

Fiat-Shamir for Proofs Lacks a Proof Even in the Presence of Shared Entanglement

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Abstract. We explore the cryptographic power of arbitrary shared physical resources. The most general such resource is access to a fresh entangled quantum state at the outset of each protocol execution. We call this the *Common Reference Quantum State (CRQS)* model, in analogy to the well-known *Common Reference String (CRS)*. The CRQS model is a natural generalization of the CRS model but appears to be more powerful: in the two-party setting, a CRQS can sometimes exhibit properties associated with a Random Oracle queried once by measuring a maximally entangled state in one of many mutually unbiased bases. We formalize this notion as a *Weak One-Time Random Oracle (WOTRO)*, where we only ask of the m -bit output to have *some* randomness when conditioned on the n -bit input.

We show that WOTRO with $n - m \in \omega(\lg n)$ is *black-box impossible* in the CRQS model, meaning that no protocol can have its security black-box reduced to a cryptographic game. We define a (inefficient) quantum adversary against any WOTRO protocol that can be efficiently simulated in polynomial time, ruling out any reduction to a secure game that only makes black-box queries to the adversary. On the other hand, we introduce a non-game quantum assumption for hash functions that implies WOTRO in the CRQ\$ model (where the CRQS consists only of EPR pairs). We first build a statistically secure WOTRO protocol where $m = n$, then hash the output.

The impossibility of WOTRO has the following consequences. First, we show the black-box impossibility of a *quantum* Fiat-Shamir transform, extending the impossibility result of Bitansky *et al.* (TCC '13) to the CRQS model. Second, we show a black-box impossibility result for a strengthened version of quantum lightning (Zhandry, Eurocrypt '19) where quantum *bolts* have an additional parameter that cannot be changed without generating new bolts.

1 Introduction

Cryptographic protocols can sometimes only be proven secure if some of their components are assumed to be ideal. For example, some protocols that make use of cryptographic hash functions can be proven secure if they are modelled as ideal random functions provided as a black box; this is called the *random oracle model (ROM)*. Another, but weaker, idealized resource is the *common random*

string model (CRS), in which the participants get a freshly generated random string at the outset of each protocol execution. Many cryptographic applications have their most efficient protocols proven secure when provided access to such extra resources, as all known protocols in the plain model are either inefficient, or do not satisfy all security requirements.

The Random Oracle Model (ROM). Introduced by Bellare and Rogaway [5] as a way to idealize cryptographic hash functions, the model has been shown to provide formal security proofs for a wide variety of cryptographic protocols that are not known to be secure under standard assumptions in the plain model. A random oracle models a hash function as one whose value for every input is chosen uniformly and independently at random and afresh before each protocol execution. This is meant to model the assumption that a hash function is random, and that looking at its source code yields nothing useful beyond its input-output behaviour. Rigorous security proofs for practical and efficient applications like *Full Domain Hash signatures (FDH-Signatures)*, *Optimal Asymmetric Encryption Padding (OAEP)*, Schnorr’s signatures [41, 43], and Fischlin’s NIZK-PoK [27] are easy to obtain in the ROM but are still missing in the plain model. The random oracle is a powerful primitive that provides all the main properties of a cryptographic hash function at once: collision resistance, preimage resistance, and pseudorandomness. It also has properties that can never be satisfied by any hash function: programmability, (query) extractability (also known as observability), and freshness.

Common Reference String Model. A CRS is nothing more than a fresh random string that materializes upon each protocol execution (freshness) and to which all players have access. This model was originally proposed by Blum, Feldman, and Micali [9] to help remove interaction in zero-knowledge proof systems. In [8], the model was shown to allow for non-interactive zero-knowledge for all NP languages. The works of [11, 13, 20] extend its use as a resource enabling universally composable cryptographic primitives. The common reference string model comes in two main flavours. The weakest consists of a random and uniform string of polynomial length (in the security parameter) while the strongest consists of a string of polynomial length picked from some efficiently sampleable distribution. The first flavour will be denoted by the *CR\$ model (i.e. the Common Random String Model)* while the second flavour will be denoted by the *CRS model (i.e. the Common Reference String Model)*.

A customary application of both the CRS model and the ROM is the removal of interaction in interactive proof systems. As mentioned above, the CRS model was originally designed for that purpose [9]. Notice that a random oracle is a much more powerful resource than a CRS, since it provides random access to an exponential number of them. However, a random oracle is an immaterial resource as its properties could never be satisfied by any efficient local process. This is in sharp contrast to a CRS, which can be implemented in practice: we only need a way to publish fresh and public random strings of polynomial length.

Unfortunately, some basic and useful cryptographic primitives are only known to be securely realizable in the ROM.

In this paper, we consider a quantum version of the CRS model, called the CRQS model, and we ask whether it could go beyond what the CRS model can provide for the safe removal of interaction in cryptographic protocols.

The CRQS and CRQ\$ Models. We consider models where a quantum state plays the role of a common random string in a situation involving two parties. In the CRQS (*Common Reference Quantum State*) model, each party receives one half of a fixed pure quantum state at the beginning of each protocol execution. The shared quantum state is of polynomial size and can be generated by some polynomial size quantum circuit. In the CRQ\$ model, each player is given halves of polynomially many (in the security parameter) maximally entangled pairs of qubits (or qudits in general). Although we could allow a CRQS or a CRQ\$ to be shared between more than two parties, in this work we only consider the two-party case. Notice that the meaning of *common* in CRQS and CRQ\$ is narrower than for a CRS and CR\$: even though a CRQS is common to both parties involved in a protocol, it is completely unknown to anybody else, as both players share a pure state. Even though a CRQS is obviously more difficult to deploy in practice than a CRS, it remains a physical resource, unlike the random oracle. Establishing limits on what a CRQS can provide would therefore contribute to a better understanding of the cryptographic power provided by the sharing of a *physical* resource between the parties involved in a protocol. In this paper, we show fundamental limits on the abilities of the CRQS to remove interaction in the two-party setting.

When a CRQ\$ Behaves Like a Random Oracle. In order to see why a CRQ\$ could outperform a CRS in some settings, consider the following scenario where a CRQ\$ seems to provide as much randomness as the random oracle. Suppose Alice and Bob are sharing a CRQ\$ made out of n EPR pairs of qubits. Each can then view their n qubits as an access to a weak random oracle implementing a random function $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$. The value $f(a)$ can be obtained the following way. To each possible value $a \in \{0, 1\}^n$, we associate a publicly known orthonormal basis θ_a for n qubits. The value of $f(a)$ is simply defined as the outcome of the measurement of the n qubits owned by each party in basis θ_a . Notice that this weak random oracle can be queried in only one place by each party, as after the measurement is performed, the entangled pairs have collapsed to a classical state. However, when both parties measure in the same basis θ_a they obtain the same uniformly distributed outcome. Moreover, when the bases $\{\theta_a\}_{a \in \{0, 1\}^n}$ are chosen to be mutually unbiased [42, 46]³, the value $f(a)$ does not provide any information about $f(a')$ for any $a \neq a'$. In this particular setting, n EPR pairs seem to contain as much randomness as a random oracle. It

³ $\{\theta_a\}_{a \in \{0, 1\}^n}$ is a set of mutually unbiased bases for n qubits if for all $|u\rangle \in \theta_a$ and $|v\rangle \in \theta_{a'}$ with $a \neq a'$, we have $|\langle u|v\rangle|^2 = 2^{-n}$. There are $2^n + 1$ mutually unbiased bases for n qubits.

is therefore tempting to believe that a CRQ\$ of polynomial size could in certain cases provide a cryptographic resource tantamount to the random oracle when only one query (or just a few) has to be made by each player.

The Fiat-Shamir Transform. One very useful primitive that needs an idealized cryptographic resource for establishing its security is the Fiat-Shamir transform, also known as the Fiat-Shamir *heuristic*, introduced in the pioneering work of Fiat and Shamir in [26] as a way to transform identification schemes of a given form into practical digital signature schemes. More generally, the FS-transform is a simple and efficient primitive allowing to convert sound interactive proof systems of a particular form into non-interactive arguments for the same language. Its primary use is to remove interaction in Σ -protocols. Σ -protocols [16, 18] are public-coin 3-message proof systems where, from public input $x \in \{0, 1\}^*$, the prover sends a *commitment* $a \in \{0, 1\}^n$ to the verifier as the first message. The verifier then replies with a random *challenge* $c \in_R \{0, 1\}^m$ (called *public coins*) before the prover sends the answer $z(x, a, c)$ that the verifier can check for consistency. Henceforth, Σ -protocols with commitments of size n and public coins of size m will be denoted by $\Sigma_{n,m}$ -protocols. These proof systems can be proofs of knowledge, like their use in identification schemes, or proofs of language membership. In this paper, Σ -protocols are always considered perfectly correct and special sound. Special soundness⁴ for proofs of knowledge means that from any two successful conversations with the same commitment $(a, c, z(x, a, c))$ and $(a, c', z(x, a, c'))$ with $c \neq c'$, one can efficiently extract a witness w for $x \in L$. For proofs of language membership, special soundness means that when $x \notin L$ and for each commitment a , there exists at most one challenge $c(a)$ for which a third message \tilde{z} can ever be found such that $(a, c(a), \tilde{z})$ is accepted by the verifier. The Fiat-Shamir transform applied to a Σ -protocol is implemented using hash function $h_r : \{0, 1\}^* \rightarrow \{0, 1\}^m$ picked according to CR\$ r . The prover then sends $(a, h_r(a), z(x, a, h_r(a)))$ to the verifier. In other words, the verifier's challenge or public coin c in the Σ -protocol is replaced by $c = h_r(a)$ ⁵. It is straightforward to see that when h_r is modelled by a random oracle, the transform applied to a Σ -protocol produces a sound argument⁶. Notice that when the hash function is modelled by a random oracle, the prover and the verifier only have to query the oracle once at the same point. The family of hash functions $\mathcal{H} = \{h_r\}_{r \in_D \{0, 1\}^{\ell(n)}}$, for D an efficiently sampleable distribution over $\{0, 1\}^{\ell(n)}$, is a *sound $\Sigma_{n,m}$ -universal instantiation* of the Fiat-Shamir transform if h_r converts the special soundness of any $\Sigma_{n,m}$ -protocol (as a proof of language membership) into a non-interactive argument.

⁴ Special soundness is called *optimal soundness* in [4].

⁵ Some works include the public instance x as input to h_r , our results remain untouched if we include it. We leave it out for simplicity.

⁶ The hard part in [38] was showing that the Fiat-Shamir transform, when applied to a Σ -protocol for a proof of knowledge, allows to extract a witness from all successful provers (i.e. thanks to their Forking Lemma).

The Fiat-Shamir Transform in the ROM and QROM. As mentioned above, the Fiat-Shamir transform was shown secure in the ROM by Pointcheval and Stern [38] in 1996. The soundness of the Fiat-Shamir transform is straightforward in the ROM. The challenging part was to show that it also provides non-interactive proofs of knowledge. The same was shown to hold in the quantum random oracle (QROM) independently and differently by Don, Fehr, Majenz, and Schaffner in [25] and by Liu and Zhandry in [34].

Classical Impossibility Results for the Fiat-Shamir Transform. The Fiat-Shamir transform does not guarantee computational soundness for all Σ -protocols in the CRS model. In particular, Goldwasser and Kalai have shown that the Fiat-Shamir transform applied to some (contrived) Σ -protocols is not sound for any instantiation of the hash function (i.e. instantiated using a CRS) [30]. However, this impossibility result requires the Σ -protocol to be a proof of knowledge. Impossibility results for Σ -protocols used as proofs of language membership are not known to be as strong as for proofs of knowledge. One reason being that for language membership, the Fiat-Shamir transform is only asked to provide computational soundness to a Σ -protocol with statistical soundness whereas for a proof of knowledge the Σ -protocol is an argument. Remember that a *cryptographic game* [31] is a standard way to define computational assumptions by requiring that no adversary can win an interactive game against a *challenger* with probability that is not overwhelmingly close to some constant value [31]. An assumption that can be formulated as a cryptographic game with an efficient challenger is called a *falsifiable assumption* [28, 35]. Known impossibility results for the Fiat-Shamir transform applied to Σ -protocols for proofs of language membership are about the impossibility of reducing its computational soundness to a cryptographic game.

In [6], Bitansky *et al.* provide two results on the impossibility of establishing the computational soundness of the Fiat-Shamir transform in the CRS model. First, if a language $L \notin BPP$ has an honest-verifier zero-knowledge (HVZK) Σ -protocol (with small enough challenges) then the soundness of the Fiat-Shamir transform applied to it cannot be established by a black-box reduction⁷ to a falsifiable assumption⁸. This impossibility result applies even to Fiat-Shamir transforms tailor-made for specific Σ -protocols. Second, they show the impossibility of black-box reducing the computational soundness of any universal instantiation of the Fiat-Shamir transform to a cryptographic game, even a non falsifiable one where the challenger is not required to run in polynomial time.

⁷ The security of protocol Π is black-box reduced to an assumption expressed as a game if there exists an oracle polynomial-time machine \mathcal{R}^{P^*} that, with oracle access to any successful adversary P^* for protocol Π , wins the game.

⁸ The reason why the result applies in the CRS model is because [6, 17] show how to get, from such a Fiat-Shamir transform, a 2-message zero-knowledge proof system for L where the verifier simply sends the identity of the hash function to the prover as first message. This is equivalent to non-interactive schemes in the CRS model. These proofs systems are shown impossible by an extension of the impossibility result for 2-round zero-knowledge for non-trivial languages by Goldreich and Oren [29].

Our main contribution consists in showing that the second impossibility result of [6] also holds in the CRQS model even though sharing an entangled quantum state seems to provide enough randomness to mimic a (classical) random oracle. In other words, the computational soundness of any universal Fiat-Shamir implementation in the CRQS model cannot be black-box reduced to any cryptographic game, just like in the CRS model.

Positive results & related work. A series of results have been focusing on achieving soundness of the Fiat-Shamir transform from a cryptographic assumptions that cannot be black-box reduced to cryptographic games. Barak, Lindell and Vadhan [4] introduce the notion of *entropy preserving* hash functions and show that their existence prevents any constant-round auxiliary-input zero-knowledge proof system for non-trivial languages. Later, Dodis, Ristenpart and Vadhan [24] gave a construction for entropy preserving hash functions from robust randomness condensers with some extra properties but without providing any construction for them. Canetti, Goldreich, and Halevi [14] introduce *correlation intractable* families of hash functions. Correlation intractability is related to entropy preservation as the latter implies the former. Therefore, a consequence of [6] is that correlation intractability cannot be proven by black-box reduction to a game. In [32], Kalai, Rothblum, and Rothblum provide a construction for correlation intractable family of hash functions from a sub-exponentially secure indistinguishability obfuscator, an exponentially secure input-hiding obfuscator for the class of multi-bit point functions, and the existence of a sub-exponentially secure puncturable PRF⁹. The sub-exponential indistinguishable security of the IO-obfuscator and the exponential security of the multi-bit point functions obfuscator allow to evade the impossibility result of [6]. In [12], Canetti, Chen, Holmgren, Lombardi, Rothblum, and Rothblum show how to construct a universal instance of the Fiat-Shamir transform using correlation intractable hash functions built from a strong version of KDM-encryption. The resulting Fiat-Shamir transform also has security black-box reducible to a cryptographic game with subexponential security.

The concept of shared entanglement as a setup was considered in previous works. In [15], Coladangelo, Vidick, and Zhang have shown how to design zero-knowledge arguments for QMA (i.e. quantum NP), with preprocessing. The preprocessing is essentially what we call here a CRQ\$. Non-interactivity is obtained from pre-shared EPR pairs used as a teleportation channel. This can be viewed as a quantum version of the work of Peikert and Shiehian [37] and, as such, is not a $\Sigma_{n,m}$ -universal instantiation of the Fiat-Shamir transform. The ability of a CRQS to provide zero-knowledge against quantum dishonest verifiers has been investigated in [19]. It was shown that a CRQS allows quantum zero-knowledge implementations of a Σ -protocols against a relaxed form of honest verifiers, called *non-oblivious*.

⁹ Notice that the result of [32] is very general as it allows to apply securely the Fiat-Shamir transform to any public-coin 3-message proof systems, not only to Σ -protocols as we define them. Some of their assumptions can be relaxed a little when the Fiat-Shamir transform is applied to Σ -protocols.

Fiat-Shamir in the CRQS model? We investigate the question of whether or not Fiat-Shamir has a secure and universal instantiation in the CRQS model. Why would it be possible given the strong impossibility results in the CRS model? For one thing, quantum entanglement is known to allow the reduction of interaction and achieve classically impossible tasks. Watrous [44] showed that every language in PSPACE has 3-message proof systems. Another example would be nonlocal games such as the magic square game [3, 2, 10], where a pair of entangled non-interacting provers can win a game that would classically require them to communicate.

The CRQS model provides quantum non-local correlations¹⁰ between the prover and the verifier. The question we are addressing here is whether these non-local quantum correlations can be harnessed to provide universal Fiat-Shamir through the use of a CRQS. As we mentioned above, n shared EPR pairs measured in one of 2^n mutually unbiased bases seems to provide a functionality reminiscent of a random oracle queried at a single point, exactly as needed for Fiat-Shamir. The correlated randomness available to the prover and the verifier seems to contain as much randomness as the random oracle.

Quantum non-local correlations provided by EPR pairs are often idealized by NL-boxes [39]. One NL-box takes the first party's input $a \in \{0, 1\}$ and the second party's input $b \in \{0, 1\}$ to provide $u \in \{0, 1\}$ and $v \in \{0, 1\}$ such that $u \oplus v = a \wedge b$ to the first and second party respectively. EPR pairs achieve this functionality with probability of success $\cos^2(\frac{\pi}{8})$ while any CRS would not be able to provide the correct answer with probability better than $\frac{3}{4}$. It is easy to see that access to sufficiently many NL-boxes provides secure and universal Fiat-Shamir (see details in Appendix D.2).

One might argue that the CRQS model is not currently realistic given the technological difficulties associated with distributing and coherently storing quantum entanglement (although this is rapidly improving). However, we ask a more fundamental question on the power of setup assumptions. Does the random oracle, an idealized mathematical object, have a *physical* instantiation that allows such applications as the Fiat-Shamir transform?

1.1 Our Contributions

We show a strong impossibility result for universal Fiat-Shamir in the CRQS model. The statement we obtain is the extension of the classical impossibility Fiat-Shamir of Bitansky *et al.* [6, 7], namely that there is no black-box reduction from the security of Fiat-Shamir to a secure cryptographic game assumption. In fact, our impossibility is more general than [6], even when restricted to the classical case. A CRQS captures asymmetric setups such as giving the verifier the

¹⁰ In quantum mechanics, a *non-local correlation* is the name given to the statistics of local measurements applied to distinct parts of a quantum states when they cannot be explained by a local realistic theory. Non-local correlations here (quantum or not) means also that they do not allow for any form of communication as they must be compatible with special relativity.

trapdoor to some primitive the prover uses or pre-computed randomized oblivious transfers. We achieve this by following a different approach than [6]. Whereas [6] shows the black-box impossibility of so-called *entropy preserving* hash functions and prove that they are necessary for a universal Fiat-Shamir transform, we introduce a cryptographic primitive called a *Weak One-Time Random Oracle*, denoted $\text{WOTRO}^{n,m}$ and defined by the box given in Fig. 1, which takes place between a “prover” who controls the interfaces on the left-hand side of the box, and a “verifier” who controls the interfaces on the right. This primitive is intended to capture the minimal functionality sufficient for Fiat-Shamir. A protocol instantiating $\text{WOTRO}^{n,m}$ is secure if for any function $f(\cdot)$, the adversary can’t produce an output of the form $(a, f(a))$ on the verifier’s interfaces. This directly implies Fiat-Shamir for Σ -protocols by having $f(\cdot)$ represent the “bad challenge” function of the Σ -protocol. Our main contribution is showing that

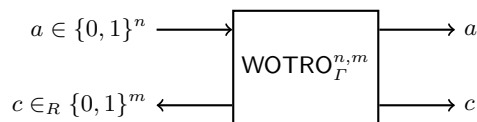


Fig. 1. Representation of the primitive WOTRO as a box. Here, the prover on the left puts a chosen $a \in \{0, 1\}^n$ into the box, the box chooses a $c \in \{0, 1\}^m$ uniformly at random, and outputs both a and c to the verifier on the right-hand side.

this primitive is black-box impossible in the CRQS model.

Theorem 1 (informal). *If $n - m \in \omega(\lg n)$, there is no protocol for $\text{WOTRO}^{n,m}$ that can have its security proven by black-box reduction to a cryptographic game assumption unless that assumption is false.*

Our impossibility uses a similar recipe as the classical impossibility [6, 7] of Fiat-Shamir, but with quantum ingredients. We construct an inefficient attacker against any protocol for WOTRO in the CRQS model and show that this attacker can be efficiently simulated. Any black-box reduction from this attacker to the security of a cryptographic game becomes an efficient algorithm for breaking the game when given the simulator instead, ruling out either the existence of such a reduction or the security of the game itself.

By applying these techniques to the Fiat-Shamir in the CRQS model, we obtain a similar impossibility result.

Theorem 2 (informal). *There is no instantiation of the Fiat-Shamir transform that maps an n -bit first message to an m -bit challenge for $n - m \in \omega(\lg n)$ whose security can be reduced by black-box to a cryptographic game assumption, unless that assumption is false.*

Studying the WOTRO primitive instead of Fiat-Shamir directly has the advantage that our black-box impossibility result also applies to any cryptographic

task which (black-box) implies WOTRO. For instance, we introduce a strengthened variant of Zhandry’s quantum lightning [47] that imply WOTRO. Quantum lightning (QL) is a primitive that produces a quantum state and an associated serial number such that no adversary can produce two states with the same serial number (hence the name “lightning”). A consequence of this property is that serial numbers are highly unpredictable. A natural question is whether some form of metadata can be embedded into quantum lightning such that changing the value of this metadata requires creating a new lightning state. This metadata could for example contain ownership information and it would thus be impossible, even to the emitter of the state, to change the owner of a state without generating an entirely new state. It could also serve to encode a denomination for quantum bank notes, such that not even the emitting bank could change the denomination of an existing quantum note.

We introduce a variant of quantum lightning that allows such metadata by adding a classical input to the state generation procedure. We call this variant *typed quantum lightning* (tQL) which is secure if the serial numbers remain unpredictable conditioned on the input. We show that this variant implies WOTRO and thus inherits the same black-box impossibility.

Theorem 3 (informal). *There is no black-box reduction from the security of a tQL scheme to the security of a cryptographic game assumption when type length n and serial length m satisfy $n - m \in \omega(\lg n)$, unless that assumption is false.*

Why would tQL be a reasonable assumption? Clearly it is a very powerful primitive, but how much of a leap is it from “vanilla” quantum lightning? While we do not have a definitive answer to that question, we can show that QL implies tQL with small types. More precisely, we construct in Appx. B a tQL scheme from regular QL for types of $O(\lg(n))$ bits.

Instantiating WOTRO from a non-game assumption. We show that it is possible to construct a WOTRO protocol for which security is based on a cryptographic assumption that does not fit the game formalism. Our result is based on a new hardness assumption on cryptographic hash functions called *collision-shelters*. Intuitively, a family of hash functions is a collision-shelter if no adversary can produce many collisions *in superposition*. As such it is an intrinsically quantum definition which cannot be framed as a game since no challenger can verify that an adversary breaks the assumption. Using this assumption, we show that the MUB-based construction for WOTRO sketched earlier can be turned into a secure protocol. To prove the security of this protocol, we give bounds on the optimal probability of distinguishing between states from many mutually unbiased bases which might be of independent interest.

Theorem 4 (informal). *Under the collision-shelter assumption, there are secure instantiations of $WOTRO^{n,m}$ in the CRQ\$ model.*

2 Technical Overview

We call *Weak One-Time Random Oracle*, denoted $\text{WOTRO}^{n,m}$, the following simple non-interactive primitive. To any $a \in \{0,1\}^n$, it provides a challenge $c \in \{0,1\}^m$ *avoiding* with good probability any function $c : \{0,1\}^n \rightarrow \{0,1\}^m$. We say that an implementation of $\text{WOTRO}^{n,m}$ *avoids* function c if no (efficient) dishonest prover is able to produce (a, c) such that $c = c(a)$. An implementation of $\text{WOTRO}^{n,m}$ is said to be κ -secure if it behaves like a random oracle when the prover is honest and avoids any function c with probability at least κ , when the prover is dishonest. It is easy to see that any non-interactive κ -secure implementation of $\text{WOTRO}^{n,m}$ can be used to implement the Fiat-Shamir transform with computational soundness error upper-bounded by $1 - \kappa$ (see Appendix D.1). Any implementation of $\text{WOTRO}^{n,m}$ that avoids any function $c(\cdot)$ would be a powerful cryptographic primitive to remove interaction. An implementation $\Pi_{\text{WRO}}^{n,m} = (\mathsf{P}', \mathsf{V}')$ of $\text{WOTRO}^{n,m}$ in the CRQS model is defined by two families of efficient POVMs $\mathsf{P}' = \{\mathcal{P}^a\}_a$ and $\mathsf{V}' = \{\mathcal{V}^{a,c,v}\}_{a,c,v}$ with $a \in \{0,1\}^n$, $c \in \{0,1\}^m$, and v is an auxiliary string announced to V' . $\Pi_{\text{WRO}}^{n,m} = (\mathsf{P}', \mathsf{V}')$ is executed as follows:

1. Upon input $a \in \{0,1\}^n$, P' applies POVM $\mathcal{P}^a := \{\mathcal{P}_{c,v}^a\}_{c,v}$ to register P of the CRQS to get classical outcome (c, v) . P' then announces (a, c, v) to V' .
2. V' applies POVM $\mathcal{V}^{a,c,v} := \{\mathcal{V}_0^{a,c,v}, \mathcal{V}_1^{a,c,v}\}$ to register V of the CRQS and accepts iff classical outcome 1 is obtained.

An adversary \mathcal{A} against $\Pi_{\text{WRO}}^{n,m}$ takes no input and applies a POVM $\mathcal{A} := \{\mathcal{A}_{a,c,v}\}_{a,c,v}$ to register P of the CRQS to obtain a along with the message (c, v) . Notice that as defined, $\Pi_{\text{WRO}}^{n,m}$ requires the message transmitted to V' to be classical. This can be done without loss of generality as a protocol asking P' to send a quantum message can be transformed into one where P' only sends a classical message by adding to the CRQS enough EPR pairs for the quantum message to be teleported. The security of the original protocol remains untouched by this transformation.

The security of $\text{WOTRO}^{n,m}$ cannot be reduced to a game. Our main contribution is that, in the CRQS model, $\text{WOTRO}^{n,m}$ for any n, m that satisfy $n - m \in \Omega(n)$ cannot be shown to avoid all functions via a black-box reduction to any cryptographic game. The proof strategy is similar to Bitansky *et al.* in [6, 7] when proving that there exists no black-box reduction from any successful adversary against the entropy preserving property of a family of hash functions to a cryptographic game. As any Σ -universal hash function was shown by Dodis, Ristenpart, and Vadhan [24] to be entropy-preserving, the impossibility result for establishing the soundness of a Σ -universal Fiat-Shamir transform (i.e a hash functions family) follows. The proof in [6, 7] uses a strategy to conclude the impossibility of a black-box reduction that was formalized by Wichs in [45], but used several times prior [23, 31, 28, 36, 22]. We proceed the same way. We first construct a possibly inefficient adversary that breaks any implementation of $\text{WOTRO}^{n,m}$, the existence of which is shown guaranteed by Ahlswede

and Winter’s Operator Chernoff bound [1]. As with Bitansky *et al.*’s adversary against any entropy preserving family of hash functions, our adversary against $\text{WOTRO}^{n,m}$ is *simulatable* by a stateful efficient quantum circuit. This prevents the security of $\text{WOTRO}^{n,m}$ to be established by a black-box reduction to any cryptographic game as if there was such a reduction, the game would also be won using the efficient simulator.

Σ –universal quantum Fiat-Shamir cannot be reduced to a game. We then show that the black-box impossibility of WOTRO (indirectly) implies that the soundness of any $\Sigma_{n,m}$ –universal quantum Fiat-Shamir transform cannot be established under the same conditions. As our basic impossibility result is about the security of a cryptographic primitive rather than a property of a family of hash functions (as in [6, 7]), we follow a different path. First, let us discuss what distinguishes $\text{WOTRO}^{n,m}$ from a $\Sigma_{n,m}$ –universal quantum Fiat-Shamir transform in the CRQS model. Consider a $\Sigma_{n,m}$ –universal Fiat-Shamir transform applied to a Σ –protocol $\Sigma = (P, V)$ for membership in language L with $x \in L$, a public input. Let $P = (P_1, P_2)$ be the prover in Σ such that $a \leftarrow P_1$ and $z \leftarrow P_2(x, a, c)$ satisfies $V(x, a, c, z) = 1$ for all $c \in \{0, 1\}^m$. A protocol for the quantum Fiat-Shamir transform in the CRQS, denoted $\Pi^{\text{QFS}} = (P^*, V^*)$, is given by two families of efficient POVMs $P^* := \{\mathcal{P}^a\}_a$ and $V^* := \{\mathcal{V}^{a,c,v}\}_{a,c,v}$, where $\mathcal{P}^a := \{\mathcal{P}_{c,v}^a\}_{c,v}$ and $\mathcal{V}^{a,c,v} := \{\mathcal{V}_0^{a,c,v}, \mathcal{V}_1^{a,c,v}\}$ exactly as it is for $\text{WOTRO}^{n,m}$. Protocol $\Pi^{\text{QFS}}[\Sigma]$ is executed as follows:

1. P^* generates $a \leftarrow P_1$, measures register P of the CRQS with POVM \mathcal{P}^a to obtain output $(c, v) \in \{0, 1\}^m \times \{0, 1\}^*$, computes $z \leftarrow P_2(x, a, c)$, and sends (a, c, z, v) to V^* .
2. V^* measures register V of the CRQS with POVM $\mathcal{V}^{a,c,v}$ and accepts upon outcome 1 and $V(x, a, c, z) = 1$.

Although Π^{QFS} is providing something very close to $\text{WOTRO}^{n,m}$ in its inner workings, it may not need to avoid all functions to be a computationally sound Σ –universal implementation of the Fiat-Shamir transform. It only needs to avoid functions $c : \{0, 1\}^n \rightarrow \{0, 1\}^m$ such that for some Σ –protocol $\Sigma = (P, V)$ for some language L , there exists $x \notin L$ for which upon commitment a , only challenge $c(a)$ has a third message z such that $V(x, a, c(a), z) = 1$. We show that this relaxation on the functions to be avoided by any $\Sigma_{n,m}$ –universal Π^{QFS} leads to the same impossibility result than for $\text{WOTRO}^{n,m}$. The proof follows from the existence of a Σ –protocol $\Sigma^g = (P, V^g)$ for membership to the empty language, where $g : \{0, 1\}^n \rightarrow \{0, 1\}^m$ is a random oracle. Although Σ^g only requires V^g to have access to the oracle $g(\cdot)$ to run the protocol honestly, the adversary \mathcal{A}^g has also access to $g(\cdot)$ to mount its attack against the soundness of $\Pi^{\text{QFS}}[\Sigma^g]$. This is essentially the same adversary defined as the one against $\text{WOTRO}^{n,m}$ described above. Notice that if the soundness of Π^{QFS} was black-box reducible to game \mathcal{G} then there would be an efficient algorithm \mathcal{B}^g , having oracle access to $g(\cdot)$, that wins game \mathcal{G} . The strategy used for $\text{WOTRO}^{n,m}$ can then be applied. A possibly inefficient adversary \mathcal{A}^g is defined that almost all the time breaks

the soundness of $\Pi^{\text{QFS}}[\Sigma^g]$. We finally show that both the adversary \mathcal{A}^g and \mathcal{V}^g can be simulated by an efficient stateful simulator. As before, this prevents the soundness of Π^{QFS} to be established by black-box reduction to a cryptographic game unless the game is trivial.

A quantum assumption allowing for WOTRO n,m . We introduce a strong variant of collision resistant families of hash functions allowing for a computationally sound Σ -universal implementation of the Fiat-Shamir transform in the CRQ\$ model. We call $\mathbb{G}^{n,m} := \{G_s^n\}_s \subset \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^m$ a *collision-shelter* if, for any target function $c : \{0,1\}^n \rightarrow \{0,1\}^m$, no efficient quantum adversary can produce any state polynomially close to a state of the form

$$|\psi\rangle_{AX} = \sum_a \alpha_a |a\rangle_A \otimes \sum_{x:G^n(a,x)=c(a)} \beta_x^a |x\rangle_X ,$$

that contains collisions to $c(a)$ when a is measured.

In order to show that collision-shelters are sufficient for a sound Σ -universal Fiat-Shamir transform in the CRQ\$ model, we start with the weak random oracle implemented using n shared EPR pairs from the introduction. We modify the scheme slightly to get an unconditionally $\frac{1}{4}$ -secure¹¹ implementation $\Pi_{\text{WRO}}^{n,n} = (\mathcal{P}', \mathcal{V}')$ of WOTRO n,n in the CRQ\$ model. This forms the basis upon which WOTRO n,m , with $m < n$, is constructed using a collision-shelter. We prove that $\Pi_{\text{WRO}}^{n,n}$ is $\frac{1}{4}$ -secure using shared maximally entangled pairs of qutrits as the CRQ\$ to allow the use of a particular set $\{\theta_a\}_{a \in \{0,1,2\}^n}$ of mutually unbiased bases, introduced by Wootters and Fields [46]. The set $\{\theta_a\}_a$ is shown to prevent any adversary $\mathcal{A} := \{\mathcal{A}_{a,c,v}\}_{a,c,v}$ from observing $\mathcal{A}_{a,c(a),v} \otimes \mathcal{V}_1^{a,c(a),v}$ with probability better than $\frac{3}{4}$ when the CRQ\$ is measured by \mathcal{P}' and \mathcal{V}' . This result may be of independent interest and is made possible as \mathcal{A} 's success probability is given by an instance of a Weil sum that can be upper bounded by Deligne's resolution of one of Weil's conjectures [21].

A protocol $\Pi_{\text{WRO}}^{n,m}[\mathbb{G}^{n,m}] = (\mathcal{P}'', \mathcal{V}'')$ for WOTRO n,m with $m < n$ can then be constructed using a collision-shelter $\mathbb{G}^{n,m}$ in the obvious way. Upon input $a \in \{0,1,2\}^n$, \mathcal{P}'' runs \mathcal{P}' upon input a to get $(c', v) \in \{0,1\}^n \times \{0,1\}^n$. \mathcal{P}'' announces (a, c', v) to \mathcal{V}'' . The challenge produced by $\Pi_{\text{WRO}}^{n,m}$ is simply set to $c := G_s^n(a, c') \in \{0,1\}^m$ for s a CRS. \mathcal{V}'' simply runs \mathcal{V}' on (a, c', v) and accepts if \mathcal{V}' accepts. It is not difficult to see that if $\mathbb{G}^{n,m}$ is a collision-shelter then no efficient adversary \mathcal{A} can do better against $\Pi_{\text{WRO}}^{n,m}$ than an unconditional adversary against $\Pi_{\text{WRO}}^{n,n}$. As a result, $\Pi_{\text{WRO}}^{n,m}$ avoids all functions with probability $\frac{1}{4}$. Negligible soundness error can then be achieved by parallel repetitions.

WOTRO and Quantum Lightning. Quantum lightning (QL), introduced by Zhandry [47], is a quantum cryptographic task allowing anyone to generate quantum states of which they can make exactly one copy. Informally, a QL scheme

¹¹ By $\frac{1}{4}$ -secure, we really mean $(\frac{1}{4} - \text{negl}(n))$ -secure.

consists of a quantum algorithm \mathcal{A} instructing how to construct bolts $|\text{bolt}\rangle$ and of a verification algorithm Ver that on input $|\text{bolt}\rangle$ returns a serial number $s \in \{0, 1\}^n$ without disturbing state $|\text{bolt}\rangle$ such that no efficient adversary can create two valid states with the same serial number. For this to hold, there must be uncertainty in the serial number of newly created bolts: for every QPT adversary \mathcal{A} , $|\text{bolt}\rangle \leftarrow \mathcal{A}(\mathcal{A})$ must satisfy $H_\infty(\text{Ver}(|\text{bolt}\rangle)) \in \omega(\lg n)$, otherwise polynomially many tries would give two bolts with the same serial number, contradicting uniqueness. Note that an efficient reduction does not necessarily exist in the other direction: an adversary could for example produce two valid states with identical serial numbers that each have maximal min-entropy. Such an adversary appears useless for producing a single lightning state with low min-entropy in the serial number.

We introduce a variant of quantum lightning where the bolt generation procedure accepts an input. *Typed quantum lightning* (tQL) is a new primitive similar to QL where \mathcal{A} takes an additional parameter (or type) $a \in \{0, 1\}^n$. Intuitively, security asks that when we fix the type a , the resulting scheme still produces unpredictable serial numbers. This is formalized by requiring that the conditional min-entropy $H_\infty(S | A)$ is large. We show that a tQL scheme with type length n and serial number length m implies the existence of a protocol for WOTRO n,m . The scheme asks the prover to generate a typed QL state with type a and teleport that state to the verifier using EPR pairs from a CRQ\$, the verifier accepts if the teleported state is a valid tQL state. A consequence of this scheme is that no tQL scheme satisfying $n - m \in \omega(\lg n)$ can have its security be black-box reducible to a cryptographic game assumption.

3 Notations & Preliminaries

We use $n \in \mathbb{N}$ as the security parameter throughout the paper. We use $\text{poly}(n)$ to denote a polynomial in n . A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is said to be *negligible* if for all polynomials $p(\cdot)$ and for $n \in \mathbb{N}$ sufficiently large, $f(n) \leq 1/p(n)$. We denote a negligible function by $\text{negl}(\cdot)$. We use ‘‘QPT’’ as a shorthand for *quantum polynomial time*. We use $\log(\cdot)$, $\ln(\cdot)$ and $\lg(\cdot)$ to respectively denote the base 10, e and 2 logarithms.

For a set A , its cardinality is denoted $|A|$ and its complement \bar{A} . We write $x \in_R A$ to indicate that x is chosen uniformly at random from A .

We often use the notation $f(\cdot)$ to denote functions as a way to differentiate them from variables. If $f(\cdot, \cdot)$ is a function of two arguments, we denote by $f(x, \cdot)$ the function of one argument defined by restricting the first argument to value x . For two sets A and B , we denote the set of functions from A to B as $A \rightarrow B$.

For a random variable X , $\mathbb{E}[X]$ denotes its expected value and for $X(r)$ a random variable function of r , $\mathbb{E}_r[X]$ denotes its expected value when r is picked at random. Let $\Delta(A, B) = \frac{1}{2} \sum_a |\Pr[A = a] - \Pr[B = a]|$ denote the statistical distance between the distribution of two random variables A and B with the same domain. For an operator $A \in \mathbb{C}^{n \times n}$, $\|A\|_1 = \text{tr}(\sqrt{A^*A})$ denotes its trace norm.

3.1 Black-Box Impossibility Results

Definition 1 ([31, 6]). A cryptographic game is a tuple $\mathcal{G} = (\Gamma, c)$ composed of an interactive Turing machine Γ and a constant $c \in [0, 1]$. On security parameter $n \in \mathbb{N}$, the challenger $\Gamma(1^n)$ interacts with an adversary \mathcal{A}_n and outputs a bit b . The output of this interaction is denoted by $\langle \mathcal{A}_n \rightleftharpoons \Gamma(1^n) \rangle$. The advantage of the family of adversaries $\mathcal{A} = \{\mathcal{A}_n\}_{n \in \mathbb{N}}$ in game \mathcal{G} is defined as

$$\text{Adv}_{\mathcal{A}, \mathcal{G}}(n) = \Pr[\langle \mathcal{A}_n \rightleftharpoons \Gamma(1^n) \rangle = 1] - c .$$

A cryptographic game \mathcal{G} is secure if for all PPT adversary \mathcal{A} , the advantage $\text{Adv}_{\mathcal{A}, \mathcal{G}}(n)$ is $\text{negl}(n)$.

A *black-box reduction* from a cryptographic task to a cryptographic game \mathcal{G} is an oracle access machine $\mathcal{R}^{(\cdot)}$ such that for any adversary \mathcal{A} that breaks the security of the task, $\mathcal{R}^{\mathcal{A}}$ has advantage at least $\frac{1}{\text{poly}(n)}$ in game \mathcal{G} .

In this paper, we show the impossibility of black-box reduction of some tasks to a cryptographic games. Our proof employs the general technique of simulatable attacks formalized by Wichs [45] and applied by [6] to the Fiat-Shamir transform. An inefficient adversary \mathcal{A} against some task is *simulatable* if there exists a simulator Sim such that no efficient algorithm can distinguish between \mathcal{A} and Sim from black-box query access. A cryptographic task that has a simulatable attack cannot be black-box reduced to a secure cryptographic game since the reduction $\mathcal{R}^{(\cdot)}$ cannot distinguish between the inefficient \mathcal{A} and the efficient Sim , which means that \mathcal{R}^{Sim} would yield an efficient algorithm for the game \mathcal{G} with non-negligible advantage, contradicting its security.

4 A Simple Non-Interactive Primitive

In this paper, we consider a simple non-interactive cryptographic primitive, called a *weak one-time random oracle* (WOTRO^{*n,m*}) and illustrated in Fig. 1 where the prover inputs $a \in \{0, 1\}^n$ into the box and gets $c \in \{0, 1\}^m$ as output while the verifier inputs nothing and gets (a, c) as output. An implementation of this primitive is a protocol taking place between the prover and the verifier. The verifier \mathbf{V} is a machine that takes no input, interacts with the prover in the way prescribed by the protocol, and either accepts and outputs (a, c) or rejects and outputs \perp . In an honest implementation, the prover is a machine \mathbf{P} taking as input an $a \in \{0, 1\}^n$ and interacts with the verifier as specified by the protocol, in such a way that the verifier accepts and outputs the same c . The strings a and c can then be determined from the transcript of the protocol. We can then view the whole protocol in the honest case as a conditional distribution $\Pi(c|a)$ that tells us the probability of getting the challenge c given that the prover was given a as input.

In a dishonest implementation, the prover $\tilde{\mathbf{P}}$ takes no input at all (so the prover is free to choose a) and might behave in a way that will cause the verifier to reject. The protocol is then simply a joint probability distribution $\tilde{\Pi}_{\tilde{\mathbf{P}}}(a, c, v)$,

representing the distribution one obtains when \tilde{P} runs the protocol with the honest verifier V , and where $v \in \{0, 1\}$ is 1 when the verifier accepts and 0 if he rejects. Note that we only consider cheating provers in this paper and do not consider cheating verifiers.

We now define correctness and security of an implementation. In a correct implementation of this primitive, $\Pi(c|a)$ will reflect exactly the same distribution over a and c given by the ideal box, namely c will be uniformly distributed and independent of a , and the verifier always accepts when the prover is honest:

Definition 2 (ϵ -correctness). *A protocol Π is a ϵ -correct implementation of $WOTRO_{\Gamma}^{n,m}$ if for all $a \in \{0, 1\}^n$ the conditional distribution $\Pi(c|a)$ is $(1 - \epsilon)$ -close (in statistical distance) to the uniform distribution over c and if V accepts with probability at least $1 - \text{negl}(n)$ when the prover is honest. Π is said to be statistically correct if it is $(1 - \text{negl}(n))$ -correct.*

As for our security definition, it will be rather weak (hence the “weak” in the name of the primitive): we will only require that in a secure implementation, a dishonest prover \tilde{P} cannot steer the choice of c towards a deterministic function of a . Rather than require that c be almost uniform and independent, we will only demand that there be *some* randomness left in this choice. We will formalize this by the notion of a probability distribution on two random variables δ -avoiding a function:

Definition 3 (δ -avoiding). *For $0 \leq \delta \leq 1$, we say that a tuple of random variables (A, C, V) taking values in $\{0, 1\}^n \times \{0, 1\}^m \times \{0, 1\}$ δ -avoids the function $c : \{0, 1\}^n \rightarrow \{0, 1\}^m$ if*

$$\Pr[V = 1 \wedge C = c(A)] \leq 1 - \delta .$$

This then leads to the following definition of security for an implementation of WOTRO.

Definition 4 (δ -security). *A protocol is a statistically (resp. computationally) δ -secure implementation of $WOTRO_{\Gamma}^{n,m}$ if for all dishonest provers (resp. all QPT dishonest provers) \tilde{P} , the random variable tuple (A, C, V) with joint distribution $\tilde{\Pi}_{\tilde{P}}(a, c, v)$ δ -avoids all functions $c : \{0, 1\}^n \rightarrow \{0, 1\}^m$. We say that a protocol for WOTRO is statistically (resp. computationally) secure if it is statistically (resp. computationally) $(1 - \text{negl}(n))$ -secure.*

Basic Facts About WOTRO. Observe that there is a trivial perfectly secure 2-message protocol for WOTRO where P sends a and V sends a uniformly random c . Therefore, we will focus on non-interactive (or 1-message) implementations of WOTRO. A secure non-interactive WOTRO protocol provides enough conditional randomness for sound instantiation of the Fiat-Shamir transform when applied to public-coin special-sound 3-message interactive proofs (Σ -protocols).

In the bare model, there is no secure WOTRO protocol as the honest prover program defines the output c as a function of a that can never be avoided. In

the CR\$ model, there exists a simple statistically $(1 - \text{negl}(m - n))$ -secure one-message protocol when $m > n$, a statistically $\frac{1}{e}$ -secure protocol when $m = n$ and there is no protocol for $m < n$ whose computational security can be black-box reduced to a cryptographic game assumption, as a consequence of [6]. A detailed examination of these facts is provided in Appx. D.

WOTRO in the CRQS Model. Since the object of study is the (im)possibility of the WOTRO primitive in the CRQS model, we present a general form for a 1-message WOTRO protocol in this model.

Definition 5 (WOTRO in the CRQS model). A $WOTRO^{n,m}$ protocol $\Pi_{WOTRO} = (P_{WOTRO}, V_{WOTRO})$ in the CRQS model consists of

- A CRQS $\Psi_{PV} \in \mathcal{D}(\mathcal{H}_{PV})$
- A mapping of $a \in \{0, 1\}^n$ to an efficient POVM $\mathcal{N}^a = \{N_{x,w}^a\}_{(x,w) \in \{0,1\}^{m \times \ell}}$ on register P .
- A mapping of $a \in \{0, 1\}^n$, $x \in \{0, 1\}^m$ and $w \in \{0, 1\}^\ell$ to an efficient POVM $\mathcal{V}^{a,x,w} = \{V_0^{a,x,w}, V_1^{a,x,w}\}$ on register V .

On input $a \in \{0, 1\}^n$:

1. P_{WOTRO} applies POVM \mathcal{N}^a on register P of Ψ_{PV} to obtain x and an auxiliary verification string w and sends (a, x, w) to the verifier.
2. V_{WOTRO} applies POVM $\mathcal{V}^{a,x,w}$ on register V of Ψ_{PV} , accepts and outputs (a, x) if the result is 1, and rejects and outputs nothing if the result is 0.

Note that requiring the auxiliary verification string w to be classical is not a restriction since the CRQS can contain EPR pairs for the teleportation of an arbitrary quantum state from the prover to the verifier.

5 Black-Box Impossibility of WOTRO in the CRQS Model

In this section, we show that there exist (inefficient) attacks against any 1-message $WOTRO^{n,m}$ protocol in the CRQS model for m (sufficiently) smaller than n . The following definition describes a general strategy for an attack against WOTRO.

Definition 6. An attack \mathcal{A}^f against a $WOTRO^{n,m}$ protocol (Def 5) is characterized by a target function $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$ and a (possibly inefficient) POVM $\{P_{a,x,w}^f\}_{(a,x,w) \in \{0,1\}^{n \times m \times \ell}}$. The adversary performs this POVM on register P of CRQS Ψ_{PV} and sends the result (A, X, W) to the verifier. We say that this attack hits function f with probability $\epsilon(\mathcal{A}^f)$ if

$$\epsilon(\mathcal{A}^f) = \Pr[X = f(A) \wedge V \text{ ACCEPTS}] = \sum_{a,w} \text{tr} \left((P_{a,f(a),w}^f \otimes V_1^{a,f(a),w}) \Psi_{PV} \right) .$$

We construct an attack whose success is based on the Chernoff bound for operators proven by Ahlswede and Winter in [1] and stated below. For operators A and B and $0 \leq \eta \leq 1$, the notation $A \in [(1 - \eta)B; (1 + \eta)B]$ means that $A \geq (1 - \eta)B$ and $A \leq (1 + \eta)B$.

Lemma 1 (“Operator Chernoff bound”). *Let X_1, \dots, X_M be i.i.d. random variables taking values in the operators $\mathcal{D}(\mathcal{H})$ on the D -dimensional Hilbert space \mathcal{H} such that $0 \leq X_j \leq \mathbb{1}$, with $A = \mathbb{E}[X_j] \geq \alpha \mathbb{1}$, and let $0 < \eta \leq 1/2$. Then*

$$\Pr \left[\frac{1}{M} \sum_{j=1}^M X_j \notin [(1 - \eta)A; (1 + \eta)A] \right] \leq 2D \exp \left(-M \frac{\alpha \eta^2}{2 \ln 2} \right). \quad (1)$$

Theorem 5. *Let $n, m \in \mathbb{N}$. Let $\Pi_{\text{WOTRO}}^{n,m}$ be an arbitrary $\text{WOTRO}^{n,m}$ protocol described by CRQS Ψ_{PV} and POVM families $\mathcal{N}^a = \{N_{x,w}^a\}_{(x,w) \in \{0,1\}^{m \times \ell}}$ and $\mathcal{V}^{a,x,w} = \{V_0^{a,x,w}, V_1^{a,x,w}\}$ for $a \in \{0,1\}^n$, $x \in \{0,1\}^m$ and $w \in \{0,1\}^\ell$. Let $f : \{0,1\}^n \rightarrow \{0,1\}^m$ be a uniformly random function and let $2^k = \dim P$ be the dimension of the prover’s register of Ψ_{PV} .*

1. *There is some constant $C > 0$ such that with probability at least $1 - \text{negl}(n - m)$ over the choice of f , the operators*

$$\left\{ P_{a,f(a),w}^f := \frac{N_{f(a),w}^a}{\frac{2^n}{2^m} + C \sqrt{k \frac{2^n}{2^m}}} \right\}_{(a,w) \in \{0,1\}^{n \times \ell}}$$

and $P_{a,x,w}^f := 0$ when $x \neq f(a)$, together with $P_\perp^f = \mathbb{1} - \sum_{a,w} P_{a,f(a),w}^f$, form a POVM on the prover’s register P .

2. *If protocol $\Pi_{\text{WOTRO}}^{n,m}$ is correct, the attack given by this POVM has average success probability $\mathbb{E}_f[\epsilon(\mathcal{A}^f)]$ that grows as $1 - \text{negl}(n - m)$.*

Proof. Let 2^k be the dimension of register P . Consider the subset of measurement operators $N_{f(a),w}^a$ from the honest POVM \mathcal{N}^a that yield the intended outcome for the cheating prover, i.e. on input a gives outcome $x = f(a)$, and define the operators $X_a^f = \sum_{w \in \{0,1\}^\ell} N_{f(a),w}^a$. We have that

$$\mathbb{E}_f[X_a^f] = \mathbb{E}_f \sum_{x,w \in \{0,1\}^{m \times \ell}} I_{\mathcal{E}_x^a}(f) \cdot N_{x,w}^a = \sum_{x,w \in \{0,1\}^{m \times \ell}} \mathbb{E}_f[I_{\mathcal{E}_x^a}(f)] \cdot N_{x,w}^a = \frac{1}{2^m} \mathbb{1}_P$$

where $I_{\mathcal{E}_x^a}$ is the indicator function for the event $\mathcal{E}_x^a = \{f \mid x = f(a)\}$ which has probability $\frac{1}{2^m}$ for any x and a since every value for $f(a)$ is equally likely.

Applying the Chernoff bound with $D = 2^k$, $M = 2^m$, and $\alpha = \frac{1}{2^m}$ to the weighted sum over a of the operators X_a^f , we have

$$\Pr_f \left[\frac{1}{2^n} \sum_{a \in \{0,1\}^n} X_a^f \not\leq (1 + \eta) \frac{1}{2^m} \mathbb{1} \right] \leq 2^{k+1} \exp \left(-\frac{1}{2 \ln 2} \cdot \frac{2^n}{2^m} \cdot \eta^2 \right).$$

This bound becomes a negligible function of $n - m$ if we choose $\eta = C \cdot \sqrt{k \frac{2^m}{2^n}}$ for some appropriate constant $C > 0$ (for example $C = \sqrt{4 \ln 2 \frac{1}{\lg e}}$). Therefore, except with probability $\text{negl}(n - m)$,

$$\frac{1}{2^n} \sum_{a \in \{0,1\}^n} X_a^f = \frac{1}{2^n} \sum_{\substack{a \in \{0,1\}^n \\ w \in \{0,1\}^\ell}} N_{f(a),w}^a \leq (1 + \eta) \frac{1}{2^m} \mathbb{1} = \left(1 + C \sqrt{k \frac{2^m}{2^n}}\right) \frac{1}{2^m} \mathbb{1} \quad (2)$$

for an appropriate constant C . Define the ensemble of operators $P_{a,x,w}^f$ by

$$P_{a,f(a),w}^f := \frac{N_{f(a),w}^a}{\frac{2^n}{2^m} + C \sqrt{k \frac{2^m}{2^n}}}$$

and $P_{a,x,w}^f := 0$ when $x \neq f(a)$. Then when (2) holds, the set of operators $\{P_{a,f(a),w}^f\}_{a \in \{0,1\}^n}$ forms a POVM completed by $P_\perp^f = \mathbb{1} - \sum_{a,w} P_{a,f(a),w}^f$.

The success probability of this attack corresponds to the probability of obtaining an outcome of the form $(a, f(a), w)$, i.e. not \perp , and of the verifier accepting that outcome:

$$\epsilon(\mathcal{A}^f) = \sum_{(a,w) \in \{0,1\}^{n \times \ell}} \text{tr} \left(\left(P_{a,f(a),w}^f \otimes V_1^{a,f(a),w} \right) \Psi_{PV} \right) .$$

Letting $c := \left(\frac{2^n}{2^m} + C \sqrt{k \frac{2^m}{2^n}} \right)$, this probability is on average over f at least

$$\begin{aligned} \mathbb{E}_f[\epsilon(\mathcal{A}^f)] &= \frac{1}{c} \cdot \mathbb{E}_f \left[\sum_{(a,w) \in \{0,1\}^{n \times \ell}} \text{tr} \left(\left(N_{a,f(a),w} \otimes V_1^{a,f(a),w} \right) \Psi_{PV} \right) \right] \\ &= \frac{1}{c} \frac{1}{2^m} \sum_{a \in \{0,1\}^n} \sum_{(x,w) \in \{0,1\}^{m \times \ell}} \text{tr} \left(\left(N_{a,x,w} \otimes V_1^{a,x,w} \right) \Psi_{PV} \right) \\ &\geq \frac{1}{c} \frac{1}{2^m} \sum_{a \in \{0,1\}^n} (1 - \text{negl}(n)) \\ &= \frac{2^n}{2^m} \left(\frac{2^n}{2^m} + C \sqrt{k \frac{2^m}{2^n}} \right)^{-1} \cdot (1 - \text{negl}(n)) \\ &= \frac{(1 - \text{negl}(n))}{(1 + C \sqrt{k 2^{m-n}})} \\ &\geq (1 - C \sqrt{k 2^{m-n}}) (1 - \text{negl}(n)) . \end{aligned}$$

Where the first inequality above follows from the correctness of protocol $\Pi_{\text{WOTRO}}^{n,m}$. The above expression scales as $1 - \text{negl}(n - m)$. \blacksquare

5.1 Simulating the Above Attack

For a WOTRO protocol to have its security reducible by black-box to a cryptographic game assumption \mathcal{G} , there must exist a *reduction* in the form of an oracle machine $\mathcal{R}^{(\cdot)}$ that, given oracle access to an adversary \mathcal{A} against WOTRO, succeeds at game \mathcal{G} . The most general attack against a WOTRO protocol in the CRQS model (Def 6) is a POVM on the prover's part of the CRQS that produces a classical message which makes the verifier accept the output $c = f(a)$ with high probability. The attack takes no input other than the prover register of the CRQS and produces a classical outcome. We therefore model (Def 7) black-box access to an adversary \mathcal{A} against WOTRO as measurement device to which an oracle access machine can send quantum queries and receive classical answers. In particular, the oracle machine cannot purify the action of \mathcal{A} .

We show that no reduction $\mathcal{R}^{(\cdot)}$ can exist to base the security of a WOTRO protocol on a game assumption \mathcal{G} . This is because the adversarial strategy described in Thm 5 is *simulatable*. This means that there is an efficient algorithm Sim such that no oracle machine can tell whether it is given oracle access to the inefficient adversarial prover \mathcal{A}^f hitting function $f(\cdot)$ or to Sim that does not know $f(\cdot)$. As mentioned in Section 3.1, this rules out any reduction from the security of a WOTRO protocol to a cryptographic game assumption \mathcal{G} since the reduction \mathcal{R} breaking game \mathcal{G} with oracle access to \mathcal{A}^f would also break game \mathcal{G} with oracle access to the efficient Sim , otherwise \mathcal{A}^f and Sim could be efficiently distinguished.

Definition 7 (Oracle Access Machine). A quantum oracle access machine $M^\mathcal{O}$ is a Turing machine that, on input 1^n , outputs the description of a quantum circuit over a universal set of quantum gates along with a special quantum gate $\mathcal{O}_{P \rightarrow Q}^n$. We say $M^\mathcal{O}$ makes $q(n)$ oracle queries if we can represent the action of circuit $M^\mathcal{O}(1^n)$ on initial state $|0\rangle_R$ as the CPTP map

$$\mathcal{M}_n^\mathcal{O}(|0\rangle\langle 0|_R) := \mathcal{M}_n^q \circ \mathcal{O}_{P_q}^n \circ \dots \circ \mathcal{M}_n^1 \circ \mathcal{O}_{P_1}^n \circ \mathcal{M}_n^0(|0\rangle\langle 0|_R)$$

where $\mathcal{M}_n^i : \mathcal{D}(\mathcal{H}_{Q_i R}) \rightarrow \mathcal{D}(\mathcal{H}_{P_{i+1} R})$ are CPTP maps representing the action of circuit $M^\mathcal{O}(1^n)$ between the calls to \mathcal{O} .

Definition 8 (Simulatable Attack for WOTRO). Let $n, m \in \mathbb{N}$ and let Π be a WOTRO n,m protocol. A $\epsilon(n)$ -simulatable attack against Π consists of an ensemble of (possibly inefficient) attacker CPTP map $\{\mathcal{A}_{P \rightarrow AXW}^{f,n}\}_{f \in \mathcal{F}, n \in \mathbb{N}}$ for a family of functions $\mathcal{F} \subseteq \{0, 1\}^n \rightarrow \{0, 1\}^m$ and a family of efficient CPTP map $\{\text{Sim}_{P \rightarrow AXW}^n\}_{n \in \mathbb{N}}$ such that

- The success probability of $\mathcal{A}^{f,n}$ is at least $1 - \text{negl}(n)$ on average over the random choice of $f \in \mathcal{F}$.
- For every (possibly inefficient) oracle access machine $M^{(\cdot)}$ making $q(n) = \text{poly}(n)$ queries to its oracle, the CPTP map $\mathcal{M}_n^{(\cdot)}$ describing the action of the circuit $M^{(\cdot)}(1^n)$ satisfies

$$\|\mathbb{E}_f[\mathcal{M}_n^{\mathcal{A}^{f,n}}(|0\rangle\langle 0|_R)] - \mathcal{M}_n^{\text{Sim}^n}(|0\rangle\langle 0|_R)\|_1 \leq q(n) \cdot \epsilon(n) \quad (3)$$

Theorem 6. *Let n, m and $\Pi_{\text{WOTRO}}^{n,m}$ be as in the statement of Thm 5. The attack $\{\mathcal{A}^f\}_{f \in \mathcal{F}}$ of Thm 5 is $\text{negl}(n-m)$ -simulatable.*

Proof (sketch). The proof is essentially a quantum version of the proof of impossibility of Fiat-Shamir of [6] as it follows very similar steps, we therefore refer to Appx. A.1 for the full proof. The simulator is constructed as follows: when queried on a quantum register P , picks a random a that wasn't seen before and applies the honest prover POVM \mathcal{N}^a . The objective is to show that the action of this simulator is close to that of the adversarial prover \mathcal{A}^f for a random f for any $q = \text{poly}(n)$ quantum queries. This is achieved using a hybrid argument where q hybrid simulators $\{\text{Sim}^i\}_{i \in [q]}$ differ only in the i th query and such that $\text{Sim}^0 = \mathcal{A}^f$ and $\text{Sim}^q = \text{Sim}$. The i th hybrid acts as Sim for the first i queries, and acts as \mathcal{A}^f for the remainder where f is picked to be consistent with the answers to the first i queries. Indistinguishability of the hybrids Sim^i and Sim^{i+1} follows from the fact that the function f picked to answer the remaining queries has essentially the same distribution in both cases and that the simulator's POVM and the adversary's POVM are close on average over this f . ■

Thm 5 and Thm 6 give a simulatable attack against any $\text{WOTRO}^{n,m}$ protocol where $n-m$ is superlogarithmic in the security parameter n . This attack succeeds in hitting a function $f(\cdot)$ with probability at least $1 - \text{negl}(n)$ which rules out black-box reduction of δ -security of a WOTRO protocol for any non-negligible δ .

Corollary 1. *Let $\mathcal{G} = (\Gamma, c)$ be a cryptographic game assumption and let $\Pi^{n,m}$ be a $\text{WOTRO}^{n,m}$ protocol with $n-m \in \omega(\lg n)$. For any $\delta \geq 1/\text{poly}(n)$ if there is a black-box reduction showing that $\Pi^{n,m}$ δ -avoids all functions from the security of the game \mathcal{G} , then assumption \mathcal{G} is false.*

6 Black-Box Impossibility of Fiat-Shamir in the CRQS Model

6.1 Σ -protocols and the Fiat-Shamir Transform.

Let $R \subseteq \{0,1\}^* \times \{0,1\}^*$ denote an arbitrary efficiently computable binary relation such that if $(x, w) \in R$ then $|w| \leq p(|x|)$ for some polynomial $p(\cdot)$. We call x a *public instance* and w a *witness* for x . The condition above ensures that the witness of any public instance can be conveyed efficiently. From the binary relation R , we define the language $L_R = \{x \mid (\exists w)[(x, w) \in R]\} \in \mathbf{NP}$ of public instances with witnesses for them.

Definition 9 (Σ -protocol [18]). *A Σ -protocol $\Sigma = (\mathsf{P}, \mathsf{V})$ for a binary relation R is a 3-message protocol with conversation alphabet $\{0,1\}$. On public input $x \in L_R$ and on private input w to P such that $(x, w) \in R$, the protocol structure is as follows:*

- The prover sends a message $a = \mathsf{P}_1(x, w) \in \{0,1\}^n$ called the commitment.

- The verifier sends a challenge $c \in \{0, 1\}^m$.
- The prover sends a reply $z = P_2(a, x, w, c) \in \{0, 1\}^*$, and the verifier outputs $V(x, a, c, z) \in \{\text{accept}, \text{reject}\}$.

Moreover, the protocol satisfies the following requirements:

Random public coins: The challenge $c \in \{0, 1\}^m$ is chosen uniformly at random in $\{0, 1\}^m$ without any extra processing (i.e. no need for private information to generate c).

Perfect correctness: When $x \in L_R$, V accepts P with probability 1.

Special soundness: When $x \in L_R$, given two accepting conversations for the same commitment (a, c, z) and (a, c', z') with $c \neq c'$, there exists a PPT algorithm W such that $(x, W(a, c, z, c', z')) \in R$.

We should mention here that Σ -protocols are also often used as a synonym of 3-message public-coins protocols (as in [32, 37], for instance) irrespectively of whether the proof system satisfies perfect correctness or special soundness. However, since we are proving a negative result, there is no loss in generality in adopting the more restrictive definition of [18].

By special soundness, if $x \notin L_R$ then for any commitment $a \in \{0, 1\}^n$, there is at most one challenge $c \in \{0, 1\}^m$ such that for some response z , (a, c, z) is an accepting conversation. For some Σ -protocol Σ_L for a language L and some $x \notin L$, we call the function that maps a to this one challenge c the *bad challenge function*.

In the ROM, the Fiat-Shamir transform $\Pi^{\text{FS}}[\Sigma] = (P^{\text{FS}}, V^{\text{FS}})$ applied to a Σ -protocol $\Sigma = (P, V)$ with first message length n and challenge length m for a proof of language membership is a non-interactive argument where, on public input $x \in L$ and random oracle $H : \{0, 1\}^n \rightarrow \{0, 1\}^m$,

1. P^{FS} runs $a = P(x, w)$ computes $c = H(a)$ and $z = P_2(a, x, w, c)$, and sends (a, c, z) to V^{FS} .
2. V^{FS} rejects if $c \neq H(a)$, otherwise outputs $V(x, a, c, z)$.

In the CRS model, the protocol is the same with the random oracle replaced with a family of cryptographic hash functions $\mathcal{H} = \{h_r\}_r$ where $h_r : \{0, 1\}^n \rightarrow \{0, 1\}^m$ is sampled using a CRS.

In this section, we consider the natural extension of the Fiat-Shamir transform in the CRQS model where the prover and verifier share an arbitrary entangled state $|\varphi_{n,m}\rangle$, the prover performs some measurement specified by a on its part of the CRQS, sends the result to the verifier who performs its own measurement based on the prover's message. Since a universal instantiation of the Fiat-Shamir is required to transform any Σ -protocol into a sound argument, the CRQS $|\varphi_{n,m}\rangle$, as well as the measurement operators of the prover and verifier must be independent of the actual Σ -protocol and of the statement x . The quantum Fiat-Shamir transform proceeds as follows:

1. P^{FS} computes $a = P(x, w)$ and performs some measurement \mathcal{N}^a on its part of $|\varphi_{n,m}\rangle$ that yield classical outcomes (c, v) . It computes $z = P_2(a, x, w, c)$, and sends (a, c, v, z) to V^{FS} .

2. V^{FS} performs a binary-outcome measurement $\mathcal{V}^{a,c,v}$ on its part of $|\varphi_{n,m}\rangle$ and rejects if the outcome is 0, and otherwise outputs $V(x, a, c, z)$.

We consider without loss of generality that all communication remains classical, since the CRQS could contain polynomially many EPR pairs allowing for the teleportation of quantum states from the prover to the verifier.

An *abstract* Fiat-Shamir transform that captures all of the above would look like the following. Since we are proving a negative result, we only ask that a universal instantiation of the Fiat-Shamir transform has constant soundness error (instead of $\text{negl}(n)$).

Definition 10. *The Fiat-Shamir transform is given by $\Pi_{\text{FS}}^{n,m} = (\text{P}_{\text{FS}}, \text{V}_{\text{FS}})$ where P_{FS} takes as input the commitment $a \in \{0, 1\}^n$ and outputs a challenge $c \in \{0, 1\}^m$ and a auxiliary verification information v . V_{FS} takes input (a, c, v) and outputs **accept** or **reject**. For a Σ -protocol $\Sigma = (\text{P}_{\Sigma}, \text{V}_{\Sigma})$, the Fiat-Shamir transform applied to Σ is the non-interactive protocol $\Pi_{\text{FS}}^{n,m}[\Sigma] = (\text{P}, \text{V})$ defined as*

1. P computes $a = \text{P}_{\Sigma}^1(x, w)$ and runs $(c, v) \leftarrow \text{P}_{\text{FS}}(a)$. It computes $z = \text{P}_{\Sigma}^2(a, x, w, c)$, and sends (a, c, v, z) to V .
2. V runs $\text{V}_{\text{FS}}(a, c, v)$ and rejects if V_{FS} rejects, and otherwise outputs $\text{V}_{\Sigma}(x, a, c, z)$.

The Fiat-Shamir transform $\Pi_{\text{FS}}^{n,m}$ is (n, m) -universal if for any Σ -protocol Σ , $\Pi_{\text{FS}}^{n,m}[\Sigma]$ is an argument with soundness error bounded above by some constant greater than zero.

Note that an instantiation of the Fiat-Shamir transform is also one for WOTRO (and vice-versa). More precisely, the WOTRO protocol implied by Fiat-Shamir is the protocol where P_{WOTRO} invokes P_{FS} , sends (a, c, v) to V_{WOTRO} that outputs (a, c) if $\text{V}_{\text{FS}}(a, c, v)$ accepts. The main distinction between the two is that a secure protocol for WOTRO needs to avoid all functions, whereas a universal instantiation of Fiat-Shamir only needs to avoid functions that are “bad challenges” functions for some Σ -protocol for language membership to L upon some public parameter $x \notin L$.

6.2 Black-Box Impossibility of Universal Fiat-Shamir in the CRQS Model.

We begin by defining what is a black-box reduction from FS to a cryptographic game assumption following along the lines of [6].

Definition 11 (Black-Box Reduction for Quantum Fiat-Shamir). *Let $\mathcal{G} = (G, c)$ be a cryptographic game assumption and let $\Pi_{\text{FS}}^{n,m}$ be an instantiation of the Fiat-Shamir transform in the CRQS model. A black-box reduction showing the (n, m) -QFS-universality of $\Pi_{\text{FS}}^{n,m}$ under the assumption \mathcal{G} in the CRQS model is an oracle-access machine $\mathcal{B}^{(\cdot, \cdot)}$ such that the following holds. Let*

1. $\Sigma = (\text{P}, \text{V})$ be a Σ -protocol for a language L with commitment length n and challenge length m that has perfect completeness and special soundness, and

2. \mathcal{A} be a (possibly inefficient) attacker that breaks the computational soundness of the non-interactive proof system $\Pi_{\text{FS}}^{n,m}[\Sigma]$ with advantage $1 - \text{negl}(n)$.

The reduction \mathcal{B} has black-box access to P , V and \mathcal{A} , runs in time polynomial in the running times of P , V and \mathcal{A} , and $\mathcal{B}^{\text{P},\text{V},\mathcal{A}}$ has advantage at least $1/\text{poly}(n)$ in game \mathcal{G} .

As mentioned previously, a FS protocol is essentially a WOTRO protocol, albeit satisfying a weaker notion of security. In particular, a WOTRO protocol avoiding *only* the “bad challenge” functions of Σ -protocols would be enough for FS. The impossibility to black-box reduce the security of WOTRO to a cryptographic game, as expressed in Corollary 1, does not apply directly to Fiat-Shamir.

To show black-box impossibility of FS in the CRQS model, we construct a family of Σ -protocols $\{\Sigma^f\}_{f:\{0,1\}^n \rightarrow \{0,1\}^m}$ such that Σ^f has bad challenge function $f(\cdot)$ for any f . The verifier V^f in Σ^f is not necessarily efficient, but we again exploit the simulation paradigm, where the inefficient adversary is replaced by an efficient indistinguishable simulator, to simulate this verifier in a way that is consistent with the adversarial prover. By definition of the reduction $\mathcal{B}^{(\cdot,\cdot)}$, if an adversary \mathcal{A}^f breaks the soundness of $\Pi_{\text{FS}}[\Sigma^f]$, $\mathcal{B}^{\text{P},\text{V}^f,\mathcal{A}^f}$ wins game \mathcal{G} . By replacing $(\text{V}^f, \mathcal{A}^f)$ with a pair of simulators $(\text{Sim}_{\text{V}}, \text{Sim}_{\mathcal{A}})$ such that no $\text{poly}(n)$ -query machine can distinguish between the two pairs, we obtain an efficient algorithm $\mathcal{B}^{\text{P},\text{Sim}_{\text{V}},\text{Sim}_{\mathcal{A}}}$ breaking the security of \mathcal{G} . We formalize this joint simulation below and then prove the black-box impossibility result using the strategy outlined above and pictured in Fig. 6.2.

Definition 12 (Joint Simulatability). A family of (possibly inefficient) algorithms $\{(\mathcal{A}^f, \text{V}^f)\}_f$ that have access to the same (possibly inefficient) resource $f : \{0,1\}^n \rightarrow \{0,1\}^m$ are jointly simulatable if there exist two QPT stateful algorithms Sim_1 and Sim_2 that share a common state and such that for any $\text{poly}(n)$ -query oracle access machine $M^{(\cdot,\cdot)}$,

$$\left| \Pr_f[M^{(\mathcal{A}^f, \text{V}^f)} = 1] - \Pr[M^{(\text{Sim}_1, \text{Sim}_2)} = 1] \right| \leq \text{negl}(n) .$$

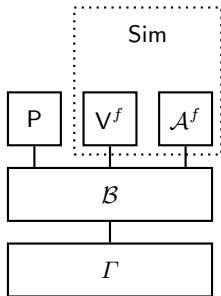


Fig. 2. Visualization of the proof of Thm 7. The black-box reduction $\mathcal{B}^{(\cdot,\cdot)}$ wins the game $\mathcal{G} = (\Gamma, c)$ if (P, V^f) forms a Σ -protocol Σ^f and \mathcal{A}^f breaks the soundness of $\Pi_{\text{FS}}[\Sigma^f]$. Since $\text{Sim} = (\text{Sim}_{\text{V}}, \text{Sim}_{\mathcal{A}})$ jointly simulates V^f and \mathcal{A}^f , neither \mathcal{B} nor Γ can distinguish if Sim or $(\text{V}^f, \mathcal{A}^f)$ is being used. Since Sim is efficient, this means $\mathcal{B}^{(\text{P}, \text{Sim}_{\text{V}}, \text{Sim}_{\mathcal{A}})}$ is an efficient machine that wins game \mathcal{G} .

Theorem 7. Let $\mathcal{G} = (\Gamma, c)$ be a cryptographic game assumption, let n, m be such that $n - m \in \omega(\lg n)$ and let $\Pi_{\text{FS}}^{n,m}$ be a Fiat-Shamir instantiation in the CRQS model. There does not exist a black-box reduction $\mathcal{B}^{(\cdot, \cdot)}$ showing the (n, m) -universality of $\Pi_{\text{FS}}^{n,m}$ from the security of game \mathcal{G} , unless assumption \mathcal{G} is false.

Proof (sketch). The full proof can be found in Appx. A.2. The main idea behind the proof is to construct a family of Σ -protocols that has bad challenge function f for any function $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$, but where the challenger is not necessarily efficient. The Σ -protocol Σ^f defined below is an interactive proof of language membership for the empty language where the verifier has oracle access to the function $f(\cdot)$. On public input x ,

1. P: does nothing.
2. V^f : interact with a potentially malicious prover in the following way.
 - (a) On first message $a \in \{0, 1\}^n$, pick $c \in_R \{0, 1\}^m$ uniformly at random and send c to the prover.
 - (b) On response z from the prover, accept iff $c = f(a)$.

Next, we use the impossibility of WOTRO (Theorem 5) to build an inefficient adversary \mathcal{A}^f against Σ^f . It follows that the reduction $\mathcal{B}^{(\cdot, \cdot)}$ of Def 11 wins game \mathcal{G} when given oracle access to P, V^f and \mathcal{A}^f . The rest of the proof consists of showing that V^f and \mathcal{A}^f can be jointly simulated (Def 12) by an efficient simulator that does not know $f(\cdot)$. The existence of the simulator is mostly a consequence of Theorem 6. Finally, replacing oracle access to V^f and \mathcal{A}^f with the corresponding interfaces of the simulator in $\mathcal{B}^{(\cdot, \cdot)}$ yields an efficient algorithm against game \mathcal{G} . An overview of the proof is presented in Fig. 6.2. ■

7 A Quantum Assumption Allowing for WOTRO ^{n, m}

In [4], Barak, Lindell, and Vadhan introduce a computational assumption allowing for Σ -universal Fiat-Shamir in the CRS model. It assumes the existence of a family of *entropy preserving* hash functions. In [24], Dodis, Ristenpart, and Vadhan showed that a family of entropy preserving hash functions is necessary for a Σ -universal implementation of Fiat-Shamir in the CRS model. Of course, it follows from [4, 6] that this assumption cannot be black-box reduced to any cryptographic game. In this section, we define a different computational assumption allowing for WOTRO ^{n, m} in the CRQ\$ model (and therefore allowing for Σ -universal Fiat-Shamir). Our assumption is a quantum assumption on hash functions called a *collision-shelter*. We first show in Sect. 7.1 how to construct WOTRO ^{n, n} with unconditional security in the CRQ\$ model. In Sect. 7.2, we define the collision-shelter assumption and we show how to use it to convert WOTRO ^{n, n} into a computationally secure WOTRO ^{n, m} as long as $m \in \Omega(n)$. We conclude in Sect. 7.3 by a short discussion about some relations and distinctions between collision-shelters and collision resistant families of hash functions.

7.1 Unconditionally Secure WOTRO^{n,n} in the CRQ\$ Model

Let us get back to the implementation of WOTRO^{n,n} roughly described in the introduction. The result stated in Thm 8 requires the set of MUB to be the one introduced by Wootters and Fields in [46]. These bases are for the tensor product of n Hilbert spaces, each of odd prime dimension p . Let $\Gamma = \{0, \dots, p-1\}$ denote the elements of the finite field \mathbb{F}_p for $p \geq 3$ prime. We refer to the Wootters and Fields MUB for Γ^n as $\Theta_{\text{WF}}^{p,n} = \{\theta_a\}_{a \in \Gamma^n}$ where $\theta_a = \{|x_a\rangle\}_{x \in \Gamma^n}$ is an orthonormal basis for Γ^n that, by virtue of mutual unbiasedness, satisfies $|\langle x_a | x_{a'} \rangle| = p^{-\frac{n}{2}}$ when $a \neq a'$. The formal definition of $\Theta_{\text{WF}}^{p,n}$ can be seen in Appx. C. The CRQ\$ we use to implement WOTRO^{n,n} is composed of $3n$ p -dimensional EPR pairs, each denoted by $|\text{EPR}_\Gamma\rangle_{PV} := \frac{1}{\sqrt{p}} \sum_{j \in \Gamma} |jj\rangle_{PV}$. The CRQ\$ is then set to $|\text{EPR}_\Gamma^{3n}\rangle_{PV} := |\text{EPR}_\Gamma\rangle^{\otimes 3n}$. Henceforth, we denote by WOTRO^{n,n} _{Γ} the primitive WOTRO^{n,n} where both the input and the output are in Γ^n .

Before giving our protocol $\Pi_{\text{WRO}}^{n,n} = (P', V')$ for WOTRO^{n,n} _{Γ} , we first consider a simpler (but insecure) version of it where the CRQ\$ is $|\text{EPR}_\Gamma^n\rangle_{PV}$ rather than $|\text{EPR}_\Gamma^{3n}\rangle_{PV}$. Upon input $a \in \Gamma^n$, the simpler scheme asks the prover to measure register P of the CRQ\$ in basis $\theta_a \in \Theta_{\text{WF}}^{p,n}$ to obtain outcome $c \in \Gamma^n$. The prover then announces (a, c) to the verifier who verifies that when measuring register V of the CRQ\$ the outcome c is obtained. If the test is performed with success then the output of the primitive is set to c . This simple protocol cannot be proven secure as it stands. Instead, $\Pi_{\text{WRO}}^{n,n}$ asks P' to measure 3 batches of EPR pairs $|\text{EPR}_\Gamma^n\rangle_{PV}$ in the same basis θ_a to get outcomes $x_1, x_2, x_3 \in \Gamma^n$. The challenge produced by the primitive is then $c = x_3(x_1 + x_2)^{-1}$ (where the operations are done in \mathbb{F}_{p^n}). This choice for determining c follows from our proof technique. P' announces (a, x_1, x_2, x_3) that is checked by V' after measuring register V for each of the three instances of $|\text{EPR}_\Gamma^n\rangle_{PV}$ in basis θ_a . If the test is successful then the output of the primitive is set to c .

Protocol $\Pi_{\text{WRO}}^{n,n}$ for WOTRO^{n,n} _{Γ}

Setup: A CRQ\$ $|\text{EPR}_\Gamma^{3n}\rangle_{PV}$.

Prover: On input $a \in \Gamma^n$,

1. Measures its part of $|\text{EPR}_\Gamma^{3n}\rangle$ in basis $\theta_a^{\otimes 3}$, let $x = (x_1, x_2, x_3) \in \Gamma^{3n}$ be the result.
2. If $x_1 + x_2 = 0$, set $c = 0$. Otherwise, output $c := x_3(x_1 + x_2)^{-1}$ and sends (a, x) to verifier.

Verifier: Upon reception of (a, x) ,

1. Measure its part of $|\text{EPR}_\Gamma^{3n}\rangle$ in basis $\theta_a^{\otimes 3}$, let $x' = (x'_1, x'_2, x'_3) \in \Gamma^{3n}$ be the result.
 2. Output **reject** if $x \neq x'$ and output (a, c') where $c' = x'_3(x'_1 + x'_2)^{-1}$ otherwise.
-

Next theorem establishes that $\Pi_{\text{WRO}}^{n,n}$ is $\frac{1}{4}$ -secure against all adversaries. The proof is given in Appx. C and may be of independent interest. It consists in showing that the best measurement to distinguish the state transmitted by a quantum source that selects a basis $a \in_R \Gamma^n$ at random and sends $|x(a)_a\rangle$ for any set $\{(a, x(a))\}_{a \in \Gamma^n}$ cannot be recognized with probability better than $\frac{3}{4}$. Wootters and Fields' MUBs are useful here as this probability is given by a Weil sum that can be bounded by Deligne's resolution of Weil third conjecture¹² [21].

Theorem 8. *Let $\Gamma = \{0, \dots, p-1\}$ be the set of elements in finite field \mathbb{F}_p for $p \geq 3$ a prime number. Protocol $\Pi_{\text{WRO}}^{n,n}$, presented above, is a statistically correct and statistically $(\frac{1}{4} - \text{negl}(n))$ -secure implementation of $\text{WOTRO}_\Gamma^{n,n}$.*

7.2 Collision-Shelters

We are now ready to define a quantum computational assumption that allows for a secure implementation of $\text{WOTRO}^{n,m}$ for $m < n$. A collision-shelter for security parameter n , is a family $\mathbb{G}^{n,m} = \{G_s^n : \Gamma^n \times \Gamma^n \rightarrow \Gamma^m\}_{s \in \{0,1\}^{\ell(n)}}$ of hash functions that exhibits a strong quantum flavour of collision resistance. Intuitively, $\mathbb{G}^{n,m}$ is a *collision-shelter* if, for any function $c : \Gamma^n \rightarrow \Gamma^m$, no QPT adversary can produce a state *close* to

$$|\psi_s\rangle = \sum_a \alpha_a |a\rangle_A \otimes \sum_{x:G_s^n(a,x)=c(a)} \beta_x^a |x\rangle_X \otimes |\varphi(a,x)\rangle_W, \quad (4)$$

for $s \in_R \{0,1\}^{\ell(n)}$ and in average over outcome a when register A is measured in the computational basis, $\sum_{x:G_s^n(a,x)=c(a)} \beta_x^a |x\rangle_X \otimes |\varphi(a,x)\rangle_W$ contains collisions in superposition. Notice that no such state can be produced efficiently when the number of possible a is in $O(\lg n)$ and G_s^n is collision resistant, as the generation of 2 such states would provide a collision for G_s^n with good probability.

Definition 13. *Let $c : \Gamma^n \rightarrow \Gamma^m$ be arbitrary and let $G_s^n \in \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^m$. Let*

$$|\psi\rangle = \sum_a \alpha_a |a\rangle_A \otimes \sum_{x:G_s^n(a,x)=c(a)} \beta_x^a |x\rangle_X \otimes |\varphi(a,x)\rangle_W$$

be a state. Let $x^(a)$ be such that $|\beta_{x^*(a)}^a| = \max_x \{|\beta_x^a|\}$ for every $a \in \{0,1\}^n$ and let $|\tilde{\psi}^*\rangle = \sum_a \alpha_a |a\rangle_A \otimes \beta_{x^*(a)}^a |x^*(a)\rangle_X \otimes |\varphi(a, x^*(a))\rangle_W$ be the corresponding sub-normalized state obtained from $|\psi\rangle$. If $|\langle \psi | \tilde{\psi}^* \rangle|^2 < 1 - \delta$ then $|\psi\rangle$ is said to be δ -colliding to $c(\cdot)$ under G_s^n .*

A collision-shelter is a family of hash functions that prevents any QPT adversary from generating a state *close* to be δ -colliding to some target function $c(\cdot)$ under G_s^n when $s \in_R \{0,1\}^n$. By *close* here, we simply mean a state ρ_{AXW} such that $\langle \psi | \rho | \psi \rangle \geq \frac{1}{\text{poly}(n)}$ for some state $|\psi\rangle$ that is δ -colliding to $c(\cdot)$ under G_s^n .

¹² Weil's third conjecture is analogue to the Riemann hypothesis over finite fields and is called as such.

Definition 14. The hash function $\mathbb{G}^{n,m}$ is a collision-shelter if, for all $\delta > 0$, all functions $\mathbf{c} : \Gamma^n \rightarrow \Gamma^m$, and all QPT adversaries $\mathcal{A} = \{\mathcal{A}_n\}$ where \mathcal{A}_n has input $s \in_R \{0,1\}^{\ell(n)}$ and produces a state ρ_{AXW} , there exists no δ -colliding state $|\psi\rangle$ to $\mathbf{c}(\cdot)$ under $G_s^n \in \mathbb{G}^{n,m}$ such that $\langle \psi | \rho | \psi \rangle \geq \frac{1}{\text{poly}(n)}$. The collision-shelter assumption simply posits the existence of a collision-shelter $\mathbb{G}^{n,m}$ for $m \leq (1 - \alpha)n$ with $0 < \alpha < 1$.

We now consider the obvious implementation of $\text{WOTRO}_\Gamma^{n,m}$ using $\Pi_{\text{WRO}}^{n,n}$ using a function-shelter $\mathbb{G}^{n,m}$ that simply sets the challenge $\hat{c} \in \Gamma^m$ as $\hat{c} = G_s^n(a, c)$ where $c \in \Gamma^n$ is the challenge produced in $\Pi_{\text{WRO}}^{n,n}$ and s is a CR\$. Let us denote this implementation of $\text{WOTRO}_\Gamma^{n,m}$ by $\Pi_{\text{WRO}}^{n,m}[\mathbb{G}^{n,m}] = (\mathbf{P}'', \mathbf{V}'')$. The following theorem is an easy consequence of Def 14 and Thm 8.

Theorem 9. Assuming that $\mathbb{G}^{n,m} = \{G_s^n\}_s$ is a collision-shelter, $\Pi_{\text{WRO}}^{n,m}[\mathbb{G}^{n,m}]$ is a $\text{negl}(n)$ -correct and computationally $(\frac{1}{4} - o(1))$ -secure implementation of $\text{WOTRO}_\Gamma^{n,m}$.

Proof (sketch). Let us view the adversary \mathcal{A}_n against $\Pi_{\text{WRO}}^{n,m}[\mathbb{G}^{n,m}]$ as a QPT algorithm (an isometry) applied to the P -register of the CRQ\$ followed by measurements in the computational basis of registers A and X . Let $|\psi_{\mathcal{A}}\rangle$ be the state generated before the final measurements:

$$|\psi_{\mathcal{A}}\rangle_{AXX'V} = \sqrt{1 - |\epsilon|^2} |\psi_s\rangle_{AXX'V} + \epsilon |\Xi\rangle_{AXX'V} ,$$

where

$$|\psi_s\rangle_{AXX'V} = \sum_a \alpha_a |a\rangle_A \otimes \sum_{c: G_s^n(a,c)=\mathbf{c}(a)} \beta_c^a |c\rangle_X \otimes \sum_{x: x_3(x_1+x_2)^{-1}=c} \gamma_x^{a,c} |x\rangle_{X'} \otimes |(x)_a\rangle_V ,$$

and $|\Xi\rangle_{AXX'V}$ always results in either a rejection by \mathbf{V}'' or an acceptance upon challenge $G_s^n(a, c) \neq \mathbf{c}(a)$. Above, $|(x)_a\rangle$ denotes $|(x_1)_a\rangle \otimes |(x_2)_a\rangle \otimes |(x_3)_a\rangle \in \theta_a^{\otimes 3}$. Since $\mathbb{G}^{n,m}$ is a collision-shelter, Def 14 ensures that

$$|\psi_s\rangle_{AXX'V} = |\tilde{\psi}^*\rangle_{AXX'V} + |\mathfrak{R}\rangle_{AXX'V} ,$$

with $|\langle \mathfrak{R} | \mathfrak{R} \rangle|^2 \in o(1)$ and $\langle \mathfrak{R} | \tilde{\psi}^* \rangle = 0$. A state $|\psi\rangle$ of that form can easily be shown to provide a probability of success no better than an additional $o(1)$ to the probability of success when using $|\psi_{\mathcal{A}}\rangle_{AXX'V}$ in $\Pi_{\text{WRO}}^{n,n}$ to hit function $\mathbf{x}^*(\cdot)$. Thm 8 then guarantees a probability of success smaller than $\frac{3}{4} + \text{negl}(n)$. Putting things together, the result follows easily. \blacksquare

7.3 Is the Collision-Shelter Assumption Realistic?

While $h : \Gamma^{\ell(n)} \times \Gamma^n \rightarrow \Gamma^m$ is *entropy-preserving* if no efficient adversary can, given the first argument $\mathbf{s} \in_R \Gamma^{\ell(n)}$ picked uniformly at random, find $x \in \Gamma^n$ such that $h(\mathbf{s}, x)$ has almost no entropy (when \mathbf{s} has been forgotten), collision-shelters prevent efficient quantum adversaries from preparing a state with entropy in the second argument when the output of the hash function applied to

both arguments is fixed to a function of its first argument. Why would it be possible for collision-shelters to exist?

Suppose that for all $a \in \Gamma^n$, the hash function $G_s^n(a, \cdot)$ is collision-resistant against quantum adversaries. Let $c(a) \in \Gamma^m$ be arbitrary. It follows that for any $a \in \Gamma^n$, no efficient quantum adversary can produce a state of the form $|\psi_a\rangle = \sum_{x:G_s^n(a,x)=c(a)} \beta_x^\alpha |x\rangle_X \otimes |\varphi(a, x)\rangle$ where $|\langle \psi_a | \tilde{\psi}_a^* \rangle|^2 < 1 - \delta$ since two states of that form would allow to find a collision with non-negligible probability. This, of course, does not imply that $\{G_s^n(\cdot, \cdot)\}_s$ is a collision-shelter as $G_s^n(a, \cdot) = G_s^n(a', \cdot) =: h_s(\cdot)$ for all $a, a' \in \Gamma^n$ is such that $G_s^n(a, \cdot)$ is collision-resistant when $h_s(\cdot)$ is collision-resistant but the following easy-to-generate state is $o(1)$ -colliding to $c(a) = a_1 \dots a_m$ when $m < n$:

$$\begin{aligned} p^{-n/2} \sum_x |x\rangle_X \otimes |h_s(x)\rangle &\rightarrow p^{-n+\frac{m}{2}} \sum_a \sum_{x:h_s(x)=a_1 \dots a_m} |x\rangle_X \otimes |a\rangle_A \\ &= p^{-n/2} \sum_a |a\rangle_A \otimes p^{-\frac{n+m}{2}} \sum_{x:G_s^n(a,x)=c(a)} |x\rangle_X . \end{aligned}$$

Such an attack seems difficult to conduct when $\{G_s^n(a, \cdot)\}_a$ is a set of collision resistant hash functions that *appear independent* of each other as far as collisions are concerned. What it means exactly for hash functions in $\{G_s^n(a, \cdot)\}_a$ to *appear independent* is unclear. However, if $G_s^n(a, \cdot)$ is modelled by a random oracle for each $a \in \Gamma^n$ then it can easily be shown to be a collision-shelter.

Notice that while any secure universal Fiat-Shamir transform in the CRS model requires the existence of an entropy-preserving family of hash functions[24], this does not seem to be the case for collisions-shelters with respect to WOTRO in the CRQS model.

8 Black-Box Impossibility of a Flavour of Quantum Lightning

In this section, we show that a secure WOTRO can be constructed from a quantum lightning scheme that satisfies a slightly stronger security notion.

Quantum lightning was introduced by Zhandry in [47] as a primitive allowing for publicly verifiable quantum money schemes, provable randomness and blockchain-less cryptocurrencies.

Definition 15. A quantum lightning *scheme* is a tuple of algorithms (QLSetup, QLGen, QLVer) where

- QLSetup(1^n) samples a quantum storm $\overset{\heartsuit}{\text{storm}}$. A quantum storm is a classical description of a procedure for producing and verifying lightning states.
- QLGen($\overset{\heartsuit}{\text{storm}}$) generates a new quantum bolt $|\overset{\heartsuit}{\text{bolt}}\rangle$ in a quantum register L .
- QLVer($\overset{\heartsuit}{\text{storm}}, |\overset{\heartsuit}{\text{bolt}}\rangle$) returns the serial number s or \perp if the state is invalid, and a leftover quantum register Q .

A quantum lightning scheme is *correct* if 1- the serial number is deterministic given a quantum bolt issued by QLGen, i.e. $\mathbb{E}[H_\infty(S|Q)_\rho] \leq \text{negl}(n)$, and 2- bolt verification does not noticeably affect the state: $\mathbb{E}[\langle \rho_Q | \rho_Q \rangle] \geq 1 - \text{negl}(n)$

A quantum lightning scheme has *uniqueness* if no QPT quantum algorithm \mathcal{A} can produce two lightning states with the same serial number, for any $\rho_{L_0 L_1}^A \leftarrow \mathcal{A}(\overset{\circlearrowleft}{\text{storm}})$ and $\rho_{S_i Q_i} \leftarrow \text{Ver}(\overset{\circlearrowleft}{\text{storm}}, \rho_{L_i}^A)$, $\Pr[S_0 = S_1 \neq \perp] \leq \text{negl}(n)$. Uniqueness implies that bolt serial numbers are unpredictable. Let $\overset{\circlearrowleft}{\text{storm}} \leftarrow \text{QLSetup}(1^n)$, $|\rho\rangle \leftarrow \mathcal{A}(\overset{\circlearrowleft}{\text{storm}})$ and $\rho_{SQ} \leftarrow \text{Ver}(\overset{\circlearrowleft}{\text{storm}}, |\rho\rangle)$, then for any polynomial $p(\cdot)$, w

$$\Pr[H_\infty(S | S \neq \perp) \leq \log p(n)] \leq \text{negl}(n) .$$

8.1 Typed Quantum Lightning.

Quantum lightning provides some fresh randomness that even an adversarial procedure cannot bias towards a certain value. We present a strengthened version of this property that requires that this randomness remains in the presence of an input to the lightning generation procedure. This notion is sufficiently strong to provide a secure WOTRO protocol.

Definition 16. A typed quantum lightning scheme is a tuple of QPT algorithms $(\text{tQLSetup}, \text{tQLGen}, \text{tQLVer})$ where

- $\text{tQLSetup}(1^n)$ produces a storm $\overset{\circlearrowleft}{\text{storm}}$.
- $\text{tQLGen}(\overset{\circlearrowleft}{\text{storm}}, a)$ takes an additional parameter $a \in \{0, 1\}^n$, and produces a lightning state $|\rho_a\rangle$.
- $\text{tQLVer}(\overset{\circlearrowleft}{\text{storm}}, |\rho_a\rangle)$ returns the type a , a serial number s or \perp if the state is not valid, and a leftover quantum register.

Correctness is defined similarly to regular QL: serial numbers are deterministic for honestly generated bolts and verification does not noticeably affect the bolt. The security properties of a tQL scheme are as follows: For any QPT adversary \mathcal{A} producing $|\rho\rangle \leftarrow \mathcal{A}(\overset{\circlearrowleft}{\text{storm}})$, if we let $\rho_{QSA} = \text{tQLVer}(\overset{\circlearrowleft}{\text{storm}}, |\rho\rangle)$, then

$$\Pr[H_\infty(S | A \wedge (S \neq \perp)) \leq \log p(n)] \leq \text{negl}(n) .$$

Based on Def 16, typed quantum lightning provides randomness in the serial number conditioned on the type. A natural WOTRO protocol in the CRS+CRQ\$ model based on this new primitive is presented below.

Protocol $\Pi_{\text{WRO}}^{\text{tQL}}$ for WOTRO n,m

Setup: A CRS containing $\overset{\circlearrowleft}{\text{storm}} \leftarrow \text{tQLSetup}(1^n)$ for a tQL scheme with n -bit types and m -bit serial numbers. A CRQ\$ containing $|\text{EPR}\rangle^{\otimes q}$ where q is the qubit size of a tQL state.

1. On input $a \in \{0, 1\}^n$, P calls $|\rho_a\rangle \leftarrow \text{tQLGen}(\overset{\circlearrowleft}{\text{storm}}, a)$, sets $\rho_{QSA} = \text{tQLVer}(\overset{\circlearrowleft}{\text{storm}}, |\rho_a\rangle)$, teleports register Q to V using the EPR pairs and sends (A, S) to V.

2. Upon reception of (a, s, ρ_Q) , V calls $\sigma_{Q'S'A'} \leftarrow \text{tQLVer}(\overset{\curvearrowright}{\text{tQL}}, \rho_Q)$ and tests that $A' = a$ and $S' = s$. V aborts if the tests failed, otherwise V sets $c = s$ and outputs (a, c) .

Theorem 10. *The above protocol is a secure instantiation of $\text{WOTRO}^{n,m}$.*

The proof is a direct consequence of the security of the tQL primitive.

Corollary 2. *There is no black-box reduction from the security of a tQL scheme with type length n and serial length m satisfying $n - m \in \omega(\lg n)$ to the security of a cryptographic game assumption, unless the assumption is false.*

9 Conclusion and Open Problems

We have shown that in the CRQS model, the security of any implementation of $\text{WOTRO}^{n,m}$ cannot be established by a black-box reduction to any cryptographic game as soon as $n - m \in \omega(\lg n)$. We proved that this implies the impossibility to conclude, in the same model, the soundness of any $\Sigma_{n,m}$ -universal implementation of the Fiat-Shamir transform when $n - m \in \omega(\lg n)$. We defined the collision-shelter assumption that allows for $\Sigma_{n,m}$ -universal implementation of the Fiat-Shamir transform in the CRQ\$ model. The collision-shelter assumption is a quantum assumption of the same flavour than the assumption that entropy-preserving hash functions exist in the CRS model. However, while entropy-preserving hash functions have been shown to be required for a $\Sigma_{n,m}$ -universal implementation of the Fiat-Shamir transform in the CRS model, we do not expect this to be true for the collision-shelter assumption in the CRQS model. It would be interesting to define a quantum assumption that allows for WOTRO and that has to be true given any secure implementation of WOTRO . Finally, we show that a variant of quantum lightning cannot have its security black-box reduced to cryptographic games. This maybe an indication of why we are still looking to find a satisfactory assumption under which vanilla quantum lightning is possible[40].

Protocol $\Pi_{\text{WRO}}^{n,m}[\mathbb{G}^{n,m}]$ implementing $\text{WOTRO}_F^{n,m}$ from a collision-shelter $\mathbb{G}^{n,m}$ does not seem to require more than a single hash function $G^n(\cdot, \cdot)$ for its security to hold, even though a single function cannot be a collision-shelter. We could define a single function as a collision-shelter if for any target function $c(\cdot)$, no efficient adversary can produce two copies of the same δ -colliding state to $c(\cdot)$ under G^n . We don't know whether such definition for a single function collision-shelter makes $\Pi_{\text{WRO}}^{n,m}[\{G^n\}]$ a secure implementation of $\text{WOTRO}_F^{n,m}$.

It would be of interest to define some flavours of the collision-shelter assumption using extensions of the cryptographic games defined in Def 1. In [32] and [12], secure universal Fiat-Shamir transforms are constructed from essentially standard primitives with security captured by cryptographic games with (sub)exponential security rather than polynomial. As a result, some known constructions of these primitives can be assumed to satisfy the security criteria expressed by those extended games. A similar approach to implement $\text{WOTRO}^{n,m}$

in the CRQS model under simpler quantum assumptions would be a valuable contribution.

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A Additional Proofs

A.1 Proof of Theorem 6

Fix $n \in \mathbb{N}$, for improved readability we will omit specifying n below unless when necessary. Let $\mathcal{A}^f = \{P_{a,x,w}^f\}_{a,x,w} \cup \{P_{\perp}^f\}$ be the POVM operators for the attack \mathcal{A}^f against $\Pi_{\text{WOTRO}}^{n,m}$ with success probability $1 - \text{negl}(n - m)$ from Thm 5. Recall the definition of these POVM operators:

$$P_{a,f(a),w}^f := \frac{N_{f(a),w}^a}{\frac{2^n}{2^m} + C\sqrt{k\frac{2^n}{2^m}}}$$

and $P_{a,x,w}^f = 0$ for $x \neq f(a)$ where $C > 0$ is some constant and $k = \lg \dim P$ is the qubit size of the prover register of the CRQS and where $\mathcal{N}^a = \{N_{x,w}^a\}_{x,w \in \{0,1\}^{m \times \ell}}$ are the measurement operators for the honest prover in protocol $\Pi_{\text{WOTRO}}^{n,m}$ on input $a \in \{0,1\}^n$. Instead of giving $\mathcal{M}^{(i)}$ oracle access to \mathcal{A}^f for a random f , we will give it oracle access to a stateful adversary \mathcal{A} that picks the function that it will hit at random before the first query. The action of this adversary on an arbitrary state of register P , conditioned on the overwhelmingly probable event that the attack succeeds, is captured by the following CPTP map:

- On first query, pick a random function $f : \{0,1\}^n \rightarrow \{0,1\}^m$.
- When queried on a quantum register P , apply the POVM \mathcal{A}^f to register P and return the outcome (a, x, w) .

We construct a simulator Sim for this attack and show that eq. (3) is satisfied by this simulator. The simulator's action on a state σ_P is described as follows:

- On first query, initialize an empty set $S = \emptyset$
- When queried on a quantum register P , pick $a \in_R \{0,1\}^n \setminus S$ and apply the POVM \mathcal{N}^a of the honest prover to get classical outcome (x, w) . Add a to S and return (a, x, w) .

To show black-box indistinguishability of this simulator with the attacker \mathcal{A} , we consider hybrid simulations $\text{Sim}^{(i)}$ that apply the strategy of Sim for the first i queries and answer the remaining $q - i$ queries using the strategy of \mathcal{A} in a way that is consistent with the first i queries. These hybrid simulators $\text{Sim}^{(i)}(\sigma_P)$ act on a register P prepared by the distinguisher \mathcal{M} as follows:

- On first query, initialize an initially empty set S and a partial mapping $f|_S : \{0,1\}^n \rightarrow \{0,1\}^m$.
- Answer the first i queries as Sim : pick $a \in_R \{0,1\}^n \setminus S$ and apply POVM \mathcal{N}^a to register P . Let (x, w) be the result, add a to S , set $f|_S(a) = x$ and output (a, x, w) .
- Answer the rest of the queries as \mathcal{A} : pick a function f uniformly from the set of functions that extend $f|_S$, apply the POVM \mathcal{A}^f to register P and return the result (a, x, w) .

By construction, we have that $\text{Sim}^{(q)}$ acts exactly as Sim and $\text{Sim}^{(0)}$ exactly as \mathcal{A} . By the triangle inequality, these hybrid simulations allow us to bound eq. (3) for the adversary and simulator described above.

$$\left\| \mathcal{M}^{\mathcal{A}}(|0\rangle\langle 0|_Q) - \mathcal{M}^{\text{Sim}}(|0\rangle\langle 0|_Q) \right\|_1 \quad (5)$$

$$\leq \sum_{i=0}^{q-1} \left\| \mathcal{M}^{\text{Sim}^{(i+1)}}(|0\rangle\langle 0|) - \mathcal{M}^{\text{Sim}^{(i)}}(|0\rangle\langle 0|) \right\|_1 \quad (6)$$

Since \mathcal{A} , Sim and every hybrid $\text{Sim}^{(i)}$ return classical variables, the above quantity can be bounded by looking only at the statistical distance $\Delta(\cdot, \cdot)$ between the values returned on each query. For $i, j \in [q]$, let $(A_j^{(i)}, X_j^{(i)}, W_j^{(i)})$

denote the random variables returned by the j th query to $\text{Sim}^{(i)}$. Note that the answer to the j th query depends on the answers to queries $1, \dots, j$ in a way that also depends on the distinguisher \mathcal{M} . Then the i th term of the above sum can be bounded above by

$$\begin{aligned} & \left\| \mathcal{M}^{\text{Sim}^{(i+1)}}(|0\rangle\langle 0|) - \mathcal{M}^{\text{Sim}^{(i)}}(|0\rangle\langle 0|) \right\|_1 \\ & \leq \Delta \left(\{(A_j^{(i)}, X_j^{(i)}, W_j^{(i)})\}_{j \in [q]}, \{(A_j^{(i+1)}, X_j^{(i+1)}, W_j^{(i+1)})\}_{j \in [q]} \right). \end{aligned}$$

Since $\text{Sim}^{(i)}$ and $\text{Sim}^{(i+1)}$ act identically on queries $j = 1, \dots, i$, we have that

$$\Delta \left((A_j^{(i)}, X_j^{(i)}, W_j^{(i)}), (A_j^{(i+1)}, X_j^{(i+1)}, W_j^{(i+1)}) \right) = 0 \text{ for } j = 1, \dots, i.$$

Therefore we can fix the values of the first i queries and bound the statistical distance of the remaining $q - i$ query answers conditioned on those i values.

Let $S \subset \{0, 1\}^n$ and $f|_S$ be the set and partial mapping determined by the answers to the first i queries of $\text{Sim}^{(i)}$. Then on the $i + 1$ th query, the behaviour of $\text{Sim}^{(i)}$ and $\text{Sim}^{(i+1)}$ differs:

- $\text{Sim}^{(i)}$ picks f uniformly at random from the set of functions that extend $f|_S$ and answers query $i + 1$ and all remaining queries by applying POVM \mathcal{A}^f on register P .
- $\text{Sim}^{(i+1)}$ picks $a \in \{0, 1\}^n \setminus S$, applies POVM \mathcal{N}^a to get (x, w) and answers the $i + 1$ th query with (a, x, w) . All remaining queries are answered using \mathcal{A}^f for f picked uniformly from the set of functions that extend $f|_S$ restricted to $f(a) = x$.

Since $\text{Sim}^{(i)}$ and $\text{Sim}^{(i+1)}$ answer queries $i + 2, \dots, q$ in the same way, but with different distributions for the function f , how these answers differ depends only on the choice of f and on the answer to query $i + 1$. Let $F^{(i)}$ and $F^{(i+1)}$ denote the random variables for the choice of f on the hybrids (i) and $(i + 1)$, respectively.

$$\begin{aligned} & \Delta \left(\{(A_j^{(i)}, X_j^{(i)}, W_j^{(i)})\}_{j > i+1}, \{(A_j^{(i+1)}, X_j^{(i+1)}, W_j^{(i+1)})\}_{j > i+1} \right) \\ & \leq \Delta \left((F^{(i)}, A_{i+1}^{(i)}, X_{i+1}^{(i)}, W_{i+1}^{(i)}), (F^{(i+1)}, A_{i+1}^{(i+1)}, X_{i+1}^{(i+1)}, W_{i+1}^{(i+1)}) \right) \end{aligned}$$

We now turn to calculating the distance between $(F^{(i)}, A_{i+1}^{(i)}, X_{i+1}^{(i)}, W_{i+1}^{(i)})$ and $(F^{(i+1)}, A_{i+1}^{(i+1)}, X_{i+1}^{(i+1)}, W_{i+1}^{(i+1)})$. Let σ_P be the state of the register P on query $i + 1$. Then

$$\Pr[(F^{(i)}, A_{i+1}^{(i)}, X_{i+1}^{(i)}, W_{i+1}^{(i)}) = (f, a, f(a), w)] = \left(\frac{1}{2^m} \right)^{2^n - i} \cdot \text{tr} \left(\frac{N_{f(a), w}^a}{\frac{2^n}{2^m} + C \sqrt{k \frac{2^n}{2^m}}} \cdot \sigma \right)$$

and the probability is zero for any $x \neq f(a)$ and any f that doesn't extend $f|_S$. There is also the possibility of an outcome \perp capturing the event that the attack

\mathcal{A}^f fails (corresponding to the POVM element P_{\perp}^f). Since the probability of this outcome is negligible (Thm 5), we will condition on the outcome being different than \perp on every query. For $\text{Sim}^{(i+1)}$, these variables are distributed as

$$\Pr[(F^{(i+1)}, A_{i+1}^{(i+1)}, X_{i+1}^{(i+1)}, W_{i+1}^{(i+1)}) = (f, a, x, w)] = \left(\frac{1}{2^m}\right)^{2^n-i-1} \cdot \text{tr}\left(\frac{N_{x,w}^a}{2^n-i} \cdot \sigma\right)$$

where $a \notin S$, f extends $f|_S$ and $f(a) = x$. The probability is zero elsewhere. Note that in both cases, $x = f(a)$, but the distribution of f differs on how its value x at point a is chosen.

Let $\text{ext}(S)$ be the set of functions that extends $f|_S$. The statistical distance between both distributions is

$$\begin{aligned} & \sum_{\substack{a \in \{0,1\}^n \\ w \in \{0,1\}^\ell \\ x \in \{0,1\}^m}} \sum_{f \in \text{ext}(S)} \left| \Pr[(F^{(i)}, A_{i+1}^{(i)}, X_{i+1}^{(i)}, W_{i+1}^{(i)}) = (f, a, x, w)] \right. \\ & \quad \left. - \Pr[(F^{(i+1)}, A_{i+1}^{(i+1)}, X_{i+1}^{(i+1)}, W_{i+1}^{(i+1)}) = (f, a, x, w)] \right| \\ = & \sum_{\substack{a \in \{0,1\}^n \setminus S \\ w \in \{0,1\}^\ell \\ x \in \{0,1\}^m \\ f \in \text{ext}(S) \wedge f(a)=x}} \left| \left(\frac{1}{2^m}\right)^{2^n-i} \cdot \text{tr}\left(\frac{N_{f(a),w}^a}{\frac{2^n}{2^m} + C\sqrt{k}\frac{2^n}{2^m}} \cdot \sigma\right) - \left(\frac{1}{2^m}\right)^{2^n-i-1} \cdot \text{tr}\left(\frac{N_{f(a),w}^a}{2^n-i} \cdot \sigma\right) \right| \\ & + \sum_{\substack{a \in \{0,1\}^n \setminus S \\ w \in \{0,1\}^\ell \\ x \in \{0,1\}^m \\ f \in \text{ext}(S) \wedge f(a)=x}} \left| \left(\frac{1}{2^m}\right)^{2^n-i} \cdot \text{tr}\left(\frac{N_{f(a),w}^a}{\frac{2^n}{2^m} + C\sqrt{k}\frac{2^n}{2^m}} \cdot \sigma\right) \right| \end{aligned} \quad (7)$$

Above, we use the fact that for both distributions, it holds that $f(a) = x$ for any (f, a, x, w) with non-zero probability, and that only the variables corresponding to the hybrid (i) have non-zero probability when $a \in S$. By replacing $f(a)$ with x , the terms are now independent of f , so we evaluate the sum over functions f that extend a partial function defined on the $i+1$ points $S \cup \{a\}$. There are $(2^m)^{2^n-i-1}$ such functions.

$$\begin{aligned} (7) = & (2^m)^{2^n-i-1} \sum_{\substack{a \in \{0,1\}^n \setminus S \\ w \in \{0,1\}^\ell \\ x \in \{0,1\}^m}} \left| \left(\frac{1}{2^m}\right)^{2^n-i} \cdot \text{tr}\left(\frac{N_{x,w}^a}{\frac{2^n}{2^m} + C\sqrt{k}\frac{2^n}{2^m}} \cdot \sigma\right) - \left(\frac{1}{2^m}\right)^{2^n-i-1} \cdot \text{tr}\left(\frac{N_{x,w}^a}{2^n-i} \cdot \sigma\right) \right| \\ & + (2^m)^{2^n-i-1} \sum_{\substack{a \in S \\ w \in \{0,1\}^\ell \\ x \in \{0,1\}^m}} \left| \left(\frac{1}{2^m}\right)^{2^n-i} \cdot \text{tr}\left(\frac{N_{x,w}^a}{\frac{2^n}{2^m} + C\sqrt{k}\frac{2^n}{2^m}} \cdot \sigma\right) \right| \end{aligned} \quad (8)$$

Distributing the factors and using the fact that $\sum_{x,w} \text{tr}(N_{x,w}^a \cdot \sigma) = 1$ for any a since $\{N_{x,w}^a\}_{x,w}$ forms a POVM, we obtain

$$\begin{aligned}
(8) &= \sum_{\substack{a \in \{0,1\}^n \setminus S \\ w \in \{0,1\}^\ell \\ x \in \{0,1\}^m}} \left| \left(\frac{1}{2^m} \right) \cdot \text{tr} \left(\frac{N_{x,w}^a}{\frac{2^n}{2^m} + C\sqrt{k\frac{2^n}{2^m}}} \cdot \sigma \right) - \text{tr} \left(\frac{N_{x,w}^a}{2^n - i} \cdot \sigma \right) \right| \\
&\quad + \sum_{\substack{a \in S \\ w \in \{0,1\}^\ell \\ x \in \{0,1\}^m}} \left| \left(\frac{1}{2^m} \right) \cdot \text{tr} \left(\frac{N_{x,w}^a}{\frac{2^n}{2^m} + C\sqrt{k\frac{2^n}{2^m}}} \cdot \sigma \right) \right| \\
&= \sum_{a \in \{0,1\}^n \setminus S} \left| \left(\frac{1}{2^m} \right) \cdot \frac{1}{\frac{2^n}{2^m} + C\sqrt{k\frac{2^n}{2^m}}} - \frac{1}{2^n - i} \right| \sum_{x,w} \text{tr}(N_{x,w}^a \cdot \sigma) \\
&\quad + \sum_{a \in S} \left| \left(\frac{1}{2^m} \right) \cdot \frac{1}{\frac{2^n}{2^m} + C\sqrt{k\frac{2^n}{2^m}}} \right| \sum_{x,w} \text{tr}(N_{x,w}^a \cdot \sigma) \\
&\leq \left| \left(\frac{2^n}{2^m} \right) \cdot \frac{1}{\frac{2^n}{2^m} + C\sqrt{k\frac{2^n}{2^m}}} - \frac{2^n}{2^n - i} \right| + \left(\frac{i}{2^m} \right) \cdot \frac{1}{\frac{2^n}{2^m} + C\sqrt{k\frac{2^n}{2^m}}} \quad (9)
\end{aligned}$$

By simplifying the above expression and using the bounds $1/(1-x) < 1+2x$ when $0 < x < 0.5$ and $1-1/(1+x) < x$ when $x > 0$, we obtain the following upper-bound on the statistical distance.

$$\begin{aligned}
(9) &= \left| \frac{1}{1 + C\sqrt{k\frac{2^n}{2^m}}} - \frac{1}{1 - \frac{i}{2^n}} \right| + \frac{i}{2^n + C\sqrt{k2^{n+m}}} \\
&\leq \frac{1}{1 - \frac{i}{2^n}} - \frac{1}{1 + C\sqrt{k\frac{2^n}{2^m}}} + i2^{-n} \\
&\leq C\sqrt{k2^{m-n}} + 3i2^{-n}
\end{aligned}$$

We complete the hybrid argument by bounding above every i by q and taking the sum over the q hybrids:

$$\left\| \mathcal{M}^{\mathcal{A}}(|0\rangle\langle 0|_Q) - \mathcal{M}^{\text{Sim}}(|0\rangle\langle 0|_Q) \right\|_1 \leq q \cdot \left(C\sqrt{k2^{m-n}} + 3q2^{-n} \right) .$$

which is negligible in $n - m$. ■

A.2 Proof of Theorem 7

Assume there exists a black-box reduction $\mathcal{B}^{(\cdot, \cdot)}$ showing the (n, m) -universality of $\Pi_{\text{FS}}^{n,m}$ from the security of game \mathcal{G} . We will show that game \mathcal{G} is insecure.

We begin by constructing a family of Σ -protocols that has bad challenge function f for any function $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$. The Σ -protocol Σ^f defined below is an interactive proof of language membership for the empty language.

On public input x ,

1. P : does nothing.
2. V^f : interact with a potentially malicious prover in the following way.
 - (a) On first message $a \in \{0, 1\}^n$, pick $c \in_R \{0, 1\}^m$ uniformly at random and send c to the prover.
 - (b) On response z from the prover, accept iff $c = f(a)$.

This is indeed a Σ -protocol as it satisfies perfectly correctness and special soundness.

Next, we build a dishonest prover that breaks the soundness of the QFS transform $\Pi_{\mathsf{FS}}^{n,m}[\Sigma^f]$ of this Σ -protocol. Since Π_{FS} naturally implies a WOTRO protocol, by Thm 6 there exists a $\text{negl}(n-m)$ -simulatable attack $\{\mathcal{A}_{\text{WOTRO}}^f\}_f$ such that $\mathcal{A}_{\text{WOTRO}}^f$ produces $(a, f(a), v)$ that V_{FS} accepts with probability $1 - \text{negl}(n-m)$. Let $\text{Sim}_{\text{WOTRO}}$ be the simulator for $\{\mathcal{A}_{\text{WOTRO}}^f\}_f$. For a function $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$, define the adversarial prover \mathcal{P}^f that attacks protocol $\Pi_{\mathsf{FS}}^{n,m}[\Sigma^f]$ as follows:

1. Invoke $\mathcal{A}_{\text{WOTRO}}^f$ on register P of the CRQS to obtain (a, c, v) .
2. Send a, c, v and $z = \perp$ to the verifier.
3. Recall that the verifier for $\Pi_{\mathsf{FS}}^{n,m}[\Sigma^f]$ runs V_{FS} of Π_{FS} with message (a, c, v) on register V of the CRQS and then runs V^f of Σ^f on input (a, c, v, z) .

The probability that the verifier accepts in protocol $\Pi_{\mathsf{FS}}^{n,m}[\Sigma^f]$ is equal to the probability that V_{FS} accepts and that $c = f(a)$, which by construction of $\mathcal{A}_{\text{WOTRO}}^f$ happens with probability at least $1 - \text{negl}(n-m)$.

Plugging P , V^f and \mathcal{P}^f into the reduction $\mathcal{B}^{(\cdot, \cdot, \cdot)}$ gives an algorithm $\mathcal{B}^{\mathsf{P}, \mathsf{V}^f, \mathcal{P}^f}$ that breaks the security of game \mathcal{G} , and yet that is not efficient. Using the simulator $\text{Sim}_{\text{WOTRO}}$ for the adversary $\mathcal{A}_{\text{WOTRO}}^f$ allows us to replace the inefficient malicious prover \mathcal{P}^f against the QFS transform with an indistinguishable efficient simulator, but V^f is still not efficiently computable.

We now show how \mathcal{P}^f and V^f can be jointly simulated (Def 12) using the stateless simulator $\text{Sim}_{\text{WOTRO}}$ for $\{\mathcal{A}_{\text{WOTRO}}^f\}_f$. The two stateful algorithms $\text{Sim}_{\mathcal{P}}$ and Sim_{V} are defined as follows

1. **Common State:** a partial function $f_A : \{0, 1\}^n \rightarrow \{0, 1\}^m$ defined on an initially empty set $A = \emptyset$.
2. $\text{Sim}_{\mathcal{P}}$: when invoked on a quantum register P , call the simulator $\text{Sim}_{\text{WOTRO}}$ for the family of adversaries $\{\mathcal{A}_{\text{WOTRO}}^f\}_f$. Let $(a, c, v) \leftarrow \text{Sim}_{\text{WOTRO}}$, set $A \leftarrow A \cup \{a\}$ and $f_A(a) = c$, and return (a, c, v, \perp) . If $\text{Sim}_{\text{WOTRO}}$ produces an a that is already in A , the simulation fails.
3. Sim_{V} : when invoked on classical message (a, c, v, z) and quantum register V , run V_{FS} on register V of the CRQS with input (a, c, v) . If $a \notin A$, pick $x \in_R \{0, 1\}^m$ uniformly at random, set $A \leftarrow A \cup \{a\}$ and $f_A(a) = x$. Output **reject** if V_{FS} rejects or if $c \neq f_A(a)$, otherwise output **accept**.

Claim. The pair of stateful (with common state) algorithms $(\text{Sim}_{\mathcal{P}}, \text{Sim}_{\mathcal{V}})$ jointly simulates $\{(\mathcal{P}^f, \mathcal{V}^f)\}_f$.

Proof. Let $M^{(\cdot, \cdot)}$ be an oracle-access machine and let $q = \text{poly}(n)$ be an upper-bound on the number of queries made by M to either of its oracles. We first bound the probability that the simulation fails and then condition on the simulation succeeding. Let α denote the random variable of the value a produced by $\text{Sim}_{\text{WOTRO}}$. Since α is uniformly distributed (by the definition of $\text{Sim}_{\text{WOTRO}}$ in the proof of Thm 6), on any given query, the probability that $\text{Sim}_{\text{WOTRO}}$ produces a that is already in the set A is upper-bounded by

$$\Pr[\alpha \in A] = \sum_{a \in A} \Pr[\alpha = a] \leq q \cdot 2^{-n} .$$

A union bound over the q queries allows us to upper-bound the probability that any of the queries returns an a that was already in A by $q^2 \cdot 2^{-n}$ which is $\text{negl}(n)$.

Conditioned on the event that $\text{Sim}_{\text{WOTRO}}$ never produces $a \in A$, we show that black-box query access to $(\text{Sim}_{\mathcal{P}}, \text{Sim}_{\mathcal{V}})$ is indistinguishable on average over f from black-box query access to $(\mathcal{P}^f, \mathcal{V}^f)$. First, observe that $\text{Sim}_{\mathcal{P}}$ behaves exactly as \mathcal{P}^f , except that it invokes $\text{Sim}_{\text{WOTRO}}$ instead of $\mathcal{A}_{\text{WOTRO}}^f$. Therefore the BB-indistinguishability of $\text{Sim}_{\mathcal{P}}$ and \mathcal{P}^f follows from that of $\text{Sim}_{\text{WOTRO}}$ and $\mathcal{A}_{\text{WOTRO}}^f$. Second, we note that $\text{Sim}_{\mathcal{V}}$ picks each new point of the partial function f_A uniformly at random, so that f_A is identically distributed to a random function f restricted to A . As a result, $\text{Sim}_{\mathcal{V}}$ perfectly simulates \mathcal{V}^f . Finally, since we condition on the event $\alpha \notin A$ at every call of $\text{Sim}_{\text{WOTRO}}$, the answers of $\text{Sim}_{\mathcal{P}}$ and $\text{Sim}_{\mathcal{V}}$ are always consistent with the same function f (i.e. the simulation doesn't fail).

Therefore, the probability that $M^{(\cdot, \cdot)}$ distinguishes $(\mathcal{P}^f, \mathcal{V}^f)$ from $(\text{Sim}_{\mathcal{P}}, \text{Sim}_{\mathcal{V}})$ is at most the probability that $\text{Sim}_{\text{WOTRO}}$ and $\mathcal{A}_{\text{WOTRO}}^f$ can be distinguished plus the probability that the simulation fails, which sum to at most $\text{negl}(n - m)$. ■

We are now ready to conclude the proof. Given the reduction $\mathcal{B}^{(\cdot, \cdot)}$ we construct an efficient algorithm for winning game \mathcal{G} as follows. The machine $\mathcal{B}^{(\mathcal{P}, \text{Sim}_{\mathcal{V}}, \text{Sim}_{\mathcal{P}})}$ either:

1. wins game \mathcal{G} , or
2. if it does not, allows to distinguish $(\text{Sim}_{\mathcal{V}}, \text{Sim}_{\mathcal{P}})$ from $\{(\mathcal{V}^f, \mathcal{P}^f)\}_f$.

Since we have established the black-box indistinguishability of $(\text{Sim}_{\mathcal{V}}, \text{Sim}_{\mathcal{P}})$ and $\{(\mathcal{V}^f, \mathcal{P}^f)\}_f$, we conclude that a BB-reduction $\mathcal{B}^{(\cdot, \cdot)}$ from the QFS-universality of $\Pi_{\text{WOTRO}}^{n, m}$ to game \mathcal{G} would allow to win the game.

B Justification for the tQL assumption.

Why is typed quantum lightning a realistic assumption? It turns out that the tQL primitive can be built from “vanilla” QL for types of length $O(\lg n)$. We present a construction of a tQL scheme for $\lg p(n)$ bits types for any polynomial $p(\cdot)$ from an arbitrary (regular) QL scheme.

Prerequisite: A QL scheme (QLSetup, QLGen, QLVer). A family of $n \cdot p(n)$ -wise independent hash functions $\mathcal{H} \subset \{0, 1\}^n \rightarrow \{0, 1\}^{\lg p(n)}$.

- tQLSetup(1^n): Let $\mathcal{H}' \leftarrow \text{QLSetup}(1^n)$ and $h \leftarrow_s \mathcal{H}$, output $\mathcal{H}' = (\mathcal{H}', h)$
 - tQLGen(\mathcal{H}' , a): Parse \mathcal{H}' as (\mathcal{H}, h) . Do $|s\rangle \leftarrow \text{QLGen}(\mathcal{H})$ until $s = \text{QLVer}(\mathcal{H}, |s\rangle)$ satisfies $h(s) = a$ and output $|s\rangle$.
 - tQLVer(\mathcal{H}' , $|s\rangle$): Parse \mathcal{H}' as (\mathcal{H}, h) . Compute $\rho_{SQ} \leftarrow \text{QLVer}(\mathcal{H}, |s\rangle)$ and set $A = h(S)$. Output ρ_{ASQ} .
-

Theorem 11. (tQLSetup, tQLGen, tQLVer) is a tQL scheme of $\lg p(n)$ bits types.

Proof. Correctness follows from that of the underlying QL scheme: a state produced by tQLGen will be recognized as a valid state by tQLVer if QLGen produces valid states.

The expected running time of tQLGen is exponential in $\lg p(n)$ and thus polynomial in n . Since h is sampled from a family of $n \cdot p(n)$ -pairwise independent hash functions, the probability that tQLGen does not produce an output after $n \cdot p(n)$ steps is at most

$$\Pr[h(s_1) \neq a \wedge \dots \wedge h(s_{n \cdot p(n)}) \neq a] = \left(1 - \frac{1}{p(n)}\right)^{n \cdot p(n)} \leq e^{-n}$$

For security (Def 16), let \mathcal{A} be an attacker against the min-entropy of the tQL scheme, i.e. \mathcal{A} produces with inverse polynomial probability a state $|s\rangle$ such that $\rho_{ASQ} \leftarrow \text{tQLVer}(\rho)$ has logarithmic min-entropy in S conditioned on A :

$$\Pr \left[H_\infty(S \mid A \wedge (S \neq \perp)) \leq \lg n^r \right] \geq \frac{1}{n^k} \quad (10)$$

for some $r, k > 0$. We construct an adversary \mathcal{B} against the uniqueness of the original lightning scheme from this \mathcal{A} . The strategy of \mathcal{B} is as follows: call $\mathcal{A}(1^n)$ twice to obtain $|s_1\rangle$ and $|s_2\rangle$, if $\text{QLVer}(|s_1\rangle) = \text{QLVer}(|s_2\rangle)$ halt and output $|s_1\rangle$ and $|s_2\rangle$, otherwise repeat. We now show that this strategy will produce a collision for the underlying QL scheme with an expected polynomial number of calls to \mathcal{A} .

Let \bar{a} be such that $H_\infty(S \mid (A = \bar{a}) \wedge (S \neq \perp)) \leq \lg n^r$ and such that $\Pr[A = \bar{a}] \geq \frac{1}{q(n)}$ for some polynomial $q(\cdot)$ when ρ_{ASQ} is obtained from $\text{tQLVer}(\mathcal{A}(\mathcal{H}'))$. Note that since a is $\lg p(n)$ in length, such an \bar{a} must exist for (10) to hold (otherwise all a that have low conditional min-entropy have negligible probability of being produced by \mathcal{A}). Then for each pair of invocations of \mathcal{A} , the following

holds with probability at least $\frac{1}{n^k}$:

$$\begin{aligned}
& \Pr[\text{QLVer}(|\psi_1\rangle) = \text{QLVer}(|\psi_2\rangle)] \\
& \geq \frac{1}{q(n)^2} \Pr[\text{QLVer}(|\psi_1\rangle) = \text{QLVer}(|\psi_2\rangle) \mid A_1 = \bar{a} \wedge A_2 = \bar{a}] \\
& \geq \frac{1}{q(n)^2} 2^{-H_2(\text{QLVer}(\rho) \mid A = \bar{a})} \\
& \geq \frac{1}{q(n)^2} 2^{-H_\infty(\text{QLVer}(\rho) \mid A = \bar{a})} \\
& \geq \frac{1}{q(n)^2} \frac{1}{n^r} .
\end{aligned}$$

The probability that \mathcal{B} halts and succeeds is therefore at least $(q(n)^2 n^{r \cdot k})^{-1}$. ■

C Proof of Theorem 8

We use a set of mutually unbiased bases (MUBs) introduced by Wootters and Fields in [46]. These bases of dimension p^n are for n instances of p -level quantum mechanical systems with $p \geq 3$ prime. The construction is as follows:

Definition 17 (Mutually Unbiased Bases of [46]). *Let $p \geq 3$ be prime. Define the set of mutually unbiased bases $\Theta[\mathbb{F}_{p^n}] = \{\theta_a\}_{a \in \mathbb{F}_{p^n}}$ for a Hilbert space of dimension p^n where $\theta_a = \{|u\rangle_a\}_{u \in \mathbb{F}_{p^n}}$ is composed of vectors $|u\rangle_a$ expressed in the computational basis as*

$$|u\rangle_a = p^{-\frac{n}{2}} \sum_{x \in \mathbb{F}_{p^n}} \exp\left(\frac{2\pi i}{p} \cdot \text{tr}(ax^2 + ux)\right) |x\rangle , \quad (11)$$

where $\text{tr} : \mathbb{F}_{p^n} \rightarrow \mathbb{F}_p$ denotes the field trace $\text{tr}(x) := x + x^p + x^{p^2} + \dots + x^{p^{n-1}}$.

Notice that Klappenecker and Rötteler in [33] have shown a very similar construction for the case $p = 2$ (mutually unbiased bases of qubits). Unfortunately, our results do not apply to this construction as Weil sums need a field of odd characteristics.

Theorem 12. *Let $n \geq 1$ and let $\Gamma = \mathbb{F}_p$ for prime $p \geq 3$. The protocol $\Pi^{n,n}$ presented in Section 7 with the family of MUBs of Def 17 is a $\text{negl}(n)$ -correct and statistically $(\frac{1}{4} - \text{negl}(n))$ -secure implementation of $\text{WOTRO}^{n,n}$.*

Proof. For correctness, observe that if both parties are honest, their measurement triplets X and X' will be uniformly distributed and perfectly correlated unless $X_1 + X_2 = 0$. Since $X_1 + X_2$ is a random element of Γ^n due to it being the result of the measurement of EPR pairs, it holds that this event occurs with probability at most $|\Gamma|^{-n}$, which is negligible in n .

Now onto security. Let $c : \Gamma^n \rightarrow \Gamma^n$ be an arbitrary target function. In order to cheat, i.e. to bias the output challenge towards $c(a)$, a dishonest prover must produce a basis selected by a (the commitment) and measurement outcome x_1, x_2, x_3 such that

1. $x_3(x_1 + x_2)^{-1} = c(a)$ and
2. V obtains the same outcomes x_1, x_2, x_3 when he measures his part of $|\text{EPR}_\Gamma^{3n}\rangle$ in basis $\theta_a^{\otimes 3}$.

We say that x is a *bad* outcome if $x_3(x_1 + x_2)^{-1} = c(a)$. Let $\mathcal{B}(a) \subseteq \Gamma^{3n}$ denote the set of bad outcomes for commitment a . Note that $|\mathcal{B}(a)| = p^{2n}$ for any $a \in \Gamma^n$.

The most general strategy for the prover is to apply a POVM $\{M_{a,x}\}_{a \in \Gamma^n, x \in \Gamma^{3n}}$ to its part of the EPR pairs to determine its message to V. The probability that P can bias the output towards $c(a)$ when V accepts is then the probability that it can produce a commitment (i.e. a basis) such that a bad outcome will be observed by V in that basis.

$$P_w = \Pr[X \in \mathcal{B}(A)] \quad (12)$$

$$= \sum_{a \in \Gamma^n, x \in \mathcal{B}(a)} \text{tr}(|x\rangle\langle x|_a \otimes M_{a,x}) \cdot |\text{EPR}_\Gamma^{3n}\rangle\langle \text{EPR}_\Gamma^{3n}| \quad (13)$$

$$= \frac{1}{p^{3n}} \sum_{a \in \Gamma^n, x \in \mathcal{B}(a)} \text{tr}(M_{a,x}|x\rangle\langle x|_a) \quad (14)$$

To simplify our computations, we have slightly abused notation by writing $|x\rangle_a := |x_1\rangle_a \otimes |x_2\rangle_a \otimes |x_3\rangle_a$ when $x \in \Gamma^{3n}$ and $x_1, x_2, x_3 \in \Gamma^n$. Using this notation, for $x, y \in \Gamma^{3n}$ we have $|\langle x|_a|y\rangle_b|^2 = p^{-3n}$ whenever $a \neq b$.

The optimal cheating strategy for P can be framed as the solution to the following semidefinite program (SDP):

$$\begin{aligned} \max_{\{M_{a,x}\}} \quad & \frac{1}{p^{3n}} \sum_{a \in \Gamma^n} \sum_{x \in \mathcal{B}(a)} \text{tr}(M_{a,x}|x\rangle\langle x|_a) \\ \text{s.t.} \quad & \sum_{a \in \Gamma^n} \sum_{x \in \mathcal{B}(a)} M_{a,x} \leq \mathbb{1} \quad . \end{aligned} \quad (15)$$

The dual of this SDP is:

$$\begin{aligned} \min_{Z \geq 0} \quad & \frac{1}{p^{3n}} \text{tr}(Z) \\ \text{s.t.} \quad & \forall a \in \Gamma^n, x \in \mathcal{B}(a) \quad |x\rangle\langle x|_a \leq Z \quad . \end{aligned} \quad (16)$$

By the duality of semidefinite programming, a feasible solution to the dual will yield an upper-bound on the optimal solution of the primal. We now show how to construct a feasible solution that has constant value for $p^{-3n} \text{tr}(Z)$.

Let $S = \sum_{a \in \Gamma^n, x \in \mathcal{B}(a)} |x\rangle\langle x|_a$ and define $f_\alpha(x) = \frac{x}{\alpha+x}$ for $\alpha \in \mathbb{R}$. Since f_α is an operator monotone function (meaning that $A \leq B \Rightarrow f_\alpha(A) \leq f_\alpha(B)$ for A, B positive semidefinite), we have that $\frac{1}{\alpha+1}|x\rangle\langle x|_a \leq f_\alpha(S)$ for any $0 < \alpha \leq 1$. The operator $Z = (\alpha+1)f_\alpha(S)$ is thus a feasible solution to the dual with associated value $\frac{\alpha+1}{p^{3n}} \text{tr}(f_\alpha(S))$.

We now proceed to upper-bound this probability. Since f_α is difficult to deal with directly, we will bound it using Taylor's theorem, yielding powers of Z that

will then be easier to compute. To get a good bound, we will use a third degree Taylor bound for f_α centered around $\lambda \in \mathbb{R}$:

$$f_\alpha(x) \leq \frac{\lambda}{\alpha + \lambda} + \frac{\alpha}{(\alpha + \lambda)^2}(x - \lambda) - \frac{\alpha}{(\alpha + \lambda)^3}(x - \lambda)^2 + \frac{\alpha}{(\alpha + \lambda)^4}(x - \lambda)^3 .$$

Using the Taylor approximation defined above,

$$\begin{aligned} \frac{1}{p^{3n}} \operatorname{tr}(Z) &\leq \frac{\alpha + 1}{p^{3n}} \operatorname{tr}(f_\alpha(S)) \\ &\leq \frac{\alpha + 1}{p^{3n}} \left(\frac{\lambda}{\alpha + \lambda} \operatorname{tr}(\mathbb{1}) + \frac{\alpha}{(\alpha + \lambda)^2} \operatorname{tr}(S - \lambda \mathbb{1}) \right. \\ &\quad \left. - \frac{\alpha}{(\alpha + \lambda)^3} \operatorname{tr}((S - \lambda \mathbb{1})^2) + \frac{\alpha}{(\alpha + \lambda)^4} \operatorname{tr}((S - \lambda \mathbb{1})^3) \right) . \end{aligned} \quad (17)$$

We can rewrite the above traces in the powers of $S - \lambda \mathbb{1}$ in the following way.

$$\left. \begin{aligned} \operatorname{tr}(\mathbb{1}) &= p^{3n} , \\ \operatorname{tr}(S - \lambda \mathbb{1}) &= \operatorname{tr}(S) - \lambda p^{3n} , \\ \operatorname{tr}((S - \lambda \mathbb{1})^2) &= \operatorname{tr}(S^2) - 2\lambda \operatorname{tr}(S) + \lambda^2 p^{3n} , \\ \operatorname{tr}((S - \lambda \mathbb{1})^3) &= \operatorname{tr}(S^3) - 3\lambda \operatorname{tr}(S^2) + 3\lambda^2 \operatorname{tr}(S) - \lambda^3 p^{3n} . \end{aligned} \right\} \quad (18)$$

We refer to Lemmas 2, 3 and 4 below for the computation of $\operatorname{tr}(S)$, $\operatorname{tr}(S^2)$ and $\operatorname{tr}(S^3)$ and we use here the values obtained:

$$\operatorname{tr}(S) = p^{3n}, \operatorname{tr}(S^2) = 2 \cdot p^{3n} - p^{2n} \text{ and } \operatorname{tr}(S^3) \leq 4p^{3n} + p^{2n} .$$

Choosing to center the Taylor approximation around $\lambda = 1$ gives the following bounds for (18):

$$\begin{aligned} \operatorname{tr}(\mathbb{1}) &= p^{3n} , \\ \operatorname{tr}(S - \lambda \mathbb{1}) &= 0 , \\ \operatorname{tr}((S - \lambda \mathbb{1})^2) &= 2p^{3n} - p^{2n} - 2p^{3n} + p^{3n} \\ &= p^{3n} - p^{2n} , \text{ and} \\ \operatorname{tr}((S - \lambda \mathbb{1})^3) &\leq 4p^{3n} + p^{2n} - 3(2p^{3n} - p^{2n}) + 3p^{3n} - p^{3n} \\ &= 4p^{2n} . \end{aligned}$$

Substituting these values into (17), we get

$$P_w \leq \frac{1}{p^{3n}} \operatorname{tr}(Z) \leq \frac{\alpha + 1}{p^{3n}} \left(\frac{p^{3n}}{\alpha + 1} - \frac{\alpha(p^{3n} - p^{2n})}{(\alpha + 1)^3} + \frac{\alpha \cdot 4p^{2n}}{(\alpha + 1)^4} \right) .$$

Looking only at the non-negligible terms, we have

$$P_w \leq 1 - \frac{\alpha}{(\alpha + 1)^2} + \operatorname{negl}(n)$$

which is minimized at $\alpha = 1$ with value $P_w \leq \frac{3}{4} + \text{negl}(n)$. Since this probability is the same for all functions $c(\cdot)$, it follows that the protocol $(\frac{1}{4} - \text{negl}(n))$ -avoids all functions. ■

We now proceed to compute the $\text{tr}(S)$, $\text{tr}(S^2)$ and $\text{tr}(S^3)$ values used in the proof of Thm 8.

Lemma 2. $\text{tr}(S) = p^{3n}$.

Proof. Since $|\mathcal{B}(a)| = p^{2n}$,

$$\text{tr}(S) = \sum_{a \in \Gamma^n} \sum_{x \in \mathcal{B}(a)} \text{tr}(|x\rangle\langle x|_a) = p^{3n} .$$
■

Lemma 3. $\text{tr}(S^2) = 2p^{3n} - p^{2n}$.

Proof.

$$\begin{aligned} \text{tr}(S^2) &= \sum_{a, b \in \Gamma^n} \sum_{x \in \mathcal{B}(a), y \in \mathcal{B}(b)} \text{tr}(|x\rangle\langle x|_a |y\rangle\langle y|_b) \\ &= \sum_{a \in \Gamma^n} \left(\sum_{x \in \mathcal{B}(a)} 1 + \sum_{b \neq a} \sum_{x \in \mathcal{B}(a), y \in \mathcal{B}(b)} |\langle x|_a |y\rangle_b|^2 \right) \\ &= \sum_{a \in \Gamma^n} \left(|\mathcal{B}(a)| + p^{-3n} \sum_{b \neq a} |\mathcal{B}(a)| \cdot |\mathcal{B}(b)| \right) \\ &= \sum_{a \in \Gamma^n} \left(p^{2n} + p^{-3n} \sum_{b \neq a} p^{4n} \right) \\ &= \sum_{a \in \Gamma^n} \left(p^{2n} + p^n \sum_{b \neq a} 1 \right) \\ &= p^n (p^{2n} + p^n (p^n - 1)) \\ &= 2p^{3n} - p^{2n} \end{aligned}$$
■

Upper-bounding $\text{tr}(S^3)$ will require a little more machinery. We introduce a theorem of Deligne [21] and some of its corollaries before proceeding with the proof.

Theorem 13 ([21], Theorem 8.4). *Let Q be a polynomial of n variables x_1, \dots, x_n and of degree d on \mathbb{F}_q , let Q_d be the homogeneous part of degree d of Q and let $\psi : \mathbb{F}_q \rightarrow \mathbb{C}^*$ be an additive non-trivial character on \mathbb{F}_q . Assume that*

1. d is coprime with p , the characteristic of \mathbb{F}_q , and
2. the hypersurface H_0 of $\mathbb{P}_{\mathbb{F}_q}^{n-1}$ defined by Q_d is smooth,

then

$$\left| \sum_{x_1, \dots, x_n \in \mathbb{F}_q} \psi(Q(x_1, \dots, x_n)) \right| \leq (d-1)^n q^{n/2} .$$

In the above, the second condition boils down to ensuring that there is no point at which the $\frac{\partial Q}{\partial x_i}$ all vanish simultaneously. Here is a version that is closer to what we will need:

Corollary 3. *Let $m \leq k$, A a $k \times m$ matrix with rank m in \mathbb{F}_q , and let C be a $k \times k$ matrix in \mathbb{F}_q . Then, if $A^\top C A$ is non-singular,*

$$\left| \sum_{\mathbf{v}, \mathbf{x} = A\mathbf{v}} \psi(\mathbf{x}^\top C \mathbf{x}) \right| \leq q^{m/2} .$$

In other words, we take the sum over all (x_1, \dots, x_k) that satisfy a system of $k - m$ independent linear equations.

Proof. Let $Q = \mathbf{x}^\top C \mathbf{x} = \mathbf{v}^\top A^\top C A \mathbf{v}$, and observe that

$$\frac{\partial Q}{\partial v_i} = e_i^\top A^\top C A \mathbf{v} + \mathbf{v}^\top A^\top C A e_i = 2e_i^\top A^\top C A \mathbf{v} .$$

Condition 2 of Thm 13 is thus equivalent to

$$A^\top C A \mathbf{v} = 0 \Leftrightarrow \mathbf{v} = 0 ,$$

which amounts to saying that $A^\top C A$ is non-singular. ■

Here is now a version that is more directly relevant to our case.

Corollary 4. *Let $m \leq k$ and let $B \in \mathbb{F}_q^{(k-m) \times k}$ and $C \in \mathbb{F}_q^{k \times k}$ be full rank matrices. Then,*

$$\left| \sum_{\mathbf{x}: B\mathbf{x} = 0} \psi(\mathbf{x}^\top C \mathbf{x}) \right| \leq q^{m/2} .$$

Proof. Let $B^c \in \mathbb{F}_q^{m \times k}$ such that $M := \begin{bmatrix} B \\ B^c \end{bmatrix} \in \mathbb{F}_q^{k \times k}$ has full rank. Then condition $B\mathbf{x} = 0$ is equivalent to $\mathbf{x} = M^{-1} \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix}$ for some $\mathbf{v} \in \mathbb{F}_q^m$. We can thus define $P := \begin{bmatrix} 0 \\ \mathbb{1} \end{bmatrix}$ and apply corollary 3 with $A = M^{-1}P$, while observing that $P^\top M^{-1 \top} C M^{-1} P$ has full rank, since $M^{-1 \top} C M^{-1}$ also has full rank. ■

Lemma 4. $\text{tr}(S^3) \leq 4p^{3n} + p^{2n}$

Proof. Let's first write out the expression of interest:

$$\begin{aligned} \text{tr}(S^3) &= \sum_{a,b,c \in \Gamma^n} \sum_{x \in \mathcal{B}(a)} \sum_{y \in \mathcal{B}(b)} \sum_{z \in \mathcal{B}(c)} \text{tr}(|x\rangle\langle x|_a |y\rangle\langle y|_b |z\rangle\langle z|_c) \\ &= \sum_{a=b=c} \sum_x 1 + 3 \sum_{a \neq b} \sum_{x,y} |\langle x|_a |y\rangle_b|^2 + \sum_{a \neq b \neq c} \sum_{x,y,z} \langle x|_a |y\rangle_b \langle y|_b |z\rangle_c \langle z|_c |x\rangle_a \end{aligned} \quad (19)$$

where the middle term groups the three cases $a \neq b$, $a \neq c$ and $b \neq c$ that all have the same value. We know how to upper-bound the first two sums using the same techniques as Lemma 3. Most of the proof is dedicated to finding an upper-bound to the third term.

Recall our construction of mutually unbiased bases θ_a presented in Def 17. For $r \in \mathbb{F}_{p^n}$ and $a \in \mathbb{F}_{p^n}$:

$$|r\rangle_a = p^{-\frac{n}{2}} \sum_{u \in \mathbb{F}_{p^n}} \exp\left(\frac{2\pi i}{p} \cdot \text{tr}(au^2 + ru)\right) |u\rangle .$$

Extending this basis to 3 systems through $\theta_a^{\otimes 3}$ yields vectors of the form

$$|x\rangle_a = p^{-3n/2} \sum_{u \in \mathbb{F}_{p^3}^3} \exp\left(\frac{2\pi i}{p} \text{tr}(au^\top u + x^\top u)\right) |u\rangle ,$$

where x^\top denotes the transpose of $x \in \mathbb{F}_{p^3}^3 \simeq \Gamma^{3n}$. Here, we slightly abuse notation by writing $|x\rangle_a$ for a vector in basis $\theta_a^{\otimes 3}$.

The inner product of two such vectors is given by the expression

$$\langle y|_b |x\rangle_a = p^{-3n} \sum_{u \in \mathbb{F}_{p^3}^3} \exp\left(\frac{2\pi i}{p} \text{tr}((a-b)u^\top u + (x-y)^\top u)\right) .$$

Combining the three inner products in the expression of interest (19), we have

$$\langle x|_a |y\rangle_b \langle y|_b |z\rangle_c \langle z|_c |x\rangle_a = p^{-9n} \sum_{u,v,w \in \mathbb{F}_{p^3}^3} \exp\left(\frac{2\pi i}{p} \text{tr}\begin{pmatrix} (a-b)u^\top u + (x-y)^\top u \\ +(b-c)v^\top v + (y-z)^\top v \\ +(c-a)w^\top w + (z-x)^\top w \end{pmatrix}\right)$$

We introduce some notation that will allow us to present the above expression in a more compact, albeit more complicated form. Let $\mathbf{c} : \mathbb{F}_{p^n} \rightarrow \mathbb{F}_{p^m}$ and for $a \in \mathbb{F}_{p^n}$, define

$$B_a = \begin{bmatrix} 1 & 0 & \mathbf{c}(a) \\ 0 & 1 & \mathbf{c}(a) \end{bmatrix} \in \mathbb{F}_{p^n}^{2 \times 3} \quad (20)$$

such that for $x_1, x_2 \in \mathbb{F}_{p^n}$, the expression

$$[x_1, x_2] \cdot B_a^\top = [x_1, x_2, \mathbf{c}(a)(x_1 + x_2)]^\top \in \mathbb{F}_{p^n}^3 \quad (21)$$

is a sequence of measurement outcomes that leads to the bad outcome $\mathbf{c}(a)$ in the protocol.

For $a, b, c \in \mathbb{F}_{p^n}$, write

$$B_{a,b,c} := \begin{bmatrix} -B_a & 0 & B_a \\ 0 & B_b & -B_b \\ B_c & -B_c & 0 \end{bmatrix} \in \mathbb{F}_{p^n}^{6 \times 9} ,$$

and

$$C_{a,b,c} := \begin{bmatrix} (c-a)\mathbb{1}_{\mathbb{F}_{p^n}^{3 \times 3}} & 0 & 0 \\ 0 & (b-c)\mathbb{1}_{\mathbb{F}_{p^n}^{3 \times 3}} & 0 \\ 0 & 0 & (a-b)\mathbb{1}_{\mathbb{F}_{p^n}^{3 \times 3}} \end{bmatrix} \in \mathbb{F}_{p^n}^{9 \times 9} . \quad (22)$$

The previous operators are defined such that

$$\begin{aligned} & \sum_{\substack{x \in \mathcal{B}(a) \\ y \in \mathcal{B}(b) \\ z \in \mathcal{B}(c)}} \langle x|_a \rangle_y \langle y|_b \rangle_z \langle z|_c \rangle_x \\ &= p^{-9n} \sum_{\varrho \in \mathbb{F}_{p^n}^6} \sum_{\xi \in \mathbb{F}_{p^n}^9} \exp\left(\frac{2\pi i}{p} \operatorname{tr}(\xi^\top C_{a,b,c} \xi + \varrho^\top B_{a,b,c} \xi)\right) \end{aligned}$$

with the goal of bounding above the right-hand side using Corollary 4. The construction of $B_{a,b,c}$ appears more complex than necessary because we want it to have a large rank.

Equipped with the above, we are now ready to upper-bound the third term in (19) with Corollary 4.

$$\begin{aligned} & \sum_{a \neq b \neq c} \sum_{\substack{x \in \mathcal{B}(a) \\ y \in \mathcal{B}(b) \\ z \in \mathcal{B}(c)}} \langle x|_a \rangle_y \langle y|_b \rangle_z \langle z|_c \rangle_x \\ &= p^{-9n} \sum_{a \neq b \neq c} \sum_{\varrho \in \mathbb{F}_{p^n}^6} \sum_{\xi \in \mathbb{F}_{p^n}^9} \exp\left(\frac{2\pi i}{p} \operatorname{tr}(\xi^\top \cdot C_{a,b,c} \cdot \xi + \varrho^\top \cdot B_{a,b,c} \cdot \xi)\right) \\ &= p^{-9n} \sum_{a \neq b \neq c} \sum_{\varrho \in \mathbb{F}_{p^n}^6} \sum_{\substack{\xi \in \mathbb{F}_{p^n}^9 \\ B_{a,b,c} \cdot \xi = 0}} \exp\left(\frac{2\pi i}{p} \operatorname{tr}(\xi^\top \cdot C_{a,b,c} \cdot \xi)\right) \end{aligned} \quad (23)$$

$$\begin{aligned} & \leq p^{-9n} \sum_{a \neq b \neq c} \sum_{\varrho \in \mathbb{F}_{p^n}^6} p^{2n} \\ &= p^{-9n} \sum_{a \neq b \neq c} p^{6n} p^{2n} \\ &= p^{-n} (p^n)(p^n - 1)(p^n - 2) . \end{aligned} \quad (24)$$

Equality (23) above follows from the observation that once ξ is fixed, if $B_{a,b,c} \cdot \xi$ is non-zero then the sum over ϱ will span all p th roots of unity in equal proportions

which sums to 0. In more details, letting $\alpha = \xi^\top \cdot C_{a,b,c} \cdot \xi \in \mathbb{F}_{p^n}$ and $0 \neq v = B_{a,b,c} \cdot \xi \in \mathbb{F}_{p^n}^6$,

$$\begin{aligned} \sum_{\varrho \in \mathbb{F}_{p^n}^6} \exp\left(\frac{2\pi i}{p} \operatorname{tr}(\alpha + \varrho^\top \cdot v)\right) &= p^{5n} \sum_{\beta \in \mathbb{F}_{p^n}} \exp\left(\frac{2\pi i}{p} \operatorname{tr}(\alpha + \beta)\right) \\ &= p^{6n-1} \sum_{\gamma \in \mathbb{F}_p} \exp\left(\frac{2\pi i}{p} \gamma\right) = 0 . \end{aligned}$$

Inequality (24) follows from Corollary 4 by observing that $\operatorname{rank}(B_{a,b,c}) \geq 4$. To see this, note that by removing columns 3, 6 and 9 from $B_{a,b,c}$ (those corresponding to $c(a), c(b)$ or $c(c)$), we are left with the matrix

$$\begin{bmatrix} \mathbb{1} & 0 & -\mathbb{1} \\ 0 & \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & -\mathbb{1} & 0 \end{bmatrix} .$$

Taking linear combinations of the above we can obtain

$$\begin{bmatrix} \mathbb{1} & 0 & -\mathbb{1} \\ 0 & \mathbb{1} & -\mathbb{1} \\ 0 & 0 & 0 \end{bmatrix}$$

and hence $B_{a,b,c}$ has rank at least that of the above matrix, which is equal to 4 since each of the identities act on $\mathbb{F}_{p^n}^2$.

We can now complete the proof by taking the expected value over g . Continuing from (19),

$$\begin{aligned} \operatorname{tr}(S^3) &= \sum_{a=b=c} \sum_{x \in \mathcal{B}(a)} 1 + 3 \sum_{a \neq b} \sum_{\substack{x \in \mathcal{B}(a) \\ y \in \mathcal{B}(b)}} |\langle x|_a|y\rangle_b|^2 + \sum_{a \neq b \neq c} \sum_{\substack{x \in \mathcal{B}(a) \\ y \in \mathcal{B}(b) \\ z \in \mathcal{B}(c)}} \langle x|_a|y\rangle_b \langle y|_b|z\rangle_c \langle z|_c|x\rangle_a \\ &\leq p^{3n} + 3p^{2n}(p^n - 1) + (p^n - 1)(p^n - 2) \leq 4p^{3n} + p^{2n} . \end{aligned}$$

■

D Properties of WOTRO

Proposition 1. *The 2-message protocol in which P sends $a \in \Gamma^n$ directly to V , and V then chooses $c \in_R \Gamma^m$ at random, sends it to P and always accepts is a correct and δ -secure implementation of $\text{WOTRO}_\Gamma^{n,m}$ for $\delta = 1 - \frac{1}{|\Gamma|^m}$ and for any alphabet Γ and $n, m \geq 1$.*

Proof. Let $\Pi(c|a)$ denote the conditional distribution of the protocol output. Indeed, correctness is obvious as a and c are correctly distributed with $\Pi(c|a) =$

$\frac{1}{\#\Gamma^m}$. For security, let A be the random variable produced by \tilde{P} and C be the random variable produced by V , and let $c : \Gamma^n \rightarrow \Gamma^m$ be some function. Then,

$$\begin{aligned} \Pr [V = 1 \wedge C = c(A)] &= \Pr [C = c(A)] \\ &= \frac{1}{|\Gamma|^m} . \end{aligned}$$

■

Proposition 2. *Let Γ be an arbitrary finite alphabet, let $m, n \geq 1$ and let $0 < \delta \leq 1$. There is no correct and δ -secure 1-message implementation of $WOTRO_{\Gamma}^{n,m}$ in the bare model. Moreover, there is no such δ -secure non-interactive implementation of $WOTRO_{\Gamma}^{n,m}$ common random string (resp. random oracle) model if the function c from Def 4 can depend on the CR\$ r (resp. the random oracle \mathcal{O}).*

Proof. Consider the message sent from the prover P to the verifier V . Without loss of generality, it is of the form (a, c, w) where a is P 's input, c is the joint output and w is additional information for V to decide whether to accept or reject. Let $\Pi = (P, V)$ (resp. $\Pi^r = (P^r, V^r)$ and $\Pi^{\mathcal{O}} = (P^{\mathcal{O}}, V^{\mathcal{O}})$) be a correct implementation of $WOTRO$ in the bare model (resp. CR\$ model and ROM). Define the first message of the prover in each model by

$$\begin{aligned} P(a, s) &:= (a, c(a, s), w(a, s)) && \text{(bare)} \\ P^r(a, s) &:= (a, c^r(a, s, r), w^r(a, s, r)) && \text{(CR\$)} \\ P^{\mathcal{O}}(a, s) &:= (a, c^{\mathcal{O}}(a, s, v), w^{\mathcal{O}}(a, s, v)) && \text{(ROM)} \end{aligned}$$

where s is the random tape of the prover, r is the value of the CR\$ and $v = (\mathcal{O}(a_1), \mathcal{O}(a_2), \dots, \mathcal{O}(a_{\kappa(n)}))$ where $a_1, \dots, a_{\kappa(n)} \in \Gamma^n$ are chosen using s for some upper bound $\kappa(n)$ on the number of oracle queries performed by P in $\Pi^{\mathcal{O}}$.

Since the protocol is correct, it must hold that

$$\Pr[V(P(A, S)) = 1] = \Pr[V^r(P^r(A, S)) = 1] = \Pr[V^{\mathcal{O}}(P^{\mathcal{O}}(A, S)) = 1] = 1 \quad (25)$$

where the probability is taken over the values of A and S . Then for each a with non zero probability, there exist a value $s(a)$, $s^r(a)$ and $s^{\mathcal{O}}(a)$ such that

$$V(P(a, s(a))) = V^r(P^r(a, s^r(a))) = V^{\mathcal{O}}(P^{\mathcal{O}}(a, s^{\mathcal{O}}(a))) = 1 \quad (26)$$

Define malicious prover \tilde{P} (resp. \tilde{P}^r and $\tilde{P}^{\mathcal{O}}$) that on input a uses random tape value $s(a)$ (resp. $s^r(a)$ and $s^{\mathcal{O}}(a)$). Then the protocol Π (resp. Π^r and $\Pi^{\mathcal{O}}$) does not avoid the functions $c(a) := c(a, s(a))$ (resp. $c^r(a) := c^r(a, s^r(a), r)$ and $c^{\mathcal{O}}(a) := c^{\mathcal{O}}(a, s^{\mathcal{O}}(a), v)$). ■

Proposition 3. *Let $m > n$. The protocol for $WOTRO_{\Gamma}^{n,m}$ in the CR\$ model where both parties output the CR\$ $r \in \Gamma^m$ for any $a \in \Gamma^n$ and V always accepts is correct and δ -secure, for $\delta = 1 - |\Gamma|^{n-m}$.*

Proof. Correctness is obvious, and security is easy to prove as well: suppose that $\tilde{\text{P}}$ wants to steer the output of the protocol towards some function \mathbf{c} . He must then look at the CR\$ r , and announce an a such that $\mathbf{c}(a) = r$. Hence, r must happen to be in the image of \mathbf{c} . However, since \mathbf{c} is a function from Γ^n to Γ^m and $m > n$, there are at most $|\Gamma|^n$ strings in the image of \mathbf{c} , and the probability that a uniformly chosen r falls into that set is at most $|\Gamma|^{n-m}$. ■

Proposition 4. *Let Γ be an arbitrary finite alphabet of size $q \geq 2$. Then, for any m, n with $m \leq n$, there exists no $\exp(-q^{n-m})$ -secure implementation of $\text{WOTRO}_\Gamma^{n,m}$ in the ROM.*

Proof. We will show that a cheating prover that is unbounded in time can search for an a that will satisfy \mathbf{V} . Consider a dishonest prover $\tilde{\text{P}}$ who uses the following strategy: run the honest prover P on all possible inputs a in lexicographic order, and declare victory if it ever outputs $\mathbf{c}(a)$. We will also assume that function $\mathbf{c}(\cdot)$ is chosen uniformly at random, and show that the expected winning probability of the cheating prover is at least $1 - \exp(-q^{n-m})$. We have the following:

$$\begin{aligned}
\Pr_{\mathcal{O}, \mathbf{c}}[\tilde{\text{P}} \text{ loses}] &= \Pr_{\mathcal{O}, \mathbf{c}}[\tilde{\text{P}} \text{ loses at step } a = 0 \wedge \tilde{\text{P}} \text{ loses at } a = 1 \wedge \dots] \\
&= \prod_{a \in \Gamma^n} \Pr_{\mathcal{O}, \mathbf{c}}[\tilde{\text{P}} \text{ loses at step } a \mid \tilde{\text{P}} \text{ loses at all steps before } a] \\
&= \prod_{a \in \Gamma^n} \Pr_{\mathcal{O}, \mathbf{c}}[\text{P does not output } \mathbf{c}(a) \text{ on input } a \mid \tilde{\text{P}} \text{ loses at all steps before } a] \\
&= \prod_{a \in \Gamma^n} \frac{q^m - 1}{q^m} \\
&= \left(1 - \frac{1}{q^m}\right)^{q^n} \\
&= \left[\left(1 - \frac{1}{q^m}\right)^{q^m}\right]^{q^{n-m}} \\
&< \exp(-q^{n-m}).
\end{aligned}$$

since $\mathbf{c}(a)$ is chosen uniformly at random for each a . Hence, $\tilde{\text{P}}$'s winning probability is at least $1 - \exp(-q^{n-m})$ as advertised, and there must exist a choice of function $\mathbf{c}(\cdot)$ that achieves this bound. ■

Proposition 5. *The protocol for $\text{WOTRO}_\Gamma^{n,m}$ in the ROM model where both parties output the $\mathcal{O}(a)$ for any $a \in \Gamma^n$ and \mathbf{V} always accepts is correct and statistically δ -secure, for $\delta = 1 - |\Gamma|^{n-m}$.*

The proof is identical to that of Proposition 3 by considering $r = \mathcal{O}(a)$.

Proposition 6. *The protocol described in Proposition 5 is $1 - \text{negl}(n)$ -secure in the ROM against polynomial-time provers as long as m is at least linear in n .*

Proof. Let $\ell(n)$ be a polynomial which bounds the number of oracle queries that $\tilde{\text{P}}$ can make. Furthermore, without loss of generality we will assume that $\tilde{\text{P}}$ never makes the same oracle call twice. Then, given any function $\mathbf{c} : \Gamma^n \rightarrow \Gamma^m$, in order to cheat successfully, $\tilde{\text{P}}$ must be able to find an a such that $\mathcal{O}(a) = \mathbf{c}(a)$.

Now, let $A_1, \dots, A_{\ell(n)}$ be random variables taking values in Γ^n where A_i represents the i th query to the oracle (if $\tilde{\text{P}}$ makes fewer than $\ell(n)$ queries, let A_i be any string that was not queried so far). These random variables are functions of the oracle \mathcal{O} , in that they can depend on the results of previous queries. We then have by the union bound that

$$\begin{aligned} \Pr_{\mathcal{O}} \left[\tilde{\text{P}} \text{ wins} \right] &\leq \Pr \left[\mathcal{O}(A_1) = \mathbf{c}(A_1) \vee \mathcal{O}(A_2) = \mathbf{c}(A_2) \vee \dots \vee \mathcal{O}(A_{\ell(n)}) = \mathbf{c}(A_{\ell(n)}) \right] \\ &\leq \sum_{i=1}^{\ell(n)} \Pr \left[\mathcal{O}(A_i) = \mathbf{c}(A_i) \right] \\ &= \ell(n)q^{-m} \\ &\leq \text{negl}(n). \end{aligned}$$

■

Proposition 7. *There are one-message implementations of $\text{WOTRO}_{\Gamma}^{n,n}$ arbitrarily close to be $\frac{1}{e}$ -avoiding against unbounded provers in the CR\$ model.*

Proof. Let $\ell(n)$ be the length of the CR\$ (i.e. $r \in_R \Gamma^{\ell(n)}$) upper bounded by some polynomial. Let $\text{P}^r : \Gamma^n \rightarrow \Gamma^m \times \Gamma^*$ denote P 's message to V upon CR\$ r and input $a \in \Gamma^n$. For $a \in \Gamma^n$ and CR\$ $r \in \Gamma^{\ell(n)}$, we have $\text{P}^r(a) = (c(r, a), v(r, a))$ which defines announcement $(a, c(r, a), v(r, a))$ to V . The verifier's algorithm $\text{V}^r : \Gamma^n \times \Gamma^n \times \Gamma^* \rightarrow \{0, 1\}$ upon CR\$ r accepts (α, β, γ) when $\text{V}^r(\alpha, \beta, \gamma) = 1$. The prover's algorithm can be considered deterministic given r , all randomness being provided by r . For $\{1, \dots, p^n\} = \Gamma^n$ an enumeration of all elements in Γ^n , let

$$C^r := c(r, 1) \| c(r, 2) \| c(r, 3) \| \dots \| c(r, p^n)$$

be the sequence of all challenges announced by P upon CR\$ r , one for each possible input $a \in \Gamma^n$. Let $\mathcal{C} := \{C^r\}_{r \in \Gamma^{\ell(n)}}$. For $\omega \in (\Gamma^n)^{p^n}$, we define

$$H_\omega := \{C \in \mathcal{C} \mid (\exists j \in [p^n]) [C_j = \omega_j]\}$$

as the set of sequences containing challenges *hitting* ω somewhere. If Π is δ -avoiding then for all $\omega \in (\Gamma^n)^{p^n}$, $|H_\omega| \leq \delta \cdot p^{\ell(n)}$.

We define Π and then show it is $\frac{3}{4}$ -avoiding using a CR\$ $r \in (\Gamma^n)^2$. Π is simply defined from $r = r_1 \| r_2 \in (\Gamma^n)^2$ as

$$C^r = \underbrace{r_1, r_1, \dots, r_1}_{\frac{p^n}{2} \text{ times}}, \underbrace{r_2, r_2, \dots, r_2}_{\frac{p^n}{2} \text{ times}} \cdot$$

We denote the elements of Γ^n by $\{1, 2, \dots, p^n\}$. Let $\omega^* \in (\Gamma^n)^{p^n}$ be defined as

$$\omega^* := 1, 2, 3, \dots, \frac{p^n}{2}, 1, 2, 3, \dots, \frac{p^n}{2} .$$

It is not difficult to see that ω^* maximizes the probability to be hit by C^R . We have,

$$\begin{aligned} \Pr[C^R \in H_{\omega^*}] &= \Pr \left[\left(R_1 \leq \frac{p^n}{2} \right) \vee \left(R_2 \leq \frac{p^n}{2} \right) \right] \\ &= 1 - \Pr \left[\left(R_1 > \frac{p^n}{2} \right) \wedge \left(R_2 > \frac{p^n}{2} \right) \right] \\ &= 1 - \frac{1}{4} = \frac{3}{4} . \end{aligned}$$

By considering longer CR\$ $r = r_1, r_2, \dots, r_{\ell(n)}$ where $r_i \in \Gamma^n$, it is possible to get arbitrarily close to a correct $\frac{1}{e}$ -avoiding scheme with

$$C^r = \underbrace{r_1, r_1, \dots, r_1}_{\frac{p^n}{\ell(n)} \text{ times}}, \underbrace{r_2, r_2, \dots, r_2}_{\frac{p^n}{\ell(n)} \text{ times}}, \dots, \underbrace{r_{\ell(n)}, r_{\ell(n)}, \dots, r_{\ell(n)}}_{\frac{p^n}{\ell(n)} \text{ times}} .$$

■

D.1 WOTRO to Implement the Fiat-Shamir Heuristic

Let R_L be a relation for a language L and let $\Sigma_L = (\mathsf{P}_L, \mathsf{V}_L)$ be a Σ -protocol for R_L with commitments in Γ^n and challenges in Γ^m . Consider a secure implementation $\Pi_{\text{WOTRO}} = (\mathsf{P}_{\text{WOTRO}}, \mathsf{V}_{\text{WOTRO}})$ of $\text{WOTRO}_{\Gamma}^{n,m}$. We construct a non-interactive zero-knowledge proof (argument) system for L by applying the Fiat-Shamir transform to Σ_L using the protocol Π_{WOTRO} as the instantiation of the hash function.

Protocol $\Pi_{\text{WOTRO}}[\Sigma_L]$

Setup: A Σ -protocol $\Sigma_L = (\mathsf{P}_L, \mathsf{V}_L)$ where $\mathsf{P}_L = (\mathsf{P}_L^1, \mathsf{P}_L^2)$ with commitments of size n and challenges of size m and a protocol $\Pi_{\text{WOTRO}} = (\mathsf{P}_{\text{WOTRO}}, \mathsf{V}_{\text{WOTRO}})$ for $\text{WOTRO}_{\Gamma}^{n,m}$.

Prover message: on public input $x \in L$ and private input w

1. compute $a \leftarrow \mathsf{P}_L^1(x, w)$,
2. compute $c \leftarrow \mathsf{P}_{\text{WOTRO}}(a)$,
3. compute $z = \mathsf{P}_L^2(a, x, w, c)$ and
4. send z to the verifier.

Verification: on public input $x \in L$ and upon reception of z ,

1. compute $(a, c) \leftarrow \mathsf{V}_{\text{WOTRO}}()$
 2. if $\mathsf{V}_{\text{WOTRO}}$ rejected, output **reject** else output $\mathsf{V}_L(x, a, c, z)$.
-

Theorem 14. *If Σ_L is a Σ -protocol for language L and if Π_{WOTRO} is a statistically (resp. computationally) $(1 - \delta)$ -secure and correct implementation of WOTRO, then $\Pi_{\text{WOTRO}}[\Sigma_L]$ is a statistically (resp. computationally) sound (with soundness error δ) and perfectly correct non-interactive proof system for language membership in L .*

Proof. We first show correctness. By the correctness of Π_{WOTRO} , it holds that the challenge $c \in \Gamma^m$ produced by Π_{WOTRO} is uniformly distributed. When both parties are honest, the probability that V_L accepts when c is taken as the output of Π_{WOTRO} in protocol $\Pi_{\text{WOTRO}}[\Sigma_L]$ is the same as the probability that V_L accepts in an execution of Σ_L . Since Σ -protocols are perfectly correct by definition, this probability is one.

Now for soundness, again by the definition of Σ -protocols, protocol Σ_L satisfies special soundness. That is, for $x \notin L$, for any commitment $a \in \Gamma^n$, there exist at most one challenge $c \in \Gamma^m$ that leads to an accepting conversation. Let $c : \Gamma^n \rightarrow \Gamma^m$ be the function that maps commitment a to this unique challenge c that makes V_L accept. If Π_{WOTRO} is a statistically $(1 - \delta)$ -secure implementation of $\text{WOTRO}^{n,m}$, then the output of Π_{WOTRO} $(1 - \delta)$ -avoids any function for any dishonest \tilde{P}_{WOTRO} . The probability that V for protocol $\Pi_{\text{WOTRO}}[\Sigma_L]$ accepts when $x \notin L$ is equal to the probability that V_{WOTRO} accepts output (A, C) and that V_L accepts on input (x, A, C, Z) for some Z . By special soundness, this probability is at most the probability that \tilde{P}_{WOTRO} can make V_{WOTRO} accept the output $(A, c(A))$. By the statistical $(1 - \delta)$ -security of Π_{WOTRO} , this probability is at most δ .

The reasoning for computational soundness is the same, but where we instead restrict to QPT adversarial provers \tilde{P}_{WOTRO} against Π_{WOTRO} . ■

D.2 WOTRO from Non-Local Correlations

A non-local box (NL-box) is a hypothetical device distributed between two parties such that party A inputs $x \in \{0, 1\}$ into the device and gets an output $u \in \{0, 1\}$ and party B inputs $y \in \{0, 1\}$ and gets $v \in \{0, 1\}$. The input/output behaviour of the NL-box is described by

$$\Pr[u, v \mid x, y] = \begin{cases} \frac{1}{2} & \text{if } u \oplus v = x \wedge y \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{C} : \{0, 1\}^n \rightarrow \{0, 1\}^N$ be an error correcting code with minimum distance ϵn (for any distinct $x, x' \in \{0, 1\}^n$, the Hamming distance between $\mathcal{C}(x)$ and $\mathcal{C}(x')$ is at least ϵn). Let $\{h_r : \{0, 1\}^N \rightarrow \{0, 1\}^m\}_{r \in \mathcal{R}}$ be a universal₂ family of hash functions. The $\text{WOTRO}^{n,m}$ protocol is as follows:

1. On CR\$ r , and using N NL boxes,
2. Prover: on input $a \in \{0, 1\}^n$, compute codeword $x := \mathcal{C}(a)$ and input x into its interface of the N NL boxes. Let $u \in \{0, 1\}^N$ be the result. Send (a, x, u) to the verifier and use $(a, h_r(u))$ as output.

3. Verifier: On reception of a, x , check that $x = \mathcal{C}(a)$. Pick $y \in \{0, 1\}^N$ uniformly at random and input y into its interface of the N NL boxes. Let $v \in \{0, 1\}^N$ be the result. Check that $u \oplus v = x \wedge y$. If any of the checks failed, output \perp , otherwise output $(a, h_r((x \wedge y) \oplus v))$.

Theorem 15. *The above protocol avoids every function $c : \{0, 1\}^n \rightarrow \{0, 1\}^m$.*

Proof. We begin by describing the most general strategy for an adversary \mathcal{A} against the protocol. \mathcal{A} can input arbitrary values in the NL boxes in any order and such that input bits can depend on the CRS r and on the boxes' outputs to previous inputs. Let $\hat{x} \in \{0, 1\}^N$ and $\hat{u} \in \{0, 1\}^N$ denote the input and output bits to the N NL boxes, respectively. \mathcal{A} is then free to choose a, x and u adaptively based on \hat{x} and \hat{u} and send (a, x, u) to the verifier. Since the verifier checks that $x = \mathcal{C}(a)$ and aborts otherwise, we can assume that x is indeed the codeword that corresponds to a .

We show that \mathcal{A} has little freedom in the choice of a due to the error-correcting code and input/output behaviour of the NL boxes. Since \mathcal{C} has minimal distance ϵn , there is at most one codeword x_0 such that $d(x_0, \hat{x}) \leq \frac{\epsilon}{2}n$. Let $a_0 = \mathcal{C}^{-1}(x_0)$. If \mathcal{A} tries to send $(a, x = \mathcal{C}(a), u)$ for any $a \neq a_0$, then the verifier will abort with overwhelming probability as the following argument shows. Let (y, v) denote the input/output pair of the verifier. Then,

$$\begin{aligned} & \Pr[x \wedge y = u \oplus v] \\ &= \Pr[x \wedge y = u \oplus (\hat{u} \oplus \hat{x} \wedge y)] \\ &= \Pr[x \wedge y \oplus u = \hat{x} \wedge y \oplus \hat{u}] \\ &= \prod_i \Pr[x_i \wedge y_i \oplus u_i = \hat{x}_i \wedge y_i \oplus \hat{u}_i] . \end{aligned}$$

Now, consider the set of positions where \hat{x} and x differ: $\mathcal{S} = \{i : \hat{x}_i \neq x_i\}$. For any $i \in \mathcal{S}$,

- When $y_i = 0$, the expression becomes $u_i = \hat{u}_i$.
- When $y_i = 1$, the expression becomes $x_i \oplus u_i = \hat{x}_i \oplus \hat{u}_i$ and it is satisfied when $u_i \neq \hat{u}_i$.

Since y is chosen independently and uniformly at random by the verifier, for every $i \in \mathcal{S}$, the expression $x_i \wedge y_i \oplus u_i = \hat{x}_i \wedge y_i \oplus \hat{u}_i$ has probability $\frac{1}{2}$ of not being satisfied. Therefore since $|\mathcal{S}| \geq \frac{\epsilon}{2}n$ whenever $x \neq x_0$, the verifier rejects with probability at least $2^{-\frac{\epsilon}{2}n}$.

Finally, since \mathcal{A} is obligated to send a_0 and x_0 as described above and u that satisfies $u \oplus v = x_0 \wedge y$ as argued above, the output of the verifier satisfies

$$\begin{aligned} \Pr[c = c(a_0)] &= \Pr[h_r(u) = c(a_0)] = \Pr[h_r(v \oplus x_0 \wedge y) = c(a_0)] \\ &= \Pr[v \oplus x_0 \wedge y \in h_r^{-1}(c(a_0))] = \frac{|h_r^{-1}(c(a_0))|}{2^{-N}} \end{aligned}$$

since $v \oplus x_0 \wedge y$ is uniformly distributed. On average over the choice of h_r , the above expression equals 2^{-m} because the universal₂ condition implies $\mathbb{E}_r |h_r^{-1}(z)| = 2^{N-m}$ for any $z \in \{0, 1\}^m$. \blacksquare