a \cdot \prod_{i \in \mathcal{J}} e_i + b \cdot (h-h') \cdot (\sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j)} \\ &= \left(Y^a \cdot (S/S')^b \right)^{\prod_{i \in \mathcal{J}} e_i}, \end{aligned} \tag{5}$$

where Eq. (5) follows from Eq. (4). Denote $W = Y^a \cdot (S/S')^b$ and $X_{-i^*} = \prod_{j \in [n] \setminus \{i^*\}} X_j$. Then, plugging in $Y = Z^{\prod_{j \in [n] \setminus \{i^*\}} e_j} \cdot \left(\prod_{j \in [n] \setminus \{i^*\}} X_j \right)^{\prod_{\ell \in [n]} e_\ell}$ and rearranging, we obtain that

$$W^{\prod_{i \in \mathcal{J}} e_i} = Z^{\prod_{j \in [n] \setminus \{i^*\}} e_j} \cdot X_{-i^*}^{\prod_{\ell \in [n]} e_\ell}.$$

Rearranging,

$$\left(\frac{W}{X_{-i^*}^{\prod_{\ell \in [n] \setminus \mathcal{J}} e_\ell}} \right)^{\prod_{\ell \in \mathcal{J}} e_\ell} = Z^{\prod_{j \in [n] \setminus \{i^*\}} e_j}.$$

Denote $T = \frac{W}{X_{-i^*}^{\prod_{\ell \in [n] \setminus \mathcal{J}} e_\ell}}$. Since, by definition, $\{e_i\}_{i \in [n]}$ are coprime to the order of \mathbb{G} , this implies that

$$T^{e_{i^*}} = Z^{\prod_{j \in [n] \setminus \mathcal{J}} e_j}.$$

Since $i^* \in \mathcal{J}$, it holds that $\text{GCD}(e_{i^*}, \prod_{j \in [n] \setminus \mathcal{J}} e_j) = 1$. Hence, using the extended Euclid's algorithm, we can find integers a' and b' such that

$$Z = Z^{a' \cdot e_{i^*} + b' \cdot \prod_{j \in [n] \setminus \mathcal{J}} e_j} = \left(Z^{a'} \cdot T^{b'} \right)^{e_{i^*}}.$$

Hence, \mathcal{B} computes $U = Z^{a'} \cdot T^{b'}$ and outputs (U, e_{i^*}) as its output to the strong RSA problem. By the above reasoning, whenever \mathcal{B} outputs an output other than \perp it succeeds in solving the strong RSA problem. By Lemma 1, this happens with probability at least

$$\Pr[\mathcal{B}_0 \text{ succeeds}] \cdot \left(\frac{\Pr[\mathcal{B}_0 \text{ succeeds}]}{q_{\text{sign}} + q_{\text{chal}}} - \frac{1}{2^\lambda} \right).$$

Using Eq. (1), we obtain Theorem 3.

We conclude by proving Claim 6.

Proof (Claim 6). Note that $0 < h - h' < 2^\lambda < \min\{e_i\}_{i \in \mathcal{J}}$ and hence

$$\text{GCD} \left(\prod_{i \in \mathcal{J}} e_i, h - h' \right) = 1.$$

We are left with arguing that $\text{GCD} \left(\prod_{i \in \mathcal{J}} e_i, \sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j \right) = 1$. Assume towards contradiction that this is not the case. Then, there exists some $\ell \in \mathcal{J}$ such that e_ℓ divides $\sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j$. However, we can write

$$\alpha = \frac{\sum_{i \in \mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{i\}} e_j}{e_\ell} = \sum_{i \in \mathcal{J} \setminus \{\ell\}} \prod_{j \in \mathcal{J} \setminus \{i, \ell\}} e_j + \frac{\prod_{j \in \mathcal{J} \setminus \{\ell\}} e_j}{e_\ell}$$

This implies that e_ℓ divides $\prod_{j \in \mathcal{J} \setminus \{\ell\}} e_j$ in contradiction to the fact that $\{e_i\}_{i \in \mathcal{J}}$ are distinct primes.

B.4 Proof of Lemma 2

Proof. Consider the following adversary \mathcal{B}_1 playing the game $\mathbf{G}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}$. \mathcal{B}_1 can issue signature queries and random oracle queries to its BLS challenger. To distinguish the random oracle in $\mathbf{G}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}$ from the one in the 1-epoch BLS ATS, we denote queries to the former by $\hat{\mathbf{H}}(m)$.

On getting a challenge public key pk^* from its BLS challenger, the adversary \mathcal{B}_1 invokes $\mathcal{A}(\mathcal{G})$ and simulates $\mathbf{G}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}$ as follows:

1. Receive (n, t) from \mathcal{A} .
2. Guess $i^* \leftarrow \$_[n]$. Sample $\{\alpha_1, \dots, \alpha_{i^*-1}, \alpha_{i^*+1}, \dots, \alpha_n\} \leftarrow \$_{\mathbb{Z}_p}$. Compute $\text{pk}_j \leftarrow g^{\alpha_j}$ for all $i \neq i^*$.
3. Set $\text{pk}_{i^*} \leftarrow \text{pk}^*$, i.e. \mathcal{B}_1 embeds the challenge public key in position i^* .
4. Send $\text{pk} = \{\text{pk}_1, \dots, \text{pk}_n\}$, $\text{pkc} = \perp$ to \mathcal{A} .

Next, \mathcal{A} issues a sequence of queries, and \mathcal{B}_1 replies as follows:

- $\mathbf{H}(m)$: \mathcal{B}_1 queries its challenger for $h \leftarrow \hat{\mathbf{H}}(m)$, sets $\mathbf{H}(m) \leftarrow h$ and returns h to \mathcal{A} .
- $\text{skO}(1, i)$: If $i = i^*$, \mathcal{B}_1 aborts. Otherwise, \mathcal{B}_1 returns α_i .
- $\text{SignO}(m, 1, i)$: We assume without loss of generality that the adversary always queries $\mathbf{H}(m)$ before any signing query on m . If $i = i^*$, \mathcal{B}_1 answers by querying its signing oracle for a signature. Otherwise, \mathcal{B}_1 returns $\mathbf{H}(m)^{\alpha_i}$.

Eventually, \mathcal{A} outputs a forgery $(m^*, (\mathcal{J}^*, \sigma^*))$. If $i^* \notin \mathcal{J}^*$ or if $i^* \in \mathcal{Q}_1^{\text{sig}}(m^*)$, then \mathcal{B}_1 aborts (outputs \perp and terminates). Let α^* denote the secret key corresponding to the BLS challenge pk^* (that is, $\text{pk}^* = g^{\alpha^*}$). Observe that if the forgery outputted by \mathcal{A} is valid and if \mathcal{B}_1 hasn't aborted, it means that

$$\sigma^* = (\mathbf{H}(m)^{\alpha^* \lambda_{i^*}}) \prod_{i \in \mathcal{J}^*, i \neq i^*} \mathbf{H}(m)^{\lambda_i \alpha_i}$$

where $\lambda_j = \prod_{i \in \mathcal{J}^* \setminus \{j\}} \frac{i}{i-j}$.

Then, \mathcal{B}_1 computes

$$\sigma' \leftarrow \left(\frac{\sigma^*}{\prod_{i \in \mathcal{J}^*, i \neq i^*} \mathbf{H}(m)^{\lambda_i \alpha_i}} \right)^{\frac{1}{\lambda_{i^*}}}$$

and outputs the pair (m^*, σ') . Note that if \mathcal{A} successfully outputted a forgery, then (m^*, σ') is a valid message-signature pair with respect to the challenge public key pk^* . Also note that if \mathcal{A} did not query for a signature share on m^* with respect to user i^* , then by definition of \mathcal{B}_1 , $i^* \notin \mathcal{Q}_1^{\text{sig}}(m^*)$.

Finally, note that whenever \mathcal{A} wins in the unforgeability game $\mathbf{G}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}$, there must be an index $j^* \in \mathcal{J}^*$ for which \mathcal{A} never queries the secret key or a signature share for the forgery message m^* . If \mathcal{B}_1 correctly guesses this value ($i^* = j^*$), then it also outputs a successful forgery. Overall, since \mathcal{B}_1 perfectly simulates $\mathbf{G}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}$ to \mathcal{A} and the view of \mathcal{A} is independent of i^* , it holds that

$$\text{Adv}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}(\mathcal{B}_1) \geq \frac{1}{n_{\max}} \cdot \text{Adv}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}(\mathcal{A}).$$

This proves the lemma.

B.5 Proof of Lemma 3

Proof. Consider the following adversary \mathcal{B}_2 playing the game $\mathbf{G}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}$. \mathcal{B}_2 can issue signature queries and random oracle queries to its 1-epoch BLS ATS challenger. To distinguish the random oracle in $\mathbf{G}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}$ from the one in the general BLS ATS, we denote queries to the former by $\hat{\text{H}}(m)$.

\mathcal{B}_2 invokes $\mathcal{A}(\mathcal{G})$ and simulates $\mathbf{G}_{\text{BLSPR}[\mathcal{G}]}^{\text{sa-uf-0}}$ as follows:

1. Receive (n, t, E) from \mathcal{A} , and forward (n, t) to the challenger in $\mathbf{G}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}$.
2. Receive $\text{pk} = (\text{pk}_1, \dots, \text{pk}_n), \text{pkc}$ from the challenger and forward them to \mathcal{A} .
3. Reply to oracle queries by \mathcal{A} as follows:
 - Whenever \mathcal{A} queries H on a message m : \mathcal{B}_2 queries $h \leftarrow \hat{\text{H}}(m)$, sets $\text{H}(m) \leftarrow h$ and returns h .
 - Whenever \mathcal{A} queries its secret-key oracle on (k, i) (that is, the key of user i in epoch k): If $k = 1$, then \mathcal{B}_2 forwards this query to its own secret-key oracle, sets sk_i^1 as the response. If $k > 1$, then \mathcal{B}_2 samples a random secret key $\text{sk}_i^k \leftarrow \mathbb{Z}_p$. For each epoch k , \mathcal{B}_2 maintains a set R_k , encoding all the signers for which \mathcal{B}_2 determined the secret key in epoch k . \mathcal{B}_2 sets $R_k \leftarrow R_k \cup \{i\}$ and returns sk_i^k to \mathcal{A} .
 - Signing queries: Assume without loss of generality that \mathcal{A} always queries the random oracle for $\text{H}(m)$ before issuing a signing query of the form (m, k, i) (a signature share on m relative to i 's secret key in the k th epoch). Whenever \mathcal{A} issues a signing query (m, k, i) , if $k = 1$ then \mathcal{B}_2 forwards the query (m, i) to its signing oracle, and replies with the response. Otherwise, if $k > 1$, \mathcal{B}_2 decides on its response as follows:
 - If $i \in R_k$, then \mathcal{B}_2 replies with $\text{H}(m)^{\text{sk}_i^k}$.
 - If $i \notin R_k$ and $|R_k| < t - 1$, then \mathcal{B}_2 samples $\text{sk}_i^k \leftarrow \mathbb{Z}_p$, updates $R_k \leftarrow R_k \cup \{i\}$, and replies with $\text{H}(m)^{\text{sk}_i^k}$.
 - If $i \notin R_k$ and $|R_k| = t - 1$, then we use the fact that $\forall j \in R_k, \text{sk}_j^k = \text{sk}_j^1 + \delta_j^k$ for some δ_j^k , where $\{\delta_j^k\}$ are Shamir secret shares of 0. In other words, in a random execution of the protocol, there should be a polynomial f of degree $t - 1$ such that $\delta_j^k = f(j) \forall j \in R_k$ and $f(0) = 0$. Hence, by Lagrange interpolation, $\delta_i^k = \sum_{j \in R_k} \lambda_j(i) \cdot (\text{sk}_j^k - \text{sk}_j^1)$, where $\lambda_j(i) = \prod_{\ell \in R_k \setminus \{j\}} \frac{\ell - i}{\ell - j}$.
 To compute the response, \mathcal{B}_2 computes $a \leftarrow \text{H}(m)^{\sum_{j \in R_k} \lambda_j(i) \cdot \text{sk}_j^k}$. For each $j \in (R_k \cup \{i\}) \cap R_1$ it then computes a term b_j as $b_j \leftarrow \text{H}(m)^{\text{sk}_j^1}$. For each $j \in (R_k \cup \{i\}) \setminus R_1$, it requests a signature share on m with respect to signer j from its own signing oracle, and sets b_j to be the response. Finally, \mathcal{B}_2 replies to \mathcal{A} with the partial signature $a \cdot b_i \cdot \prod_{j \in R_k} b_j^{-\lambda_j(i)}$.
 The above discussion shows that this preserves the perfect simulation of $\mathbf{G}_{\text{BLSPR}[\mathcal{G}]}^{\text{sa-uf-0}}$.

Eventually, \mathcal{A} outputs a forgery $(m^*, (\mathcal{J}^*, \sigma^*))$. \mathcal{B}_2 outputs this forgery as well. We claim that if the output of the simulated $\mathbf{G}_{\text{BLSPR}[\mathcal{G}]}^{\text{sa-uf-0}}$ is 1 then so is the output of $\mathbf{G}_{\text{BLSPR-1}[\mathcal{G}]}^{\text{sa-uf-0}}$. This is because in this case it holds that

- $\text{Vf}(\text{pk}, m^*, \sigma^*) = 1$.
- For each $k \in [E]$ it holds that $|\mathcal{Q}_k^{\text{sk}}| < t$. Hence, $|\mathcal{Q}_1^{\text{sk}}| < t$. Also, \mathcal{B}_2 makes no secret key queries for epochs $k > 1$.

- For all $k \in [E]$ it holds that $\mathcal{Q}_k^{\text{sig}}(m^*) = \emptyset$. In particular, this means that \mathcal{B}_2 forwards no signing queries for m^* to its signing oracle.

Hence, all the requirements are satisfied for the output $\mathbf{G}_{\text{BLS-PR-1}[\mathcal{G}]}^{\text{sa-uf-0}}$ to be 1. Since

$$\text{Adv}_{\text{BLS-PR}[\mathcal{G}]}^{\text{sa-uf-0}}(\mathcal{A}) \leq \text{Adv}_{\text{BLS-PR-1}[\mathcal{G}]}^{\text{sa-uf-0}}(\mathcal{B}_2).$$

This completes the proof of the lemma.

B.6 Proof of Lemma 4

Proof. Consider the following adversary \mathcal{B} playing the game $\mathbf{G}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}$. We use α to denote the challenge secret key in the BLS game $\mathbf{G}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}$. On getting a challenge public key $\text{pk}^* \leftarrow g^\alpha$ from its BLS challenger, \mathcal{B} invokes $\mathcal{A}(\mathcal{G})$ and simulates $\mathbf{G}_{\text{BLS-PR}[\mathcal{G}]}^{\text{sa-acc-0}}$ as follows:

- Receive (n, t, E) from \mathcal{A} .
- Guess $i^* \leftarrow_{\$} [n]$. Sample $\{\alpha_1, \dots, \alpha_{i^*-1}, \alpha_{i^*+1}, \dots, \alpha_n\} \leftarrow_{\$} \mathbb{Z}_p$. Compute $\text{pk}_j \leftarrow g^{\alpha_j}$ for all $i \neq i^*$.
- Set $\text{pk}_{i^*} \leftarrow \text{pk}^*$ (i.e. \mathcal{B} embeds the challenge public key at position i^*).
- For all $k \in \{2, \dots, E\}$, sample $a_{1,k}, \dots, a_{t-1,k} \leftarrow_{\$} \mathbb{F}_p$. Then, for all $i \in [n]$ compute $\delta_i^k \leftarrow \sum_{l=1}^{t-1} a_{l,k} \cdot i^l$. (i.e. we generate new Shamir secret shares of 0 for every epoch).
- Send $\text{pk} = \{\text{pk}_1, \dots, \text{pk}_n\}$, $\text{pkc} = \perp$ to \mathcal{A} .

Next, \mathcal{A} issues a sequence of queries. We assume without loss of generality that \mathcal{A} always queries $\text{H}(m)$ before issuing any signing query on m . We now specify how \mathcal{B} answers each of those queries:

- $\text{H}(m)$: \mathcal{B} queries the random oracle of its BLS challenger and forwards the answer to \mathcal{A} . \mathcal{B} also maintains a mapping of messages to their corresponding hashes.
- $\text{skO}(k, i)$: If $i = i^*$, \mathcal{B} aborts. Otherwise, if $k = 1$, \mathcal{B} returns α_i , and if $k > 1$, \mathcal{B} returns $\alpha_i + \sum_{j=2}^k \delta_i^j$.
- $\text{SignO}(m, k, i)$: If $i \neq i^*$, \mathcal{B} returns $\text{H}(m)^{\alpha_i^k}$, where $\alpha_i^k = \alpha_i$ if $k = 1$ and $\alpha_i^k = \alpha_i + \sum_{j=2}^k \delta_i^j$ if $k > 1$. If $i = i^*$, \mathcal{B} queries its BLS challenger for $\sigma_{m, i^*} \leftarrow_{\$} \text{Sign}(m)$. Then, if $k = 1$, \mathcal{B} returns σ_{m, i^*} , and if $k > 1$, \mathcal{B} returns $\sigma_{m, i^*} \cdot \text{H}(m)^{\sum_{j=2}^k \delta_{i^*}^j}$.

Eventually, \mathcal{A} outputs a forgery $(m^*, (\mathcal{J}^*, \sigma^*))$. \mathcal{B} aborts if $i^* \notin \mathcal{J}^*$ or if there exists some $k \in [E]$ such that $i^* \in \mathcal{Q}_k^{\text{sig}}(m^*)$. Observe that by a similar analysis to that found in the proof of Lemma 2, if \mathcal{B} does not abort, it wins the unforgeability game of BLS. Moreover, by the same analysis, it holds that

$$\text{Adv}_{\text{BLS}[\mathcal{G}]}^{\text{uf}}(\mathcal{B}) \geq \frac{1}{n_{\text{max}}} \cdot \text{Adv}_{\text{BLS-PR}[\mathcal{G}]}^{\text{sa-acc-0}}(\mathcal{A}).$$

This concludes the lemma.

B.7 Proof of Theorem 5

The proof of Theorem 5 follows the same roadmap as the proof for our BLS-based scheme, proving unforgeability and accountability separately. Let us use **Schnorr3-PR-1** to denote a 1-epoch variant of our Schnorr-based 3-round ATS-PR scheme (obtained by fixing the number E of epochs to 1). Similar to the BLS based scheme, unforgeability is proven in two steps. First, we reduce the unforgeability of **Schnorr3-PR-1** to that of Schnorr signatures in Lemma 7. Then, in Lemma 8, we reduce the unforgeability of our multi-epoch scheme to the 1-epoch variant.

Lemma 7. For every adversary \mathcal{A} for the game $\mathbf{G}_{\text{Schnorr3-PR-1}[\text{pp}]}^{\text{sa-uf-0}}$, there exists an adversary \mathcal{B}_1 , such that for all groups \mathbb{G} ,

$$\text{Adv}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{A}) \leq n_{\text{max}} \cdot (q_H + q_S) \cdot \left(\text{Adv}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}(\mathcal{B}_1) + \frac{q_S \cdot (q_H + q_S)}{p} + \frac{2q_C^2}{2^\lambda} \right).$$

Lemma 8. For every adversary \mathcal{A} , there exists an adversary \mathcal{B}_2 such that,

$$\text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{A}) \leq (q_S + q_H) \cdot \left(\text{Adv}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{B}_2) + \frac{q_S(q_S + q_H)}{p} + \frac{2q_C^2}{2^\lambda} \right).$$

Lemma 9. For every adversary \mathcal{A} there exists an adversary \mathcal{B} , such that,

$$\text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}(\mathcal{A}) \leq n_{\text{max}} \cdot (q_S + q_H) \cdot \left(\text{Adv}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}(\mathcal{B}) + \frac{q_S(q_S + q_H)}{p} + \frac{2q_C^2}{2^\lambda} \right).$$

Proof of Lemma 7

Proof. We assume without loss of generality that \mathcal{A} does not issue the same query to H_{com} or to H_{chal} more than once.

Consider the following adversary \mathcal{B}_1 playing the game $\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}$. \mathcal{B}_1 can issue signature queries and random oracle queries. To distinguish the random oracle in $\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}$ from the one in the 1-epoch Schnorr ATS, we denote queries to the former by $\hat{\text{H}}_{\text{sig}}(m, \text{pk}, R)$.

On getting a challenge public key pk^* from its challenger, \mathcal{B}_1 invokes $\mathcal{A}(\mathbb{G})$ and simulates $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}$ as follows:

- Receive (n, t) from \mathcal{A} .
- Guess $i^* \leftarrow_{\$} [n]$. Then, sample $\{\alpha_1, \dots, \alpha_{i^*-1}, \alpha_{i^*+1}, \dots, \alpha_n\} \leftarrow_{\$} \mathbb{Z}_p$. Compute $\text{pk}_j \leftarrow g^{\alpha_j} \forall j \neq i^*$.
- Set $\text{pk}_{i^*} \leftarrow \text{pk}^*$. Send $(\text{pk} = \{\text{pk}_1, \dots, \text{pk}_n\}, \text{pkc} = \perp)$ to \mathcal{A} .

Next, \mathcal{A} issues a sequence of queries. We use q_S, q_H, q_C to denote a bound on the number of signing queries, random oracle queries on H_{chal} and random oracle queries on H_{com} , issued by \mathcal{A} across all signing sessions, respectively.

\mathcal{B}_1 also samples $j^* \leftarrow_{\$} [q_H + q_S]$, guessing which of the $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ queries will correspond to the forgery that \mathcal{A} will return. Note that j^* is sampled from $[q_H + q_S]$ (rather than from $[q_H]$), because we assume w.l.o.g. that \mathcal{A} always queries H_{chal} before sending a corresponding Sign_3O query, meaning that the total number of H_{chal} queries is upper bounded by $q_H + q_S$.

\mathcal{B}_1 initializes (the simulated oracles) $\text{H}_{\text{com}}(R) \leftarrow \perp, \text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow \perp$ for all values of m, R and all possible subsets of signers \mathcal{J} . \mathcal{B}_1 also maintains (i) a counter t to track the number of H_{chal} queries, initialized with 0, and (ii) a dictionary Q_{chal} to track auxiliary information used to answer H_{chal} queries. It is initialized as empty, i.e. $Q_{\text{chal}} \leftarrow \emptyset$.

For signing queries, for each signer $j \in [n]$, \mathcal{B}_1 maintains four mappings: (i). $R_j : [q_S] \rightarrow \{0, 1\}^\lambda$ to record nonces generated to respond to $\text{Sign}_1\text{O}(1, j, \mathcal{J}, m)$ queries, (ii) S_j to store auxiliary information used for answering signing queries, and (iii) $M_{j,1}$ to record all the messages sent by \mathcal{A} in a $\text{Sign}_2\text{O}(1, j, \text{sid}, \{\text{msg}_{i,1}\})$ query, (iv) S'_j to store the (m, \mathcal{J}, e) values for every signing session.

All these mappings are initialized as empty at the beginning of simulation. These are in addition to the variables $\text{sid}_i, \mathcal{S}_{i,1}, \mathcal{S}_{i,2}$ defined and initialized as per the security game in Figure 2.

We now discuss how \mathcal{B}_1 responds to each of \mathcal{A} 's queries.

$\text{H}_{\text{com}}(R)$. If $\text{H}_{\text{com}}(R)$ has been determined, i.e. if $\text{H}_{\text{com}}(R) \neq \perp$, then \mathcal{B}_1 returns $\text{H}_{\text{com}}(R)$. Otherwise, \mathcal{B}_1 samples a random string $c \leftarrow_{\$} \{0, 1\}^\lambda$ and returns c . \mathcal{B}_1 sets $\text{H}_{\text{com}}(R) \leftarrow c$.

$\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$.

- If $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ has been determined, i.e. $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \neq \perp$, then \mathcal{B}_1 returns the value $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$.
- Otherwise, \mathcal{B}_1 sets $t \leftarrow t + 1$. And, if $i^* \notin \mathcal{J}$, then (i) if this is the j^* th query (i.e. $t = j^*$), \mathcal{B}_1 aborts, (ii) else, \mathcal{B}_1 samples a random element $h \leftarrow_{\$} \mathbb{Z}_p$. \mathcal{B}_1 sets $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow h$ and returns h to \mathcal{A} .
- Or, if $i^* \in \mathcal{J}$ and $t = j^*$ then \mathcal{B}_1 samples a random message $m' \leftarrow_{\$} \mathcal{M}$, and queries its Schnorr challenger for $h \leftarrow \hat{\text{H}}_{\text{sig}}(m', \text{pk}^*, R^{\frac{1}{\lambda_{i^*}^{\mathcal{J}}}})$, where $\lambda_{i^*}^{\mathcal{J}} = \prod_{j \in \mathcal{J} \setminus \{i^*\}} \frac{j}{j - i^*}$. \mathcal{B}_1 then sets $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow h$ and returns h to \mathcal{A} . \mathcal{B}_1 also records the value $Q_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow m'$.
- Otherwise, if $i^* \in \mathcal{J}$ and $t \neq j^*$, then, \mathcal{B}_1 iterates over all session ids in $\mathcal{S}_{i^*, 2}$. For each $\text{sid} \in \mathcal{S}_{i^*, 2}$, \mathcal{B}_1 first checks if (m, \mathcal{J}) match the values in $\mathcal{S}'_{i^*}(\text{sid})$. Next, for all $j \in \mathcal{J} \setminus \{i^*\}$, \mathcal{B}_1 finds a value R_j s.t. $\text{H}_{\text{com}}(R_j) = \text{msg}_{j,1}$, with $\text{msg}_{j,1}$ stored in $M_{i^*, 1}(\text{sid})$. If there's more than one such R_j for some j , \mathcal{B}_1 aborts. If there is exactly one such value for all j then, \mathcal{B}_1 calculates $R(\text{sid}) \leftarrow R_{i^*}(\text{sid}) \cdot \prod_{j \in \mathcal{J} \setminus \{i^*\}} R_j$. If $R = R(\text{sid})$, then, \mathcal{B}_1 uses h stored in $S_{i^*}(\text{sid})$, sets $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow h$, and returns this h .
- If there is no such sid that meets all the conditions above, then \mathcal{B}_1 samples $h \leftarrow_{\$} \mathbb{Z}_p$, sets $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow h$ and returns h .

$\text{skO}(1, i)$. If $i = i^*$ then \mathcal{B}_1 aborts. Otherwise, \mathcal{B}_1 returns α_i .

$\text{Sign}_1\text{O}(1, i, \mathcal{J}, m)$. \mathcal{B}_1 sets $\text{sid}_i \leftarrow \text{sid}_i + 1$, $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \cup \{\text{sid}_i\}$, $\mathcal{S}'_i(\text{sid}_i) \leftarrow (m, \mathcal{J}, 1)$.

- If $i \neq i^*$, then \mathcal{B}_1 samples a random $r_i \leftarrow_{\$} \mathbb{Z}_p$ and sets $R_i(\text{sid}_i) \leftarrow g^{r_i}$. If $\text{H}_{\text{com}}(R_i(\text{sid}_i)) \neq \perp$, then \mathcal{B}_1 returns $\text{msg}_{i,1} \leftarrow \text{H}_{\text{com}}(R_i(\text{sid}_i))$. Otherwise, \mathcal{B}_1 samples a random string $c \leftarrow_{\$} \{0, 1\}^\lambda$, sets $\text{H}_{\text{com}}(R_i(\text{sid}_i)) \leftarrow c$. \mathcal{B}_1 returns c and sets $S_i(\text{sid}_i) \leftarrow (r_i, c)$.
- If $i = i^*$, then \mathcal{B}_1 samples a random string c_{i^*} and returns this. \mathcal{B}_1 also stores $S_{i^*}(\text{sid}_{i^*}) \leftarrow c_{i^*}$.

$\text{Sign}_2\text{O}(1, i, \text{sid}, \{\text{msg}_{j,1}\}_{j \in \mathcal{J} \setminus \{i\}})$. Assuming that the session is valid, i.e. $\text{sid} \in \mathcal{S}_{i,1}$, and denoting $(m, \mathcal{J}, 1) \leftarrow \mathcal{S}'_i(\text{sid})$, for each $j \in \mathcal{J}$, \mathcal{B}_1 finds the element R_j such that $\text{H}_{\text{com}}(R_j) = \text{msg}_{j,1}$. If for some j , there is more than one such, then \mathcal{B}_1 aborts. If for some $j \neq i^*$ this value is undefined, \mathcal{B}_1 sets a flag $\text{flag} \leftarrow 1$, returns $R_i(\text{sid})$, and stores all the messages received, $M_{i,1}(\text{sid}) \leftarrow \{\text{msg}_{j,1}\}_{j \in \mathcal{J} \setminus \{i\}}$. Also, for $j = i^*$, if there is no $\text{sid}_{i^*, i}$ s.t. $S_{i^*}(\text{sid}_{i^*, i}) = \text{msg}_{i^*, 1}$, then \mathcal{B}_1 sets $\text{flag} \leftarrow 1$.

Otherwise, if $i \neq i^*$, \mathcal{B}_1 simply returns $R_i(\text{sid})$. But if $i = i^*$, \mathcal{B}_1 first samples $s_{i^*} \leftarrow_{\$} \mathbb{Z}_p$, and a random $h \leftarrow_{\$} \mathbb{Z}_p$. Then, \mathcal{B}_1 sets $R_{i^*}(\text{sid}) \leftarrow \frac{g^{s_{i^*}}}{(\text{pk}^*)^{h \cdot \lambda_{i^*}^{\mathcal{J}}}}$, and sets $\text{H}_{\text{com}}(R_{i^*}(\text{sid})) \leftarrow c_{i^*}$, where c_{i^*} is stored in $S_{i^*}(\text{sid})$. \mathcal{B}_1 additionally sets $S_{i^*}(\text{sid}) \leftarrow (S_{i^*}(\text{sid}), s_{i^*}, h)$.

Let $(m, \mathcal{J}, 1) \leftarrow \mathcal{S}'_i(\text{sid})$.

Then, \mathcal{B}_1 computes $R = R_{i^*}(\text{sid}) \prod_{j \in \mathcal{J} \setminus \{i^*\}} R_j$, wherein $\text{H}_{\text{com}}(R_j) = \text{msg}_{j,1} \forall j \in \mathcal{J} \setminus \{i^*\}$. Then, If $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ was ever queried by \mathcal{A} before or was programmed by \mathcal{B}_1 in a sign query in another session, then \mathcal{B}_1 aborts.

In all cases, \mathcal{B}_1 stores all the messages received, $M_{i,1}(\text{sid}) \leftarrow \{\text{msg}_{j,1}\}_{j \in \mathcal{J} \setminus \{i\}}$. Similar to the game, \mathcal{B}_1 sets $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \setminus \{\text{sid}\}$, $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \cup \{\text{sid}\}$.

$\text{Sign}_3\text{O}(1, i, \text{sid}, \{\text{msg}_{j,2}\}_{j \in \mathcal{J} \setminus \{i\}})$. Assuming that the session is valid, i.e. $\text{sid} \in \mathcal{S}_{i,2}$, and denoting $(m, \mathcal{J}, 1) \leftarrow S'_i(\text{sid})$, \mathcal{B}_1 first verifies all commitments that were sent by \mathcal{A} in signing rounds 1 and 2. If, for some $j \in \mathcal{J} \setminus \{i\}$, $\text{H}_{\text{com}}(\text{msg}_{j,2}) = \perp$, then \mathcal{B} samples a random number $c \leftarrow_{\$} \{0, 1\}^\lambda$ and sets $\text{H}_{\text{com}}(\text{msg}_{j,2}) \leftarrow c$. Next, If for some $j \in \mathcal{J} \setminus \{i\}$, $\text{H}_{\text{com}}(\text{msg}_{j,2}) \neq \text{msg}_{j,1}$ (using the $\text{msg}_{j,1}$ stored in $M_{i,1}(\text{sid})$), then \mathcal{B}_1 returns \perp . \mathcal{B}_1 also checks that for $\text{msg}_{i^*,1}$, if there is any other value $R \neq \text{msg}_{i^*,2}$, s.t. $\text{H}_{\text{com}}(R) = \text{msg}_{i^*,1}$, then \mathcal{B}_1 aborts.

If $\text{flag} = 1$, \mathcal{B}_1 aborts. If not aborted or returned \perp , \mathcal{B}_1 computes $R = R_i(\text{sid}) \prod_{j \in \mathcal{J} \setminus \{i\}} \text{msg}_{j,2}$.

If the j^* th H_{chal} query has already been made, then \mathcal{B}_1 checks if the j^* th H_{chal} query had inputs $(m, \text{pk}, \mathcal{J}, R)$, if so, then \mathcal{B}_1 aborts since for \mathcal{A} to return a valid uf-0 forgery on $(m, \text{pk}, \mathcal{J}, R)$, \mathcal{A} is not allowed to query partial signature of any signer on these inputs.

Note that by our assumption that \mathcal{A} always queries $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ before calling Sign_3O , $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \neq \perp$.

\mathcal{B}_1 then returns partial signatures as follows:

- If $i \neq i^*$, \mathcal{B}_1 returns $s_i = r_i + \lambda_i^{\mathcal{J}} \cdot \alpha_i \cdot \text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ (using $r_i \leftarrow S_i(\text{sid})$).
- Otherwise, if $i = i^*$, \mathcal{B}_1 returns s_{i^*} from $S_{i^*}(\text{sid})$.

Eventually, \mathcal{A} outputs a forgery $(m^*, \mathcal{J}^*, z^*, R^*)$. If $i^* \notin \mathcal{J}^*$ or if $m^*, \text{pk}, \mathcal{J}^*, R^*$ are not the j^* th H_{chal} query, then \mathcal{B}_1 aborts. We assume w.l.o.g that $\text{H}_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*) \neq \perp$, and denote it as $h^* \leftarrow \text{H}_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*)$. \mathcal{B}_1 gets $m'^* \leftarrow Q_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*)$, which was used to program $\text{H}_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*)$.

If not aborted, \mathcal{B}_1 outputs

$$\left(m'^*, R'^* = (R^*)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}}, z'^* = \frac{1}{\lambda_{i^*}^{\mathcal{J}^*}} \cdot \left(z^* - h^* \cdot \left(\sum_{j \in \mathcal{J}^* \setminus \{i^*\}} \lambda_j^{\mathcal{J}^*} \cdot \alpha_j \right) \right) \right).$$

We claim that this is a valid Schnorr forgery. This is because, (i). \mathcal{B}_1 never sent a Sign query to the Schnorr challenger, and (ii). $\text{Vf}(\text{pk}, m^*, (\mathcal{J}^*, R^*, z^*)) = 1$ which means that,

$$g^{z^*} = R^* \cdot \left(\prod_{j \in \mathcal{J}^*} X_j^{h^* \cdot \lambda_j} \right)$$

This means that,

$$\begin{aligned}
g^{z'^*} &= \left(g^{z^* - h^* \cdot (\sum_{j \in \mathcal{J}^* \setminus \{i^*\}} \lambda_j^{\mathcal{J}^*} \cdot \alpha_j)} \right)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}} \\
&= \left(\frac{g^{z^*}}{g^{h^* (\sum_{j \in \mathcal{J}^* \setminus \{i^*\}} \lambda_j^{\mathcal{J}^*} \cdot \alpha_j)}} \right)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}} \\
&= \left(\frac{R^* \cdot \prod_{j \in \mathcal{J}^*} X_j^{h^* \lambda_j}}{\prod_{j \in \mathcal{J}^* \setminus \{i^*\}} (X_j)^{h^* \lambda_j}} \right)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}} \\
&= \left(R^* X_{i^*}^{h^* \lambda_{i^*}} \right)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}} \\
&= (R^*)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}} \cdot X_{i^*}^{h^*} \\
&= R'^* \cdot X_{i^*}^{h^*}
\end{aligned}$$

Lastly, \mathcal{B}_1 programmed $h^* = \text{H}_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*)$ to be equal to the value of $\hat{\text{H}}_{\text{sig}}$ on input $\left(m'^*, \text{pk}^*, (R^*)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}} \right)$. This follows from how \mathcal{B}_1 responds to all H_{chal} queries with $i^* \in \mathcal{J}$, and how \mathcal{B}_1 programs the j^* th H_{chal} query from within the signing queries. This gives us:

$$g^{z'^*} = R'^* \cdot X_{i^*}^{\hat{\text{H}}_{\text{chal}}\left(m'^*, \text{pk}^*, (R^*)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}}\right)}$$

Hence, $\left(R'^* = \frac{R^*}{\lambda_{i^*}^{\mathcal{J}^*}}, z'^* = \frac{1}{\lambda_{i^*}^{\mathcal{J}^*}} \cdot (z^* - h^* \cdot (\sum_{j \in \mathcal{J}^* \setminus \{i^*\}} \lambda_j^{\mathcal{J}^*} \cdot \alpha_j)) \right)$ is a valid Schnorr signature for message m'^* .

\mathcal{B}_1 is able to produce a valid forgery whenever it does not abort. Let **Abort** denote the event in which \mathcal{B}_1 prematurely aborts the simulation. This means that $\Pr[\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}} = 1 \mid \overline{\text{Abort}}] = \text{Adv}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{A})$. Hence, we get:

$$\begin{aligned}
\text{Adv}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}(\mathcal{B}_1) &= \Pr[\overline{\text{Abort}} \wedge (\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}} = 1)] + \Pr[\text{Abort} \wedge (\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}} = 1)] \\
&= \Pr[\overline{\text{Abort}} \wedge (\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}} = 1)] \\
&= \Pr[\overline{\text{Abort}}] \cdot \Pr[\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}} = 1 \mid \overline{\text{Abort}}]
\end{aligned}$$

Let us use E_{1a} to refer to the event that \mathcal{B}_1 guessed i^* correctly (i.e. $i^* \in \mathcal{J}^*$ and $i^* \notin Q_1^{SK}$), let E_{1b} represent the event that \mathcal{B}_1 correctly guessed j^* . Let E_{2a} be the event that, for all q_S signing queries, \mathcal{A} had not queried $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ before calling $\text{Sign}_2\text{O}(m, 1, i, \mathcal{J})$ for some i , and E_{2b} be the event that, for all q_S signing queries, \mathcal{B}_1 had not already programmed $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ before \mathcal{A} calls $\text{Sign}_2\text{O}(\cdot)$. Let E_3 denote the event in which \mathcal{B}_1 does not abort due to a collision in H_{com} and E_4 the event in which \mathcal{B}_1 does not abort due to **flag** being equal to 1.

Next, observe that $\Pr[\text{E}_{1a}] \geq \frac{1}{n}$, $\Pr[\text{E}_{1b}] \geq \frac{1}{q_H + q_S}$, and the probability that \mathcal{B}_1 guesses both i^*, j^* correctly is atleast $\frac{1}{n(q_S + q_H)}$. Also, $\Pr[\overline{\text{E}_{2b}}] \leq \frac{q_S}{p}$, since for any signing query, a collision can occur

with probability $\max q_S/p$. Next, $\Pr[E_{2a}] \geq (1 - \frac{q_S}{p})^{q_H}$. This is true since, conditioned on \mathcal{A} not observing an H_{com} query whose output was $\hat{H}_{\text{com}}(R_i(\text{sid}))$, its view is independent of the value R in the input $(m, \text{pk}, \mathcal{J}, R)$ to H_{chal} . We simplify the expression to get $\Pr[E_{2a}] \geq 1 - \frac{q_S q_H}{p}$, meaning $\Pr[\overline{E_{2a}}] \leq \frac{q_S q_H}{p}$. Similarly $\Pr[\overline{E_3}] \leq q_C^2/2^\lambda$ and $\Pr[\overline{E_4}] \leq q_C/2^\lambda$, since the responses of \mathcal{B}_1 to H_{com} queries are uniformly random in $\{0, 1\}^\lambda$. Then, we have,

$$\begin{aligned} \Pr[\overline{\text{Abort}}] &= \Pr[E_{1a} \wedge E_{1b} \wedge E_{2a} \wedge E_{2b} \wedge E_3 \wedge E_4] \\ &\geq \Pr[E_{1a} \wedge E_{1b}] - \Pr[\overline{E_{2a}}] - \Pr[\overline{E_{2b}}] - \Pr[\overline{E_3}] - \Pr[\overline{E_4}] \\ &\geq \frac{1}{n(q_H + q_S)} - \frac{q_S q_H}{p} - \frac{q_S^2}{p} - \frac{q_C^2}{2^\lambda} - \frac{q_C}{2^\lambda} \\ &\geq \frac{1}{n(q_H + q_S)} - \frac{q_S(q_S + q_H)}{p} - \frac{2q_C^2}{2^\lambda}. \end{aligned}$$

Since conditioned on $\overline{\text{Abort}}$, \mathcal{B}_1 perfectly simulates the game to \mathcal{A} , we get,

$$\begin{aligned} \text{Adv}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}(\mathcal{B}_1) &\geq \left(\frac{1}{n(q_H + q_S)} - \frac{q_S(q_H + q_S)}{p} - \frac{2q_C^2}{2^\lambda} \right) \cdot \text{Adv}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{A}) \\ &\geq \frac{\text{Adv}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{A})}{n(q_H + q_S)} - \frac{q_S \cdot (q_H + q_S)}{p} - \frac{2q_C^2}{2^\lambda}. \end{aligned}$$

This completes the proof.

Proof of Lemma 8

Proof. Consider the following adversary \mathcal{B}_2 playing the game $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}$. \mathcal{B}_2 can issue signature queries and random oracle queries to its challenger. To distinguish the random oracles in $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}$ from the ones in the Schnorr3-PR game, we denote queries to the former by $\hat{H}_{\text{com}}(R)$ and $\hat{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$. \mathcal{B}_2 invokes $\mathcal{A}(\mathbb{G})$ and simulates $\mathbf{G}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-uf-0}}$ as follows:

1. Receive (n, t, E) from \mathcal{A} , and forward (n, t) to the challenger in the game $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}$
2. Receive $\text{pk} = (\text{pk}_1, \dots, \text{pk}_n), \text{pkc}$ from the challenger and forward them to \mathcal{A} .
3. Initialize (the simulated oracles) $H_{\text{com}}(R) \leftarrow \perp, H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow \perp$ for all values of m, R and all possible subsets of signers \mathcal{J} . \mathcal{B}_2 also maintains a counter t to track the number of H_{chal} queries, initialized with 0.
4. For signing queries, for each signer $j \in [n]$, \mathcal{B}_2 maintains the following mappings: (i) $\mathcal{R}_j : [S] \rightarrow \{0, 1\}^\lambda$ to record nonces generated to respond to $\text{Sign}_1\text{O}(m, e, j, \mathcal{J})$ queries, (ii) $Q_{j,2} : [S] \rightarrow \mathcal{M} \times [E] \times 2^N \times \{\{0, 1\}^\lambda\}$ to record all the Sign_2O queries (with their corresponding inputs) that \mathcal{A} sends to \mathcal{B}_2 , (iii) $\mathcal{S}'_j : [S] \rightarrow \mathcal{M} \times [E] \times 2^N$ to track Sign_1O queries, and (iv) $Q'_{j,1}, Q'_{j,2}$ to store auxiliary information for answering Sign_1O and Sign_2O queries respectively. All these mappings are initialized as empty at the beginning of simulation. These are in addition to the variables $\text{sid}_j, \mathcal{S}_{i,1}, \mathcal{S}_{i,2}$ defined and initialized as per the security game in Figure 2.
5. We use q_S, q_H, q_C to denote a bound on the number of signing queries, random oracle queries on H_{chal} and random oracle queries on H_{com} , issued by \mathcal{A} across all epochs and signing sessions, respectively. \mathcal{B}_2 guesses $j^* \leftarrow_{\$} [q_H + q_S]$, denoting which of the $H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ queries will correspond to the forgery that \mathcal{A} will return. j^* is upper bounded by $q_H + q_S$, because we assume w.l.o.g. that \mathcal{A} queries H_{chal} before calling Sign_3O .

6. Reply to oracle queries by \mathcal{A} as follows:

- $H_{\text{com}}(R)$: \mathcal{B}_2 forwards query $\hat{H}_{\text{com}}(R)$ to the challenger in $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]^{\text{sa-uf-0}}}$, and sends the response to \mathcal{A} , where we use \hat{H}_{com} to denote the corresponding random oracle in $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]^{\text{sa-uf-0}}}$. \mathcal{B}_2 sets $H_{\text{com}}(R) \leftarrow \hat{H}_{\text{com}}(R)$.
- $H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$: If $H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \neq \perp$, then \mathcal{B}_2 just returns that. Otherwise, \mathcal{B}_2 increments $t \leftarrow t + 1$.

If $t = j^*$, then \mathcal{B}_2 queries its oracle, and sets $H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow \hat{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$.

Otherwise, \mathcal{B}_2 iterates over all the Sign_2O queries that \mathcal{A} has made so far, i.e. for all j , for all $\text{sid}_j \in Q_{j,2}$, with $(m_j, e_j, \mathcal{J}_j, \{\text{msg}_{k,1}^j\}) \leftarrow Q_{j,2}(\text{sid}_j)$ s.t. $j \notin Q_{e_j}^{SK} \cup Q_{e_j}^{SK'}$ and $m_j = m, \mathcal{J}_j = \mathcal{J}$, \mathcal{B}_2 does the following: for all $k \in \mathcal{J}_j \setminus \{j\}$, \mathcal{B}_2 finds the element $R_{k,j}$ s.t. $H_{\text{com}}(R_{k,j}) = \text{msg}_{k,1}^j$. For any k if there's more than one such, \mathcal{B}_2 aborts. Otherwise, if there's exactly one $R_{k,j}$ value for all k , then \mathcal{B}_2 computes $R^j = \mathcal{R}_j(\text{sid}_j) \prod_{k \in \mathcal{J} \setminus \{j\}} R_{k,j}$, and if $R^j = R$, then \mathcal{B}_2 sets $H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow h$, where h is taken from $Q'_{j,2}(\text{sid}_j)$.

Lastly, if no such Sign_2O query is found, \mathcal{B}_2 sends query $\hat{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ to the challenger in $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]^{\text{sa-uf-0}}}$, and sets $H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow \hat{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$.

In all cases, \mathcal{B}_2 sends $H_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ to \mathcal{A} , after setting its value.

- $\text{skO}(e, i)$: For $e = 1$, \mathcal{B}_2 forwards this to the challenger in $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]^{\text{sa-uf-0}}}$, and sends the response to \mathcal{A} . For $e > 1$, \mathcal{B}_2 samples a random $\text{sk}_i^e \leftarrow_{\$} \mathbb{Z}_p$, and returns sk_i^e to \mathcal{A} . \mathcal{B}_2 sets $Q_e^{SK} \leftarrow Q_e^{SK} \cup \{i\}$.
- $\text{Sign}_1\text{O}(e, i, \mathcal{J}, m)$: For $e = 1$, \mathcal{B}_2 forwards this to the challenger in $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]^{\text{sa-uf-0}}}$, and sends the response to \mathcal{A} . For $e > 1$, if this is the first signing query for this epoch, and if \mathcal{A} has made less than $t - 1$ skO queries for this epoch, then \mathcal{B}_2 samples a random set $Q_e^{SK'} \subset [n] \setminus Q_e^{SK}$ of signers of size $t - 1 - |Q_e^{SK}|$, and for all $j \in Q_e^{SK'}$, \mathcal{B}_2 samples a random $\text{sk}_j^e \leftarrow_{\$} \mathbb{Z}_p$. From now on, we can use lagrange interpolation over the elements in $Q_e^{SK} \cup Q_e^{SK'}$ to respond to signing queries for epoch e .

Then, \mathcal{B}_2 sets $\text{sid}_i \leftarrow \text{sid}_i + 1$, $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \cup \{\text{sid}_i\}$ and $\mathcal{S}'_i(\text{sid}_i) \leftarrow (m, \mathcal{J}, e)$. \mathcal{B}_2 samples a random element, $c_i \leftarrow_{\$} \{0, 1\}^\lambda$, sets $Q'_{i,1}(\text{sid}_i) \leftarrow c_i$ and returns $\text{msg}_{i,1} \leftarrow c_i$ and sid_i to \mathcal{A} .

- $\text{Sign}_2\text{O}(e, i, \text{sid}_i, \{\text{msg}_{j,1}\})$: If $\text{sid}_i \notin \mathcal{S}_{i,1}$ or if e does not match the epoch number in $\mathcal{S}'_i(\text{sid}_i)$, then \mathcal{B}_2 returns \perp . Otherwise, we denote $(m, \mathcal{J}, e) \leftarrow \mathcal{S}'_i(\text{sid}_i)$. For each $j \in \mathcal{J}$, \mathcal{B}_2 finds the element R_j such that $H_{\text{com}}(R_j) = \text{msg}_{j,1}$. If, for some j , there is more than one such value, then \mathcal{B}_2 aborts.

If $e = 1$, then \mathcal{B}_2 just forwards this to its challenger.

Otherwise, \mathcal{B}_2 sets $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \setminus \{\text{sid}_i\}$, $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \cup \{\text{sid}_i\}$. Next, if $i \in Q_e^{SK} \cup Q_e^{SK'}$, then \mathcal{B}_2 samples $r_i \leftarrow_{\$} \mathbb{Z}_p$, sets $R_i \leftarrow g^{r_i}$, and saves $Q'_{i,2}(\text{sid}_i) \leftarrow (r_i, R_i)$.

Otherwise, \mathcal{B}_2 calculates $\text{pk}_i^e \leftarrow \text{pk}_i \prod_{j \in Q_e^{SK} \cup Q_e^{SK'}} \left(\frac{g^{\text{sk}_j^e}}{\text{pk}_j}\right)^{\lambda_j(i)}$, where $\lambda_j(i) = \prod_{k \in Q_e^{SK} \cup Q_e^{SK'} \setminus \{j\}} \frac{k-i}{k-j}$.

Next, for each $j \in \mathcal{J} \setminus \{i\}$, \mathcal{B}_2 finds the $\text{sid}_{j,i}$ s.t. $Q'_{j,1}(\text{sid}_{j,i}) = \text{msg}_{j,1}$.

- If for any j , there is no such $\text{sid}_{j,i}$, or if $\mathcal{S}'_j(\text{sid}_{j,i}) \neq (m, \mathcal{J}, e)$, \mathcal{B}_2 samples $s_i \leftarrow_{\$} \mathbb{Z}_p$, $h_i \leftarrow_{\$} \mathbb{Z}_p$ and computes $R_i \leftarrow \frac{g^{s_i}}{(\text{pk}_i^e)^{h_i \lambda_i^{\mathcal{J}}}}$, where $\lambda_i^{\mathcal{J}} = \prod_{j \in \mathcal{J} \setminus \{i\}} \frac{j}{j-i}$. \mathcal{B}_2 then sets $Q'_{i,2}(\text{sid}_i) \leftarrow (s_i, h_i, R_i)$. If no such $\text{sid}_{j,i}$ is found for some j , then \mathcal{B}_2 sets $\text{flag} \leftarrow 1$.
- Otherwise, for all j for which \mathcal{B}_2 found a $\text{sid}_{j,i}$, \mathcal{B}_2 checks if \mathcal{A} has called $\text{Sign}_2\text{O}(e, j, \text{sid}_{j,i}, \{\text{msg}_{k,1}^j\})$ yet, i.e. \mathcal{B}_2 checks if $\text{sid}_{j,i} \in Q_{j,2}$. If this has not been called for any j i.e. $\text{sid}_{j,i} \notin Q_{j,2} \forall j$, then, \mathcal{B}_2 samples $s_i \leftarrow_{\$} \mathbb{Z}_p$, $h_i \leftarrow_{\$} \mathbb{Z}_p$ and computes $R_i \leftarrow \frac{g^{s_i}}{(\text{pk}_i^e)^{h_i \lambda_i^{\mathcal{J}}}}$. \mathcal{B}_2 stores $Q'_{i,2}(\text{sid}_i) \leftarrow (s_i, h_i, R_i)$.

- If there is a j such that $\text{sid}_{j,i} \in Q_{j,2}$ i.e. \mathcal{A} has already called $\text{Sign}_2\text{O}(m, e, j, \mathcal{J}, \{\text{msg}_{k,1}^j\})$ such that $j \notin Q_e^{SK} \cup Q_e^{SK'}$, then, \mathcal{B}_2 compares $\text{msg}_{k,1}$ with $\text{msg}_{k,1}^j$ for all $k \in \mathcal{J} \setminus \{i, j\}$. If they are all equal, and if $\text{msg}_{i,1}^j = Q'_{i,1}(\text{sid}_i)$, then \mathcal{B}_2 samples a random $s_i \leftarrow \$_\mathbb{Z}_p$ but uses the same h_j from $Q'_{j,2}(\text{sid}_{j,i})$. \mathcal{B}_2 computes $R_i \leftarrow \frac{g^{s_i}}{(\text{pk}_i^e)^{h_j \lambda_i^{\mathcal{J}}}}$. \mathcal{B}_2 stores $Q'_{i,2}(\text{sid}_i) \leftarrow (s_i, h_j, R_i)$.
- If there is no such j (i.e. for which all the messages in the $\text{Sign}_2\text{O}(m, e, j, \mathcal{J}, \text{sid}_{j,i}, \{\text{msg}_{k,1}^j\})$ call match this $\text{Sign}_2\text{O}(m, e, i, \mathcal{J}, \text{sid}_i, \{\text{msg}_{j,1}\})$ call, and which is also not in $Q_e^{SK} \cup Q_e^{SK'}$), then \mathcal{B}_2 samples a random $s_i \leftarrow \$_\mathbb{Z}_p, h_i \leftarrow \$_\mathbb{Z}_p$, and sets $R_i \leftarrow \frac{g^{s_i}}{(\text{pk}_i^e)^{h_i \lambda_i^{\mathcal{J}}}}$. \mathcal{B}_2 stores $Q'_{i,2}(\text{sid}_i) \leftarrow (s_i, h_i, R_i)$.

In all cases, \mathcal{B}_2 sends R_i to \mathcal{A} , and sets $Q_{i,2}(\text{sid}_i) \leftarrow (m, e, \mathcal{J}, \{\text{msg}_{j,1}\})$, $\text{H}_{\text{com}}(R_i) \leftarrow Q'_{i,1}(\text{sid}_i)$, $\mathcal{R}_i(\text{sid}_i) \leftarrow R_i$.

Lastly, \mathcal{B}_2 iterates over all the Sign_2O queries that have been made so far, i.e. for all j , for all $\text{sid}_j \in Q_{j,2}$, with $Q_{j,2}(\text{sid}_j) = (m_j, e_j, \mathcal{J}_j, \{\text{msg}_{k,1}^j\})$ s.t. $j \notin Q_{e_j}^{SK} \cup Q_{e_j}^{SK'}$ and $i \in \mathcal{J}_j$ and $\text{msg}_{i,1}^j = Q'_{i,1}(\text{sid}_i)$, \mathcal{B}_2 does the following: for all $k \in \mathcal{J}_j \setminus \{j\}$, \mathcal{B}_2 finds the element $R_{k,j}$ s.t. $\text{H}_{\text{com}}(R_{k,j}) = \text{msg}_{k,1}^j$. For any k if there's more than one such, \mathcal{B}_2 aborts. Otherwise, if there's exactly one $R_{k,j}$ value for all k , then \mathcal{B}_2 computes $R^j = \mathcal{R}_j(\text{sid}_j) \prod_{k \in \mathcal{J} \setminus \{j\}} R_{k,j}$, and does the following:

- If $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R^j)$ has been set before, and was not the j^* th H_{chal} query, then \mathcal{B}_2 aborts.
- $\text{Sign}_3\text{O}(e, i, \text{sid}_i, \{\text{msg}_{j,2}\})$: If $e = 1$, \mathcal{B}_2 forwards this to its challenger. If $\text{sid}_i \notin \mathcal{S}_{i,2}$, then \mathcal{B}_2 returns \perp . Otherwise, let us denote $(m, \mathcal{J}, e) \leftarrow \mathcal{S}'_i(\text{sid}_i)$.

Upon receiving $\text{msg}_{j,2}$ for all $j \in \mathcal{J} \setminus \{i\}$, \mathcal{B}_2 first checks that all the commitments are valid.

If, for some $j \in \mathcal{J} \setminus \{i\}$, $\text{H}_{\text{com}}(\text{msg}_{j,2}) = \perp$, then \mathcal{B}_2 sets $\text{H}_{\text{com}}(\text{msg}_{j,2}) \leftarrow \hat{\text{H}}_{\text{com}}(\text{msg}_{j,2})$.

Next, if for some $j \in \mathcal{J} \setminus \{i\}$, $\text{H}_{\text{com}}(\text{msg}_{j,2}) \neq \text{msg}_{j,1}$ (\mathcal{B}_2 uses the $\text{msg}_{j,1}$ stored in $Q_{i,2}(\text{sid}_i)$ to compare), then \mathcal{B}_2 returns \perp . Note that this perfectly simulates the game $\mathbf{G}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{uf-0}}$, since the protocol returns null if the messages received are malformed.

If $\text{flag} = 1$, \mathcal{B}_2 aborts.

If not aborted or returned \perp up to this point, \mathcal{B}_2 computes

$$R \leftarrow \mathcal{R}_i(\text{sid}_i) \prod_{j \in \mathcal{J} \setminus \{i\}} \text{msg}_{j,2}.$$

\mathcal{B}_2 denotes $h_{m,\mathcal{J},e} \leftarrow \text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$. Note that this should not correspond to the j^* th query since \mathcal{A} is not allowed to query any partial sign on m^* for its forgery to be valid. i.e. if $(m, \text{pk}, \mathcal{J}, R)$ are the inputs to the j^* th H_{chal} query, then \mathcal{B}_2 aborts.

Note that $h_{m,\mathcal{J},e} \neq \perp$ due to our assumption that \mathcal{A} always calls H_{chal} oracle before the corresponding Sign_3O query. Then, if $i \in Q_e^{SK} \cup Q_e^{SK'}$, \mathcal{B}_2 returns $r_i + h_{m,\mathcal{J},e} \cdot \text{sk}_i^e \cdot \lambda_i^{\mathcal{J}}$, using r_i from $Q'_{i,2}(\text{sid}_i)$. Otherwise, \mathcal{B}_2 uses the s_i value in $Q'_{i,2}(\text{sid}_i)$ and returns it as the partial signature.

Lastly, \mathcal{B}_2 does $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \setminus \{\text{sid}_i\}$.

\mathcal{B}_2 preserves perfect simulation of $\mathbf{G}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-uf-0}}$.

Eventually, \mathcal{A} outputs a forgery $(m^*, (\mathcal{J}^*, R^*, \sigma^*))$. If $(m^*, \text{pk}, \mathcal{J}^*, R^*)$ are not the j^* th H_{chal} query, then \mathcal{B}_2 aborts. Otherwise, \mathcal{B}_2 outputs this forgery as well. We claim that if the output of

the simulated $\mathbf{G}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-uf-0}}$ is 1, then so is the output of $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}$, assuming \mathcal{B}_2 does not abort. This is because,

- $\text{Vf}(\text{pk}, m^*, (\mathcal{J}^*, R^*, \sigma^*)) = 1$ implies that this forgery will be valid w.r.t. the 1-epoch game as well.
- $(m^*, (\mathcal{J}^*, R^*, \sigma^*))$ being a valid forgery means that \mathcal{A} didn't query any partial sign on m^* , in any epoch. Since \mathcal{B}_2 forwards signing queries to its challenger only in the first epoch, this means that \mathcal{B}_2 made no signing queries on m^* as well.
- Since \mathcal{A} returns a valid UF-0 forgery, for each $k \in [E]$, it holds that $|Q_e^{SK}| < t$. This implies, $|Q_1^{SK}| < t$. Also, \mathcal{B}_2 forwards no secret key queries in any epoch $e > 1$.

Hence, all the requirements are satisfied for the output of $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}$ to be 1. Using law of total probability and Bayes theorem, we get:

$$\text{Adv}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{B}_2) \geq \Pr[\overline{\text{Abort}}] \cdot \text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{A})$$

Let us use \mathbf{E}_1 to denote the event that \mathcal{B}_2 guessed j^* correctly, \mathbf{E}_{2a} to denote the event that for all q_S signing queries, \mathcal{A} had not queried $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$, before \mathcal{A} calls $\text{Sign}_2\text{O}(m, e, i, \mathcal{J}, \cdot)$ for some e, i , and \mathbf{E}_{2b} to denote the event that, for all signing queries, \mathcal{B}_2 had not already programmed $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$, before \mathcal{A} calls $\text{Sign}_2\text{O}(m, e, i, \mathcal{J}, \cdot)$ for some e, i . Let \mathbf{E}_3 be the event that \mathcal{B}_2 doesn't abort due to a collision in H_{com} (which is checked in $\text{Sign}_2\text{O}(\cdot)$ queries), and \mathbf{E}_4 be the event that \mathcal{B}_2 doesn't abort because $\text{flag} = 1$. Then, we have,

$$\begin{aligned} \Pr[\overline{\text{Abort}}] &= \Pr[\mathbf{E}_1 \wedge \mathbf{E}_{2a} \wedge \mathbf{E}_{2b} \wedge \mathbf{E}_3 \wedge \mathbf{E}_4] \\ &\geq \Pr[\mathbf{E}_1] - \Pr[\overline{\mathbf{E}_{2a}}] - \Pr[\overline{\mathbf{E}_{2b}}] - \Pr[\overline{\mathbf{E}_3}] - \Pr[\overline{\mathbf{E}_4}]. \end{aligned}$$

Similar to Proof B.7, $\Pr[\mathbf{E}_1] \geq \frac{1}{q_S + q_H}$. $\Pr[\overline{\mathbf{E}_3}] \leq q_C^2/2^\lambda$ and $\Pr[\overline{\mathbf{E}_4}] \leq q_C/2^\lambda \leq q_C^2/2^\lambda$, since the responses of \mathcal{B}_1 to H_{com} queries are uniformly random in $\{0, 1\}^\lambda$.

$\Pr[\overline{\mathbf{E}_{2b}}] \leq \frac{q_S^2}{p}$, since for every signing query, a collision can occur with probability atmost q_S/p . Next, $\Pr[\mathbf{E}_{2a}] \geq (1 - \frac{q_S}{p})^{q_H}$. This is true since conditioned on \mathcal{A} not observing an H_{com} query whose output was $\hat{\text{H}}_{\text{com}}(R_i(m, \mathcal{J}, e))$, its view is independent of the value R in the input $(m, \text{pk}, \mathcal{J}, R)$ to H_{chal} . We simplify the expression to get $\Pr[\mathbf{E}_{2a}] \geq 1 - \frac{q_S q_H}{p}$, meaning $\Pr[\overline{\mathbf{E}_{2a}}] \leq \frac{q_S q_H}{p}$.

Combining everything, we get,

$$\begin{aligned} &\text{Adv}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{B}_2) \\ &\geq \left(\frac{1}{q_S + q_H} - \frac{q_S q_H}{p} - \frac{q_S^2}{p} - \frac{q_C^2}{2^\lambda} - \frac{q_C^2}{2^\lambda} \right) \cdot \text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{A}) \\ &\geq \frac{\text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-uf-0}}(\mathcal{A})}{q_S + q_H} - \frac{q_S(q_S + q_H)}{p} - \frac{2q_C^2}{2^\lambda} \end{aligned}$$

This completes the proof.

Proof of Lemma 9

Proof. Consider the following adversary \mathcal{B} playing the game $\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}$, wherein \mathcal{B} can issue signature queries and random oracle queries. To distinguish the random oracle in $\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}$ from the one in the accountability game of 3-round Schnorr ATS-PR, we denote queries to the former by $\hat{\text{H}}_{\text{sig}}(m, \text{pk}, R)$. We use α to denote the challenge secret key in the Schnorr game $\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}$. On getting the challenge public key $\text{pk}^* \leftarrow g^\alpha$ from its Schnorr challenger, \mathcal{B} invokes $\mathcal{A}(\mathbb{G})$ and simulates $\mathbf{G}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}$ as follows:

- Receive (n, t, E) from \mathcal{A} .
- Guess $i^* \leftarrow_{\$} [n]$. Sample $\{\alpha_1, \dots, \alpha_{i^*-1}, \alpha_{i^*+1}, \dots, \alpha_n\} \leftarrow_{\$} \mathbb{Z}_p$. Compute $\text{pk}_j \leftarrow g^{\alpha_j}$ for all $j \neq i^*$.
- Set $\text{pk}_{i^*} \leftarrow \text{pk}^*$ (i.e. \mathcal{B} embeds the challenge public key at position i^*)
- For all $k \in \{2, \dots, E\}$, sample $a_{1,k}, \dots, a_{t-1,k} \leftarrow_{\$} \mathbb{F}_p$. Then, for all $i \in [n]$, compute $\delta_i^k \leftarrow \sum_{l=1}^{t-1} a_{l,k} \cdot i^l$ (i.e. we generate new Shamir secret shares of 0 for every epoch)
- Send $\text{pk} = \{\text{pk}_1, \dots, \text{pk}_n\}$, $\text{pkc} = \perp$ to \mathcal{A} .

Next, \mathcal{A} issues a sequence of queries. We use q_S, q_H, q_C to denote a bound on the number of signing queries, random oracle queries on H_{chal} and random oracle queries on H_{com} across all epochs and signing sessions respectively. \mathcal{B} guesses j^* , denoting which of the $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ queries will correspond to the forgery that \mathcal{A} will return. j^* is upper bounded by $q_H + q_S$, because we assume w.l.o.g. that \mathcal{A} always queries H_{chal} before sending a corresponding Sign_3O query.

\mathcal{B} initializes (the simulated oracles) $\text{H}_{\text{com}}(R) \leftarrow \perp$, $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow \perp$ for all values of m, R and all possible subsets of signers \mathcal{J} . \mathcal{B} also maintains (i) a counter t to track the number of H_{chal} queries, and (ii) a dictionary Q_{chal} to track auxiliary information used to answer H_{chal} queries. It is initialized as empty, i.e. $Q_{\text{chal}} \leftarrow \emptyset$.

For signing queries, for each signer $j \in [n]$, \mathcal{B} maintains four mappings: (i). $\mathcal{R}_j : [q_S] \rightarrow \{0, 1\}^\lambda$ to record nonces generated to respond to $\text{Sign}_1\text{O}(e, j, \mathcal{J}, m)$ queries, (ii) Next, \mathcal{S}_j to store auxiliary information for answering signing queries, (iii) $M_{j,1}$ to record all the messages sent by \mathcal{A} in a $\text{Sign}_2\text{O}(e, j, \text{sid}, \{\text{msg}_{i,1}\})$ query and (iv) \mathcal{S}'_j , to store (m, \mathcal{J}, e) corresponding to session ids. All these mappings are initialized as empty at the beginning of simulation. We now specify how \mathcal{B} answers each of \mathcal{A} 's queries:

- $\text{H}_{\text{com}}(R)$: If $\text{H}_{\text{com}}(R)$ has been determined, i.e. if $\text{H}_{\text{com}}(R) \neq \perp$, then \mathcal{B}_1 returns $\text{H}_{\text{com}}(R)$. Otherwise, \mathcal{B}_1 samples a random string $c \leftarrow_{\$} \{0, 1\}^\lambda$ and returns c . \mathcal{B}_1 sets $\text{H}_{\text{com}}(R) \leftarrow c$.
- $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$: If $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ has been determined, i.e. it holds that $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \neq \perp$, then \mathcal{B} returns $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$. Otherwise, \mathcal{B} increments the counter $t \leftarrow t + 1$. Next, if $i^* \notin \mathcal{J}$, then, if $t = j^*$, then \mathcal{B} aborts, otherwise, \mathcal{B} samples a random element $h \leftarrow_{\$} \mathbb{Z}_p$, sets $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow h$ and returns h to \mathcal{A} . If $i^* \in \mathcal{J}$, and if $t \neq j^*$, \mathcal{B} iterates through all the $\text{Sign}_2\text{O}(\cdot, i^*, \cdot, \cdot)$ queries that \mathcal{A} has made so far, i.e. for all $\text{sid} \in \mathcal{S}'_{i^*}$, with $(m_{i^*}, \mathcal{J}_{i^*}, e) \leftarrow \mathcal{S}'_{i^*}(\text{sid})$, if $m_{i^*} = m, \mathcal{J}_{i^*} = \mathcal{J}$, then, for each $j \in \mathcal{J} \setminus \{i^*\}$, \mathcal{B} finds R_j s.t. $\text{H}_{\text{com}}(R_j) = \text{msg}_{j,1}$, where $\text{msg}_{j,1}$ is stored in $M_{j,1}(\text{sid})$. If there's exactly one such R_j for all $j \in \mathcal{J} \setminus \{i^*\}$, then \mathcal{B} computes $R^{i^*, \text{sid}} = R_{i^*}(\text{sid}) \prod_{j \in \mathcal{J} \setminus \{i^*\}} R_j$, and if $R^{i^*, \text{sid}} = R$, then \mathcal{B} sets $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow h$, using h stored in $\mathcal{S}_{i^*}(\text{sid})$. In all other cases (including the case $i^* \in \mathcal{J}$ and $t = j^*$), \mathcal{B} samples a random message $m' \leftarrow_{\$} \mathcal{M}$, queries its Schnorr challenger for $h \leftarrow \hat{\text{H}}_{\text{sig}}(m', \text{pk}^*, R^{\frac{1}{\lambda_{i^*}}})$. \mathcal{B} sets $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow h$, and returns h to \mathcal{A} . \mathcal{B} also stores $Q_{\text{chal}}(m, \text{pk}, \mathcal{J}, R) \leftarrow m'$.
- $\text{skO}(k, i)$: if $i = i^*$, \mathcal{B} aborts. Otherwise, if $k = 1$, \mathcal{B} returns α_i , and if $k > 1$, \mathcal{B} returns $\alpha_i + \sum_{j=2}^k \delta_i^j$.

– $\text{Sign}_1\text{O}(e, i, \mathcal{J}, m)$: \mathcal{B} first sets $\text{sid}_i \leftarrow \text{sid}_i + 1$, $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \cup \{\text{sid}_i\}$ and $\mathcal{S}'_i(\text{sid}_i) \leftarrow (m, \mathcal{J}, e)$. If $i \neq i^*$, then \mathcal{B} samples random $r \leftarrow_{\$} \mathbb{Z}_p$ and computes $R_i(\text{sid}_i) \leftarrow g^r$. Also, \mathcal{B}_1 sets $S_i(\text{sid}_i) \leftarrow r$. If $\text{H}_{\text{com}}(R_i(\text{sid}_i)) \neq \perp$, then \mathcal{B}_1 returns $\text{msg}_{i,1} \leftarrow \text{H}_{\text{com}}(R_i(\text{sid}_i))$. Otherwise, \mathcal{B} samples a random string $c \leftarrow_{\$} \{0, 1\}^\lambda$, sets $\text{H}_{\text{com}}(R_i(\text{sid}_i)) \leftarrow c$, and returns $\text{msg}_{i,1} \leftarrow c$ to \mathcal{A} .

If $i = i^*$, \mathcal{B} samples a random $c_{i^*} \leftarrow_{\$} \{0, 1\}^\lambda$, stores $S_{i^*}(\text{sid}_{i^*}) \leftarrow c_{i^*}$ and returns $\text{msg}_{i^*,1} \leftarrow c_{i^*}$.

In all cases, \mathcal{B} also sends sid_i to \mathcal{A} .

– $\text{Sign}_2\text{O}(e, i, \text{sid}_i, \{\text{msg}_{j,1}\})$: \mathcal{B} first checks if $\text{sid}_i \in \mathcal{S}_{i,1}$ and the epoch number in $\mathcal{S}'_i(\text{sid}_i)$ is e , if not, it returns \perp . Otherwise, denoting $(m, \mathcal{J}, e) \leftarrow \mathcal{S}'_i(\text{sid}_i)$, for each $j \in \mathcal{J} \setminus \{i\}$, \mathcal{B} finds the element R_j such that $\text{H}_{\text{com}}(R_j) = \text{msg}_{j,1}$. If for some j there is more than one such, then \mathcal{B} aborts. If for some $j \neq i^*$ this value is undefined, \mathcal{B} sets a flag $\text{flag} \leftarrow 1$, returns $R_i(\text{sid}_i)$, and stores all the messages received, $M_{i,1}(\text{sid}_i) \leftarrow \{\text{msg}_{j,1}\}_{j \in \mathcal{J} \setminus \{i\}}$. Also, for $i \neq i^*, j = i^*$, if there is no $\text{sid}_{i^*,i}$ s.t. $S_{i^*}(\text{sid}_{i^*,i}) = \text{msg}_{i^*,1}$, then \mathcal{B}_1 sets $\text{flag} \leftarrow 1$.

Otherwise, if $i \neq i^*$, \mathcal{B} simply returns $R_i(\text{sid}_i)$ and stores all the messages received, $M_{i,1}(\text{sid}_i) \leftarrow \{\text{msg}_{j,1}\}_{j \in \mathcal{J} \setminus \{i\}}$. But if $i = i^*$, \mathcal{B} first samples $s_{i^*} \leftarrow_{\$} \mathbb{Z}_p$, and a random $h \leftarrow_{\$} \mathbb{Z}_p$. Then, \mathcal{B} computes $\text{pk}_{i^*}^e \leftarrow \text{pk}^* \cdot g^{\sum_{k=2}^e \delta_{i^*}^k}$ and sets $R_{i^*}(\text{sid}_{i^*}) \leftarrow \frac{g^{s_{i^*}}}{(\text{pk}_{i^*}^e)^{h \cdot \lambda_{i^*}^e}}$, and sets $\text{H}_{\text{com}}(R_{i^*}(\text{sid}_{i^*})) \leftarrow c_{i^*}$,

where c_{i^*} is stored in $S_{i^*}(\text{sid}_{i^*})$. \mathcal{B} additionally sets $S_{i^*}(\text{sid}_{i^*}) \leftarrow (S_{i^*}(\text{sid}_{i^*}), s_{i^*}, h)$.

Next, \mathcal{B} computes $R = R_{i^*}(\text{sid}_{i^*}) \prod_{j \in \mathcal{J} \setminus \{i^*\}} R_j$, wherein $\text{H}_{\text{com}}(R_j) = \text{msg}_{j,1} \forall j \in \mathcal{J} \setminus \{i^*\}$.

- If $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$ was ever queried by \mathcal{A} before, or was programmed by \mathcal{B} while responding to another $\text{Sign}_2\text{O}(m, e', i', \mathcal{J}, \{\text{msg}_{j,1}\})$ query in a different epoch or signing session, then \mathcal{B} aborts.

\mathcal{B} returns $R_{i^*}(\text{sid}_{i^*})$ and stores all the messages received, $M_{i^*,1}(\text{sid}_{i^*}) \leftarrow \{\text{msg}_{j,1}\}_{j \in \mathcal{J} \setminus \{i^*\}}$.

In all cases where \mathcal{B} doesn't abort or send \perp , \mathcal{B} sets $\mathcal{S}_{i,1} \leftarrow \mathcal{S}_{i,1} \setminus \{\text{sid}_i\}$, $\mathcal{S}_{i,2} \leftarrow \mathcal{S}_{i,2} \cup \{\text{sid}_i\}$.

– $\text{Sign}_3\text{O}(e, i, \text{sid}_i, \{\text{msg}_{j,2}\})$: \mathcal{B} first checks that all the commitments sent by \mathcal{A} are valid. We denote $(m, \mathcal{J}, e) \leftarrow \mathcal{S}'_i(\text{sid}_i)$. If, for some $j \in \mathcal{J} \setminus \{i\}$, $\text{H}_{\text{com}}(\text{msg}_{j,2}) = \perp$, then \mathcal{B} samples a random number $c \leftarrow_{\$} \{0, 1\}^\lambda$ and sets $\text{H}_{\text{com}}(\text{msg}_{j,2}) \leftarrow c$.

Next, if for some $j \in \mathcal{J} \setminus \{i\}$, $\text{H}_{\text{com}}(\text{msg}_{j,2}) \neq \text{msg}_{j,1}$ (using $\text{msg}_{j,1}$ stored in $M_{i,1}(\text{sid}_i)$), then \mathcal{B} returns \perp . Also, if $\text{flag} = 1$, then \mathcal{B} aborts. Note that this perfectly simulates the game $\mathbf{G}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{uf-0}}$, since the protocol returns null if the messages received are malformed.

If not aborted or returned \perp yet, \mathcal{B} computes $R = R_i(\text{sid}_i) \prod_{j \in \mathcal{J} \setminus \{i\}} \text{msg}_{j,2}$.

- If $i = i^*$ and if the j^* th H_{chal} query has already been made, then \mathcal{B} checks if the j^* th H_{chal} query had inputs $(m, \text{pk}, \mathcal{J}, R)$, if so, then \mathcal{B} aborts since for \mathcal{A} to return a valid acc-0 forgery on $(m, \text{pk}, \mathcal{J}, R)$, \mathcal{A} is not allowed to query partial signature of the i^* signer on the forgery message.
- Note that $\text{H}_{\text{chal}}() \neq \perp$ because of our assumption that \mathcal{A} always queries H_{chal} before calling the corresponding Sign_3O query.

Then, if $i \neq i^*$, \mathcal{B} returns $s_i \leftarrow r_i + \lambda_i^{\mathcal{J}} (\alpha_i + \sum_{k=2}^e \delta_i^k) \cdot \text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$, where $r_i \leftarrow S_i(\text{sid}_i)$. If $i = i^*$, \mathcal{B} returns s_{i^*} from $S_{i^*}(\text{sid}_{i^*})$.

It can be seen that \mathcal{B} is able to perfectly simulate the game $\mathbf{G}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}$.

Eventually, \mathcal{A} outputs a forgery $(m^*, \mathcal{J}^*, z^*, R^*)$. If $i^* \notin \mathcal{J}^*$ or if $m^*, \text{pk}, \mathcal{J}^*, R^*$ are not the j^* th H_{chal} query, then \mathcal{B} aborts. Otherwise, We assume w.l.o.g. that $\text{H}_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*) \neq \perp$ and denote it as $h^* \leftarrow \text{H}_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*)$.

\mathcal{B} finds $(m'^*) \leftarrow Q_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*)$ which was used to program $\text{H}_{\text{chal}}(m^*, \text{pk}, \mathcal{J}^*, R^*) \leftarrow \hat{\text{H}}_{\text{sig}}(m'^*, \text{pk}^*, (R^*)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}})$.

\mathcal{B} returns $(m^{i^*}, (R^*)^{\frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}}, \frac{1}{\lambda_{i^*}^{\mathcal{J}^*}}(z^* - h^* \cdot \sum_{j \in \mathcal{J}^* \setminus \{i^*\}} \lambda_j^{\mathcal{J}^*} \alpha_j))$ as a Schnorr signature forgery.

Similar to the proof of Lemma 7, it can be seen that this is a valid forgery. Hence, \mathcal{B} is able to produce a valid forgery using \mathcal{A} whenever it doesn't abort. Let Abort denote the event in which \mathcal{B} prematurely aborts the simulation. This means that $\Pr[\mathbf{G}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}} = 1 \mid \overline{\text{Abort}}] = \text{Adv}_{\text{Schnorr3-PR-1}[\mathbb{G}]}^{\text{sa-acc-0}}(\mathcal{A})$. Hence we get (similar to B.7),

$$\text{Adv}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}(\mathcal{B}) = \Pr[\overline{\text{Abort}}] \cdot \text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}(\mathcal{A})$$

Let us use \mathbf{E}_{1a} to denote the event that \mathcal{B} guesses i^* correctly (i.e. $i^* \in \mathcal{J}^*$ and $i^* \notin Q_e^{SK} \cup Q_e^{\text{Sig}}(m^*)$ for all epochs e) and \mathbf{E}_{1b} denotes the event that \mathcal{B} guesses j^* correctly. Also, let \mathbf{E}_{2a} to denote the event that for all q_S signing queries, \mathcal{A} had not queried $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$, before \mathcal{A} calls $\text{Sign}_2\text{O}(m, e, i, \mathcal{J})$ for some e, i , and \mathbf{E}_{2b} to denote the event that, for all signing queries, \mathcal{B} had not already programmed $\text{H}_{\text{chal}}(m, \text{pk}, \mathcal{J}, R)$, before \mathcal{A} calls $\text{Sign}_2\text{O}(m, e, i, \mathcal{J})$ for some e, i . Let \mathbf{E}_3 be the event that \mathcal{B}_2 doesn't abort due to a collision in H_{com} (which is checked in $\text{Sign}_2\text{O}(\cdot)$ queries), and \mathbf{E}_4 be the event that \mathcal{B}_2 doesn't abort because $\text{flag} = 1$.

We have $\Pr[\mathbf{E}_{1a}] \geq \frac{1}{n}$, $\Pr[\mathbf{E}_{1b}] \geq \frac{1}{q_H + q_S}$ (similar to B.7) and, $\Pr[\overline{\mathbf{E}_{2b}}] \leq \frac{q_S^2}{p}$, $\Pr[\overline{\mathbf{E}_{2a}}] \leq \frac{q_S q_H}{p}$, $\Pr[\overline{\mathbf{E}_3}] \leq q_C^2 / 2^\lambda$ and $\Pr[\overline{\mathbf{E}_4}] \leq q_C / 2^\lambda \leq q_C^2 / 2^\lambda$, similar to B.7.

Then, similar to the analysis in B.7, we get,

$$\begin{aligned} \text{Adv}_{\text{Schnorr}[\mathbb{G}]}^{\text{uf}}(\mathcal{B}) &= \Pr[\overline{\text{Abort}}] \cdot \text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}(\mathcal{A}) \\ &= (\Pr[\mathbf{E}_{1a} \wedge \mathbf{E}_{1b} \wedge \mathbf{E}_{2a} \wedge \mathbf{E}_{2b} \wedge \mathbf{E}_3 \wedge \mathbf{E}_4]) \cdot \text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}(\mathcal{A}) \\ &\geq (\Pr[\mathbf{E}_{1a} \wedge \mathbf{E}_{1b}] - \Pr[\overline{\mathbf{E}_{2a}}] - \Pr[\overline{\mathbf{E}_{2b}}] - \Pr[\overline{\mathbf{E}_3}] - \Pr[\overline{\mathbf{E}_4}]) \\ &\quad \cdot \text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}(\mathcal{A}) \\ &\geq \left(\frac{1}{n(q_S + q_H)} - \frac{q_S^2}{p} - \frac{q_S q_H}{p} - \frac{q_C^2}{2^\lambda} - \frac{q_C^2}{2^\lambda} \right) \cdot \text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}(\mathcal{A}) \\ &\geq \frac{\text{Adv}_{\text{Schnorr3-PR}[\mathbb{G}]}^{\text{sa-acc-0}}(\mathcal{A})}{n(q_S + q_H)} - \frac{q_S(q_S + q_H)}{p} - \frac{2q_C^2}{2^\lambda} \end{aligned}$$

This completes the proof.