

On Zero-Knowledge Proofs over the Quantum Internet

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Abstract

This paper presents a new method for quantum identity authentication (QIA) protocols. The logic of classical zero-knowledge proofs (ZKPs) due to Schnorr [7] is applied in quantum circuits and algorithms. This novel approach gives an exact way with which a prover P can prove they know some secret without transmitting that directly to a verifier V by means of a quantum channel - allowing for a ZKP wherein an eavesdropper or manipulation can be detected with a ‘fail safe’ design. With the anticipated advent of a ‘quantum internet’, such protocols and ideas may soon have utility and execution in the real world.

1 Introduction

With the advent of Quantum Computing comes with it the idea of the Quantum Internet - the ability to transfer a quantum state $|\Psi\rangle$ from one quantum computer/device to another. There are many challenges with this kind of networking [1], as well as many benefits. As Cacciapuoti [1] points out, with a quantum internet we get Quantum Key Distribution ‘for free’, a major benefit to quantum communications infrastructure. There are many existing Quantum Identity Authentication (QIA) protocols [3] and this paper adds a new approach to the collection.

Existing approaches make use of various features of QKD, quantum teleportation techniques, Physically Unclonable Functions (PUFs), distributed Bell states, quantum private queries, quantum secure direct communications, etc. Many of these details may be found in [3].

Schnorr introduced in [7] the idea of efficient identification signatures, initially designed for use with smart cards. This method of ‘proving’ your identity without disclosing a secret became known as ‘zero-knowledge proofs’ and have recently found much use in many cryptographic protocols [4].

The benefits of ZKPs over other past approaches

are that there needs be no prior exchange or other pre-sharing, nor any explicit statement of what the hidden information is. The proof system itself carries the correctness and soundness that guarantees the validity of a proof presented by the prover to the verifier, and that the claim by the prover to know such a secret is ‘true’.

ZKPs have been used to create quantum proof systems that have also been shown to be possible in a quantum setting [8]. These make use of graph isomorphism problems, which this approach does not. The method herein takes advantage of a quantum communications network to reduce the number of quantum and classical transmissions down to 1 and 2 respectively.

The work presented here aims to demonstrate how a quantum ZKP protocol might look by coding Schnorr’s original method into quantum states. Some benefits and restrictions of this approach are included.

2 Schnorr ZKP Protocol

In its simplest form, a zero-knowledge proof is a method for a prover P to provide a way of showing that they know some secret x to a verifier V , but without exposing the secret at any point, hence ‘zero-knowledge’.

The following algorithm is the usual presentation of Schnorr’s work. P wants to prove that they know x such that $Y = g^x \pmod p$, for prime p and generator g , with g and Y public. The following method is presented:

1. $\underline{P \rightarrow V}$: P chooses some r and sends $t = g^r \pmod p$ to V .
2. $\underline{V \rightarrow P}$: V sends a random c to P .
3. $\underline{P \rightarrow V}$: P sends $s = r + cx$ to V
4. $\underline{V \text{ checks}}$ that $g^s \equiv t \times Y^c \pmod p$.

This works as

$$\begin{aligned} t \times Y^c &\equiv g^r \times (g^x)^c \pmod{p} \\ &\equiv g^{r+cx} \pmod{p} \\ &\equiv g^s \pmod{p} \end{aligned} \quad (1)$$

This very neat scheme was a very important development in authentication schemes, and will form the basis for the quantum protocol presented next.

3 Quantum Preliminaries

This protocol utilises a single qubit, and only one quantum gate. Qubits are assumed to be initialised in $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Quantum gates are formed from collections of unitary matrices, relying on the fact that composing unitary matrices with each other preserves this property [5].

Define the R_x gate as [5]:

$$\begin{aligned} R_x(\theta) &= e^{i\theta X/2} \\ &= \cos(\theta/2)I + i \sin(\theta/2)X \\ &= \begin{pmatrix} \cos(\theta/2) & -i \sin(\theta/2) \\ -i \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \end{aligned} \quad (2)$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. With the representation of the Bloch sphere, this gate is usually interpreted as a rotation along the x axis.

The following gate $G_p(g^a)$ shall be utilised, defined as follows:

$$G_p(g^a) = R_x\left(\left(g^a \pmod{p}\right) \times \frac{\pi}{p}\right) \quad (3)$$

Intuitively, we split the π rotation about the x axis on the Bloch sphere into p many steps, and then apply a rotation on our qubit, moving that number of steps around.

4 Quantum Internet ZKPs

This section brings these two domains together to propose an authentication scheme that makes use of a quantum internet with additional classical channel.

4.1 Q-ZKP Protocol

The Quantum Internet, loosely defined, is a quantum communications protocol that permits the transfer of some quantum state $|\Psi\rangle$ from one quantum computer to another. Utilising this property, the following zero-knowledge proof can be constructed.

As before, $Y = g^x \pmod{p}$ is known to all parties, and P wishes to prove they know x to V , only this time they have a quantum internet.

1. $\underline{P \rightarrow V}$: P chooses some r and sends $|G_p(t)\rangle$ to V , with (as before) $t = g^r \pmod{p}$
2. $\underline{V \rightarrow P}$: V sends a random c to P .
3. $\underline{P \rightarrow V}$: P sends $s = r + cx$ to V
4. $\underline{V \text{ checks}}$ using the state from 1. that

$$|G_p(t)\rangle \cdot |G_p(Y^c)\rangle \cdot |G_p(p - g^s)\rangle = 1$$

under the normal z axis measurement.

4.2 Correctness

Theorem 4.1. *When V checks*

$$|G_p(t)\rangle \cdot |G_p(Y^c)\rangle \cdot |G_p(p - g^s)\rangle$$

the output will always be a 1 if and only if V agrees that P has a valid proof that they know x .

Proof. (\leftarrow) Start by re-asserting equation 1, namely that for a valid proof it follows that

$$g^s \equiv t \times Y^c \pmod{p}$$

With this, it then follows that $|G_p(t)\rangle \cdot |G_p(Y^c)\rangle$ should be the same R_x rotation as $|G_p(g^s)\rangle$.

Under multiplication the rotation is completed by $G_p(p - g^s)$, as

$$g^s + (p - g^s) \equiv p \equiv 0 \pmod{p} \quad (4)$$

Given the protocol only divides a half, not a full, qubit rotation by p this completion should always send the qubit to have a 1 measurement.

(\rightarrow) If the measurement output is always 1 then the $|G_p(t)\rangle$ state received from P by V matches the $|G_p(g^s)\rangle$ state that V can construct, when composed with the $|G_p(Y^c)\rangle$ state that V wants to verify.

By equation 4 a 1 measurement means that everything required to line up has done so, and P 's proof is correct. \square

Note that the proof of soundness is given by the original protocol from [7].

4.3 Remarks

4.3.1 Repetitions

Of course, just one exchange between P and V will likely not be convincing enough. Numerous exchanges and measurements will build the statistical significance that P is indeed correct according to V . This significance will additionally grow as p and the resolution of the qubit increase.

As the protocol is quite efficient compared to others (there is only one quantum and two classical

transmissions), the overhead for multiple transmissions should be linear. The values for r and c may also change each time without affecting the overall validity of the proofs.

Future developments may involve multi-party distributed entangled states (such as GHZ states) where multiple verifiers can receive the same $|G_p(g^r)\rangle$, issue different challenges, and all verify proofs from P .

4.3.2 Error Rate

It should be noted that the error rate introduced by someone listening in on the quantum channel will also be evident very quickly, as the error rate will rise significantly when an eavesdropper tries to guess the random r chosen by P . This is congruous with how QKD protocols add security using quantum states. The quantum channel, as with other quantum communications protocols [1], offers some significant added protection there.

It should also be noted that, because the security relies on the statistical likelihood of zero measurements, the protocol effectively fails safe for sufficiently high introductions of error.

4.3.3 Hardware

There are several constraints on current hardware that would preclude this from being immediately practical. Namely, the need for a very high precision on the qubit in use, and a likewise minimal amount of noise required to not skew the results.

Error corrected qubits and quantum communication channels are required to deal with the second part of these issues [6]. The resolution of the qubits and their longevity is taken into account by some benchmarks, such as ‘Quantum Volume’ [2]. Therefore, as quantum computers grow in reliability and complexity, and quantum networks begin to be tested and deployed and improve, we might consider such high enough resolutions, error correction, and reliability to one day be attainable.

4.3.4 Underlying Group and Shor

Shor’s algorithm (see [5, Appendix 4]) would allow an attacker to solve the discrete logarithm problem that our exchange relies on to be secure. There are however two impediments for this attack to be viable.

Firstly, note that the state $|G_p(g^r)\rangle$ would have to be measured a large number of times in order to work out what g^r is with any certainty. This is mitigated by r being only used once per state. Secondly, the detection of such interceptions would very quickly cause the verifier to denounce the invalid proofs from P .

5 Conclusion

This paper hopes to have shown that there is another possibility for performing zero-knowledge proofs using quantum algorithms over quantum communications networks. This system has been shown to have some additional benefits over purely classical approaches, despite its classical basis.

This work thereby adds to the collection of proposals for QIA and quantum zero-knowledge proofs that might help shape future quantum communications.

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