Lattice-Based Signature with Efficient Protocols, Revisited

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Abstract. Digital signature is an essential primitive in cryptography, which can be used as the digital analogue of handwritten signatures but also as a building block for more complex systems. In the latter case, signatures with specific features are needed, so as to smoothly interact with the other components of the systems, such as zero-knowledge proofs. This has given rise to so-called *signatures with efficient protocols*, a versatile tool that has been used in countless applications. Designing such signatures is however quite difficult, in particular if one wishes to withstand quantum computing. We are indeed aware of only one post-quantum construction, proposed by Libert et al. at Asiacrypt'16, yielding very large signatures and proofs.
In this paper, we propose a new construction that can be instantiated in both standard lattices and structured ones, resulting in each case in dramatic performance improvements. In particular, the size of a proof of message-signature possession, which is one of the main metrics for such schemes, can be brought down to less than 650 KB. As our construction retains all the features expected from signatures with efficient protocols, it can be used as a drop-in replacement in all systems using them, which mechanically improves their own performance, and has thus an impact on many applications.

Keywords: Lattice-Based Cryptography · Signature · Efficient Protocols · Privacy

1 Introduction

Electronic authentication massively relies on digital signatures, a cryptographic primitive that can be traced back to the Diffie-Hellman seminal paper [DH76]. The strong point of digital signatures is that they act in the digital world in the same way as handwritten signatures do in the real world: they add a short element S to some data m attesting that m has been validated by the signer and that it has not been modified afterwards. By emulating handwritten signatures, they position themselves as the perfect electronic counterpart and they are indeed ubiquitous today.

However, for several decades, cryptographers have questioned this hegemony in some situations as these signatures may give rise to many privacy issues. Typically, presentation of the same certificate[3] S each time m needs to be authenticated allows tracing S and hence its owner. Moreover, if m is a set of elements m_i, then verification of S requires knowledge of all these elements even if they are irrelevant for the current authentication.

For example, let us consider the classical use-case of age control (e.g., to check that a customer is an adult) where some customer owns a digital certificate (embedded in some ID document) authenticating his attributes (name, birthdate, address, etc). With standard digital signature, this customer has no other choice than providing the full set of attributes to the controller as they are required to run the verification algorithm. This is clearly a significant privacy issue but here one could argue that the situation already occurs in the real world: it is indeed quite common to present an ID document displaying many personal information to a cashier that needs to control your age.

This apparent paradox epitomizes the differences between the real world and the digital one. In the former, it is natural to assume that the cashier will not memorize all the information contained in the document for further commercial exploitation or identity theft. This does not hold true in the digital world where the users definitely lose control of their data as soon as they reveal them and it is very likely that the same customer will be much more reluctant to provide the same information to a website that needs to verify that he is an adult.

1.1 Related Works

Since the problems of the two worlds are different it is actually logical that standard digital signatures are not best suited for all use-cases. In particular, the fact that electronic data can no longer be controlled once they are revealed calls for solutions disclosing as few information as possible during authentication. This has given rise to countless advanced cryptographic primitives, tailored to very specific use-cases, such as anonymous credentials [Cha85,CL01,FHS19], group signatures [CvH91,BSZ05], Direct Anonymous Attestations (DAA) [BCC04], EPID [BL07], etc. Far from simply being theoretical constructions, some of them have been included in standards (e.g., [ISO13a,ISO13b]) and even embedded in billions of devices (e.g., [TCG15,Int16]).

Surprisingly, the diversity of use-cases addressed by these privacy-preserving authentication mechanisms contrasts with the very few mathematical settings allowing efficient designs. A closer look at these standards indeed shows that all of them make use of RSA moduli or cyclic groups and thus cannot withstand the power of quantum computing. The emerging success of such systems is thus based on foundations that will crumble as soon as a sufficiently powerful quantum computer appears.

This unsatisfying state of affairs clearly calls for the design of post-quantum alternatives to such systems. However, when we look at the cryptographic litera-

[3] All along this paper, the words *signature* and *certificate* will be used interchangeably.

ture on this topic, it is striking to see that the existing post-quantum solutions are not only much less efficient than their classical[4] counterparts but also extremely rare. Typically, we are not aware of any explicit post-quantum anonymous credentials system. Even when we consider popular primitives such as group signatures, we note that the most efficient solutions [dPLS18,LNPS21] depart from the traditional model [BSZ05] as they do not achieve non-frameability, a property implying that the certificate issuer does not know users' secret keys and that is thus incompatible with their construction. Although this might seem to be a minor restriction for group signatures, this has very important consequences on their industrial variants such as DAA and EPID. Indeed, for the latter, the knowledge of the users' secret keys allows one to break anonymity, which makes the whole construction totally pointless.

To understand the contrasting situations of classical constructions and post-quantum ones in the area of privacy-preserving authentication mechanisms, it is important to recall that all of them require, at some point, to prove knowledge of a signature on some (potentially secret) attributes. For example, in an anonymous credential system, the user generally receives a signature on their attributes and some secret key at the time of issuance. To show their credentials they then reveal the requested attributes and prove knowledge of the signature, the hidden attributes and the secret key so as to remain anonymous. In non-frameable group signatures, DAA or EPID schemes, the user first receives a certificate C on a secret key s and then generates their own signatures by including a zero-knowledge proof that C is valid on s. Of course, the resulting signatures also contains additional elements that define the specificity of each primitive but the point is that the common core is this proof of knowledge which essentially needs two kinds of building blocks: a "signature scheme with efficient protocols" as coined by Camenisch and Lysyanskaya [CL02] and an associated zero-knowledge (ZK) proof system.

The latter notion is well-known and has seen several advances over the past few years, in particular in the lattice setting, e.g., [BLS19,YAZ$^+$19,LNP22]. The former notion has not been properly formalized but it usually refers to a digital signature scheme with some specific features such as the ability to sign committed (hidden) messages and to prove knowledge of a signature on such messages. This places some restrictions on the design of the signature scheme as it for example proscribes hash functions and hence most popular paradigms such as Hash-and-Sign and Fiat-Shamir. Yet, several extremely efficient constructions from number theoretic assumptions exist, in particular in bilinear (pairing) environments [CL04,BB08,PS16]. They constitute a very powerful and simple-to-use building block which explains the countless applications using them.

This situation stands in sharp contrast with the one of post-quantum cryptography where we are aware of only one lattice-based construction [LLM$^+$16] with such features. Moreover the latter was designed with Stern's proof of knowledge in mind and thus does not leverage the recent advances in the area of lattice-

[4] In this paper, we use "classical" to denote cryptographic constructions that rely on computational assumptions broken by quantum algorithms.

based zero-knowledge proofs. The original paper only provides asymptotic estimation but our thorough analysis (deferred in Appendix F) shows that, even with the recent ZK protocol from [YAZ^{+}19], a proof of knowledge of a signature is still, at best, 550 MB large, which is far too high for practical applications. This leaves designers of privacy-preserving systems with no other solution than constructing the whole system from scratch, which requires skills in many different areas and thus limits the number of contributions.

1.2 Our Contributions

The goal of our paper is to propose the lattice counterpart of [CL04,BB08,PS16], that is, a signature scheme with efficient protocols that is specifically designed to smoothly and efficiently interact with the most recent lattice-based zero-knowledge proof systems. More precisely, we provide a lattice-based signature scheme for which we can (1) obtain signatures on potentially hidden (in a commitment) messages, and (2) prove in zero-knowledge the possession of a message-signature pair. Compared to the only such construction [LLM^{+}16], our scheme is not only much more efficient but also transposes well to an algebraically structured setting which leads to further performance improvements, as summarized in Table 1.1.

Our natural starting point is [LLM^{+}16] which consists in a Boyen signature [Boy10] on a randomly chosen tag $\tau \in \{0,1\}^{\ell}$ and for a syndrome shifted by the binary decomposition of the commitment $\mathbf{c} = \mathbf{D}_0\mathbf{r} + \mathbf{D}_1\mathbf{m}$ to a binary message $\mathbf{m}$, the commitment scheme being implicit in [Ajt96]. At first sight, this scheme perfectly fits the recent zero-knowledge proof system proposed by Yang et al. [YAZ^{+}19] but yet leads to an extremely large proof of knowledge as explained above (a thorough complexity analysis is provided in Appendix D and Table F.1). We then undertake a complete overhaul of this scheme, pointing out at the same time the reasons of such a high complexity.

The main novelty is that we adopt a much more global approach as we look simultaneously at the three components of such systems, namely the commitment scheme (necessary to obtain signature on hidden messages), the signature scheme and the zero-knowledge proof systems, and the possible synergies. We, in particular, emphasize that the design choices we made for each component were not driven by the will to improve the latter individually but rather by their impact on the whole system. Typically, some of the modifications we introduce in the signature scheme itself has almost no impact on its complexity but yet results in very significant gains when it comes to proving knowledge of a signature. More generally, our approach leads to a series of contributions that we regroup in three main parts.

The signature scheme. One of the first consequences of having to sign committed messages is that the signature must now include the randomness added to the commitment by the signer. In [LLM^{+}16], this randomness has the same dimension as the one of the Boyen signature but a much larger width (see Table F.1) and thus represents the largest part of the signature. This is amplified by the proof of knowledge, which explains in part the high complexity of the latter. One of the reasons of such a large width is that the security proof requires to embed a hidden relation in the matrix $\mathbf{D}$ that is applied to the binary decomposition of the Ajtai commitment $\mathbf{c}$. More precisely, it defines $\mathbf{D} = \mathbf{AU}$ for the matrix $\mathbf{A}$ from the Boyen public key and some short matrix $\mathbf{U}$. This (along with other design choices discussed below) deteriorates the quality of the SIS solution extracted during the security proof and thus leads to large parameters.

To address this issue, we depart from [LLM+16] by generating conjointly the parameters of the signature scheme and the ones of the commitment scheme and in particular by re-using parts of the former in the latter. More specifically, in our construction, a commitment to $\mathbf{m}$ is $\mathbf{c} = \mathbf{Ar} + \mathbf{Dm}$, for a Gaussian randomness $\mathbf{r}$, where $\mathbf{A}$ is a matrix from the signer's public key and $\mathbf{D}$ is a public random matrix. From the efficiency standpoint, this has two important effects. First, this allows merging the randomness $\mathbf{r}$ with the other parts of the signatures, as we explain below, and thus to reduce the number of elements that we have to prove knowledge of. Second, as $\mathbf{A}$ is no longer hidden by a matrix $\mathbf{U}$, this significantly reduces the discrepancy between the adversary output and the extracted SIS solution in the security proof, leading to much better parameters.

Obviously, this has important consequences on the construction as the commitment matrix $\mathbf{A}$ is now selected by the signer, which is usually embodied by the adversary in privacy security games. To ensure that $\mathbf{A}$ is random to make the Ajtai commitment hiding, we need to generate it as a hash output. This solution is then totally incompatible with the [LLM+16] approach where the signer needs to generate $\mathbf{A}$ together with an associated trapdoor.

Instead of Boyen's signature, we then choose to use the trapdoors of [MP12], which interface well with the Ajtai commitment. More precisely, our public key is composed of a random matrix $\mathbf{A}$, a matrix $\mathbf{B} = \mathbf{AR}$ and a random syndrome $\mathbf{u}$, and the secret key is a random ternary matrix $\mathbf{R}$. In order to sign a binary message $\mathbf{m}$ hidden in a commitment $\mathbf{c} = \mathbf{Ar} + \mathbf{Dm}$, we select a random tag τ in a tag space $\mathcal{T} \subseteq \mathbb{Z}_q^\times$ and use pre-image sampling to sample a Gaussian vector $\mathbf{v}'$ such that $[\mathbf{A}|\tau\mathbf{G} - \mathbf{B}]\mathbf{v}' = \mathbf{u} + \mathbf{c}$, where $\mathbf{G}$ is the gadget matrix from [MP12]. As $\mathbf{A}$ is involved in both the left hand side of the equation and in $\mathbf{c}$, we can set the signature as $(\tau, \mathbf{v} = \mathbf{v}' - [\mathbf{r}^T|\mathbf{0}]^T)$. Verification consists in checking

$$[\mathbf{A}|\tau\mathbf{G} - \mathbf{B}]\mathbf{v} = \mathbf{u} + \mathbf{Dm} \bmod q \text{ and } \|\mathbf{v}\|_\infty \text{ small.} \tag{1}$$

One can note that we have removed in the process the binary decomposition of $\mathbf{c}$. We indeed choose a very different approach in the security proof which shows that this step is actually not necessary. Removing this decomposition is also crucial in order to compact the commitment randomness $\mathbf{r}$ with the pre-image $\mathbf{v}'$. It avoids further intermediate steps that deteriorate the SIS solution extracted from the forgery, as explained above, which leads to better parameters overall. Moreover, when it comes to proving knowledge of the signature, each intermediate step makes the whole statement harder to prove and requires to create additional witnesses, i.e., each bit of $\mathbf{c}$, that must be committed, whose membership in $\{0, 1\}$ must be proven, etc. The same holds true with the tag τ which is an element of $\mathbb{Z}_q$ in our case, contrarily to [LLM+16] where τ is in $\{0, 1\}^\ell$.

	setting	λ	$\lvert\mathsf{pk}\rvert$ (MB)	$\lvert\mathsf{sk}\rvert$ (MB)	$\lvert\mathsf{sig}\rvert$ (KB)	$\lvert\pi\rvert$ (KB)
[LLM+16] (exact proof)	stand.	128	$867 \cdot 10^3$	$138 \cdot 10^2$	$809 \cdot 10^1$	$958 \cdot 10^4$
[LLM+16] (fast mode)	stand.	128	$205 \cdot 10^4$	$326 \cdot 10^2$	$132 \cdot 10^2$	$566 \cdot 10^3$
Sec. 3 (exact proof)	stand.	128	$116 \cdot 10^1$	898	262	$309 \cdot 10^3$
Sec. 3 (fast mode)	stand.	128	$299 \cdot 10^1$	$231 \cdot 10^1$	420	$178 \cdot 10^2$
Sec. 6 (exact proof)	module	128	8.1	9.1	275	638

Table 1.1. Comparison of efficiency estimates of the signature schemes of [LLM+16], of Section 3 and of Section 6 for $\lambda = 128$ bits of quantum security, with the size of zero-knowledge proof of possession of a message-signature pair. In the setting column, *stand.* stands for standard lattices, as opposed to the ring setting of our last construction. The proofs for [LLM+16] and Section 3 are either exact proofs or approximate ones using the *fast mode* of Section 5.1 and described in the technical overview. The complete analysis and parameter sets used for these estimates can be found in Appendix F.

As $\ell < \log_2 q$, this might look like a downside for the signature itself but this is the exact opposite in the ZK proof. Each bit i of the tag τ in [LLM+16] indeed constitutes a witness on its own and additionally yields a full witness vector $\mathbf{w}_i = \tau[i]\mathbf{v}_2$, where $\mathbf{v}_2$ is half of the Boyen signature, and associated quadratic relations. Our point here is that each seemingly innocent modification we introduce is considerably amplified when considering the full protocol and therefore results in major gains.

So far, we have essentially discussed improvements of both the commitment and the signature schemes. The comparison provided in Table 1.1 shows that our resulting signature is between 30 and 40 times smaller than the one of [LLM+16] when considering the same setting (standard lattices). However, this gain is still not sufficient to lead to practical proofs as ZK lattice proofs are still complex, even with the recent framework of [YAZ+19]. We now focus on the proofs of knowledge necessary for our protocol and explain how we can modify the previous framework for a better efficiency.

Efficient Protocols and Zero-Knowledge Arguments. A "signature scheme with efficient protocols" requires two kinds of protocols, one to get a signature on a committed message and one for proving possession of a message-signature pair. Regarding the former, the problem is rather simple as the message $\mathbf{m}$ to sign is already embedded in a commitment $\mathbf{c} = \mathbf{A}\mathbf{r} + \mathbf{D}\mathbf{m}$. However, we have to slightly modify this construction because both the user requesting the signature and the signer must contribute to the randomness of the commitment. This leads to a commitment $\mathbf{c} = \mathbf{A}(\mathbf{r}' + \mathbf{r}'') + \mathbf{D}\mathbf{m}$ where $\mathbf{r}'$ is added by the user to enforce the hiding property of $\mathbf{c}$ and $\mathbf{r}''$ is added by the signer to be able to handle any query in the security proof. Only the former needs to prove knowledge of $\mathbf{r}'$ and $\mathbf{m}$ so as to rely on the EUF-CMA property of the signature scheme we introduced. In all cases, the user ends up with a signature $(\tau, \mathbf{v})$ on a binary $\mathbf{m}$ verifying (1) and needs to prove it in a zero-knowledge way.

For that, we employ the recent zero-knowledge framework proposed by Yang et al. [YAZ+19] which can be used to prove linear relations with quadratic constraints. The latter feature is very useful in our case as our verification equation (1) is quadratic in $(\mathbf{m}; (\tau, \mathbf{v}))$ because of the term $\tau\mathbf{G}\mathbf{v}_2$ (where $\mathbf{v}_2$ is the bottom part of $\mathbf{v}$). Moreover, this allows one to prove that an element is short by first writing its binary decomposition and then proving that each resulting component x is indeed binary through the quadratic equation $x(x-1) = 0$.

Unfortunately, this nice feature comes at a price as this decomposition procedure entails a $(\log_2 B)$-fold increase of the size of the witness $\mathbf{v}$, where B is a bound on $\|\mathbf{v}\|_\infty$. For a high dimensional vector $\mathbf{v}$ in $\mathbb{Z}^m$, this results in a very large proof which has led the authors of [YAZ+19] to propose a so-called *fast mode* for their protocol. In a nutshell, this variant relies on the observation that the norm of $\mathbf{H}\mathbf{v}$, for a random short matrix $\mathbf{H}$ of dimension $k \times m$, implies some bound on the norm of $\mathbf{v}$, even when the latter is chosen by the adversary. As $\mathbf{H}\mathbf{v}$ must be hidden, one must still use the quadratic relation above to prove shortness but on a witness with a much smaller dimension as k is in practice much smaller than m. The efficiency gains are very significant but we point out several shortcomings with the solution proposed in [YAZ+19]. First, contrarily to the claim in [YAZ+19], this fast mode *cannot* be used to prove that $\mathbf{v}$ is positive and we provide a concrete counter-example in Section 5. This is not a problem in our case as we only want to prove results on the infinity norm of $\mathbf{v}$ but this can be a problem for specific applications such as the e-cash system considered in [YAZ+19]. Second, the authors in [YAZ+19] make use of a binary matrix $\mathbf{H}$ which significantly deteriorates the overall statement as one must set a bound $m\beta$ on the norm of $\mathbf{H}\mathbf{v}$, when $\|\mathbf{v}\|_\infty$ is bounded by β. Although this soundness gap seems unavoidable with this mode, we show that we can do better with a matrix $\mathbf{H} \in \{-1, 0, 1\}^{k\times m}$, which allows selecting better parameters and thus leads to more efficient protocols.

Finally, we also propose in Appendix E a series of optimizations for the Yang et al. protocol that range from better parameters selection to compression of the commitments, resulting in further efficiency improvements. For a fair comparison, the figures in Table 1.1 take into account these improvements for both our scheme and the one from [LLM+16]. This table shows that our contributions reduce the size of a proof of knowledge (using the fast mode) to roughly 18 MB, which can be interpreted in two ways. On the one hand, this is a dramatic improvement over [LLM+16]. On the other hand, this is still large and probably impractical for many applications. The last part of our contributions thus investigates how to instantiate our construction in another setting to further reduce this size.

Extending to Structured Lattices. Our construction extends to the module setting where we replace the integers by polynomials with integer coefficients. More concretely, we consider a power-of-two cyclotomic ring, i.e., $R = \mathbb{Z}[X]/\langle X^n + 1\rangle$ with n a power-of-two. The additional structure yields more efficient computations, as well as more compact keys. The trapdoors of [MP12] have already been used over such algebraic rings, e.g., [DM14,dPLS18,BEP+21], which makes our module construction very similar to the one based on standard lattice assumptions. All the tools required to prove the security of our scheme also have a ring counterpart, which therefore leads to almost no differences in the security proofs either. The main difference comes when considering exact zero-knowledge proofs over algebraic rings. Our verification equation, once translated into the module setting, is

$$[\mathbf{A}|\tau\mathbf{G}-\mathbf{B}]\mathbf{v}=\mathbf{u}+\mathbf{D}\mathbf{m} \bmod qR \text{ and } \mathbf{v} \text{ short}. \tag{2}$$

Proving knowledge of (2) requires to prove that (1) τ is indeed in the specified tag space, (2) $\mathbf{v}$ is a short vector, (3) $\mathbf{m}$ is a vector of binary polynomials, and (4) that the quadratic equation is verified. Based on state-of-the-art proof systems, (1) constrains which tag space to choose so that we can efficiently prove membership, while ensuring that a difference of tags is invertible in R/qR as needed per the security proofs. Statement (2) requires to define a notion of shortness over the ring, which is usually defined based on the size of the coefficients of the polynomials. Up until recently, exact proofs performing the latter task [BLS19,ENS20] (which can also be used for (3)) interpreted the coefficients of $\mathbf{v}$ as the NTT (Number Theoretic Transform) of another vector $\mathbf{v}'$, which are most efficient when X^n+1 splits into low-degree irreducible factors modulo q. This splitting makes it harder to choose a proper tag space for which differences are always invertible. Finally, (4) requires a proof system able to deal with quadratic equations. Similar relations [dPLS18,LNPS21] were handled by transforming the relation quadratic in the witnesses into a linear relation in the commitment of the witnesses. Since efficient proofs of commitment opening rely on relaxed openings, this solution introduces a soundness gap in the proven statement, which we would like to avoid.

Instead, we use the very recent framework of Lyubashevsky et al. [LNP22] which provides a unified method to prove all our statements. It extends the previous works of [BLS19,ENS20] and enables proving quadratic relations exactly, as well as quadratic evaluations. The latter can be used to prove exact bounds directly in the Euclidean norm, which leads to more efficient proofs than proving bounds in the infinity norm.

In the module setting, we therefore end up with a signature scheme that is efficient on all metrics, as highlighted in Table 1.1. In particular, we manage to keep our proofs of knowledge of a message-signature pair below 640 KB. As these proofs are one of the main building blocks of privacy-preserving protocols, these efficiency gains readily translate to the latter and thus should have a significant impact on the area. More generally, our construction is designed to be used as a black box, which should foster many applications, as was the case with the pairing-based signatures with efficient protocols [CL04,BB08,PS16].

1.3 Organization

We provide the necessary notions and background in Section 2, before introducing our signature scheme and security in Section 3. Then, Section 4 presents the privacy-enhancing efficient protocols that accompany our signature scheme. We detail out the needed zero-knowledge arguments in Section 5. Finally, we present our signature on structured lattices along with the associated zero-knowledge arguments in Section 6.

2 Preliminaries

Throughout this paper, for two integers $a \leq b$, we define $[a, b] = \{k \in \mathbb{Z} : a \leq k \leq b\}$. When $a = 1$ and $b \geq 1$, we simply use $[b]$ to denote $[1, b]$. For a positive integer q, we define $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. In this work, we consider q to be an odd prime (or product of odd primes), and we sometimes identify $\mathbb{Z}_q$ with the set of representatives $[-(q-1)/2, (q-1)/2]$. The vectors are written in bold lowercase letters $\mathbf{a}$, while the matrices are in bold uppercase letters $\mathbf{A}$. The transpose operator is denoted with the superscript T. The identity matrix of size $n \times n$ is denoted by $\mathbf{I}_n$. For any $\mathbf{a} \in \mathbb{R}^n$, we define its Euclidean norm as $\|\mathbf{a}\|_2 = (\sum_{i \in [n]} |a_i|^2)^{1/2}$ and its infinity norm as $\|\mathbf{a}\|_\infty = \max_{i \in [n]} |a_i|$. For a matrix $\mathbf{A} = [\mathbf{a}_i]_{i \in [m]} \in \mathbb{R}^{n \times m}$, we define $\|\mathbf{A}\|_{\max} = \max_{i \in [m]} \|\mathbf{a}_i\|_\infty$, and $\|\mathbf{A}\|_2 = \max_{\mathbf{x} \neq \mathbf{0}} \|\mathbf{A}\mathbf{x}\|_2 / \|\mathbf{x}\|_2$. We denote by λ the security parameter.

2.1 Lattices

A (full-rank) *lattice* Λ of rank n is a discrete additive subgroup of $\mathbb{R}^n$. Each lattice can be represented by a basis $\mathbf{B} = [\mathbf{b}_i]_{i \in [n]} \in \mathbb{R}^{n \times n}$ as the set of all integer linear combinations of the $\mathbf{b}_i$, i.e., $\Lambda = \mathbf{B}\mathbb{Z}^n$. The *dual lattice* of a lattice Λ is defined by $\Lambda^* = \{\mathbf{x} \in \mathrm{Span}_{\mathbb{R}}(\Lambda) : \forall \mathbf{y} \in \Lambda, \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{Z}\}$. In this work, we consider the following family of q-ary lattices.

Definition 2.1. *Let n, m, q be positive integers. Let $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$. We define the lattice $\Lambda_q^\perp(\mathbf{A}) = \{\mathbf{e} \in \mathbb{Z}^m : \mathbf{A}\mathbf{e} = \mathbf{0} \bmod q\}$.*

2.2 Probabilities

For a finite set S, we define $|S|$ to be its cardinality, and $U(S)$ to be the uniform probability distribution over S. The action of sampling $x \in S$ from a probability distribution P is denoted by $x \hookleftarrow P$. We use $x \sim P$ to say that the random variable x follows the distribution P. The *statistical distance* between two discrete probability distributions P and Q over a countable set S is defined as $\Delta(P, Q) = \frac{1}{2} \sum_{x \in S} |P(x) - Q(x)|$.

We recall here the leftover hash lemma from [HILL99] for universal hash functions, which we write to match our context and notations. In particular, the following shows that when $\mathbf{A}$ has sufficiently many columns, then $\mathbf{AR}$ is statistically close to a uniform matrix where $\mathbf{R}$ is a uniform ternary matrix. This is a requirement for the correct distribution of the signature keys.

Lemma 2.1 (Adapted from [HILL99,DORS08]). *Let n, m, q be positive integers such that q is an odd prime. For $\mathbf{A} \sim U(\mathbb{Z}_q^{n \times m})$, $\mathbf{x} \sim U(\{-1, 0, 1\}^m)$, and $\mathbf{u} \sim U(\mathbb{Z}_q^n)$, it holds that $\Delta((\mathbf{A}, \mathbf{A}\mathbf{x}), (\mathbf{A}, \mathbf{u})) \leq \frac{1}{2}\sqrt{q^n / 3^m}$. In particular, whenever $m \log_2 3 \geq n \log_2 q + \omega(\log_2 \lambda)$, the statistical distance is negligible.*

For any *center* vector $\mathbf{c} \in \mathbb{R}^n$, and *Gaussian width* $\sigma > 0$, we define the Gaussian function $\rho_{\sigma,\mathbf{c}} : \mathbf{x} \in \mathbb{R}^n \mapsto \exp(-\pi\|\mathbf{x} - \mathbf{c}\|_2^2/\sigma^2)$. For a lattice Λ of rank n, we define the *discrete Gaussian distribution* $\mathcal{D}_{\Lambda,\sigma,\mathbf{c}}$ of support Λ, width σ and center $\mathbf{c}$ by $\mathcal{D}_{\Lambda,\sigma,\mathbf{c}} : \mathbf{x} \in \Lambda \mapsto \rho_{\sigma,\mathbf{c}}(\mathbf{x})/\rho_{\sigma,\mathbf{c}}(\Lambda)$, where $\rho_{\sigma,\mathbf{c}}(\Lambda) = \sum_{\mathbf{x}\in\Lambda} \rho_{\sigma,\mathbf{c}}(\mathbf{x})$. When $\mathbf{c} = \mathbf{0}$, we omit it in the notations. We then use it to define the *smoothing parameter* of a lattice Λ [MR07], parameterized by a real $\varepsilon > 0$, by $\eta_\varepsilon(\Lambda) = \inf\{s > 0 : \rho_{1/s}(\Lambda^* \setminus \{\mathbf{0}\}) \leq \varepsilon\}$. If the standard deviation is wider than the smoothing parameter, the discrete Gaussian distribution benefits from properties that are similar to the ones of the continuous Gaussian distribution. In particular, the sum of two independent discrete Gaussians is a discrete Gaussian.

Lemma 2.2 (Adapted from [Reg05, Claim 3.9][MP13, Thm. 3.3]). *Let Λ be lattice of rank n. Let $r, s > 0$ and $t = \sqrt{r^2 + s^2}$ be such that $rs/t \geq \eta_\varepsilon(\Lambda)$ for some $\varepsilon \in (0, 1/2]$. Then, we have $\Delta(\mathcal{D}_{\Lambda,r} + \mathcal{D}_{\Lambda,s}, \mathcal{D}_{\Lambda,t}) \leq 7\varepsilon/4$. The condition on r, s is satisfied for example when $r, s \geq \sqrt{2}\eta_\varepsilon(\Lambda)$.*

When centered around $\mathbf{0}$, the discrete Gaussian distribution benefits from tail bounds similar to the standard Gaussian distribution. In this work, we use tail bounds on the Euclidean and infinity norms. We also recall the result of [Lyu12] bounding the magnitude of $\langle \mathbf{x}, \mathbf{v} \rangle$ for a discrete Gaussian $\mathbf{x}$ and an arbitrary vector $\mathbf{v}$. Although the tail bound on the infinity norm follows directly from the latter, it was first proven in [Pei08, Cor. 5.3].

Lemma 2.3 ([Ban93, Lem. 1.5][Pei08, Cor. 5.3][Lyu12, Lem 4.3]). *Let Λ be a lattice of rank n. Let $\sigma > 0$ and $\mathbf{v} \in \mathbb{R}^n$. Then, for all $t > 0$, it holds that*

1. $\mathbb{P}_{\mathbf{x}\sim\mathcal{D}_{\Lambda,\sigma}}\left[\|\mathbf{x}\|_2 > \sigma\sqrt{n}\right] < 2^{-2n}$,
2. $\mathbb{P}_{\mathbf{x}\sim\mathcal{D}_{\Lambda,\sigma}}\left[\|\mathbf{x}\|_\infty > \sigma \log_2 n\right] \leq 2ne^{-\pi \log_2^2 n}$,
3. $\mathbb{P}_{\mathbf{x}\sim\mathcal{D}_{\Lambda,\sigma}}\left[|\langle \mathbf{x}, \mathbf{v} \rangle| > \sigma t \|\mathbf{v}\|_2\right] \leq 2e^{-\pi t^2}$.

We also use the following bound on the spectral norm of a matrix with independent sub-Gaussian entries. We recall the definition of a sub-Gaussian random vector.

Definition 2.2 (Sub-Gaussian Distribution). *Let n be a positive integer, and $\mathbf{x}$ a (discrete or continuous) random vector over $\mathbb{R}^n$. We say that $\mathbf{x}$ is sub-Gaussian with sub-Gaussian moment s if for all unit vector $\mathbf{u} \in \mathbb{R}^n$ and all $t \in \mathbb{R}$, we have $\mathbb{E}[\exp(t\langle \mathbf{x}, \mathbf{u} \rangle)] \leq e^{s^2t^2/2}$.*

Lemma 2.4 ([Ver12]). *Let ℓ, m be two positive integers, and $\mathcal{P}$ a sub-Gaussian distribution of moment s. There exists a universal constant $C > 0$ such that for all $t > 0$, $\mathbb{P}_{\mathbf{U}\hookleftarrow\mathcal{P}^{\ell\times m}}[\|\mathbf{U}\|_2 \geq Cs(\sqrt{\ell} + \sqrt{m} + t)] \leq 2e^{-\pi t^2}$.*

By noticing that $\mathcal{P} = U([-1, 1])$ is sub-Gaussian with moment $\sqrt{2/3}$, we can bound the spectral norm of ternary uniform matrix by $C\sqrt{2/3}(\sqrt{\ell} + \sqrt{m} + t)$ except with probability $2e^{-\pi t^2}$, for some constant $C > 0$ that does not depend on the dimensions. We can verify experimentally that in this case $C\sqrt{2/3} \leq 1$, and we thus omit it in the rest of the paper for clarity. The security proof of our signature requires a bound on $\|\mathbf{Um}\|_2$ for an arbitrary message $\mathbf{m} \in \{0,1\}^m$ and uniform ternary $\mathbf{U}$. When m is small, Lemma 2.4 gives a close to optimal bound by $\|\mathbf{Um}\|_2 \leq \|\mathbf{U}\|_2\sqrt{m}$. However, when m is large, we expect a tighter bound. By using the fact that the square of a sub-Gaussian random variable is sub-exponential and tail bounds on sub-exponential distributions, we get the following lemma. The proof and associated definitions are provided in Appendix A.

Lemma 2.5. *Let ℓ, m be two positive integers and $x > 0$. We assume that $\ell > x \cdot 10/\log_2 e$. Let $\mathbf{m} \in \{0,1\}^m$. We have $\mathbb{P}_{\mathbf{U} \hookleftarrow U([-1,1])^{\ell \times m}}[\|\mathbf{Um}\|_2 \geq 2\sqrt{\ell m}] \leq 2^{-x}$.*

In our situation, $x = \Theta(\lambda)$ with λ the security parameter, and $\ell = O(n \log_2 q + \omega(\log_2 \lambda))$. The condition $\ell > 10x/\log_2 e$ is then verified. Note that this condition is necessary only to obtain the simple bound $2\sqrt{\ell m}$ with probability 2^{-x}, but one could use a different bound or different probability to avoid this condition. Combining both lemmas gives the following

$$\mathbb{P}_{\mathbf{U} \hookleftarrow U([-1,1]^{\ell \times m})}[\|\mathbf{Um}\|_2 \geq \min(2\sqrt{\ell}, \sqrt{\ell} + \sqrt{m} + t)\sqrt{m}] \leq 2^{-2\lambda} + 2e^{-\pi t^2}, \quad (3)$$

whenever $\ell \geq 20\lambda/\log_2 e$ which is the case in our context. The spectral bound of Lemma 2.4 is also necessary to set the correct parameters to sample Gaussian vectors $\mathbf{v}$ verifying $[\mathbf{A}|\tau\mathbf{G} - \mathbf{AR}]\mathbf{v} = \mathbf{u}$, where $\mathbf{A}$ is a uniform matrix, $\mathbf{R}$ a short random matrix and $\mathbf{G}$ the gadget matrix of [MP12] used for efficient pre-image sampling. Our signature uses the following pre-image sampling algorithm.

Lemma 2.6 ([MP12]). *There exists an algorithm* SampleD *that takes as input a trapdoor matrix $\mathbf{R} \in \mathbb{Z}^{m_1 \times n\lceil \log_2 q \rceil}$, a partial parity-check matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m_1}$, a invertible tag matrix $\mathbf{H} \in \mathbb{Z}_q^{n \times n}$, a syndrome $\mathbf{u} \in \mathbb{Z}_q^n$ and a standard deviation $\sigma \geq \eta_\varepsilon(\mathbb{Z})\sqrt{7}\sqrt{1 + \|\mathbf{R}\|_2^2}$, and that outputs $\mathbf{v}$ that is statistically close to $\mathcal{D}_{\mathbb{Z}^{m_1 + n\lceil \log_2 q \rceil}, \sigma}$ conditioned on $[\mathbf{A}|\mathbf{HG} - \mathbf{AR}]\mathbf{v} = \mathbf{u} \bmod q$, with $\mathbf{G} = \mathbf{I}_n \otimes \mathbf{g}$ and $\mathbf{g} = [1 \ldots 2^{\lceil \log_2 q \rceil - 1}]$.*

2.3 Hardness Assumption

The security of our signature scheme relies on the *Short Integer Solution* (SIS) problem [Ajt96], which we recall here.

Definition 2.3 (Short Integer Solution). *Let n, m, q be positive integers, and $\beta_2 \geq \beta_\infty \geq 1$. The* Short Integer Solution *problem $\mathrm{SIS}^{\infty,2}_{n,m,q,\beta_\infty,\beta_2}$ consists in finding $\mathbf{x} \in \Lambda_q^\perp(\mathbf{A})$ given $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{n \times m})$ such that $0 < \|\mathbf{x}\|_\infty \leq \beta_\infty$ and $0 < \|\mathbf{x}\|_2 \leq \beta_2$.*

Note that the original formulation of SIS considers a single bound β on the Euclidean norm. There is a trivial reduction from the latter to $\mathrm{SIS}^{\infty,2}_{n,m,q,\beta_\infty,\beta_2}$ by setting $\beta = \min(\beta_\infty\sqrt{m}, \beta_2)$. As discussed by Micciancio and Peikert [MP13, Thm. 1.1], using both norm bounds leads to more precise hardness results, and sometimes smaller approximation factors when relating the problem to worst-case problems on lattices. Moreover, it seems to be relevant for the concrete hardness of the problem as well. Indeed, most lattice reduction algorithms aim at finding vectors in the ball of radius β_2 but without constraining the magnitude of the coefficients. Finding a lattice vector in the intersection of the ball of radius β_2 and the hypercube of half side β_∞ is at least as hard as the same task without the β_∞ bound. When $\beta_\infty \ll \beta_2$, it may even be substantially harder.

2.4 Signature Scheme

A signature scheme is defined by four algorithms. The Setup algorithm is a probabilistic algorithm that, on input a security parameter λ, outputs the public parameters pp that will be common to all users. The key generation algorithm KeyGen is a probabilistic algorithm that, on input pp, outputs a secret signing key sk and a public verification key pk. The signing algorithm Sign is a probabilistic algorithm which, on inputs sk and a message $\mathbf{m}$ (and pk, pp), outputs a signature sig. Finally, the verification algorithm Verify is a deterministic algorithm that, on inputs $\mathsf{pk}, \mathbf{m}, \mathsf{sig}$ (and pp), outputs 1 if sig is a valid signature on $\mathbf{m}$ under pk, and 0 otherwise. We use the *Existential Unforgeability against Chosen Message Attacks* (EUF-CMA) security model, which we formally recall in Appendix B along with the security proofs of our signature scheme.

3 A Lattice-Based Signature Scheme

We present here our signature scheme which interfaces smoothly with privacy-enhancing protocols. It provides an alternative to the only such scheme based on lattices due to Libert et al. [LLM+16].

One of the main differences between their construction and ours is that we aim at optimizing the interactions between the commitment scheme implicitly used by such kind of protocols and the signature scheme itself. In [LLM+16], the public parameters of these two components were indeed generated independently. We depart completely from this approach by generating these parameters conjointly and even by using a common matrix $\mathbf{A}$ for these two parts. Besides the natural reduction of the public key size, this strategy allows one to merge different components of the signature itself. In particular, compared to [LLM+16], our signature no longer has to include the commitment opening, which significantly reduces its size.

Obviously, this has important consequences on the design of the scheme itself. One of them is that it forbids to re-use the approach of [LLM+16], inherited from the Boyen's signature [Boy10], where $\mathbf{A}$ was generated together with a trapdoor, because it would clearly break the hiding property of the commitment scheme. We instead rely on a $\mathbf{G}$-trapdoor $\mathbf{R}$ of size $m_1 \times m_2$ in the sense of [MP12] and then use a matrix $[\mathbf{A}|\tau\mathbf{G} - \mathbf{A}\mathbf{R}]$ where τ is a tag from $\mathbb{Z}_q^\times$. We can therefore generate $\mathbf{A}$ as a random matrix[5] of size $n \times m_1$, where m_1 is the dimension of the commitment randomness. We then use it to construct the commitment $\mathbf{c}$ to a message $\mathbf{m} \in \{0,1\}^{m_3}$ as $\mathbf{c} = \mathbf{Ar} + \mathbf{Dm} \bmod q$, where $\mathbf{D}$ is a random matrix of size $n \times m_3$ and m_3 is the dimension of the message. The randomness $\mathbf{r}$ can then be merged with the short vector $\mathbf{v}$ generated thanks to the trapdoor, as mentioned above.

An interesting side effect of moving from Boyen matrix $[\mathbf{A}|\mathbf{A}_0 + \sum_{j\in[\ell]} \tau[j]\mathbf{A}_j]$ to the one described above is more subtle as it only appears when considering proofs of knowledge of the signature. The fact that each bit $\tau[i]$ of the tag appears separately in the Boyen's matrix may indeed look harmless when we only consider the signature because it does not increase the size of the vector $\mathbf{v}$. However, when plugged in the Yang et al. ZK framework [YAZ+19] this creates a set of 2ℓ intermediary witnesses $(\mathbf{w}_j = \mathbf{A}_j\mathbf{v}, \tau[j]\mathbf{w}_j)$ and ℓn quadratic relations that significantly increase the size of the proof. Treating τ monolithically saves a factor ℓ in both the number of extra witness vectors and quadratic relations, which correspondingly improves complexity.

In the same vein, the authors of [LLM+16] had to first compute a binary decomposition $\mathbf{c}'$ of the commitment $\mathbf{c}$ to the message before generating a short pre-image of $\mathbf{u} + \mathbf{Dc}'$ where $\mathbf{u}$ (resp. $\mathbf{D}$) was some public vector (resp. matrix). Here again, the impact on the complexity might not seem obvious when only considering the signature scheme but this is no longer true when it comes to prove knowledge of a signature: this replaces one secret vector $\mathbf{c}$ by $\log_2 q$ ones and makes the overall statement to prove more complex. To remove this binary decomposition we revisit the security proof and show how to avoid it by using an argument based on the Rényi Divergence. Additionally, this change seems necessary to extend our construction to polynomial rings, as described in Section 6.

More generally, all the modifications we introduce have a second positive effect on complexity. In both our security proof and the one of [LLM+16], it is necessary to generate the public matrices with hidden relations, usually by multiplying one by some low-norm matrix $\mathbf{U}$ to generate the other one. This has an impact on the norm of the extracted solutions, which grows with the number of such matrices and computational steps, and therefore on the system parameters. By reusing $\mathbf{A}$ for different purposes and by removing some computational steps (e.g., multiplication by $\mathbf{D}$), we manage to significantly reduce the discrepancy between the adversary output and the resulting SIS solution, leading to much better parameters.

3.1 Description of the Signature

We now describe the four algorithms that define our signature scheme. The signature is designed to sign a binary message $\mathbf{m}$. We present our scheme for the more general case of a message with variable length rather than a variable number of blocks of fixed length which may require unnecessary padding.

[5] In our protocol for signing hidden messages, we will have to enforce this requirement but this can be done easily by setting $\mathbf{A}$ as some hash output.

Algorithm 1 Setup

```
Input: Security parameter $\lambda$.
Choose a positive integer $n$.
Choose a prime integer $q$.
Choose a positive integer $q' \leq q$.                                   ▷ Bound on the tags.
$\mathcal{T} \leftarrow \mathbb{Z}_{q'} \setminus \{0\}$.                ▷ Tag space
Choose $f(\lambda) \leftarrow \omega(\log_2 \lambda)$.
$m_1 \leftarrow \lceil (n \log_2 q + f(\lambda)) / \log_2 3 \rceil$.     ▷ Commitment randomness dimension
$m_2 \leftarrow n \lceil \log_2 q \rceil$.
$m \leftarrow m_1 + m_2$.                                                ▷ Signature dimension
Choose a positive integer $m_3$.                                         ▷ Maximum bit-size of $\mathbf{m}$
$\mathbf{g} = [2^0 \cdots 2^{\lceil \log_2 q \rceil - 1}] \in \mathbb{Z}_q^{1 \times \lceil \log_2 q \rceil}$.   ▷ Gadget vector
$r \leftarrow \eta_\varepsilon(\mathbb{Z})$.                             ▷ $r = 5.4$ leads to $\varepsilon \approx 2^{-131}$
Choose $t > 0$.                                                          ▷ Spectral norm slack
$\sigma \leftarrow r\sqrt{7}\sqrt{(\sqrt{m_1} + \sqrt{m_2} + t)^2 + 1}$.   ▷ Pre-image sampling width
$\sigma_2 \leftarrow \max\left(\sqrt{\min(2\sqrt{m_1}, \sqrt{m_1} + \sqrt{m_3} + t)^2 m_3 - \sigma^2}, \omega(\sqrt{\log_2 m_1})\right)$.
$\sigma_1 \leftarrow \sqrt{\sigma^2 + {\sigma_2}^2}$.
$\mathbf{D} \hookleftarrow U(\mathbb{Z}_q^{n \times m_3})$.              ▷ Message commitment key
Output: $\mathsf{pp} = (\mathbf{D}; \mathbf{g}; \lambda, n, m_1, m_2, m_3, q, \sigma, \sigma_2, \sigma_1)$.
```

Algorithm 2 KeyGen

```
Input: Public parameters $\mathsf{pp}$ as in Algorithm 1.
$\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{n \times m_1})$.
$\mathbf{R} \hookleftarrow U([-1, 1]^{m_1 \times m_2})$.
$\mathbf{B} \leftarrow \mathbf{AR} \bmod q \in \mathbb{Z}_q^{n \times m_2}$.
$\mathbf{u} \hookleftarrow U(\mathbb{Z}_q^n)$.
Output: $\mathsf{pk} = (\mathbf{A}, \mathbf{B}, \mathbf{u})$, and $\mathsf{sk} = \mathbf{R}$.
```

Algorithm 3 Sign

```
Input: Signing key $\mathsf{sk}$, Message $\mathbf{m} \in \{0,1\}^{m_3}$, Public key $\mathsf{pk}$, Public Parameters $\mathsf{pp}$.
$\mathbf{r} \hookleftarrow \mathcal{D}_{\mathbb{Z}^{m_1}, \sigma_2}$.
$\mathbf{c} \leftarrow \mathbf{Ar} + \mathbf{Dm} \bmod q$.                ▷ Commitment to $\mathbf{m}$
$\tau \hookleftarrow U(\mathcal{T})$.
$\mathbf{v} \leftarrow \mathsf{SampleD}(\mathbf{R}, \mathbf{A}, \tau\mathbf{I}_n, \mathbf{u} + \mathbf{c}, \sigma) - [\mathbf{r}^T | \mathbf{0}_{m_2}]^T$.   ▷ $\mathbf{A}_\tau = [\mathbf{A} | \tau(\mathbf{I}_n \otimes \mathbf{g}) - \mathbf{B}]$
Output: $\mathsf{sig} = (\tau, \mathbf{v})$.
```

Algorithm 4 Verify

```
Input: Public key $\mathsf{pk}$, Message $\mathbf{m} \in \{0,1\}^{m_3}$, Signature $\mathsf{sig}$, Public Parameters $\mathsf{pp}$.
$\mathbf{A}_\tau \leftarrow [\mathbf{A} | \tau(\mathbf{I}_n \otimes \mathbf{g}) - \mathbf{B}] \in \mathbb{Z}_q^{n \times m}$.
Split $\mathbf{v}$ into $\left[\mathbf{v}_1^T \ \mathbf{v}_2^T\right]^T$, with $\mathbf{v}_1 \in \mathbb{Z}^{m_1}$, $\mathbf{v}_2 \in \mathbb{Z}^{m_2}$.
if $(\mathbf{A}_\tau \mathbf{v} = \mathbf{u} + \mathbf{Dm} \bmod q) \wedge (\|\mathbf{v}_1\|_\infty \leq \sigma_1 \log_2 m_1) \wedge (\|\mathbf{v}_2\|_\infty \leq \sigma \log_2 m_2) \wedge (\tau \in \mathcal{T})$
then return 1                                                            ▷ Valid Signature
else return 0                                                            ▷ Invalid signature
```

The correctness of the signature scheme simply relies on the sum of discrete Gaussians (Lemma 2.2) and the Gaussian tail bound (Lemma 2.3). The former guarantees that $\mathbf{v}_1$ is statistically close to $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_1}$, and the latter ensures that for an honest signature it holds that $\|\mathbf{v}_1\|_\infty \leq \sigma_1 \log_2 m_1$, and $\|\mathbf{v}_2\|_\infty \leq \sigma \log_2 m_2$ with overwhelming probability. Note that the randomness $\mathbf{r}$ used to commit to the message can be drawn from a Gaussian with any width $\sigma_2 > 0$. However, the security proofs require σ_1 to be at least $\min(2\sqrt{m_1}, \sqrt{m_1} + \sqrt{m_3} + t)\sqrt{m_3}$ in order to hide the shifted center of the Gaussian vector, which in turns restrict the value of σ_2. Additionally, the goal of this signature scheme being to allow signing on committed messages, the value of σ_2 must be chosen so that the commitment scheme is statistically hiding, which is why we take it at least $\omega(\sqrt{\log_2 m_1})$. We present our signature scheme in the most general way, thus explaining the multitude of dimensions m_i and Gaussian widths. We make this distinction to highlight the fact that these parameters can be set somewhat independently, provided they verify their respective conditions. This also allows fine-tuning of the parameters depending on the specific application. Typically, an application requiring to sign only small messages of constant bit-size m_3 would be able to select a much smaller standard deviation σ_1 and would then yield smaller signatures.

We also point out the fact that we express the shortness condition on $\mathbf{v}$ in the infinity norm. This is due to the fact that the zero-knowledge argument we consider in Section 5 to prove possession of a message-signature pair allows one to prove bounds on the coefficients more naturally. As a result, we can base the security of our signature scheme on $\mathrm{SIS}^{\infty,2}$ which is at least as hard as SIS^2 as explained in Section 2.3.

An example parameter set, also taking into account the requirements of Sections 4 and 5, can be found in Appendix F, Table F.2. The scheme can also be instantiated as a standalone signature, without considering the efficient protocols and zero-knowledge proof systems. This would allow one to reduce the size of q, but at the expense of increasing n to achieve the same security, which in the end leads to similar signature and key sizes.

3.2 Security of the Signature

We distinguish two types of forgeries that an attacker can produce, which we treat separately for the sake of clarity. More precisely we distinguish between the cases depending on whether or not the tag τ^* of the forgery has been reused from the signature queries. Combining the corresponding lemmas proves the EUF-CMA security of the signature under the SIS assumption. It consists in the SIS challenger tossing a coin and proceeding as in either Lemma 3.1 or 3.2 and aborting if the forgery does not match the coin toss. The proofs are provided in Appendix B.2 and B.3 for completeness.

Lemma 3.1. *An adversary produces a* Type I *forgery* $(\tau^*, \mathbf{v}^*)$ *if the tag* τ^* *does not collide with the tags of the signing queries. If an adversary can produce a Type I forgery with advantage* δ*, then we can construct an adversary* $\mathcal{B}$ *that* *solves the* $\mathrm{SIS}^{\infty,2}_{n,m_1,q,\beta_\infty,\beta_2}$ *problem with advantage* $Adv[\mathcal{B}] \gtrsim \delta/(|\mathcal{T}|-Q)$*, for*

$$\begin{cases} \beta_\infty &= \sigma_1 \log_2 m_1 + m_2 \sigma \log_2 m_2 + m_3 + 1 \\ \beta_2 &= \sqrt{1+(\sqrt{m_1}+\sqrt{m_2}+t)^2} \cdot \sqrt{m_1(\sigma_1 \log_2 m_1)^2 + m_2(\sigma \log_2 m_2)^2} \\ & \quad + \sqrt{m_1} + \min(2\sqrt{m_1}, \sqrt{m_1}+\sqrt{m_3}+t)\sqrt{m_3}. \end{cases}$$

Lemma 3.2. *An adversary produces a* Type II *forgery* $(\tau^*, \mathbf{v}^*)$ *if the tag* τ^* *is re-used from some* i^**-th signing query* $(\tau^{(i^*)}, \mathbf{v}^{(i^*)})$*, i.e.,* $\tau^* = \tau^{(i^*)}$*. If an adversary can produce a Type II forgery with advantage* δ*, we can construct* $\mathcal{B}$ *solving* $\mathrm{SIS}^{\infty,2}_{n,m_1,q,\beta'_\infty,\beta'_2}$ *with advantage*

$$Adv[\mathcal{B}] \gtrsim \left(1 - \frac{Q^2}{2|\mathcal{T}|}\right) \cdot \frac{\delta^{\alpha^*/(\alpha^*-1)} e^{-\alpha^* \pi}}{Q},$$

for

$$\begin{cases} \beta'_\infty &= 2\sigma_1 \log_2 m_1 + m_2 \cdot 2\sigma \log_2 m_2 + m_3 \\ \beta'_2 &= \sqrt{1+(\sqrt{m_1}+\sqrt{m_2}+t)^2} \cdot \sqrt{\sigma_1^2 m_1(1+\log_2^2 m_1) + \sigma^2 m_2(1+\log_2^2 m_2)} \\ & \quad + \min(2\sqrt{m_1}, \sqrt{m_1}+\sqrt{m_3}+t)\sqrt{m_3}. \end{cases}$$

and where $\alpha^* = 1 + \sqrt{\log_2(1/\delta)/(\pi \log_2 e)}$*.*

4 Privacy-Preserving Protocols

In this section, we present two protocols that interface well with our signature scheme, following the efficient protocols from [LLM+16]. In particular, we give a first protocol in Section 4.1 which allows a signer to obliviously sign a message, by only knowing a commitment to the message. The second protocol, presented in Section 4.2, enables a user to prove the possession of a message-signature pair, where the signature has been obtained by the oblivious signing protocol. As opposed to the protocols from [LLM+16], our protocols only feature the zero-knowledge arguments for either the commitment opening or the message-signature pair possession. In particular, we remove the encryption of the witnesses that can be useful for online extraction and thus to support concurrent protocols but that deteriorates efficiency of the ZKAoKs. We indeed recall that the goal of our paper is to provide a very flexible tool for many different use-cases and we therefore prefer avoiding such specific features that may be unnecessary in some situations. Moreover, other approaches exist to achieve online extraction, such as the one from [Fis05], and we therefore let designers of privacy-preserving protocols choose the one that is the most appropriate in their context. Throughout this section, we assume the existence of a zero-knowledge proof system compatible with the relations induced by our protocols. We explain in the next section how to instantiate it concretely with the recent framework by Yang et al. [YAZ+19].

4.1 Oblivious Signing Protocol

We present here the first protocol between a signer S and a user U. The user U is interacting with S in order to obtain a signature $(\tau, \mathbf{v})$ on a message $\mathbf{m}$, by only providing S with a commitment $\mathbf{c}$ to the message $\mathbf{m}$. We assume that Algorithms 1 and 2 have been run prior to entering the protocol but with some slight modifications that we detail below. First, instead of choosing σ_2 as in Algorithm 1, it first chooses $\sigma_3 = \omega(\sqrt{\log_2 m_1}) \geq \sqrt{2}\eta_\varepsilon(\mathbb{Z}^{m_1})$ and then

$$\sigma_4 \geq \max\left(\sqrt{\min(2\sqrt{m_1}, \sqrt{m_1} + \sqrt{m_3} + t)\sqrt{m_3} + \sigma_3\sqrt{m_1})^2 - \sigma^2}, \sqrt{2}\eta_\varepsilon(\mathbb{Z}^{m_1})\right).$$

It then re-defines $\sigma_2 = \sqrt{\sigma_3^2 + \sigma_4^2}$ and $\sigma_1 = \sqrt{\sigma^2 + \sigma_2^2}$. The new widths σ_3, σ_4 are also included in pp in addition to $\sigma, \sigma_1, \sigma_2$. We explain this change in Remark 4.1. Second, as we use the public key matrix $\mathbf{A}$ as part of the commitment matrices, we must ensure that it cannot be tempered with by the attacker. As such, we generate $\mathbf{A}$ as the hash of a public string. In the random oracle model, the matrix can be assumed to follow the prescribed uniform distribution over $\mathbb{Z}_q^{n \times m_1}$.

Algorithm 5 $\mathsf{OblSign}$: Oblivious Signing Interactive Protocol

```
Input: Signer $S$ with $\mathsf{sk}, \mathsf{pk}, \mathsf{pp}$, and a user $U$ with $\mathbf{m} \in \{0,1\}^{m_3}$ and $\mathsf{pk}, \mathsf{pp}$.
    User $U$.
    $\mathbf{r}' \hookleftarrow \mathcal{D}_{\mathbb{Z}^{m_1}, \sigma_3}$.                    ▷ $\sigma_3 \geq \sqrt{2}\eta_\varepsilon(\mathbb{Z}^{m_1})$
    $\mathbf{c} \leftarrow \mathbf{A}\mathbf{r}' + \mathbf{D}\mathbf{m} \bmod q$.
    Send $\mathbf{c}$ to $S$.
    User $U \leftrightarrow$ Signer $S$.
    Interactive zero-knowledge argument between $U$ and $S$, where $U$ proves that $\mathbf{c}$ is
    commitment to $\mathbf{m}$ with randomness $\mathbf{r}'$. If $S$ is not convinced, the protocol aborts.
    Signer $S$.
    $\mathbf{r}'' \hookleftarrow \mathcal{D}_{\mathbb{Z}^{m_1}, \sigma_4}$.
    $\mathbf{c}' \leftarrow \mathbf{c} + \mathbf{A}\mathbf{r}'' \bmod q$.
    $\tau \hookleftarrow U(\mathcal{T})$.
    $\mathbf{v}' \leftarrow \mathsf{SampleD}(\mathbf{R}, \mathbf{A}, \tau\mathbf{I}_n, \mathbf{u} + \mathbf{c}', \sigma) - [\mathbf{r}''^T | \mathbf{0}]^T$.
    Send $(\tau, \mathbf{v}')$ to $U$.
    User $U$.
    $\mathbf{v} \leftarrow \mathbf{v}' - [\mathbf{r}'^T | \mathbf{0}]^T$.
    if $\mathsf{Verify}(\mathsf{pk}; \mathbf{m}; (\tau, \mathbf{v}); \mathsf{pp}) = 1$, then return $(\tau, \mathbf{v})$.      ▷ Algorithm 4
    else return $\perp$
```

Remark 4.1. Notice that Algorithm 5 does not exactly rely on the signature scheme of the previous section. This is due to the fact that the signer S also contributes to the randomness of the commitment to the message $\mathbf{m}$ via $\mathbf{r}''$. If the randomness came only from the user U, the signer, who is embodied by the SIS adversary in the security proofs, would have no control over the randomness part of the signing query. In the proof of Lemma 3.1 (and Lemma 3.2 for the i-th

Algorithm 6 Prove: Message-Signature Pair Possession

```
Input: User $U$ with $\mathsf{pk}, \mathsf{pp}, \mathbf{m}, (\tau, \mathbf{v})$, and a verifier $V$ with $\mathsf{pk}, \mathsf{pp}$.
    User $U \leftrightarrow$ Verifier $V$.
    Interactive zero-knowledge argument between $U$ and $V$, where $U$ proves knowledge
    of $(\mathbf{m}; (\tau, \mathbf{v}))$ such that $\mathsf{Verify}(\mathsf{pk}; \mathbf{m}; (\tau, \mathbf{v}); \mathsf{pp}) = 1$.
```

query with $i \neq i^+$), the randomness $\mathbf{r}$ is legitimately sampled from $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_2}$. As such, it could instead be sampled as $\mathbf{r}' + \mathbf{r}''$ with $\mathbf{r}' \hookleftarrow \mathcal{D}_{\mathbb{Z}^{m_1},\sigma_3}$ sampled by the forger $\mathcal{A}$, and $\mathbf{r}'' \hookleftarrow \mathcal{D}_{\mathbb{Z}^{m_1},\sigma_4}$ sampled by the SIS adversary, thus matching with Algorithm 5. This would restrict $\sigma_2 = \sqrt{\sigma_3^2 + \sigma_4^2}$. If $\sigma_3, \sigma_4 \geq \sqrt{2}\eta_\varepsilon(\mathbb{Z}^{m_1})$, Lemma 2.2 guarantees that $\mathbf{r}' + \mathbf{r}''$ is $7\varepsilon/4$-close to $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_2}$ as required. However, when dealing with the i^+-th query in Lemma 3.2, the SIS adversary needs to control part of the randomness. At this step of the proof, $\mathbf{r}_0$ would be distributed according to $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_4}$, and it would construct ${\mathbf{v}'_1}^{(i^+)} = \mathbf{v}_1 - (\mathbf{r}_0 - \mathbf{U}\mathbf{m}^{(i^+)} - {\mathbf{r}'}^{(i^+)})$ with ${\mathbf{r}'}^{(i^+)}$ sampled from $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_3}$ by the forger $\mathcal{A}$. The rest of the proof remains the same, but this modification introduces the condition $\sqrt{\sigma^2 + \sigma_4^2} \geq \alpha + \sigma_3\sqrt{m_1}$, where $\alpha = \min(2\sqrt{m_1}, \sqrt{m_1} + \sqrt{m_3} + t)\sqrt{m_3}$. It yields $\sigma_2 \geq \sqrt{(\alpha + \sigma_3\sqrt{m_1})^2 + \sigma_3^2 - \sigma^2}$, leading to $\sigma_1 = \sqrt{\sigma^2 + \sigma_2^2} \geq \sqrt{(\alpha + \sigma_3\sqrt{m_1})^2 + \sigma_3^2}$ instead of just α before. In most applications, m_3 would be much larger than σ_3 and therefore it would entail only a mild increase of σ_1.

4.2 Message-Signature Pair Possession Protocol

The second protocol provides a user, who obtained a certificate $\mathsf{sig} = (\tau, \mathbf{v})$ on a message $\mathbf{m}$ from the $\mathsf{OblSign}$ protocol above, with the ability to prove possession of this valid message-signature pair. For that, they only have to prove that $\mathsf{Verify}(\mathsf{pk}; \mathbf{m}; (\tau, \mathbf{v}); \mathsf{pp}) = 1$ without revealing either of $\mathbf{m}$ nor $(\tau, \mathbf{v})$. The protocol of Algorithm 6 thus simply consists in using the ZKAoK presented in Section 5.2.2 to prove this relation. The proof can be made non-interactive in the random oracle model using the Fiat-Shamir transform. This also allows one to turn it into a signature of knowledge by including the message in the challenges of the proof.

5 Zero-Knowledge Arguments of Knowledge

We now detail out the zero-knowledge arguments of knowledge (ZKAoK) that we use to instantiate the protocols from Section 4. We could have used Stern-like protocols but this would only reach constant soundness error, thus implying a large number of repetitions and hence bad performance. Additionally, the decomposition-extension methods used in the original scheme [LLM+16] make the relation to be proven much larger. To circumvent these two shortcomings, we instead use the more recent framework by Yang et al. [YAZ+19]. It combines the perks of Stern-like ZKAoK and Fiat-Shamir with Aborts ZKAoK to reach a framework with standard soundness and inverse polynomial soundness error. This requires fewer iterations as a result. Additionally, this framework avoids the extensions which were used to interface well with permutations in Stern-like ZKAoK. This limits the size of the witness. The decomposition steps are however used to prove the shortness of the witnesses. More precisely, the framework of [YAZ+19] provides a ZKAoK for the relation

$$\mathcal{R}^* = \{((\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}); \mathbf{x}) \in (\mathbb{Z}_q^{k\times L_{\mathbf{x}}} \times \mathbb{Z}_q^k \times ([L_{\mathbf{x}}]^3)^{L_{\mathcal{M}}}) \times \mathbb{Z}_q^{L_{\mathbf{x}}} : \ \overline{\mathbf{A}}\mathbf{x} = \mathbf{y} \bmod q \\ \wedge \forall (h,i,j) \in \mathcal{M}, \mathbf{x}[h] = \mathbf{x}[i]\cdot\mathbf{x}[j] \bmod q\}.$$

This relation can be used to prove that the witness vector is short, which we need for our verification equation for example. Concretely, any witness $x \in \mathbb{Z}_q$ that we need to prove smaller than some bound B is decomposed as $x_1, \ldots, x_\ell$, where $\ell = \lceil \log_2 B \rceil$, which are proved binary using the quadratic relation $x_i^2 = x_i \bmod q$. The downside of this approach is that it adds ℓ witnesses for each short element, which quickly becomes cumbersome. To address this issue, the authors of [YAZ+19] introduced a so-called *fast mode* that significantly reduces the size of the witness. We describe such a mode in Section 5.1 but also show that its analysis in [YAZ+19] is not entirely correct and thus provide a more thorough one. Then, in Section 5.2 we show how to transform our relations so that they match $\mathcal{R}^*$. In particular, Section 5.2.1 is dedicated to the proof of knowledge of a commitment opening as needed in Algorithm 5. Then, in Section 5.2.2 we instantiate it to prove knowledge of a message-signature pair for our signature scheme as required by Section 4.2. We also propose additional optimizations on the framework of [YAZ+19], which we defer in Appendix E.

5.1 Zero-Knowledge Fast Mode Revisited

As explained above, the decomposition technique entails a $(\ell+1)$-fold increase of the witness, which is prohibitive for high-dimensional vectors. This has led the authors of [YAZ+19] to sketch a so-called *fast mode* to obtain drastic efficiency gains in this case. The idea is to relax the zero-knowledge argument, thus introducing a soundness gap, and prove knowledge of a solution $\mathbf{w}'$ of $\mathbf{P}\mathbf{w}' = \mathbf{v} \bmod q$ such that $\mathbf{w}'$ is nB-bounded instead of B-bounded, where n is the dimension of $\mathbf{w}$. More precisely, they consider the following relation

$$\begin{aligned}\mathcal{R}'_{\text{short}} = \{((\mathbf{P}, \mathbf{v}, \mathbf{H}, \mathbf{c}), (\mathbf{w}, \mathbf{u}, \mathbf{r})) \in \\ (\mathbb{Z}_q^{m\times n} \times \mathbb{Z}_q^m \times [0,1]^{\lambda\times n} \times \mathtt{C}) \times (\mathbb{Z}_q^n \times [0, nB]^\lambda \times \mathtt{R}) : \\ \mathbf{P}\mathbf{w} = \mathbf{v} \bmod q \wedge \mathbf{H}\mathbf{w} - \mathbf{u} = \mathbf{0} \bmod q \wedge \mathbf{c} = \mathtt{Commit}(\mathbf{w}; \mathbf{r})\}\end{aligned}$$

The point is that the prover now only has to prove a bound on the λ elements of $\mathbf{u}$ instead of the n elements from $\mathbf{w}$, which is very interesting when $\lambda \ll n$, a condition easily met in practice. The authors argue that, if one knows a witness $(\mathbf{w}, \mathbf{u}, \mathbf{r})$ satisfying $\mathcal{R}'_{\text{short}}$, it ensures that $\mathbf{w}$ is in $[0, nB]^n$, except with negligible probability over the randomness of $\mathbf{H}$. We provide a simple counterexample to the above. For example, assume a prover knows $\mathbf{w} = [-1, 1, \ldots, 1]^T$

such that $\mathbf{P}\mathbf{w} = \mathbf{v} \bmod q$. We now consider $\mathbf{H}$ to be a random matrix whose entries are independently distributed according to $U(\{0,1\})$. We denote by $\mathbf{h}_i$ the i-th row of $\mathbf{H}$ for $i \in [\lambda]$. For all $i \in [\lambda]$, we have $\mathbb{P}_{\mathbf{h}_i}[\mathbf{h}_i^T\mathbf{w} \in [0, nB]] = 1 - 2^{-n}$ by simply conditioning on the first coefficient of $\mathbf{h}_i$. It yields $\mathbb{P}_{\mathbf{H}}[\mathbf{H}\mathbf{w} \in [0, nB]^\lambda] = (1-2^{-n})^\lambda \geq 1 - \lambda 2^{-n}$. Since the *fast mode* is only relevant when $n \geq \lambda$, it holds that $\mathbf{H}\mathbf{w} \in [0, nB]^\lambda$ with overwhelming probability. This shows that $\mathcal{R}'_{\text{short}}$ cannot be used to prove that $\mathbf{w}$ has non-negative coefficients and thus for example invalidates the use of the fast mode in the e-cash use-case in [YAZ+19].

Fortunately, a more thorough analysis shows that $\mathbf{H}\mathbf{w} \bmod q$ is in $[0, B]^\lambda$ implies that $\mathbf{w} \bmod q \in [-2B, 2B]^n$ with high probability, which would be sufficient in our case as we only need to prove bounds on the infinity norm. However, we have so far only discussed of soundness. When it comes to correctness, we note that the choices made in [YAZ+19] results in an unwieldy situation.

First, because one has to set an upper bound on $\mathbf{H}\mathbf{w}$ that will be satisfied with high probability for any $\mathbf{w}$ in $[-B, B]^n$. For a binary matrix $\mathbf{H}$, it seems hard to do much better than $[-nB, nB]^\lambda$ since we will be close to this bound for $\mathbf{w} = [B, \ldots, B]^T$, hence the factor n in the soudness gap mentioned above.

Second, because one cannot start the argument with $\mathbf{w} \in [-B, B]^n$ as it can lead to having $\mathbf{H}\mathbf{w}$ with negative coefficients. One must shift all the coefficients of $\mathbf{w}$ before running the protocol, but it results in a skewed statement on $\mathbf{w}$. Indeed, it would prove that $\mathbf{w} + B\mathbf{1}_n$ is in $[-2nB, 2nB]^n$ and therefore that $\mathbf{w} \in [-(2n+1)B, (2n-1)B]^n$, where $\mathbf{1}_n = [1 \ldots 1]^T \in \mathbb{Z}^n$.

For these reasons, we believe it is much more natural to sample the coefficients of $\mathbf{H}$ uniformly from $\{-1, 0, 1\}$. We prove below that $\mathbf{H}\mathbf{w} \bmod q$ is in $[-B, B]^\lambda$ still implies that $\mathbf{w} \bmod q \in [-2B, 2B]^n$, which avoids to shift the witness and thus the problem mentioned above. Moreover, such distribution of $\mathbf{H}$ allows us to derive much better upper bounds on $\mathbf{H}\mathbf{w}$ using for example an argument similar to the one of lemma 2.5. However, we do not study more thoroughly this general problem as we are able to derive sharp bounds for our specific use case (see remark 5.1 below).

More formally, let $\mathbf{H} \in \{-1, 0, 1\}^{k \times n}$, with $k = \lambda / \log_2(9/5)$. The following lemma, proven in Appendix C, argues that $\mathbf{H}\mathbf{w} \bmod q \in [-B, B]^k$ implies $\mathbf{w} \bmod q \in [-2B, 2B]^n$ with overwhelming probability over the choice of $\mathbf{H}$.

Lemma 5.1. *Let $B \in \mathbb{Z}$ be such that $6B < q/2$. Let k be a positive integer. Let $\mathbf{w} \in \mathbb{Z}^n$ be a vector. Assuming that $\|\mathbf{w} \bmod q\|_\infty > 2B$, it then holds that $\mathbb{P}_{\mathbf{H} \hookleftarrow U([-1,1]^{k \times n})}[\|\mathbf{H}\mathbf{w} \bmod q\|_\infty \leq B] \leq (5/9)^k$.*

The fast mode that we consider now corresponds to the following relation, where B is chosen so that $\|\mathbf{H}\mathbf{w} \bmod q\|_\infty \leq B$ with overwhelming probability for an honest witness $\mathbf{w}$.

$$\begin{aligned}\mathcal{R}''_{\text{short}} = \{&((\mathbf{P}, \mathbf{v}, \mathbf{H}, \mathbf{c}), (\mathbf{w}, \mathbf{u}, \mathbf{r})) \in \\ &(\mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^m \times \{-1, 0, 1\}^{k \times n} \times \mathtt{C}) \times (\mathbb{Z}_q^n \times [-B, B]^k \times \mathtt{R}) : \\ &\mathbf{P}\mathbf{w} = \mathbf{v} \bmod q \wedge \mathbf{H}\mathbf{w} - \mathbf{u} = \mathbf{0} \bmod q \wedge \mathbf{c} = \mathtt{Commit}(\mathbf{w}; \mathbf{r})\}\end{aligned}$$

Remark 5.1. For our relations, the vectors that we need to prove short are sampled from discrete Gaussian distributions. For example the vector $\mathbf{v}_1$ follows $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_1}$. For a fixed $\mathbf{H} \in \{-1,0,1\}^{k \times m_1}$, the third statement of Lemma 2.3 yields that $\mathbb{P}_{\mathbf{v}_1}[|\langle \mathbf{v}_1, \mathbf{h}_i \rangle| \geq \sigma_1 t \sqrt{m_1}] \leq \mathbb{P}_{\mathbf{v}_1}[|\langle \mathbf{v}_1, \mathbf{h}_i \rangle| \geq \sigma_1 t \|\mathbf{h}_i\|_2] \leq 2e^{-\pi t^2}$, where $\mathbf{h}_i$ is the i-th row of $\mathbf{H}$ and the first inequality follows by event inclusion as $\|\mathbf{h}_i\|_2 \leq \sqrt{m_1}$. The union bound yields $\mathbb{P}_{\mathbf{v}_1}[\|\mathbf{H}\mathbf{v}\|_\infty \geq \sigma_1 t \sqrt{m_1}] \leq 2ke^{-\pi t^2}$, where $k = \lambda / \log_2(9/5)$ as per Lemma 5.1. Hence, taking $t = \log_2 \lambda$ gives that $\|\mathbf{H}\mathbf{v}\|_\infty \leq \sigma_1 \sqrt{m_1} \log_2 \lambda$ with overwhelming probability. This improves on the trivial bound $\sigma_1 m_1 \log_2 m_1$. By making sure that $2\sigma_1 \sqrt{m_1} \log_2 \lambda < (q-1)/2$, which is generally the case, we have that there is no reduction modulo q in $\mathbf{H}\mathbf{v}_1$ and therefore $\|\mathbf{H}\mathbf{v}_1 \bmod q\|_\infty \leq \sigma_1 \sqrt{m_1} \log_2 \lambda$. The conditions of Lemma 5.1 allow one to choose $B = \sigma_1 \sqrt{m_1} \log_2 \lambda \ll q/12$. Then, proving that $\|\mathbf{H}\mathbf{v}_1 \bmod q\|_\infty \leq \sigma_1 \sqrt{m_1} \log_2 \lambda$ implies that $\|\mathbf{v}_1 \bmod q\|_\infty \leq 2\sigma_1 \sqrt{m_1} \log_2 \lambda$.

5.2 Instantiating the Protocols

We now give more details on how to instantiate our relations in the zero-knowledge framework of [YAZ+19]. In practice, we introduce some optimizations, consisting in compacting the commitments of the original framework and in better parameter selections, which leads to substantial efficiency improvements. However, as their description is not required to understand the protocols described in this section, we postpone it to Appendix E due to lack of space.

5.2.1 Proof of Commitment Opening. Consider a prover with private input $\mathbf{m} \in \{0,1\}^{m_3}$ and $\mathbf{r}' \sim \mathcal{D}_{\mathbb{Z}^{m_1},\sigma_3}$, and public input pp, pk. Recall that by Lemma 2.3, we have $\|\mathbf{r}'\|_\infty \leq \sigma_3 \log_2 m_1$ with overwhelming probability. We can thus define $\alpha_3 = \lceil \sigma_3 \log_2 m_1 \rceil$ and assume that $\mathbf{r}' \in [-\alpha_3, \alpha_3]^{m_1}$. The prover wishes to prove that

$$\mathbf{A}\mathbf{r}' + \mathbf{D}\mathbf{m} = \mathbf{c} \bmod q \ \wedge \ \|\mathbf{r}'\|_\infty \leq \alpha_3 \ \wedge \ \mathbf{m} \in \{0,1\}^{m_1}.$$

We thus transform this relation into one that fits the Yang et al. framework. For that, we first define $\mathbf{a}_3 = \alpha_3 \mathbf{1}_{m_1}$. Next, we define $\mathbf{r}'' = \mathbf{r}' + \mathbf{a}_3 \in [0, 2\alpha_3]^{m_1}$. Let $k_{\alpha_3} = \lfloor \log_2 2\alpha_3 \rfloor + 1$ and define[6] $\mathbf{g}_{\alpha_3} = [\lfloor (2\alpha_3 + 2^{i-1})/2^i \rfloor]_{i \in [k_{\alpha_3}]} \in \mathbb{Z}^{1 \times k_{\alpha_3}}$, and $\mathbf{G}_{\alpha_3} = \mathbf{I}_{m_1} \otimes \mathbf{g}_{\alpha_3}$. We then denote by $\overline{\mathbf{r}'} \in \{0,1\}^{m_1 k_{\alpha_3}}$ a binary decomposition of $\mathbf{r}''$ along $\mathbf{g}_{\alpha_3}$, i.e., that verifies $\mathbf{r}'' = \mathbf{G}_{\alpha_3} \overline{\mathbf{r}'}$. Such a decomposition can be efficiently computed. It now suffices to prove the following

$$\mathbf{A}\mathbf{G}_{\alpha_3} \overline{\mathbf{r}'} + \mathbf{D}\mathbf{m} = \mathbf{c} + \mathbf{A}\mathbf{a}_3 \bmod q \ \wedge \ \overline{\mathbf{r}'} \in \{0,1\}^{m_1 k_{\alpha_3}} \ \wedge \ \mathbf{m} \in \{0,1\}^{m_3}.$$

By defining $\overline{\mathbf{A}} = [\mathbf{A}\mathbf{G}_{\alpha_3} | \mathbf{D}]$, $\mathbf{x} = [\overline{\mathbf{r}'}^T | \mathbf{m}^T]^T$, $\mathbf{y} = \mathbf{c} + \mathbf{A}\mathbf{a}_3$ and $\mathcal{M} = \{(i,i,i); i \in [m_1 k_{\alpha_3} + m_3]\}$, we have $((\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}); \mathbf{x}) \in \mathcal{R}^*$. The length of the witness is $L_{\mathbf{x}} = m_1 k_{\alpha_3} + m_3$, and the size of $\mathcal{M}$ is $L_{\mathcal{M}} = L_{\mathbf{x}}$. Note that since q is prime, the constraint $\mathbf{x}[i] = \mathbf{x}[i]^2 \bmod q$ indeed implies that $\mathbf{x}[i] \in \{0,1\}$.

[6] Choosing $\mathbf{g}_{\alpha_3}$ this way ensures that for any binary vector $\mathbf{x}$, $\mathbf{g}_{\alpha_3}\mathbf{x} \in [0, 2\alpha_3]$.

When $m_3 \gg \lambda$, the *fast mode* of Section 5.1 compresses the size of the witness and the constraint set as we now prove that $\mathbf{H}\mathbf{r}'$ has coefficients bounded by $\sigma_3\sqrt{m_1}\log_2\lambda$. It yields a witness of size $L_{\mathbf{x}} = m_1 + k(\lfloor\log_2(2\sigma_3\sqrt{m_1}\log_2\lambda)\rfloor + 1) + m_3$, with $k = \lambda/\log_2(9/5)$, and $L_{\mathcal{M}} = L_{\mathbf{x}} - m_1$.

5.2.2 Proof of Message-Signature Pair Possession. Here, the prover has a private input $\mathbf{m} \in \{0,1\}^{m_3}$ and $(\tau, \mathbf{v}) \in \mathbb{Z}_q \times \mathbb{Z}^m$ and has to prove

$$\mathbf{A}\mathbf{v}_1 - \mathbf{B}\mathbf{v}_2 + \tau\mathbf{G}\mathbf{v}_2 - \mathbf{D}\mathbf{m} = \mathbf{u} \bmod q,$$

where $\mathbf{v}_1 \in \mathbb{Z}^{m_1}$ and $\mathbf{v}_2 \in \mathbb{Z}^{m_2}$, with $\|\mathbf{v}_1\|_\infty \le \sigma_1\log_2 m_1$, $\|\mathbf{v}_2\|_\infty \le \sigma\log_2 m_2$, $\tau \in \mathcal{T}$ and $\mathbf{m} \in \{0,1\}^{m_3}$. We define

$$\begin{cases} \alpha_1 = \lceil\sigma_1\log_2 m_1\rceil & k_{\alpha_1} = \lfloor\log_2 2\alpha_1\rfloor + 1 & \mathbf{g}_{\alpha_1} = \left[\lfloor(2\alpha_1 + 2^{i-1})/2^i\rfloor\right]_{i\in[k_{\alpha_1}]} \\ \alpha = \lceil\sigma\log_2 m_2\rceil & k_\alpha = \lfloor\log_2 2\alpha\rfloor + 1 & \mathbf{g}_\alpha = \left[\lfloor(2\alpha + 2^{i-1})/2^i\rfloor\right]_{i\in[k_\alpha]} \\ & k_{q'} = \lfloor\log_2 q'\rfloor + 1 & \mathbf{g}_{q'} = \left[\lfloor(q' + 2^{i-1})/2^i\rfloor\right]_{i\in[k_{q'}]} \end{cases}$$

Further, we define $\mathbf{a}_1 = \alpha_1\mathbf{1}_{m_1}$ and $\mathbf{a} = \alpha\mathbf{1}_{m_2}$. Next, we set $\mathbf{G}_{\alpha_1} = \mathbf{I}_{m_1} \otimes \mathbf{g}_{\alpha_1}$, and $\mathbf{G}_\alpha = \mathbf{I}_{m_2} \otimes \mathbf{g}_\alpha$. We define $\mathbf{v}_1' = \mathbf{v}_1 + \mathbf{a}_1$, and $\mathbf{v}_2' = \mathbf{v}_2 + \mathbf{a}$. We then denote $\overline{\mathbf{v}_j}$ their respective binary decomposition along $\mathbf{g}_{\alpha_1}, \mathbf{g}_\alpha$, i.e., such that $\mathbf{G}_{\alpha_1}\overline{\mathbf{v}_1} = \mathbf{v}_1'$, and $\mathbf{G}_\alpha\overline{\mathbf{v}_2} = \mathbf{v}_2'$. We also denote by $\overline{\tau}$ the binary decomposition of τ along $\mathbf{g}_{q'}$ such that $\tau = \mathbf{g}_{q'}\overline{\tau}$. We need however to deal with the product term $\tau\mathbf{v}_2$. We use the same idea as for subset-sums from the framework of Yang et al. [YAZ+19]. For that, we define $\mathbf{u}_2 = \mathbf{G}\mathbf{v}_2 \in \mathbb{Z}^n$, and $\mathbf{u}_2' = \tau\mathbf{u}_2$. This gives an additional linear relation, but fewer decompositions. The prover now has to prove that

$$\begin{cases} \mathbf{A}\mathbf{G}_{\alpha_1}\overline{\mathbf{v}_1} - \mathbf{B}\mathbf{G}_\alpha\overline{\mathbf{v}_2} + \mathbf{u}_2' - \mathbf{D}\mathbf{m} = \mathbf{u} + \mathbf{A}\mathbf{a}_1 - \mathbf{B}\mathbf{a} \bmod q \\ \mathbf{G}\mathbf{G}_\alpha\overline{\mathbf{v}_2} - \mathbf{u}_2 = \mathbf{G}\mathbf{a} \bmod q \\ -\tau + \mathbf{g}_{q'}\overline{\tau} = 0 \bmod q \end{cases}$$

We thus define $\mathbf{x} = [\tau|\overline{\tau}|\overline{\mathbf{v}_1}|\overline{\mathbf{v}_2}|\mathbf{m}|\mathbf{u}_2|\mathbf{u}_2'] \in \mathbb{Z}^{L_{\mathbf{x}}}$, where $L_{\mathbf{x}} = 1 + k_{q'} + m_1k_{\alpha_1} + m_2k_\alpha + m_3 + 2n$, as well as

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{0}_{n\times1} & \mathbf{0}_{n\times k_{q'}} & \mathbf{A}\mathbf{G}_{\alpha_1} & -\mathbf{B}\mathbf{G}_\alpha & -\mathbf{D} & \mathbf{0}_{n\times n} & \mathbf{I}_n \\ \mathbf{0}_{n\times1} & \mathbf{0}_{n\times k_{q'}} & \mathbf{0}_{n\times m_1k_{\alpha_1}} & \mathbf{G}\mathbf{G}_\alpha & \mathbf{0}_{n\times m_3} & -\mathbf{I}_n & \mathbf{0}_{n\times n} \\ -1 & \mathbf{g}_{q'} & \mathbf{0}_{n\times m_1k_{\alpha_1}} & \mathbf{0}_{1\times m_2k_\alpha} & \mathbf{0}_{1\times m_3} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \end{bmatrix}$$

and $\mathbf{y} = [\mathbf{u} + \mathbf{A}\mathbf{a}_1 - \mathbf{B}\mathbf{a}|\mathbf{G}\mathbf{a}'|0] \bmod q \in \mathbb{Z}_q^{2n+1}$. Finally, we define $\mathcal{M}_1 = \{(i,i,i); i \in [2, 1 + k_{q'} + m_1k_{\alpha_1} + m_2k_\alpha + m_3]\}$, which corresponds to the coefficients that need to be binary. We then need to add the relations $\mathbf{u}_2' = \tau\mathbf{u}_2$. For that, we define

$$\begin{aligned} \mathcal{M}_2 = \{(&1 + k_{q'} + m_1k_{\alpha_1} + m_2k_\alpha + m_3 + n + i, 1, \\ &1 + k_{q'} + m_1k_{\alpha_1} + m_2k_\alpha + m_3 + i);\ \ i \in [n]\}, \end{aligned}$$

and construct $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. The witness has length $L_{\mathbf{x}}$, and $\mathcal{M}$ is of size $L_{\mathcal{M}} = L_{\mathbf{x}} - n - 1$. Using the *fast mode* instead proves that $\mathbf{H}_1\mathbf{v}_1, \mathbf{H}_2\mathbf{v}_2$ have coefficients bounded by $\sigma_1\sqrt{m_1}\log_2\lambda$ and $\sigma\sqrt{m_2}\log_2\lambda$ respectively. It yields a witness of size $L_{\mathbf{x}} = 1 + k_{q'} + m_1 + m_2 + m_3 + 2n + k(\lfloor \log_2(2\sigma_1\sqrt{m_1}\log_2\lambda)\rfloor + \lfloor \log_2(2\sigma\sqrt{m_2}\log_2\lambda)\rfloor + 2)$, with $k = \lambda/\log_2(9/5)$, and $L_{\mathcal{M}} = L_{\mathbf{x}} - m_1 - m_2 - n - 1$.

Remark 5.2. In the case where $q' = q$, the tag does not need to be decomposed in binary form. However, when the proof system is run only a few number of times, we need to drastically increase the size of challenges to reach a negligible soundness error. For example, to obtain a negligible soundness error in one iteration, one needs to take challenges of size $p = 2^{\lambda}$. Because the SIS bound for the proof system is $\beta_\infty = \mathsf{poly}(\lambda) \cdot p^2$, one must take q to be polynomially larger than p^2. In Algorithm 1, choosing $q' = q$ then leads to a tag space $\mathcal{T}$ of size at least $\mathsf{poly}(\lambda)2^{2\lambda}$. As a result, the proof of Lemma 3.1 incurs an exponential reduction loss of $1/|\mathcal{T}| = 2^{-2\lambda}/\mathsf{poly}(\lambda)$. To circumvent this limitation, one can choose $q' = \mathsf{poly}(\lambda) \ll q$ to make the reduction loss acceptable. It implies that the signature verification must ensure that $\tau < q'$, which we consider when proving possession of a message-signature pair.

6 Our Signature on Modules

The results of Table 1.1 show that the performances of the signature scheme from Section 3 and associated protocols are dramatically improved over [LLM+16]. However, the complexity is still rather high and we therefore investigate in this section a way to decrease it. Concretely, we show that the signature scheme from Section 3 can be extended over the ring of integers of a number field. For the zero-knowledge arguments required by the efficient protocols, we employ the recent results of Lyubashevsky, Nguyen and Plançon [LNP22]. We use a tag space that corresponds to the identity space of their group signature construction. We also use a message space that is similar to the latter but with no restriction on the number of non-zero coefficients. We present our construction with a single power-of-two cyclotomic ring, but we note that it can be adapted to use subrings for efficiency gains. For more details on the use of subrings, we refer to [LNPS21,LNP22]. In what follows, we take n a power of two and R the $2n$-th cyclotomic ring, i.e., $R = \mathbb{Z}[X]/\langle X^n + 1\rangle$. We also define $R_q = \mathbb{Z}_q[X]/\langle X^n + 1\rangle$ for any modulus $q \geq 2$. We call θ the coefficient embedding of R, i.e., for all $r = \sum_{i\in[0,n-1]} r_i X^i \in R$, $\theta(r) = [r_0 \ldots r_{n-1}]^T$. We then define $R_2 = \theta^{-1}(\{0,1\}^n)$ and $R_{\pm 1} = \theta^{-1}(\{-1,0,1\}^n)$. We also define the usual norms $\|\cdot\|_p$ over R by $\|r\|_p := \|\theta(r)\|_p$. Finally, we define the discrete Gaussian distribution over R by $\theta^{-1}(\mathcal{D}_{\theta(R),\sigma})$, which we denote by $\mathcal{D}_{R,\sigma}$.

Remark 6.1. The Gaussian distributions are defined with respect to the coefficient embedding θ. Theoretical works usually define Gaussian distributions with respect to the Minkowski embedding (or canonical embedding) σ_H. We refer to [LPR13] for more details. In our specific case of power-of-two cyclotomic rings, it holds that $\sigma_H = \sqrt{n}\mathbf{P}\theta$ where $\mathbf{P}$ is a unitary matrix. Hence, by denoting $\mathcal{D}^{\theta}_{R,\sigma}$ (resp. $\mathcal{D}^{\sigma_H}_{R,\sigma}$) the Gaussian distribution with respect to θ (resp. σ_H), we can show that $\mathcal{D}^{\sigma_H}_{R,\sigma\sqrt{n}}$ is exactly the same distribution as $\mathcal{D}^{\theta}_{R,\sigma}$.

6.1 Description of the Signature

The description of our module signature scheme is provided in Algorithms 7, 8, 9 and 10.

Algorithm 7 Setup

```
Input: Security parameter $\lambda$.
Choose a positive integer $d$.
Choose $k \leq n$ to be a power of two.
Choose a prime integer $q$ such that $q = 2k + 1 \bmod 4k$ and $q \geq (2\sqrt{k})^k$.
Choose positive integers $w, \kappa$.
$\mathcal{T}_w \leftarrow \{\tau \in R_2 : \|\tau\|_2 = \sqrt{w}\}$.                          ▷ Tag space
$g \leftarrow \lceil q^{1/\kappa} \rfloor$.
$m_1 \leftarrow \lceil (d \log_2 q + f(\lambda))/\log_2 3 \rceil$                  ▷ $f(\lambda) = \omega(\log_2 \lambda)$
$m_2 \leftarrow d\kappa$
$m \leftarrow m_1 + m_2$.                                            ▷ Signature dimension
Choose a positive integer $m_3$.                  ▷ Maximum bit-size of $\mathbf{m}$ is $n \cdot m_3$
$\mathbf{g} = [1 \cdots g^{\kappa-1}] \in R_q^{1\times\kappa}$.                                 ▷ Gadget vector
$r \leftarrow \eta_\varepsilon(\mathbb{Z})$.                              ▷ $r = 5.4$ leads to $\varepsilon \approx 2^{-131}$
Choose $t > 0$.                                                ▷ Spectral norm slack
$\sigma \leftarrow r\sqrt{g^2+1}\sqrt{(\sqrt{nm_1}+\sqrt{nm_2}+t)^2+1}$.       ▷ Pre-image sampling width
$\sigma_2 \leftarrow \sqrt{(\sqrt{nm_1}+\sqrt{nm_3}+t)^2 \cdot nm_3 - \sigma^2}$.   ▷ Commitment randomness width
$\sigma_1 \leftarrow \sqrt{\sigma^2 + {\sigma_2}^2}$.
$\mathbf{D} \hookleftarrow U(R_q^{d\times m_3})$.                           ▷ Message Commitment Key
Output: $\mathsf{pp} = (\mathbf{D}; \mathbf{g}; \lambda, n, d, m_1, m_2, m_3, q, w, \kappa, \sigma, \sigma_2, \sigma_1)$.
```

Algorithm 8 KeyGen

```
Input: Public parameters $\mathsf{pp}$ as in Algorithm 7.
$\mathbf{A} \hookleftarrow U(R_q^{d\times m_1})$.
$\mathbf{R} \hookleftarrow U(R_{\pm1}^{m_1\times m_2})$.
$\mathbf{B} \leftarrow \mathbf{AR} \bmod qR \in R_q^{d\times m_2}$.
$\mathbf{u} \hookleftarrow U(R_q^d)$.
Output: $\mathsf{pk} = (\mathbf{A}, \mathbf{B}, \mathbf{u})$, and $\mathsf{sk} = \mathbf{R}$.
```

6.2 Security of the Signature on Modules

The security of the scheme is now based on the M-SIS$_{d,m_1,q,\beta}$ problem. It asks to find $\mathbf{w} \in R^{m_1}$ such that $\mathbf{Aw} = \mathbf{0} \bmod qR$ and $0 < \|\mathbf{w}\|_2 \leq \beta$ given $\mathbf{A} \hookleftarrow$

Algorithm 9 Sign

```
Input: Signing key sk, Message $\mathbf{m} \in R_2^{m_3}$, Public key pk, Public Parameters pp.
$\mathbf{r} \hookleftarrow \mathcal{D}_{R^{m_1},\sigma_2}$.
$\mathbf{c} \leftarrow \mathbf{Ar} + \mathbf{Dm} \bmod qR$.                              ▷ Commitment to $\mathbf{m}$
$\tau \hookleftarrow U(\mathcal{T}_w)$.
$\mathbf{v} \leftarrow \mathsf{SampleD}(\mathbf{R}, \mathbf{A}, \tau\mathbf{I}_d, \mathbf{u} + \mathbf{c}, \sigma) - [\mathbf{r}^T|\mathbf{0}_{m_2}]^T$.     ▷ $\mathbf{A}_\tau = [\mathbf{A}|\tau(\mathbf{I}_d \otimes \mathbf{g}) - \mathbf{B}]$
Output: $\mathsf{sig} = (\tau, \mathbf{v})$.
```

Algorithm 10 Verify

```
Input: Public key pk, Message $\mathbf{m} \in R_2^{m_3}$, Signature sig, Public Parameters pp.
$\mathbf{A}_\tau \leftarrow [\mathbf{A}|\tau(\mathbf{I}_d \otimes \mathbf{g}) + \mathbf{B}] \in R_q^{d\times m}$.
if $(\mathbf{A}_\tau\mathbf{v} = \mathbf{u} + \mathbf{Dm} \bmod qR) \wedge (\|\mathbf{v}\|_2 \leq \sqrt{\sigma_1^2 nm_1 + \sigma^2 nm_2}) \wedge (\tau \in \mathcal{T}_w)$
then return 1                                                   ▷ Valid Signature
else return 0                                                   ▷ Invalid signature
```

$U(R_q^{d\times m_1})$. The security proofs rigorously follow that of Lemma 3.1 and 3.2. This is due to the fact that all the tools that we use have a ring counterpart. We briefly explain what tools are needed to carry out the proofs in the module case. We stress that the construction can also be instantiated over modules of rank $d = 1$.

First, we need to ensure that a difference of distinct tags is invertible in R_q. By [LS18, Cor. 1.2], when $q = 2k+1 \bmod 4k$, a ring element r is invertible in R_q if $0 < \|r\|_\infty \leq q^{1/k}/\sqrt{k}$. We chose q so that a difference of tags $\tau_1 - \tau_2$ has infinity norm at most $2 \leq q^{1/k}/\sqrt{k}$. Hence, a difference of distinct tags is in $R_q^\times$. Then, the leftover hash lemma of Lemma 2.1 has been adapted to general rings of integer by Boudgoust et al. and further generalized in [BJRW22]. We state it here for our specific usage in power-of-two cyclotomic rings.

Lemma 6.1 ([BJRW22, Lem. 2.7]). *Let $R = \mathbb{Z}[X]/\langle X^n + 1\rangle$ with n a power of two. Further let d, m, q be positive integers with q prime. Then, it holds that $\Delta((\mathbf{A}, \mathbf{As}), (\mathbf{A}, \mathbf{u})) \leq \frac{1}{2}\sqrt{(1 + q^d/3^m)^n - 1}$, where $\mathbf{A} \sim U(R_q^{d\times m})$, $\mathbf{s} \sim U(R_{\pm 1}^d)$ and $\mathbf{u} \sim U(R_q^d)$.*

The use of the Rényi divergence in the proof of Lemma 3.2 applies on the discrete Gaussian distributions, which are defined by their embedding to $\mathbb{R}^n$. As such, the argument remains unchanged. We also need to argue that for $\mathbf{A} \hookleftarrow U(R_q^{d\times m_1+m_2})$ and $\mathbf{v} \hookleftarrow \mathcal{D}_{R^{m_1+m_2},\mathbf{\Sigma}}$ with $\mathbf{\Sigma} = \begin{bmatrix} \sigma_1\mathbf{I}_{nm_1} & \mathbf{0} \\ \mathbf{0} & \sigma\mathbf{I}_{nm_2} \end{bmatrix}$, then $\mathbf{u} = \mathbf{Av} \bmod q$ is close to uniform. For that, we use [LPR13, Thm. 7.4] which states that if $\sigma, \sigma_1 \geq 2nq^{(d+2/n)/(m_1+m_2)}$, then the public syndrome $\mathbf{u}$ is close to uniform in R_q^d. We note that this results holds when the Gaussian over R is defined with respect to the Minkowski embedding. As explained in Remark 6.1, in the case of our Gaussian distributions, we only need $\sigma, \sigma_1 \geq 2\sqrt{n}q^{\frac{d+2/n}{m_1+m_2}}$. Since $m_1 + m_2 \geq d(\log_2(q)/\log_2(3) + \kappa) + f(\lambda)/\log_2(3)$, the result holds whenever $\sigma, \sigma_1 \geq 3^{1+2/n} \cdot 2\sqrt{n}$, which is the case in our context.

Finally, we need to bound the spectral norm of structured matrices that are of size $nm_1 \times nm_2$ (or $nm_1 \times nm_3$). In power-of-two cyclotomic rings, the structured matrix considered is a block matrix whose blocks are nega-circulant matrices of size $n \times n$. The entries are thus all distributed according to $U([-1,1])$ but they are not all independent within a block. This means we cannot apply Lemma 2.4 directly. The spectral norm of such a structured matrix of size $nm_1 \times nm_2$ is proven to be the maximal spectral norm of the n complex-embedded matrices of size $m_1 \times m_2$ [BJRW22, Lem. 2.3], which all have i.i.d. entries that are sub-Gaussian of moment $\sqrt{2n/3}$. Applying Lemma 2.4 to these embedded matrices with the union bound (on half the complex embeddings) gives

$$\mathbb{P}_{\mathbf{R} \hookleftarrow R_{\pm 1}^{m_1 \times m_2}}[\|\mathbf{R}\|_2 \geq C\sqrt{2n/3}(\sqrt{m_1} + \sqrt{m_2} + t)] \leq ne^{-\pi t^2},$$

for an absolute constant $C > 0$. Although this bound is proven, we can verify experimentally that it is not tight, and rather that the original bound (when there is no structure) of $\sqrt{nm_1} + \sqrt{nm_2} + t$ for a small t (typically $6-7$) is satisfied with overwhelming probability. Further, we use the latter bound for setting parameters in the description of the signature.

Lemma 6.2. *If an adversary can produce a Type I forgery with advantage δ, then we can construct $\mathcal{B}$ that solves $\text{M-SIS}^2_{d,m_1,q,\beta}$ with advantage $Adv[\mathcal{B}] \gtrsim \delta/(|\mathcal{T}_w| - Q)$, for*

$$\begin{aligned}\beta = &\sqrt{1 + (\sqrt{nm_1} + \sqrt{nm_2} + t)^2}\sqrt{\sigma_1^2 nm_1 + \sigma^2 nm_2} + \sqrt{nm_1} \\ &+ (\sqrt{nm_1} + \sqrt{nm_3} + t)\sqrt{nm_3}.\end{aligned}$$

Lemma 6.3. *If an adversary can produce a Type II forgery with advantage δ, we can construct $\mathcal{B}$ that solves the $\text{M-SIS}^2_{d,m_1,q,\beta'}$ problem with advantage*

$$Adv[\mathcal{B}] \gtrsim \left(1 - \frac{Q^2}{2|\mathcal{T}_w|}\right) \cdot \frac{\delta^{\alpha^*/(\alpha^*-1)} e^{-\alpha^* \pi}}{Q},$$

for

$$\begin{aligned}\beta' = &\sqrt{1 + (\sqrt{nm_1} + \sqrt{nm_2} + t)^2} \cdot \sqrt{2\sigma_1^2 nm_1 + 2\sigma^2 nm_2} \\ &+ (\sqrt{nm_1} + \sqrt{nm_3} + t)\sqrt{nm_3},\end{aligned}$$

and where α^* is defined by $\alpha^* = 1 + \sqrt{\log_2(1/\delta)/(\pi \log_2 e)}$.

6.3 Privacy-Preserving Protocols and Zero-Knowledge Proofs

The privacy-preserving protocols for signing a committed message and proving the possession of a message-signature pair are exactly the same as those described in Sections 4.1 and 4.2 by considering vectors and matrices in R instead of $\mathbb{Z}$. To avoid repetition, we simply refer to these sections. We instead focus on the zero-knowledge arguments that are required by these protocols. Although the framework of [YAZ+19] straightforwardly adapts to the ring or module setting, it results in relations of the form $\mathbf{A}\mathbf{x} = \mathbf{y} \bmod qR$ and $\mathbf{x}[h] = \mathbf{x}[i]\mathbf{x}[j] \bmod qR$.

In our case, we aim to prove that the witness is short (or binary for the message part) with respect to the coefficient embedding of R. Taking the example of the message, $\mathbf{m}[i] = \mathbf{m}[i]^2 \bmod qR$ does not imply that the coefficients of the polynomial $\mathbf{m}[i]$ are binary, but only that the number theoretic transform (NTT) of $\mathbf{m}[i]$ is a binary vector. A naive alternative would be two embed the entire relation into $\mathbb{Z}$ via the coefficient embedding and applying [YAZ+19] in a non-structured way. This would indeed prove the desired relation but it would also ignore the underlying structure and all the optimizations that come with it. Instead, we use the very recent framework by Lyubashevsky, Nguyen and Plançon [LNP22], which generalizes the previous work of [BLS19] and [ENS20] used to obtain exact proofs. The advantage of this framework is that it provides a way to prove bounds on the Euclidean norm of the witness without resorting to bounds on the infinity norm. As explained in [LNP22], this leads to proving tighter bound on the Euclidean norm, and in a more efficient way as a result. We denote by σ_{-1} to be the automorphism of R_q that can be defined as $\sigma_{-1}(\sum_{i=0}^{n-1} r_i X^i) = r_0 - \sum_{i=1}^{n-1} r_i X^{n-i}$. Their proof system allows one to prove relations of the form

$$\begin{cases} \forall i \in [\rho], f_i(\mathbf{s}) = 0 \bmod qR & \forall i \in [v_e], \left\|\mathbf{E}_i^{(e)}\mathbf{s} - \mathbf{u}_i^{(e)}\right\|_2 \leq \beta_i^{(e)} \\ \forall i \in [\rho_{eval}], \widetilde{F}_i(\mathbf{s}) = 0 & \forall i \in [v_a], \left\|\mathbf{E}_i^{(a)}\mathbf{s} - \mathbf{u}_i^{(a)}\right\|_\infty \leq \beta_i^{(a)}, \end{cases}$$

where the f_i, F_i are quadratic functions in $\mathbf{s} = [\mathbf{s}_1^T, \sigma_{-1}(\mathbf{s}_1)^T]^T$ ($\mathbf{s}_1$ being the committed vector), and $\widetilde{F}_i(\mathbf{s})$ denotes the constant coefficient of the polynomial $F_i(\mathbf{s})$. The norm conditions with superscript (e) are proven exactly, while those with superscript (a) are proven approximately. We note for completeness that the considered automorphism is not necessarily σ_{-1}. We present here how our relations can be instantiated in their framework, which consists in describing the functions f_i, F_i and matrices and vectors for the norm conditions.

Let $q_1 < q$ be a prime integer such that $q_1 = 2k + 1 \bmod 4k$, and define $q_\pi = q_1 q$ as the modulus of the proof system, which is different from the modulus of our signature. We take q_1 having the same splitting as q in R to ensure the invertibility of challenge differences in R_{q_π} as discussed in [LNP22, Sec. 2.3].

6.3.1 Proof of Commitment Opening. Consider the relation

$$q_1(\mathbf{A}\mathbf{r}' + \mathbf{D}\mathbf{m}) = q_1\mathbf{c} \bmod q_\pi R \wedge \|\mathbf{r}'\|_2 \leq \sigma_3\sqrt{nm_1} =: \alpha_3 \wedge \mathbf{m} \in R_2^{m_3},$$

where the private input is $\mathbf{r}', \mathbf{m}$ and the public input is $\mathbf{A}, \mathbf{D}, \mathbf{c}$. We multiply the linear equation by q_1 to work with the proof system modulus. We now instantiate this relation in the framework of [LNP22]. Using the notations of [LNP22], we define $\mathbf{s}_1 = [\mathbf{r}'|\mathbf{m}] \in R^{m_1+m_3}$ and $\mathbf{s} = [\mathbf{s}_1|\sigma_{-1}(\mathbf{s}_1)] \in R^{2(m_1+m_3)}$.

Quadratic Equations: Define $f_i(\mathbf{s}) = (\mathbf{e}_i^T[q_1\mathbf{A}|q_1\mathbf{D}|\mathbf{0}_{d\times m_1+m_3}]) \cdot \mathbf{s} + (-\mathbf{e}_i^T q_1 \mathbf{c})$ for all $i \in [d]$, where $\mathbf{e}_i$ is the zero vector with a 1 at position i. Then, proving $f_i(\mathbf{s}) = 0 \bmod q_\pi R$ for all $i \in [d]$ yields $q_1(\mathbf{A}\mathbf{r}' + \mathbf{D}\mathbf{m}) = q_1\mathbf{c} \bmod q_\pi R$.

Quadratic Evaluations: We define $r = \sum_{j\in[0,n-1]} X^j$. For all $i \in [m_3]$, define $F_i(\mathbf{s}) = \mathbf{s}^T\mathbf{E}_{2m_1+m_3+i,m_1+i}\mathbf{s} + (-r\mathbf{e}_{2m_1+m_3+i})^T\mathbf{s} = \sigma_{-1}(\mathbf{m}[i])(\mathbf{m}[i] - r)$, where $\mathbf{E}_{k,\ell}$ denotes the zero matrix with a 1 at position (k,ℓ). Then, proving $\widetilde{F}_i(\mathbf{s}) = 0$ for all $i \in [m_3]$ implies $\mathbf{m} \in R_2^{m_3}$. This relies on the fact that for $m \in R$, the constant coefficient of $\sigma_{-1}(m)(m-r)$ is $\langle \theta(m), \theta(m) - \mathbf{1}_n \rangle$. Then, proving that this inner product is 0 over $\mathbb{Z}$ is equivalent to proving that $\theta(m) \in \{0,1\}^n$, i.e., $m \in R_2$.

Norm Conditions: We define $\mathbf{E}^{(e)} = [\mathbf{I}_{m_1} | \mathbf{0}_{m_1 \times m_1 + 2m_3}]$, $\mathbf{u}^{(e)} = \mathbf{0}_{m_1}$, and $\beta^{(e)} = \alpha_3$. Then $\left\|\mathbf{E}^{(e)}\mathbf{s} - \mathbf{u}^{(e)}\right\|_2 \leq \beta^{(e)}$ is equivalent to $\|\mathbf{r}'\|_2 \leq \alpha_3$.

Remark 6.2. The above aims at proving the relation exactly. However, we note that the commitment scheme employed in [LNP22] already contains a part $\mathbf{A}_1\mathbf{s}_1 + \mathbf{A}_2\mathbf{s}_2$. By setting the public matrices $\mathbf{A}_1, \mathbf{A}_2$ as $\mathbf{A}, \mathbf{D}$ respectively, $\mathbf{s}_2 = \mathbf{r}'$ which is chosen from a Gaussian distribution, and $\mathbf{s}_1 = \mathbf{m}$, we can directly use the protocol of [LNP22, Fig. 8]. We simply have to set $\|\mathbf{s}_1\|_2 \leq \sqrt{nm_3} =: \alpha$, and the quadratic evaluations as above to prove (exactly) that $\mathbf{s}_1 = \mathbf{m}$ is indeed in $R_2^{m_3}$. It then proves the desired relation exactly at the exception of a soundness gap on the norm of $\mathbf{r}'$.

6.3.2 Proof of Message-Signature Pair Possession.

Consider the relation

$$q_1(\mathbf{A}\mathbf{v}_1 - \mathbf{B}\mathbf{v}_2 + \tau\mathbf{G}\mathbf{v}_2 - \mathbf{D}\mathbf{m}) = q_1\mathbf{u} \bmod q_\pi R$$
$$\text{with } \|\mathbf{v}\|_2 \leq \sqrt{\sigma_1^2 nm_1 + \sigma^2 nm_2} =: \alpha \wedge \mathbf{m} \in R_2^{m_3} \wedge \tau \in \mathcal{T}_w,$$

where the private input is $\tau, \mathbf{v} = [\mathbf{v}_1^T | \mathbf{v}_2^T]^T, \mathbf{m}$ and the public input is composed of $\mathbf{A}, \mathbf{B}, \mathbf{D}, \mathbf{G}, \mathbf{u}$. We define $\mathbf{s}_1 = [\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{m} | \tau] \in R^{m_1+m_2+m_3+1}$ and $\mathbf{s} = [\mathbf{s}_1 | \sigma_{-1}(\mathbf{s}_1)] \in R^{2(m_1+m_2+m_3+1)}$.
Quadratic Equations: We define $\mathbf{A}' = q_1[\mathbf{A} | -\mathbf{B} | -\mathbf{D} | \mathbf{0}_{d \times m_1+m_2+m_3+2}]$, and for all $i \in [d]$, we define

$$\mathbf{G}_i = q_1 \begin{bmatrix} \mathbf{0}_{(m_1+m_2+m_3) \times 2(m_1+m_2+m_3+1)} \\ \mathbf{0}_{1 \times m_1} \ \mathbf{e}_i^T\mathbf{G} \ \mathbf{0}_{1 \times m_1+m_2+2(m_3+1)} \\ \mathbf{0}_{(m_1+m_2+m_3+1) \times 2(m_1+m_2+m_3+1)} \end{bmatrix}.$$

Then, for all $i \in [d]$, define $f_i(\mathbf{s}) = \mathbf{s}^T\mathbf{G}_i\mathbf{s} + (\mathbf{e}_i^T\mathbf{A}')\mathbf{s} + (-q_1\mathbf{e}_i^T\mathbf{u})$. Proving $f_i(\mathbf{s}) = 0 \bmod q_\pi R$ for all $i \in [d]$ yields $q_1(\mathbf{A}\mathbf{v}_1 - \mathbf{B}\mathbf{v}_2 + \tau\mathbf{G}\mathbf{v}_2 - \mathbf{D}\mathbf{m}) = q_1\mathbf{u} \bmod q_\pi R$.

Quadratic Evaluations: We define $r = \sum_{j \in [0,n-1]} X^j$. For all $i \in [m_3+1]$, define $F_i(\mathbf{s}) = \mathbf{s}^T\mathbf{E}_{2(m_1+m_2)+m_3+1+i, m_1+m_2+i}\mathbf{s} + (-r\mathbf{e}_{2(m_1+m_2)+m_3+1+i})^T\mathbf{s}$. We also define $F_{m_3+2}(\mathbf{s}) = \mathbf{s}^T\mathbf{E}_{2(m_1+m_2+m_3+1), m_1+m_2+m_3+1}\mathbf{s} - w = \sigma_{-1}(\tau)\tau - w$. Proving $\widetilde{F}_i(\mathbf{s}) = 0$ for $i \in [m_3]$ is equivalent to $\mathbf{m} \in R_2^{m_3}$ as before. Then, showing $\widetilde{F}_{m_3+1}(\mathbf{s}) = 0$ proves $\tau \in R_2$, while $\widetilde{F}_{m_3+2}(\mathbf{s}) = 0$ proves that $\|\tau\|_2^2 = \langle \theta(\tau), \theta(\tau) \rangle = w$, thus giving $\tau \in \mathcal{T}_w$.

Norm Conditions: We define $\mathbf{E}^{(e)} = [\mathbf{I}_{m_1+m_2} | \mathbf{0}_{m_1+m_2 \times m_1+m_2+2(m_3+1)}]$, $\mathbf{u}^{(e)} = \mathbf{0}_{m_1+m_2}$, and $\beta^{(e)} = \alpha$. Then $\left\|\mathbf{E}^{(e)}\mathbf{s} - \mathbf{u}^{(e)}\right\|_2 \leq \beta^{(e)}$ proves $\|\mathbf{v}\|_2 \leq \alpha$.

Conclusion

In this paper, we have proposed a new signature scheme with efficient protocols which is several orders of magnitude more efficient than the current state-of-the-art [LLM+16]. This improvement was obtained by revisiting the latter construction in a systematic way, considering not only the signature scheme itself but also its interactions with the other components such as the commitment scheme and the zero-knowledge proofs. In the process, we have also rectified a problem with the fast mode of the Yang et al zero-knowledge framework [YAZ+19] and introduced some optimizations, which are of independent interest.

Our construction was designed to remain as generic as possible in order to be compatible with the broadest possible spectrum of applications. In particular, it can be instantiated in both standard lattices and structured ones so as to suit any lattice-based system. Despite this versatility, the size of a proof of knowledge of a message-signature pair, one of the core component of privacy-preserving systems, can be as low as 640 KB, which should foster the development of practical post-quantum constructions in this area.

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A Proof of Lemma 2.5

We recall here the definition of a sub-exponential random variable. We say that a random variable X is sub-exponential with parameters (ν, α) if for all $t \in (-1/\alpha, 1/\alpha)$, $\mathbb{E}[\exp(t(X - \mathbb{E}[X]))] \leq \exp(t^2\nu^2/2)$. We have that a sum of m independent sub-exponential random variables with the same parameters (ν, α) is sub-exponential with parameters $(\nu\sqrt{m}, \alpha)$. Finally, it holds that for a sub-exponential random variable with parameter (ν, α)

$$\forall r > 0, \mathbb{P}[X - \mathbb{E}[X] \geq r] \leq \begin{cases} e^{-r^2/(2\nu^2)} & \text{if } 0 < r < \nu^2/\alpha \\ e^{\nu^2/(2\alpha^2) - r/\alpha} & \text{if } r \geq \nu^2/\alpha. \end{cases}$$

Proof (of Lemma 2.5). Let $\mathbf{m} \in \{0,1\}^m$ be an arbitrary vector, and we denote by $k = \|\mathbf{m}\|_1$ the number of ones in the vector. We consider the random matrix $\mathbf{U}$ whose entries are independent and identically distributed according to $U([-1,1])$, and we denote by u_{ij} the random variable representing the (i,j)-th entry of $\mathbf{U}$. For clarity, we also denote by $\mathbf{u}_i^T$ the i-th row of $\mathbf{U}$. We know that each u_{ij} is sub-Gaussian with parameter $\sqrt{2/3}$, i.e.,

$$\forall t \in \mathbb{R}, \mathbb{E}[\exp(tu_{ij})] \leq \exp(t^2/3).$$

By independence of the entries, we directly obtain for all $i \in [n]$

$$\forall t \in \mathbb{R}, \mathbb{E}[\exp(t\mathbf{u}_i^T\mathbf{m})] \leq \exp(kt^2/3).$$

Hence, each $\mathbf{u}_i^T\mathbf{m}$ is sub-Gaussian with parameter $s = \sqrt{2k/3}$. We define the random variables $x_i = \mathbf{u}_i^T\mathbf{m}$, $y_i = x_i^2$ and we also define $\mu_i = \mathbb{E}[y_i]$. Since x_i is sub-Gaussian with parameter s, we can prove that

$$\forall p \geq 1, \mathbb{E}[|x_i|^p] \leq p(\sqrt{2}s)^p\mathit{\Gamma}(p/2),$$

where $\mathit{\Gamma}$ is the Gamma function. In particular, we have $\mu_i \leq 2(\sqrt{2}s)^2\mathit{\Gamma}(1) = 4s^2 = 8k/3$. We then have

$$\begin{aligned}
\mathbb{E}[e^{t(y_i - \mu_i)}] &= 1 + t\mathbb{E}[y_i - \mu_i] + \sum_{p=2}^{\infty} t^p\mathbb{E}[(y_i - \mu_i)^p]/p! \\
&\leq 1 + \sum_{p=2}^{\infty} t^p\mathbb{E}[x_i^{2p}]/p! \\
&\leq 1 + \sum_{p=2}^{\infty} t^p(2p(\sqrt{2}s)^{2p}\mathit{\Gamma}(p))/p! \\
&= 1 + 2\sum_{p=2}^{\infty}(2s^2t)^p \\
&= 1 + 8s^4t^2/(1 - 2s^2t),
\end{aligned}$$

where we used the fact that $\Gamma(p) = (p-1)!$ and that we restrict $|t| < 1/(2s^2\beta)$ for some free variable $\beta \geq 1$. It thus follows that

$$\mathbb{E}[e^{t(y_i - \mu_i)}] \leq 1 + 8\beta s^4 t^2/(\beta - 1) \leq \exp(16\beta s^4/(\beta-1) \cdot t^2/2).$$

Hence, $y_i - \mu_i$ is a centered sub-exponential with parameters $\nu = 4s^2\sqrt{\beta/(\beta-1)}$ and $\alpha = 2s^2\beta$. We then define $y = \sum_{i\in[\ell]} y_i$ and $\mu = \sum_{i\in[\ell]} \mu_i$. It thus holds that $y - \mu$ is a centered sub-exponential with parameters $\nu\sqrt{\ell}$ and α. Using the tail bound above for a sub-exponential distribution, we have that for all $0 < r < \nu^2\ell/\alpha = 16\ell k/(3(\beta-1))$ then

$$\mathbb{P}[y - \mu \geq r] \leq \exp(-r^2/(2\ell\nu^2)).$$

Since the y_i are identically distributed, we have that $\mu = \ell\mu_1 \leq 8\ell k/3$. And we also have $y = \|\mathbf{Um}\|_2^2$. We now set the parameters β and r so that

$$\mathbb{P}[\|\mathbf{Um}\|_2 \geq 2\sqrt{\ell m}] \leq 2^{-x}.$$

In particular, we set $\beta = 1/(1 - 8x/(\ell \log_2 e))$. Assuming $\ell \geq 10x/\log_2 e$ entails $\beta \in (1, 5]$. Also, we set β this way to have $\sqrt{2\beta/((\beta-1)\log_2 e)}\sqrt{x/\ell} = 1/2$. Then, we set $r = 8k/3 \cdot \sqrt{2\beta/((\beta-1)\log_2 e)}\sqrt{\ell x} = 4\ell k/3$. We indeed have $r \leq 16\ell k/(3(\beta-1)) = \ell\nu^2/\alpha$. The way we set r, we have $\exp(-r^2/(2\nu^2)) = 2^{-x}$, and $r + \mu \leq 4\ell k/3 + 8\ell k/3 = 4\ell k$. Hence

$$\mathbb{P}[\|\mathbf{Um}\|_2 \geq 2\sqrt{\ell k}] \leq \mathbb{P}[\sqrt{y} \geq \sqrt{r+\mu}] \leq \exp(-r^2/(2\nu^2)) = 2^{-x},$$

In the worst case, we have $k = m$ which yields the claim. □

B Security Proofs

B.1 Additional Preliminaries

Probabilities. We denote by Supp(P) the support of the probability distribution P. In addition to the statistical distance, we use another measure of closeness between two probability distributions, namely the *Rényi divergence* [Ŕ61] RD. The Rényi divergence was thoroughly studied for its use in cryptography by Bai et al. [BLR+18] as it shows to be a powerful alternative to the statistical distance.

Definition B.1. *Consider two discrete probability distributions P and Q over a countable set S such that $Supp(P) \subseteq Supp(Q)$. We define the* Rényi divergence of order $\alpha > 1$ *as*

$$\mathrm{RD}_\alpha(P\|Q) = \left(\sum_{x\in Supp(P)} \frac{P(x)^\alpha}{Q(x)^{\alpha-1}}\right)^{\frac{1}{\alpha-1}}.$$

The two measures enjoy a probability preservation property, which are essential in proving our results.

Lemma B.1. *Let P, Q be two probability distributions with $Supp(P) \subseteq Supp(Q)$, and $E \subseteq Supp(Q)$ be an arbitrary event. Then, $P(E) \leq \Delta(P,Q) + Q(E)$, and $P(E)^{\frac{\alpha}{\alpha-1}} \leq \mathrm{RD}_\alpha(P\|Q) \cdot Q(E)$.*

In the security proofs, we need to compute the Rényi divergence between two shifted discrete Gaussian distributions. We use the following lemma.

Lemma B.2 ([LSS14, Lem. 4.2]). *Let Λ be a lattice of rank n, and $\mathbf{c} \in \mathbb{R}^n$. Let $\alpha > 1$. Then, for any $\sigma > 0$, it holds that*

1. $\mathrm{RD}_\alpha(\mathcal{D}_{\Lambda,\sigma}\|\mathcal{D}_{\Lambda,\sigma,\mathbf{c}}) \leq \exp(\alpha\pi\|\mathbf{c}\|_2^2/\sigma^2)$,
2. $\mathrm{RD}_\alpha(\mathcal{D}_{\Lambda,\sigma,\mathbf{c}}\|\mathcal{D}_{\Lambda,\sigma}) \leq \exp(\alpha\pi\|\mathbf{c}\|_2^2/\sigma^2) \cdot \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\alpha/(\alpha-1)}$*, if* $\sigma \geq \eta_\varepsilon(\Lambda)$.

Finally, to ensure that the syndrome generated by the SIS challenger is correctly distributed, we need to argue that $\mathbf{A}'\mathbf{v}'$ is close to uniform for a Gaussian vector $\mathbf{v}'$. We thus use the result of [MP12] which argues that the smoothing parameter of $\Lambda_q^\perp(\mathbf{A}')$ is small with high probability over the choice of $\mathbf{A}'$.

Lemma B.3 (Adapted from [MP12, Lem. 2.4]). *Let n and q be positive integers with q prime, and let $m \geq n\log_2 q + \log_2(2 + 2\varepsilon^{-1})$ for some $\varepsilon > 0$. Let $\sigma \geq 2\eta_\varepsilon(\mathbb{Z}^m)$. Then for any $\delta > 0$, it holds that $\Delta((\mathbf{A}, \mathbf{Ae} \bmod q), (\mathbf{A}, \mathbf{u})) \leq \delta + 2\varepsilon/\delta$, where $\mathbf{A} \sim U(\mathbb{Z}_q^{n\times m})$, $\mathbf{e} \sim \mathcal{D}_{\mathbb{Z}^m,\sigma}$, and $\mathbf{u} \sim U(\mathbb{Z}_q^n)$.*

In particular, choosing $\varepsilon = \delta^2/2$, $m > n\log_2 q + 2 - 2\log_2\delta$, $\sigma \geq \omega(\sqrt{\log_2 m})$ leads to a statistical distance of at most 2δ. In our case, we apply it with $m = m_1 + m_2 \gg n\log_2 q + 2\lambda + 4$, yielding a statistical distance much smaller than $2^{-\lambda}$.

Signature Security Model. The most widely used notion of security for a signature scheme is the *Existential Unforgeability against Chosen Message Attacks* (EUF-CMA) security. This captures the fact that an attacker that can obtain signatures on messages of its choosing is incapable of forging a signature on a new message. We formally define it by a game between an adversary $\mathcal{A}$ and a challenger $\mathcal{C}$ in three stages.

Setup Stage: $\mathcal{C}$ successively runs Setup and KeyGen to obtain $\mathsf{pp}, \mathsf{pk}, \mathsf{sk}$. It then gives pp, pk to $\mathcal{A}$.
Query Stage: $\mathcal{A}$ queries signatures on at most Q messages $\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(Q)}$, which are answered by $\mathcal{C}$ returning $\mathsf{sig}_i \leftarrow \mathsf{Sign}(\mathsf{sk}; \mathbf{m}^{(i)}; \mathsf{pp}, \mathsf{pk})$.
Forgery Stage: $\mathcal{A}$ then outputs a forgery $(\mathbf{m}^*, \mathsf{sig}^*)$.

The adversary wins if $\mathsf{Verify}(\mathsf{pk}; \mathbf{m}^*; \mathsf{sig}^*; \mathsf{pp}) = 1$ and if for all i in $[Q]$, it holds $\mathbf{m}^* \neq \mathbf{m}^{(i)}$. The adversary's advantage is defined as $\mathrm{Adv}[\mathcal{A}] = \mathbb{P}[\mathcal{A} \text{ wins}]$, where the probability is over all the random coins. We say that the scheme is EUF-CMA secure if for all probabilistic polynomial time (PPT) adversary $\mathcal{A}$, $\mathrm{Adv}[\mathcal{A}]$ is negligible in λ.

B.2 Proof of Lemma 3.1

Proof. Consider a PPT adversary $\mathcal{A}$ that produces Type I forgeries for the signature scheme with advantage δ. We now construct an adversary $\mathcal{B}$ that solves the $\mathrm{SIS}^{\infty,2}_{n,m_1,q,\beta}$ problem. The adversary $\mathcal{B}$ is given $\overline{\mathbf{A}} \in \mathbb{Z}_q^{n\times m_1}$ as input and is asked to find $\mathbf{w} \in \Lambda_q^{\perp}(\overline{\mathbf{A}})$ such that $0 < \|\mathbf{w}\|_\infty \leq \beta_\infty$ and $0 < \|\mathbf{w}\|_2 \leq \beta_2$.

Setup Stage: $\mathcal{B}$ first generates the cryptographic material to give to $\mathcal{A}$. We assume that the parameters $\mathbf{g}, n, m_1, m_2, m_3, q, \sigma, \sigma_2, \sigma_1$. are already set. We also define $\mathbf{G} = \mathbf{I}_n \otimes \mathbf{g}$. The adversary $\mathcal{B}$ first samples $\tau^{(1)}, \ldots, \tau^{(Q)}$ from $U(\mathcal{T})$ as the tags that will be used for the signing queries of $\mathcal{A}$. It also makes a guess $\overline{\tau} \hookleftarrow U(\mathcal{T} \setminus \{\tau^{(i)}; i \in [Q]\})$ on the tag that we be used in the adversary's forgery. In particular, we assume that $Q = \mathsf{poly}(\lambda)$ is the maximum number of signing queries that $\mathcal{A}$ is able to make.

Next, $\mathcal{B}$ samples $\mathbf{U}$ from $U([-1,1]^{m_1\times m_3})$. It then randomizes $\overline{\mathbf{A}}$ to define $\mathbf{D} = \overline{\mathbf{A}}\mathbf{U} \bmod q$, and sets $\mathsf{pp} = (\mathbf{D}; \mathbf{g}; \lambda, n, m_1, m_2, m_3, q, \sigma, \sigma_2, \sigma_1)$.

Then, $\mathcal{B}$ samples $\mathbf{R} \hookleftarrow U([-1,1]^{m_1\times m_2})$ and defines $\mathbf{B} = \overline{\mathbf{A}}\mathbf{R} + \overline{\tau}\mathbf{G} \bmod q$. It also samples $\mathbf{e_u}$ from $U([-1,1]^{m_1})$ and defines $\mathbf{u} = \overline{\mathbf{A}}\mathbf{e_u} \bmod q$. The adversary $\mathcal{B}$ then forms $\mathsf{pk} = (\overline{\mathbf{A}}, \mathbf{B}, \mathbf{u})$. From these matrices, we can define $\mathbf{A}_\tau$ for any tag $\tau \in \mathbb{Z}_{q'}$ by

$$\mathbf{A}_\tau = \left[\overline{\mathbf{A}} | \tau\mathbf{G} - \mathbf{B}\right] = \left[\overline{\mathbf{A}} | (\tau - \overline{\tau})\mathbf{G} - \overline{\mathbf{A}}\mathbf{R}\right], \tag{4}$$

Since $\overline{\tau}$ does not collide with the tags $\tau^{(1)}, \ldots, \tau^{(Q)}$ that will be used to answer the signing queries, we have $\tau^{(i)} - \overline{\tau} \in \mathbb{Z}_q^\times$ as q is prime. The matrices $\mathbf{A}_{\tau^{(i)}}$ thus have the adequate form to sample preimages using the trapdoor-based algorithms from [MP12]. Finally, $\mathcal{B}$ sends $(\mathsf{pk}, \mathsf{pp})$ to $\mathcal{A}$.

Query Stage: At the i-th signature query, $\mathcal{A}$ provides $\mathcal{B}$ with a message $\mathbf{m}^{(i)} \in \{0,1\}^{m_3}$. $\mathcal{B}$ can then faithfully run Algorithm 3 using the carefully crafted key material, and the tag $\tau^{(i)}$. More precisely, it computes $\mathbf{A}_{\tau^{(i)}}$ using Equation (4), as well as the message commitment $\mathbf{c} = \overline{\mathbf{A}}\mathbf{r}^{(i)} + \mathbf{D}\mathbf{m}^{(i)} \bmod q$ for a fresh randomness $\mathbf{r}^{(i)} \hookleftarrow \mathcal{D}_{\mathbb{Z}^{m_1},\sigma_2}$. As discussed, we can still use the $\mathbf{G}$-trapdoor $\mathbf{R}$ to sample preimages, allowing $\mathcal{B}$ to compute

$$\mathbf{v}^{(i)} = \mathsf{SampleD}(\mathbf{R}, \mathbf{A}, (\tau^{(i)} - \overline{\tau})\mathbf{I}_n, \mathbf{u} + \mathbf{c}, \sigma) - \begin{bmatrix} \mathbf{r}^{(i)} \\ \mathbf{0}_{m_2} \end{bmatrix}.$$

Note that $\mathbf{v}^{(i)}$ is correctly distributed and passes verification (with overwhelming probability by Lemma 2.2 and 2.3). The signature given to $\mathcal{A}$ is $\mathsf{sig}_i = (\tau^{(i)}, \mathbf{v}^{(i)})$.

Forgery Stage: After at most Q queries, the adversary returns a forgery $\mathsf{sig}^* = (\tau^*, \mathbf{v}^*)$ on a new message $\mathbf{m}^*$ that passes verification. If $\mathcal{A}$ fails to produce such a forgery, $\mathcal{B}$ aborts. We call this event Abort_1. We now condition on $\neg\mathsf{Abort}_1$. At this point, $\mathcal{B}$ aborts if $\tau^* \neq \overline{\tau}$. We call this event Abort_2 and further condition on $\neg\mathsf{Abort}_2$. Then, the guess was correct and therefore the contribution of $\mathbf{G}$ in $\mathbf{A}_{\tau^*}$ vanishes. Since the forgery passes verification we have $\mathbf{A}_{\tau^*}\mathbf{v}^* = \mathbf{u} + \mathbf{D}\mathbf{m}^* \bmod q$. Using the definition of the cryptographic material from the setup stage, it can be written as

$$\left[\overline{\mathbf{A}} | -\overline{\mathbf{A}}\mathbf{R}\right]\mathbf{v}^* = \overline{\mathbf{A}}\mathbf{e_u} + \overline{\mathbf{A}}\mathbf{U}\mathbf{m}^* \bmod q.$$

This means that

$$\mathbf{w} = [\mathbf{I}_{m_1}| - \mathbf{R}]\mathbf{v}^* - \mathbf{e}_{\mathbf{u}} - \mathbf{U}\mathbf{m}^* \in \mathbb{Z}^{m_1}$$

is in $\Lambda_q^{\perp}(\overline{\mathbf{A}})$. The adversary $\mathcal{B}$ thus returns $\mathbf{w}$ as a solution for $\mathrm{SIS}^{\infty,2}_{n,m_1,q,\beta_\infty,\beta_2}$.

Advantage: We now analyze the advantage of $\mathcal{B}$. We first look at the distribution of $(\mathsf{pk}, \mathsf{pp})$. Since $m_1 \log_2 3 \geq n \log_2 q + f(\lambda)$, it holds by Lemma 2.1 that

$$\begin{cases} \Delta((\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{R} \bmod q), U(\mathbb{Z}_q^{n\times m_1} \times \mathbb{Z}_q^{n\times m_2})) \leq \frac{m_2}{2}\sqrt{\frac{q^n}{3^{m_1}}} \leq m_2 2^{-f(\lambda)/2-1}, \\ \Delta((\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{U} \bmod q), U(\mathbb{Z}_q^{n\times m_1} \times \mathbb{Z}_q^{n\times m_3})) \leq \frac{m_3}{2}\sqrt{\frac{q^n}{3^{m_1}}} \leq m_3 2^{-f(\lambda)/2-1}, \\ \Delta((\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{e}_{\mathbf{u}}), U(\mathbb{Z}_q^{n\times m_1} \times \mathbb{Z}_q^{n})) \leq \frac{1}{2}\sqrt{\frac{q^n}{3^{m_1}}} \leq 2^{-f(\lambda)/2-1}) \end{cases}$$

Additionally, since $\overline{\mathbf{A}}$, $\mathbf{R}$ are independent of $\overline{\tau}\mathbf{G}$, it holds that $\Delta(\mathbf{B}, \overline{\mathbf{A}}\mathbf{R}) \leq m_2 2^{-f(\lambda)/2}$ (by the triangle inequality). The signatures that are given to $\mathcal{A}$ in the query stage are distributed according to the legitimate distribution. This means that

$$\mathbb{P}[\neg\mathsf{Abort}_1] \geq \delta - \mathsf{negl}(\lambda). \tag{5}$$

As the guess $\overline{\tau}$ is independent of $\mathcal{A}'s$ view, we directly have

$$\mathbb{P}[\neg\mathsf{Abort}_2|\neg\mathsf{Abort}_1] = \frac{1}{|\mathcal{T}| - Q}. \tag{6}$$

We now analyze the solution provided by $\mathcal{B}$. We have to show it is non-zero and have infinity norm at most β. We first focus on the former. Denote $\mathbf{e}_{\mathbf{u}}^* = [\mathbf{I}_{m_1}| - \mathbf{R}]\mathbf{v}^* - \mathbf{U}\mathbf{m}^*$. Since $\mathbf{v}^*$ and $\mathbf{m}^*$ are chosen by $\mathcal{A}$, the forger can control the value of $\mathbf{e}_{\mathbf{u}}^*$. However $\mathbf{e}_{\mathbf{u}}$ is statistically hidden by $\mathbf{u}$ making the vector $\mathbf{w}$ unpredictable. Concretely, by Lemma B.1 and 2.1, it holds

$$\begin{aligned} &\mathbb{P}_{\mathbf{e}_{\mathbf{u}}}[\mathbf{w} = \mathbf{0}|\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{e}_{\mathbf{u}} \bmod q] \\ &\qquad\qquad \leq \mathbb{P}_{\mathbf{e}_{\mathbf{u}}}[\mathbf{e}_{\mathbf{u}} = \mathbf{e}_{\mathbf{u}}^*|\overline{\mathbf{A}}, U(\mathbb{Z}_q^n)] + \Delta((\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{e}_{\mathbf{u}}), U(\mathbb{Z}_q^{n\times m_1} \times \mathbb{Z}_q^n)) \\ &\qquad\qquad \leq \mathbb{P}_{\mathbf{e}_{\mathbf{u}}}[\mathbf{e}_{\mathbf{u}} = \mathbf{e}_{\mathbf{u}}^*] + 2^{-f(\lambda)/2-1} \\ &\qquad\qquad = 3^{-m_1} + 2^{-f(\lambda)/2-1}. \end{aligned}$$

Finally, by decomposing $\mathbf{v}^*$ into $[{\mathbf{v}_1^*}^T|{\mathbf{v}_2^*}^T]^T$, with $\mathbf{v}_1^* \in \mathbb{Z}^{m_1}$ and $\mathbf{v}_2^* \in \mathbb{Z}^{m_2}$, we have

$$\begin{aligned} \|\mathbf{w}\|_\infty &\leq \|\mathbf{v}_1^*\|_\infty + m_2\|\mathbf{R}\|_{\max}\|\mathbf{v}_2^*\|_\infty + \|\mathbf{e}_{\mathbf{u}}\|_\infty + m_3\|\mathbf{U}\|_{\max}\|\mathbf{m}^*\|_\infty \\ &\leq \sigma_1 \log_2 m_1 + m_2 \cdot \sigma \log_2 m_2 + 1 + m_3 \\ &= \beta_\infty. \end{aligned}$$

Now, since $\mathbf{v}_1^*, \mathbf{v}_2^*$ correspond to the forgery that passes verification, we only know their infinity norm. In particular, we cannot apply the Gaussian tail bound to determine their Euclidean norm. Therefore, we can at best have $\|\mathbf{v}_1^*\|_2 \leq$

$\sigma_1\sqrt{m_1}\log_2 m_1$ and $\|\mathbf{v}_2^*\|_2 \leq \sigma\sqrt{m_2}\log_2 m_2$. Also, note that the spectral norm of $[\mathbf{I}_{m_1}|-\mathbf{R}]$ is exactly $\sqrt{1+\|\mathbf{R}\|_2^2}$. It follows that

$$\begin{aligned}
\|\mathbf{w}\|_2 &\leq \|[\mathbf{I}_{m_1}|-\mathbf{R}]\|_2\|\mathbf{v}^*\|_2 + \|\mathbf{e_u}\|_2 + \|\mathbf{Um}^*\|_2 \\
&\leq \sqrt{1+\|\mathbf{R}\|_2^2}\sqrt{m_1(\sigma_1\log_2 m_1)^2 + m_2(\sigma\log_2 m_2)^2} + \sqrt{m_1} \\
&\qquad + \min(2\sqrt{m_1 m_3}, (\sqrt{m_1}+\sqrt{m_3}+t)\sqrt{m_3}) \\
&\leq \sqrt{1+(\sqrt{m_1}+\sqrt{m_2}+t)^2}\sqrt{m_1(\sigma_1\log_2 m_1)^2 + m_2(\sigma\log_2 m_2)^2} + \sqrt{m_1} \\
&\qquad + \min(2\sqrt{m_1}, (\sqrt{m_1}+\sqrt{m_3}+t))\sqrt{m_3} \\
&= \beta_2,
\end{aligned}$$

where the inequalities follow from Equation (3) and Lemma 2.4 except with probability $4e^{-\pi t^2}+2^{-2\lambda}$. By defining $\mathsf{Abort}_{\{1,2\}} = \mathsf{Abort}_1 \vee \mathsf{Abort}_2$, we obtain

$$\begin{aligned}
\mathbb{P}[\mathbf{w}\text{ valid solution}|\neg\mathsf{Abort}_{\{1,2\}}] &\geq 1-3^{-m_1}-2^{-f(\lambda)/2-1}-4e^{-\pi t^2}-2^{-2\lambda} \\
&= 1-\mathsf{negl}(\lambda).
\end{aligned} \tag{7}$$

Combining Equations (5), (6) and (7) by the probability chain rule, we get

$$\mathrm{Adv}[\mathcal{B}] \geq (\delta-\mathsf{negl}(\lambda))\cdot\frac{1}{|\mathcal{T}|-Q}\cdot(1-\mathsf{negl}(\lambda)) \approx \frac{\delta}{q'-Q},$$

as claimed. □

B.3 Proof of Lemma 3.2

Proof. Consider a PPT adversary $\mathcal{A}$ that can produce a Type II forgery for the signature scheme with advantage δ. We now construct an adversary $\mathcal{B}$ that solves the $\mathrm{SIS}^{\infty,2}_{n,m_1,q,\beta'_\infty,\beta'_2}$ problem. The adversary $\mathcal{B}$ is given $\overline{\mathbf{A}} \in \mathbb{Z}_q^{n\times m_1}$ as input and is asked to find $\mathbf{w} \in \Lambda_q^\perp(\overline{\mathbf{A}})$ such that $0 < \|\mathbf{w}\|_\infty \leq \beta'_\infty$ and $0 < \|\mathbf{w}\|_2 \leq \beta'_2$.

Setup Stage: The adversary $\mathcal{B}$ first samples the tags $\tau^{(1)},\ldots,\tau^{(Q)}$ from $U(\mathcal{T})$ that will be used to answer $\mathcal{A}$'s signing queries. At this point, $\mathcal{B}$ aborts if the set of tags contains a collision[7]. We call this event Col, and further condition on $\neg\mathsf{Col}$. The adversary $\mathcal{B}$ makes a guess $i^+ \hookleftarrow U([Q])$ on the index of the tag that will be re-used by $\mathcal{A}$ in the forgery stage. Then, $\mathcal{B}$ samples $\mathbf{R} \hookleftarrow U([-1,1]^{m_1\times m_2})$, and $\mathbf{U}$ from $U([-1,1]^{m_1\times m_3})$. It then defines

$$\begin{cases} \mathbf{B} = \overline{\mathbf{A}}\mathbf{R} + \tau^{(i^+)}\mathbf{G} \bmod q \\ \mathbf{D} = \overline{\mathbf{A}}\mathbf{U} \bmod q \end{cases}$$

[7] We could see this stage as a preprocessing stage and make $\mathcal{B}$ re-draw the tags until there is no collision. This would affect the runtime instead of the advantage.

The adversary $\mathcal{B}$ samples $\mathbf{v}$ from $\mathcal{D}_{\mathbb{Z}^m,\sigma}$, $\mathbf{r}_0$ from $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_2}$, and defines

$$\mathbf{u} = \mathbf{A}_{\tau^{(i^+)}} \left(\mathbf{v} - \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{0}_{m_2} \end{bmatrix} \right) \bmod q.$$

Note that for all $i \in [Q]$, we have

$$\mathbf{A}_{\tau^{(i)}} = [\overline{\mathbf{A}} | (\tau^{(i)} - \tau^{(i^+)})\mathbf{G} - \overline{\mathbf{A}}\mathbf{R}],$$

where the contribution in $\mathbf{G}$ vanishes only for $i = i^+$, as there is no collision. The adversary $\mathcal{B}$ thus forms $\mathsf{pp} = (\mathbf{D}; \mathbf{g}; \lambda, n, m_1, m_2, m_3, m, q, \sigma, \sigma_2, \sigma_1)$ and the public key $\mathsf{pk} = (\overline{\mathbf{A}}, \mathbf{B}, \mathbf{u})$, and sends both to $\mathcal{A}$.

Query Stage: We distinguish the queries for $i \neq i^+$ from the i^+-th query. First, consider the i-th query, for $i \neq i^+$, on the message $\mathbf{m}^{(i)}$. $\mathcal{B}$ samples a fresh randomness $\mathbf{r}^{(i)}$ from $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_2}$ and computes the commitment $\mathbf{c} = \overline{\mathbf{A}}\mathbf{r}^{(i)} + \mathbf{D}\mathbf{m}^{(i)} \bmod q$. Since $\tau^{(i)} - \tau^{(i^+)} \in \mathbb{Z}_q^\times$, $\mathcal{B}$ computes

$$\mathbf{v}^{(i)} = \mathsf{SampleD}(\mathbf{R}, \mathbf{A}, (\tau^{(i)} - \tau^{(i^+)})\mathbf{I}_n, \mathbf{u} + \mathbf{c}, \sigma) - \begin{bmatrix} \mathbf{r}^{(i)} \\ \mathbf{0}_{m_2} \end{bmatrix}.$$

Note that $\mathbf{v}^{(i)}$ is correctly distributed and passes verification (with overwhelming probability by Lemma 2.2 and 2.3). The signature given to $\mathcal{A}$ is $\mathsf{sig}_i = (\tau^{(i)}, \mathbf{v}^{(i)})$.

Now consider the i^+-th query. In this case, $\mathcal{B}$ simply computes $\mathbf{v}^{(i^+)} = \mathbf{v} - \begin{bmatrix} \mathbf{r}_0 - \mathbf{U}\mathbf{m}^{(i^+)} \\ \mathbf{0}_{m_2} \end{bmatrix}$ and gives $\mathsf{sig}_{i^+} = (\tau^{(i^+)}, \mathbf{v}^{(i^+)})$ to $\mathcal{A}$. We analyze later the distribution of $\mathbf{v}^{(i^+)}$, but notice that the verification equation is verified because of the definition of $\mathbf{u}$.

$$\begin{aligned} \mathbf{A}_{\tau^{(i^+)}}\mathbf{v}^{(i^+)} &= \mathbf{u} + \mathbf{A}_{\tau^{(i^+)}} \begin{bmatrix} \mathbf{U}\mathbf{m}^{(i^+)} \\ \mathbf{0}_{m_2} \end{bmatrix} \bmod q \\ &= \mathbf{u} + \overline{\mathbf{A}}\mathbf{U}\mathbf{m}^{(i^+)} \bmod q \\ &= \mathbf{u} + \mathbf{D}\mathbf{m}^{(i^+)} \bmod q. \end{aligned}$$

Forgery Stage: After at most Q queries, $\mathcal{A}$ outputs a Type II forgery $(\tau^*, \mathbf{v}^*)$ on a new message $\mathbf{m}^*$. If $\mathcal{A}$ fails to output a valid forgery, event that we denote by Abort_1, then $\mathcal{B}$ aborts. We now condition on $\neg\mathsf{Abort}_1$. At this point, $\mathcal{B}$ checks its guess on i^+ and aborts if $\tau^* \neq \tau^{(i^+)}$. We denote this event Abort_2, and further condition on $\neg\mathsf{Abort}_2$. It holds that

$$\mathbf{A}_{\tau^{(i^+)}}\mathbf{v}^{(i^+)} - \mathbf{D}\mathbf{m}^{(i^+)} = \mathbf{u} \bmod q = \mathbf{A}_{\tau^*}\mathbf{v}^* - \mathbf{D}\mathbf{m}^* \bmod q.$$

Since $\mathbf{A}_{\tau^*} = \mathbf{A}_{\tau^{(i^+)}} = \overline{\mathbf{A}}[\mathbf{I}_{m_1} | -\mathbf{R}]$, it holds that

$$\overline{\mathbf{A}} \left([\mathbf{I}_{m_1} | -\mathbf{R}](\mathbf{v}^{(i^+)} - \mathbf{v}^*) - \mathbf{U}(\mathbf{m}^{(i^+)} - \mathbf{m}^*) \right) = \mathbf{0} \bmod q.$$

As a result, $\mathcal{B}$ forms the vector

$$\mathbf{w} = [\mathbf{I}_{m_1} | -\mathbf{R}](\mathbf{v}^{(i^+)} - \mathbf{v}^*) - \mathbf{U}(\mathbf{m}^{(i^+)} - \mathbf{m}^*) \in \mathbb{Z}^{m_1},$$

which is in $\Lambda_q^{\perp}(\overline{\mathbf{A}})$, and returns it as a solution for SIS.

Advantage: We now analyze the advantage of $\mathcal{B}$. First, a standard calculation allows one to bound $\mathbb{P}[\mathsf{Col}]$. Since the tags are independent and uniform, we have

$$\begin{aligned}\mathbb{P}[\mathsf{Col}] &= 1 - \mathbb{P}[\forall i \neq j, \tau^{(i)} \neq \tau^{(j)}] \\ &= 1 - \prod_{i \in [Q-1]} (1 - i/|\mathcal{T}|) \\ &= - \sum_{i \in [Q-1]} -i/|\mathcal{T}| \cdot \prod_{i < j \leq Q-1} (1 - j/|\mathcal{T}|) \\ &\leq Q(Q-1)/(2|\mathcal{T}|). \end{aligned} \tag{8}$$

We now focus on the distribution of $(\mathsf{pk}, \mathsf{pp})$. Since $m_1 \log_2 3 \geq n \log_2 q + f(\lambda)$, Lemma 2.1 yields

$$\begin{cases} \Delta((\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{R} \bmod q, U(\mathbb{Z}_q^{n \times m_1} \times \mathbb{Z}_q^{n \times m_2})) \leq \frac{m_2}{2}\sqrt{\frac{q^n}{3^{m_1}}} \leq m_2 2^{-f(\lambda)/2-1}, \\ \Delta((\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{U} \bmod q, U(\mathbb{Z}_q^{n \times m_1} \times \mathbb{Z}_q^{n \times m_3})) \leq \frac{m_3}{2}\sqrt{\frac{q^n}{3^{m_1}}} \leq m_3 2^{-f(\lambda)/2-1}. \end{cases}$$

As $\overline{\mathbf{A}}, \mathbf{R}$ are independent of $\tau^{(i^+)}\mathbf{G}$, it holds that $\Delta(\mathbf{B}, \overline{\mathbf{A}}\mathbf{R}) \leq m_2 2^{-f(\lambda)/2}$ (by the triangle inequality). Then, let us analyze the distribution of $\mathbf{u}$. Define $\mathbf{A}' = [\overline{\mathbf{A}} | -\overline{\mathbf{A}}\mathbf{R}] \bmod q$. By construction, we have $\mathbf{u} = \mathbf{A}'[(\mathbf{v}_1 - \mathbf{r}_0)^T | \mathbf{v}_2^T]^T \bmod q$. Fix $f(m) = \omega(\sqrt{\log_2 m})$ such that $\sigma, \sigma_1 \geq f(m)$. Lemma B.3 thus yields that $\mathbf{u}$ is within negligible statistical distance of $U(\mathbb{Z}_q^n)$, conditioning on $\mathbf{A}'$ being uniform and $\mathbf{v}_1 - \mathbf{r}_0$ being Gaussian. Lemma 2.2 yields that $\mathbf{v}_1 - \mathbf{r}_0$ is within statistical distance $7\varepsilon/4$ of $\mathcal{D}_{\mathbb{Z}^{m_1}, \sigma_1}$. Without loss of generality, we can take the same ε as in Lemma B.3. Changing $\mathbf{A}'$ back to $[\overline{\mathbf{A}} | -\overline{\mathbf{A}}\mathbf{R}]$ gives

$$\Delta(\mathbf{u}, U(\mathbb{Z}_q^n)) \leq \mathsf{negl}(\lambda) + m_2 2^{-f(\lambda)/2-1} = \mathsf{negl}(\lambda).$$

As a result, $(\mathsf{pk}, \mathsf{pp})$ is correctly distributed up to a negligible statistical distance. We now analyze the distribution of the signature that are produced by $\mathcal{B}$. For the i-th query with $i \neq i^+$, the signature is distributed exactly as in the legitimate algorithm. At the i^+-th signing query, the vector $\mathbf{v}_1^{(i^+)} = \mathbf{v}_1 - \mathbf{r}_0 + \mathbf{U}\mathbf{m}^{(i^+)}$ is within statistical distance $7\varepsilon/4$ of $\mathcal{D}_{\mathbb{Z}^{m_1}, \sigma_1, \mathbf{z}^+}$, where $\mathbf{z}^+ = \mathbf{U}\mathbf{m}^{(i^+)}$. As before, by Equation (3) obtained by combining Lemma 2.4 and 2.5, we have

$$\left\|\mathbf{z}^+\right\|_2 = \left\|\mathbf{U}\mathbf{m}^{(i^+)}\right\|_2 \leq \min(2\sqrt{m_1}, \sqrt{m_1} + \sqrt{m_3} + t)\sqrt{m_3},$$

except with probability $2e^{-\pi t^2} + 2^{-2\lambda}$. We now measure the closeness of $\mathbf{v}^{(i^+)}$ to the real distribution by using the Rényi divergence of order α for a free parameter $\alpha > 1$. By Lemma B.2 it holds that

$$\mathrm{RD}_\alpha(\mathcal{D}_{\mathbb{Z}^{m_1}, \sigma_1} \| \mathcal{D}_{\mathbb{Z}^{m_1}, \sigma_1, \mathbf{z}^+}) \leq \exp\left(\frac{\alpha\pi}{\sigma_1^2}\left\|\mathbf{z}^+\right\|_2^2\right) \leq e^{\alpha\pi},$$

as $\sigma_1 \geq \min(2\sqrt{m_1}, \sqrt{m_1} + \sqrt{m_3} + t)\sqrt{m_3}$. Combining the probabilities for the distribution of the keys and the signatures, and by the probability preservation properties of the statistical distance and Rényi divergence of Lemma B.1, we have

$$\mathbb{P}[\neg\mathsf{Abort}_1|\neg\mathsf{Col}] \geq e^{-\alpha\pi}(\delta - \mathsf{negl}(\lambda))^{\alpha/(\alpha-1)} \geq e^{-\alpha\pi}\delta^{\alpha/(\alpha-1)} - \mathsf{negl}(\lambda). \tag{9}$$

We then optimize over α. The maximum value of the right-hand side is attained for $\alpha^* = 1 + \sqrt{\log_2(1/\delta)/(\pi \log_2 e)}$. Further, since the guess i^+ is independent of $\mathcal{A}$'s view it holds that

$$\mathbb{P}[\neg\mathsf{Abort}_2|\neg\mathsf{Abort}_1 \wedge \neg\mathsf{Col}] = \frac{1}{Q}. \tag{10}$$

We now analyze the solution constructed by $\mathcal{B}$. We have to show it is non-zero and have norm at most β'. We first focus on the former. Define $\mathbf{u}^* = [\mathbf{I}_{m_1}|-\mathbf{R}](\mathbf{v}^{(i^+)} - \mathbf{v}^*)$ which can be controlled by $\mathcal{A}$. In particular, we do not exclude the fact that $\mathcal{A}$ chooses $\mathbf{v}_2^* = \mathbf{v}_2^{(i^+)}$. Hence we cannot rely on the unpredictability of $\mathbf{R}$. However, since $\mathbf{m}^* \neq \mathbf{m}^{(i^+)}$, the vector $\mathbf{w}$ features at least one column of $\mathbf{U}$. We show that this is enough to argue that $\mathbf{w} \neq \mathbf{0}$ except with negligible probability. Let $S \subseteq [m_3]$ the set of indices at which $\mathbf{m}^*$ and $\mathbf{m}^{(i^+)}$ differ. Note that since $\mathbf{U}$ is centered, we can write $\mathbf{w} = \mathbf{u}^* - \sum_{i \in S} \mathbf{u}_i$ where $\mathbf{u}_i$ are independently distributed according to $U([-1,1]^{m_1})$ (they are the columns of $\mathbf{U}$ up to a ± 1 factor). For clarity we write $\mathbf{u}_S = \sum_{i \in S} \mathbf{u}_i$. It holds that

$$\begin{aligned}
&\mathbb{P}_{\mathbf{U}}[\mathbf{w} = \mathbf{0}|\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{U} \bmod q, \mathbf{v}_1 - \mathbf{r}_0 + \mathbf{U}\mathbf{m}^{(i^+)}] \\
=&\mathbb{P}_{\mathbf{U}}[\mathbf{u}_S = \mathbf{u}^*|\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{U} \bmod q, \mathbf{v}_1 - \mathbf{r}_0 + \mathbf{U}\mathbf{m}^{(i^+)}] \\
\leq&\sqrt{\mathbb{P}_{\mathbf{U}}[\mathbf{u}_S = \mathbf{u}^*|\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{U} \bmod q, \mathcal{D}_{\mathbb{Z}^{m_1},\sigma_1}] \cdot \mathrm{RD}_2(\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_1,\mathbf{U}\mathbf{m}^{(i^+)}} \| \mathcal{D}_{\mathbb{Z}^{m_1},\sigma_1})} \\
&\quad + 7\varepsilon/4 \\
\leq&\sqrt{\mathbb{P}_{\mathbf{U}}[\mathbf{u}_S = \mathbf{u}^*|\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{U} \bmod q] \cdot \exp(2\pi\left\|\mathbf{U}\mathbf{m}^{(i^+)}\right\|_2^2/\sigma_1^2) \cdot (1+\varepsilon)^2/(1-\varepsilon)^2} \\
&\quad + 7\varepsilon/4 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (11) \\
\leq&\frac{1+\varepsilon}{1-\varepsilon}e^{\pi}\sqrt{\mathbb{P}_{\mathbf{U}}[\mathbf{u}_S = \mathbf{u}^*|\overline{\mathbf{A}}, U(\mathbb{Z}_q^{n \times m_3})] + \mathit{\Delta}((\overline{\mathbf{A}}, \overline{\mathbf{A}}\mathbf{U} \bmod q), (\overline{\mathbf{A}}, U(\mathbb{Z}_q^{n \times m_3}))} \\
&\quad + 7\varepsilon/4 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (12) \\
\leq&\frac{1+\varepsilon}{1-\varepsilon}e^{\pi}\sqrt{\mathbb{P}_{\mathbf{U}}[\mathbf{u}_S = \mathbf{u}^*] + m_3 2^{-f(\lambda)/2-1}} + 7\varepsilon/4,
\end{aligned}$$

where (11) stems from Lemma B.2 as $\sigma_1 \geq \sigma \geq \eta_\varepsilon(\mathbb{Z}^{m_1})$[8], and (12) follows from Lemma 2.1. Since the $(\mathbf{u}_i)_{i \in S}$ are independent and uniformly distributed in $[-1,1]^{m_1}$, it follows (by induction of $|S|$) that

$$\mathbb{P}_{\mathbf{U}}[\mathbf{u}_S = \mathbf{u}^*] \leq 3^{-m_1}. \tag{13}$$

[8] Note that the Rényi divergence is taken in the opposite direction than before, hence the presence of the factor $(1+\varepsilon)/(1-\varepsilon)$.

Assuming without loss of generality that $\varepsilon \leq 1/2$, we obtain that $\mathbf{w} \neq \mathbf{0}$ except with probability $3e^{\pi}\sqrt{3^{-m_1} + m_3 \cdot 2^{-f(\lambda)/2-1}} + 7\varepsilon/4 = \mathsf{negl}(\lambda)$.

Finally, note that by definition of $\mathbf{v}_1^{(i^+)}$ we can rewrite $\mathbf{w}$ as

$$\mathbf{w} = [\mathbf{I}_{m_1} | -\mathbf{R}] \begin{bmatrix} (\mathbf{v}_1 - \mathbf{r}_0) - \mathbf{v}_1^* \\ \mathbf{v}_2 - \mathbf{v}_2^* \end{bmatrix} + \mathbf{U}\mathbf{m}^*$$

Then, it holds that

$$\begin{aligned} \|\mathbf{w}\|_\infty &\leq \|(\mathbf{v}_1 - \mathbf{r}_0) - \mathbf{v}_1^*\|_\infty + m_2\|\mathbf{R}\|_{\max}\|\mathbf{v}_2 - \mathbf{v}_2^*\|_\infty + m_3\|\mathbf{U}\|_{\max}\|\mathbf{m}^*\|_\infty \\ &\leq 2\sigma_1 \log_2 m_1 + m_2 \cdot 2\sigma \log_2 m_2 + m_3 \\ &= \beta'_\infty. \end{aligned}$$

The inequality is valid if $\mathbf{v}_1 - \mathbf{r}_0$ follows $\mathcal{D}_{\mathbb{Z}^{m_1},\sigma_1}$ (Lemma 2.2) and that the Gaussian tail bound of Lemma 2.3 is verified for $\mathbf{v}_1 - \mathbf{r}_0, \mathbf{v}_2$. By the union bound, this happens with probability at least $1 - (2m_1 e^{-\pi \log_2^2 m_1} + 7\varepsilon/4) - 2m_2 e^{-\pi \log_2^2 m_2} = 1 - \mathsf{negl}(\lambda)$. As in the proof of Lemma 3.1, we cannot use the Gaussian tail bound in Euclidean norm for $\mathbf{v}^*$. Hence, we have the following

$$\begin{aligned} \|\mathbf{w}\|_2 \leq & \sqrt{1 + \|\mathbf{R}\|_2^2}\sqrt{(\sigma^2 m_1 + \sigma_2^2 m_1 + m_1\sigma_1^2 \log_2^2 m_1) + (\sigma^2 m_2 + m_2\sigma^2 \log_2^2 m_2)} \\ & \quad + \|\mathbf{U}\mathbf{m}^*\|_2 \\ & \leq \sqrt{1 + (\sqrt{m_1} + \sqrt{m_2} + t)^2}\sqrt{\sigma_1^2 m_1(1 + \log_2^2 m_1) + \sigma^2 m_2(1 + \log_2^2 m_2)} \\ & \quad + \min(2\sqrt{m_1}, \sqrt{m_1} + \sqrt{m_3} + t)\sqrt{m_3} \\ & = \beta'_2, \end{aligned}$$

where the first inequality follows from Lemma 2.3 except with probability $2 \cdot 2^{-2m_1} + 2^{-2m_2}$, and the second inequality stems from Lemma 2.4 except with probability $4e^{-\pi t^2} + 2^{-2\lambda}$. This yields

$$\mathbb{P}[\mathbf{w} \text{ valid solution} | \neg\mathsf{Abort}_{\{1,2\}} \wedge \neg\mathsf{Col}] = 1 - \mathsf{negl}(\lambda). \tag{14}$$

Combining Equations (8), (9), (10) and (14) by the probability chain rule, we get

$$\mathrm{Adv}[\mathcal{B}] \geq (1 - Q^2/(2(q'-1)))(\delta^{\alpha^*/(\alpha^*-1)} e^{-\alpha^*\pi} - \mathsf{negl}(\lambda)) \cdot \frac{1}{Q} \cdot (1 - \mathsf{negl}(\lambda)),$$

as desired. Note that the parameters and the behavior of $\mathcal{B}$ do not depend on the order α that is used to compute the advantage bound. As such, α^* can indeed depend on the forger's advantage δ. □

Proof (of (13)*).* For completeness, we detail out the proof of Equation (13) even though it is a standard calculation from probability theory. Let $(\mathbf{u}_i)_{i \in [m_3]}$ be independent random vectors distributed according to $[-1,1]^{m_1}$. For any set $S \subseteq [m_3]$, we define $\mathbf{u}_S = \sum_{i \in S} \mathbf{u}_i$. For any $k \in [m_3]$, we define the statement

$$\mathcal{P}(k) : \forall S \subseteq [m_3], |S| = k \Rightarrow \forall \mathbf{u}^* \in [-k,k]^{m_1}, \mathbb{P}_{(\mathbf{u}_i)_{i \in S}}[\mathbf{u}_S = \mathbf{u}^*] \leq 3^{-m_1}.$$

Initialization: Let $S \subseteq [m_3]$ such that $|S| = 1$. Denote by i_0 the only element of S. Then, it directly holds that

$$\forall \mathbf{u}^* \in [-1,1]^{m_1}, \mathbb{P}_{\mathbf{u}_{i_0}}[\mathbf{u}_{i_0} = \mathbf{u}^*] = 3^{-m_1} \leq 3^{-m_1}.$$

Induction: Assume that $\mathcal{P}(k)$ is verified for some $k \in [m_3]$. Let $S \subseteq [m_3]$ such that $|S| = k+1$. Let i_0 be in S (exists because S is non-empty). Let $\mathbf{u}^*$ be in $[-(k+1), k+1]^{m_1}$. It holds that

$$\begin{aligned}
\mathbb{P}_{(\mathbf{u}_i)_{i\in S}}\left[\mathbf{u}_S = \mathbf{u}^*\right] &= \sum_{\mathbf{u}'_{i_0} \in [-1,1]^{m_1}} \mathbb{P}_{\mathbf{u}_{i_0}}[\mathbf{u}_{i_0} = \mathbf{u}'_{i_0}] \cdot \mathbb{P}_{(\mathbf{u}_i)_{i\in S}}\left[\mathbf{u}_S = \mathbf{u}^* | \mathbf{u}_{i_0} = \mathbf{u}'_{i_0}\right] \\
&= \sum_{\mathbf{u}'_{i_0} \in [-1,1]^{m_1}} 3^{-m_1} \cdot \mathbb{P}_{(\mathbf{u}_i)_{i\in S\setminus\{i_0\}}}\left[\mathbf{u}_{S\setminus\{i_0\}} = \mathbf{u}^* - \mathbf{u}'_{i_0}\right].
\end{aligned}$$

Yet, by $\mathcal{P}(k)$, we have that

$$\begin{aligned}
&\mathbb{P}_{(\mathbf{u}_i)_{i\in S\setminus\{i_0\}}}[\mathbf{u}_{S\setminus\{i_0\}} = \mathbf{u}^* - \mathbf{u}'_{i_0}] \\
&\qquad = \mathbf{1}(\mathbf{u}^* - \mathbf{u}_{i_0} \in [-k,k]^{m_1}) \cdot \mathbb{P}_{(\mathbf{u}_i)_{i\in S\setminus\{i_0\}}}\left[\mathbf{u}_{S\setminus\{i_0\}} = \mathbf{u}^* - \mathbf{u}'_{i_0}\right] \\
&\qquad \leq 3^{-m_1}.
\end{aligned}$$

Hence, we obtain

$$\mathbb{P}_{(\mathbf{u}_i)_{i\in S}}\left[\mathbf{u}_S = \mathbf{u}^*\right] \leq \sum_{\mathbf{u}'_{i_0} \in [-1,1]^{m_1}} 3^{-m_1} \cdot 3^{-m_1} = 3^{-m_1},$$

thus proving $\mathcal{P}(k+1)$. By induction, $\mathcal{P}(k)$ is true for all $k \in [m_3]$. □

C Proof of Lemma 5.1

Proof. For clarity, we denote by $\overline{\mathbf{x}} = \mathbf{x} \bmod q$ the vector of representatives in $[-q/2, q/2]$ of a vector $\mathbf{x}$. With such representatives, we have $\overline{C} = C$ for any integer C in $[-q/2, q/2]$, which simplifies the notations in what follows. We assume that there exists i in $[n]$ such that $\overline{w_i} \notin [-2B, 2B]$. Let $\mathbf{h}$ be a random vector distributed according to $U([-1,1]^n)$. For clarity, we define $S = [n] \setminus \{i\}$. We now have

$$\begin{aligned}
&\mathbb{P}_{\mathbf{h}}[\overline{\mathbf{h}^T\mathbf{w}} \in [-B,B]] \\
&\quad = \sum_{C\in[-q/2,q/2]} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j\in S} h_j w_j} = C] \cdot \mathbb{P}_{\mathbf{h}}[\overline{\sum_{i\in[n]} h_j w_j} \in [-B,B] | \overline{\sum_{j\in S} h_j w_j} = C] \\
&\quad = \sum_{C\in[-q/2,q/2]} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j\in S} h_j w_j} = C] \cdot \mathbb{P}_{h_i}[\overline{h_i\overline{w_i} + C} \in [-B,B]] \\
&\quad = \sum_{C\in[-q/2,q/2]} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j\in S} h_j w_j} = C] \cdot \frac{\left|\{h_i : \overline{h_i\overline{w_i} + C} \in [-B,B]\}\right|}{3},
\end{aligned}$$

where the second equality is a consequence of h_i being uniform in $[-1, 1]$. We now split the sum indexed by C into two complementary parts $\varSigma_1$ and $\varSigma_2$ as follows.

$$\varSigma_1 = \sum_{C \in]-\frac{q}{2}+2B, \frac{q}{2}-2B[} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j \in S} h_j w_j} = C] \cdot \frac{\left|\{h_i : \overline{h_i \overline{w_i} + C} \in [-B, B]\}\right|}{3}$$

$$\varSigma_2 = \sum_{C \in [-\frac{q}{2}, -\frac{q}{2}+2B] \cup [\frac{q}{2}-2B, \frac{q}{2}]} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j \in S} h_j w_j} = C] \cdot \frac{\left|\{h_i : \overline{h_i \overline{w_i} + C} \in [-B, B]\}\right|}{3}$$

We can now focus on bounding the set $\{h_i : \overline{h_i \overline{w_i} + C} \in [-B, B]\}$ in each case. First note that for all $h_i \in [-1, 1]$, if

$$\begin{cases} \overline{h_i \overline{w_i} + C} \in [-B, B] \\ \overline{(h_i + 1)\overline{w_i} + C} \in [-B, B], \end{cases}$$

are both satisfied, then there exist $r_1, r_2 \in [-B, B]$ and $k_1, k_2 \in \mathbb{Z}$, such that

$$h_i \overline{w_i} + C = r_1 + k_1 q \quad \wedge \quad (h_i + 1)\overline{w_i} + C = r_2 + k_2 q.$$

Note that the above equations are now over $\mathbb{Z}$, not $\mathbb{Z}_q$. Combining these two equations gives us $\overline{w_i} = r_2 - r_1 + (k_2 - k_1)q$, which implies that the representative $\overline{w_i}$ is necessarily in $[-2B, 2B]$. This contradicts the original assumption of $\overline{w_i} \notin [-2B, 2B]$. In other words, we have shown that the set $\{h_i : \overline{h_i \overline{w_i} + C} \in [-B, B]\}$ cannot contain two consecutive numbers.

Now let us consider the situation where both h_i and $h_i + 2$ would be in this set. As $h_i \in [-1, 1]$, this can only occur when $h_i = -1$. This means that:

$$\begin{cases} \overline{-\overline{w_i} + C} \in [-B, B] \\ \overline{\overline{w_i} + C} \in [-B, B], \end{cases}$$

and hence there exist $r_1, r_2 \in [-B, B]$ and $k_1, k_2 \in \mathbb{Z}$, such that

$$-\overline{w_i} + C = r_1 + k_1 q \tag{15}$$

$$\overline{w_i} + C = r_2 + k_2 q. \tag{16}$$

This implies that $2\overline{w_i} = r_2 - r_1 + (k_2 - k_1)q$. As $\overline{w_i} \notin [-2B, 2B]$, these equations can be satisfied only when $\overline{w_i} = \frac{r_2 - r_1 \pm q}{2}$ with $r_2 - r_1 \in [-2B, 2B]$. The latter interval implies that $\overline{w_i} \in [-\frac{q}{2}, -\frac{q}{2} + B] \cup [\frac{q}{2} - B, \frac{q}{2}]$. But in that case, Equation (15) implies that $C = \overline{w_i} + r_1 \in [-\frac{q}{2}, -\frac{q}{2} + 2B] \cup [\frac{q}{2} - 2B, \frac{q}{2}]$. In other words, $\{h_i : \overline{h_i \overline{w_i} + C} \in [-B, B]\}$ contains at most 1 element if $C \notin [-\frac{q}{2}, -\frac{q}{2} + 2B] \cup [\frac{q}{2} - 2B, \frac{q}{2}]$ and at most two elements otherwise. We thus get the following bounds on $\varSigma_1$ and $\varSigma_2$.

$$\varSigma_1 \leq \frac{1}{3} \sum_{C \in]-\frac{q}{2}+2B, \frac{q}{2}-2B[} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j \in S} h_j w_j} = C]$$

$$\varSigma_2 \leq \frac{2}{3} \sum_{C \in [-\frac{q}{2}, -\frac{q}{2}+2B] \cup [\frac{q}{2}-2B, \frac{q}{2}]} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j \in S} h_j w_j} = C]$$

Let x denote $\sum_{C\in[-\frac{q}{2},-\frac{q}{2}+2B]\cup[\frac{q}{2}-2B,\frac{q}{2}]} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j\in S} h_j w_j} = C]$. Then, we have that $1-x$ is $\sum_{C\in]-\frac{q}{2}+2B,\frac{q}{2}-2B[} \mathbb{P}_{\mathbf{h}}[\overline{\sum_{j\in S} h_j w_j} = C]$ which yields

$$\mathbb{P}_{\mathbf{h}}[\overline{\mathbf{h}^T\mathbf{w}} \in [-B,B]] \leq \frac{1}{3} + \frac{x}{3}.$$

Our last task is then to find a suitable upper bound on x. More concretely, we want to prove that $x \leq \frac{2}{3}$. To this end, we will show that, for any vector $\mathbf{u}$ uniformly sampled from $\{-1,0,1\}^{n-1}$ and any vector $\mathbf{w} \in \mathbb{Z}_q^{n-1}$, the probability (over the choice of $\mathbf{u}$) that $\sum_j u_j w_j \in]-\frac{q}{2},-\frac{q}{2}+2B] \cup [\frac{q}{2}-2B,\frac{q}{2}]$ is at most $\frac{2}{3}$ when the requirements of our lemma are fulfilled.

In our case, we first recall that the elements of the sets $\{h_j\}_{j\in S}$ are uniformly sampled from $\{-1,0,1\}^{n-1}$ that we identify to $\mathbb{Z}_3$ seen as an additive group. Let $\mathbf{t} = [1,\ldots,1]^T \in \mathbb{Z}_3^{n-1}$ and let $T = \mathbb{Z}_3^{n-1}/\langle\mathbf{t}\rangle$. Any element $\mathbf{u} \in T$ then has exactly 3 representatives in $\mathbb{Z}_3^{n-1}$ that we note $\mathbf{u},\mathbf{u}',\mathbf{u}'' \in \{-1,0,1\}^{n-1}$. We then have exactly

$$\mathbf{u} + \mathbf{u}' + \mathbf{u}'' = \mathbf{0}$$

where the previous equation holds in $\mathbb{Z}^{n-1}$ because, for any $j \in [n-1]$, it holds $\{u_j, u_j', u_j''\} = \{-1,0,1\}$. For all $\mathbf{w} \in \mathbb{Z}_q^{n-1}$, we define

$$\varSigma_{\mathbf{u}} = \sum_{j=1}^{n-1} u_j w_j,\ \varSigma_{\mathbf{u}'} = \sum_{j=1}^{n-1} u_j' w_j,\ \varSigma_{\mathbf{u}''} = \sum_{j=1}^{n-1} u_j'' w_j.$$

We then know that $\varSigma_{\mathbf{u}} + \varSigma_{\mathbf{u}'} + \varSigma_{\mathbf{u}''} = 0$ in $\mathbb{Z}$ and hence that $\overline{\varSigma_{\mathbf{u}}} + \overline{\varSigma_{\mathbf{u}'}} + \overline{\varSigma_{\mathbf{u}''}} = 0$. What remains to prove is that this implies that at least one of these representatives is not in $]-\frac{q}{2},-\frac{q}{2}+2B] \cup [\frac{q}{2}-2B,\frac{q}{2}]$. Let us assume, without loss of generality that both $\overline{\varSigma_{\mathbf{u}}}$ and $\overline{\varSigma_{\mathbf{u}'}}$ are in this set (else, we are done). This means (over $\mathbb{Z}$) that

$$\overline{\varSigma_{\mathbf{u}}} + \overline{\varSigma_{\mathbf{u}'}} = r + kq$$

for some integer k and some $r \in [-4B,4B]$. Therefore $\overline{\varSigma_{\mathbf{u}''}} = -r-kq$ and, as it is an element of $]-\frac{q}{2},\frac{q}{2}[$ (as any representative), this means that $\overline{\varSigma_{\mathbf{u}''}} = -r \in [-4B,4B]$. Since the lemma assumes $\frac{q}{2} > 6B$, this means that $\overline{\varSigma_{\mathbf{u}''}} \notin]-\frac{q}{2},-\frac{q}{2}+2B] \cup [\frac{q}{2}-2B,\frac{q}{2}]$. In other words, for all $\mathbf{w} \in \mathbb{Z}_q^{n-1}$, among the 3 representatives of any element of T, at least one is such that $\overline{\sum_j u_j w_j}$ is not in this set, which proves that $x \leq \frac{2}{3}$ and hence our lemma.

□

D Instantiating [LLM$^+$16] with $\boldsymbol{\mathcal{R}}^*$

The original construction by Libert et al. [LLM$^+$16] uses the binary decomposition of the commitment $\mathbf{c}$ instead of using the commitment itself. It additionally bases itself on the Boyen signature scheme, and involves an extra matrix $\mathbf{D} \in \mathbb{Z}_q^{n \times 2nk}$, where $k = \lceil \log_2 q \rceil$. We analyze here the impact the construction of [LLM+16] using the Yang et al. framework from [YAZ+19] for a fair comparison. For this section only, we set the parameters differently according to [LLM+16]. We thus have $m = 2nk$, $\sigma_1 = \sigma\sqrt{1 + 8(N+1)^2 m^3}$. Also, prior to being signed, the message blocks are encoded using $b \mapsto (1-b, b)$. This means that although the relevant message information is of mN bits, it is treated as a message of $2mN$ bits. To be thorough, one would need to prove that the message is properly encoded in addition to proving that the message is binary. This can be done by proving the additional relation $(\mathbf{I}_{mN} \otimes [1\ 1])\mathbf{m} = \mathbf{1}_{mN}$ which proves that the consecutive bits $b, 1-b$ indeed sum to 1. Since the relation is proven modulo q, one must make sure that the coefficients are $\mathbf{m}$ are also proven binary. For simplicity, we do not take this into account in the estimations of Table F.1. The matrices $\mathbf{A}_i$ are uniform in $\mathbb{Z}_q^{n \times m}$, but the commitment key matrices $\mathbf{D}_i$ are uniform in $\mathbb{Z}_q^{2n \times 2m}$. We define $\mathbf{H} = \mathbf{I}_{2n} \otimes [2^0 \ldots 2^{k-1}]$. Since the binary decomposition operator is non-linear, the verification equation has to be splitted into two equations as follows.

$$\begin{cases} \mathbf{A}\mathbf{v}_1 + \mathbf{A}_0\mathbf{v}_2 + \sum_{i\in[\ell]} \mathbf{A}_i(\tau[i]\mathbf{v}_2) - \mathbf{D}\mathbf{w} = \mathbf{u} \bmod q, \\ \mathbf{H}\mathbf{w} = \mathbf{D}_0\mathbf{r} - \sum_{i\in[N]} \mathbf{D}_i\mathbf{m}_i \bmod q, \end{cases}$$

with $\|\mathbf{v}_1\|_\infty, \|\mathbf{v}_2\|_\infty \leq \sigma \log_2 m$, $\|\mathbf{r}\|_\infty \leq \sigma_1 \log_2 2m$ as well as $\tau \in \{0,1\}^\ell, \mathbf{m} \in \{0,1\}^{2mN}$, and $\mathbf{w} \in \{0,1\}^{2nk}$. We define $\alpha, \alpha_1, k_\alpha, k_{\alpha_1}, \mathbf{g}_\alpha, \mathbf{g}_{\alpha_1}$ in a similar way as Section 5.2. We then define $\mathbf{a} = \alpha\mathbf{1}_m$, $\mathbf{a}_1 = \alpha_1\mathbf{1}_{2m}$ and set $\mathbf{G}_\alpha = \mathbf{I}_m \otimes \mathbf{g}_\alpha$ and $\mathbf{G}_{\alpha_1} = \mathbf{I}_{2m} \otimes \mathbf{g}_{\alpha_1}$. Then, we define $\mathbf{u}_i = \mathbf{A}_i\mathbf{v}_2 \in \mathbb{Z}^n$, as well as $\mathbf{u}'_i = \tau[i]\mathbf{u}_i$. The verification equations thus become

$$\begin{cases} \mathbf{A}\mathbf{G}_\alpha\overline{\mathbf{v}_1} + \mathbf{A}_0\mathbf{G}_\alpha\overline{\mathbf{v}_2} + \sum_{i\in[\ell]} \mathbf{u}'_i - \mathbf{D}\mathbf{w} = \mathbf{u} + (\mathbf{A} + \mathbf{A}_0)\mathbf{a} \bmod q, \\ \mathbf{D}_0\mathbf{G}_{\alpha_1}\overline{\mathbf{r}} + \sum_{i\in[N]} \mathbf{D}_i\mathbf{m}_i - \mathbf{H}\mathbf{w} = \mathbf{D}_0\mathbf{a}_1 \bmod q, \\ \mathbf{A}_i\mathbf{G}_\alpha\overline{\mathbf{v}_2} - \mathbf{u}_i = \mathbf{A}_i\mathbf{a} \text{ for all } i \in [\ell], \end{cases}$$

We thus define $\mathbf{x} = [\tau|\overline{\mathbf{v}_1}|\overline{\mathbf{v}_2}|\mathbf{u}_1|\ldots|\mathbf{u}_\ell|\mathbf{u}'_1|\ldots|\mathbf{u}'_\ell|\mathbf{w}|\overline{\mathbf{r}}|\mathbf{m}_1|\ldots|\mathbf{m}_N] \in \mathbb{Z}^{L_\mathbf{x}}$, where $L_\mathbf{x} = \ell + 2mk_\alpha + 2\ell n + m + 2mk_{\alpha_1} + 2mN$. We then define

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{A}\mathbf{G}_\alpha & \mathbf{A}_0\mathbf{G}_\alpha & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I}_n & \cdots & \mathbf{I}_n & -\mathbf{D} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ & & & & & & & & & -\mathbf{H} & \mathbf{D}_0\mathbf{G}_{\alpha_1} & \mathbf{D}_1 & \cdots & \mathbf{D}_N \\ & & \mathbf{A}_1\mathbf{G}_\alpha & -\mathbf{I}_n & & & & & & & & & & \\ & & \vdots & & \ddots & & & & & & & & & \\ & & \mathbf{A}_\ell\mathbf{G}_\alpha & & & -\mathbf{I}_n & & & & & & & & \end{bmatrix},$$

and

$$\mathbf{y} = \begin{bmatrix} \mathbf{u} + (\mathbf{A} + \mathbf{A}_0)\mathbf{a} \\ \mathbf{D}_0\mathbf{a}_1 \\ \mathbf{A}_1\mathbf{a} \\ \vdots \\ \mathbf{A}_\ell\mathbf{a} \end{bmatrix}.$$

Finally, we define $\mathcal{M}_1 = \{(i,i,i); i \in [\ell + 2mk_\alpha] \cup [\ell + 2mk_\alpha + 2\ell n + 1, L]\}$, which corresponds to having all the coefficients of $\mathbf{x}$ to be binary, except for the $\mathbf{u}_i, \mathbf{u}'_i$. We then need to add the relations $\mathbf{u}'_i = \tau[i]\mathbf{u}_i$ for $i \in [\ell]$. For that, we define

$$\mathcal{M}_2 = \{(\ell + 2mk_\alpha + \ell n + n(i-1) + j, i, \ell + 2mk_\alpha + n(i-1) + j); (i,j) \in [\ell] \times [n]\},$$

and construct $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. The witness has length $L_{\mathbf{x}}$, and $\mathcal{M}$ is of size $L_{\mathbf{x}} - \ell n$ as well. The fast mode of Section 5.1 reduces the witness and relation set sizes to

$$\begin{cases} L_{\mathbf{x}} = \ell(2n+1) + m(5+2N) + k(2\lfloor \log_2(2\sigma\sqrt{m}\log_2 \lambda)\rfloor \\ \qquad\qquad\qquad\qquad\qquad\qquad + \lfloor \log_2(2\sigma_1\sqrt{2m}\log_2 \lambda)\rfloor + 3) \\ L_{\mathcal{M}} = L_{\mathbf{x}} - \ell n - 4m, \end{cases}$$

where $k = \lambda / \log_2(9/5)$ according to Lemma 5.1.

E Optimizing the Zero-Knowledge Framework

We detail here three independent optimizations of the framework from [YAZ$^+$19]. We note that the first two optimizations apply as is to the original framework, while the third involves further changes.

Concrete Hardness Assumptions. The first one consists in changing the underlying hardness assumptions. Instead of using worst-case to average-case connections to standard lattice problems, we use slightly overstretched parameters for which the hardness of LWE is only based on concrete hardness arguments. The goal is to change the distribution of the randomness used to commit to the witness $\mathbf{x}$ in the original proof so that it leads to smaller elements. More precisely, we sample the randomness from a ternary distribution instead of a discrete Gaussian. Additionally, we add an extra verification step in order to rely on the HNF-SIS problem with two norm bounds β_∞, β_2 on the infinity norm and L^2 norm respectively (Definition 2.3). Again, now relying on concrete hardness arguments, we obtain an improved condition on q only depending on β_∞, which is usually much smaller than β_2. Moreover, as discussed after Definition 2.3, constraining the magnitude of the solution's coefficients seems to be beneficial for both theoretical and concrete hardness.

Rejection Sampling. The second modification we make is to better leverage the rejection sampling result from Lemma E.2 adapted from [Lyu12]. This step ensures that a discrete Gaussian sample shifted by a small enough vector is statistically close to the original discrete Gaussian distribution, thus masking the shift. However, this fact is not used to its full potential in the proof of [YAZ$^+$19]. Doing so leads to smaller bounds in the verification equations and SIS norm bounds as a result. Additionally, we note when computing the size of the proof (in the non-interactive version), the authors treat the discrete Gaussian vectors as mere vectors over $\mathbb{Z}_q$. We can thus reduce the amount of storage needed as they have small coefficients with overwhelming probability, which results in smaller proofs by up to 20%.

Compacted Commitments. The final optimization regards the case when the proof system is run only once. It is sometimes better to increase the size p of the challenges rather than re-iterate the proof several times in order to achieve negligible soundness error. When running it once, we can compact the commitments thus limiting the number of elements to send and the size of the proof as a consequence. We note that compacting the commitments may not be desirable when the proof system is run multiple times as it would involve committing to the witness $\mathbf{x}$ multiple times.

E.1 Preliminaries

In what follows we sample the commitment randomness from a small distribution in order to optimize the parameters and the efficiency. For that, we employ the following ternary distribution which we denote by ψ_1, instead of Gaussian distributions. It outputs 0 with probability 6/16 and $-1, 1$ both with probability 5/16. This distribution has the advantage of being very efficiently sampleable as it only requires the sampling of 4 uniformly random bits to output a sample of ψ_1. For $\mathbf{x}$ sampled from ψ_1^n, it holds that $\|\mathbf{x}\|_2^2$ is distributed according to a binomial distribution with parameter $(n, 5/8)$. As such, Hoeffding's inequality gives the following.

Lemma E.1. *Let n be a positive integer. Then for all $\delta > 0$ it holds*

$$\mathbb{P}_{\mathbf{x} \hookleftarrow \psi_1^n}\left[\|\mathbf{x}\|_2 \geq \sqrt{(1+\delta)\frac{5}{8}n}\right] \leq \exp\left(-\frac{25}{32}\delta^2 n\right).$$

The above probability becomes 0 *when* $\delta > 3/5$.

Proof. Let $\mathbf{x}$ be a random vector whose coefficient are independent and identically distributed according to ψ_1. Therefore, for all $i \in [n]$, x_i^2 follows a Bernoulli distribution with parameter 5/8. Then, define the random variable $X = \|\mathbf{x}\|_2^2 = \sum_{i\in[n]} x_i^2$. Hence, since X follows a binomial distribution $\mathcal{B}(n, 5/8)$. By Hoeffding's inequality, for all $t > 0$, we have

$$\mathbb{P}[X - \mathbb{E}[X] \geq t] \leq e^{-2t^2/n}.$$

Since $\mathbb{E}[X] = 5n/8$, then it holds that for all $\delta > 0$, setting $t = 5n\delta/8 > 0$ gives

$$\mathbb{P}[X \geq (1+\delta)5n/8] \leq e^{-25\delta^2 n/32}.$$

Therefore, it holds

$$\forall \delta > 0, \mathbb{P}_{\mathbf{x} \hookleftarrow \psi_1^n}\left[\|\mathbf{x}\|_2 \geq \sqrt{(1+\delta)\frac{5}{8}n}\right] \leq \exp(-\frac{25}{32}\delta^2 n).$$

□

We also recall the rejection sampling result from [Lyu12], which we adapt to our modified definition of Gaussian distributions. We reformulate it to give the freedom to choose the repetition rate M and the tail bound error. Note that in [Lyu12], M is determined by T, σ and the tail bound error. We instead choose M and the tail bound error, and then determine the minimal σ needed. Although it is a direct application of [Lyu12, Lem. 4.7], we provide the proof for completeness.

Lemma E.2 (Adapted from [Lyu12, Thm. 4.6]). *Let n be a positive integer and $\varLambda$ a lattice of rank n. Let V be a subset of $\mathbb{R}^n$ and define $T = \max_{\mathbf{v}\in V}\|\mathbf{v}\|_2$. Let h be a probability distribution over V. Let $M > 1$ and $t > 0$. Then, define $\sigma_{\min} = (-t + \sqrt{t^2 + \ln(M)/\pi})^{-1} \cdot T$. Let $\sigma \geq \sigma_{\min}$. We now define two distributions*

$\mathcal{P}_1$**:** *Sample $\mathbf{v} \hookleftarrow h$ and $\mathbf{y} \hookleftarrow \mathcal{D}_{\varLambda,\sigma}$. Define $\mathbf{z} = \mathbf{y} + \mathbf{v}$. Output $(\mathbf{v}, \mathbf{z})$ with probability $\min(1, \frac{\mathcal{D}_{\varLambda,\sigma}(\mathbf{z})}{M\cdot\mathcal{D}_{\varLambda,\sigma}(\mathbf{z}-\mathbf{v})})$.*

$\mathcal{P}_2$**:** *Sample $\mathbf{v} \hookleftarrow h$ and $\mathbf{z} \hookleftarrow \mathcal{D}_{\varLambda,\sigma}$. Output $(\mathbf{v}, \mathbf{z})$ with probability $1/M$.*

Then, it holds that $\mathcal{P}_1$ outputs something with probability at least $(1-2e^{-\pi t^2})/M$, and that $\varDelta(\mathcal{P}_1, \mathcal{P}_2) \leq 2e^{-\pi t^2}/M$.

Proof. The proof is a direct application of [Lyu12, Lem. 4.7] with $f = \mathcal{D}_{\varLambda,\sigma}$ and $(g_{\mathbf{v}})_{\mathbf{v}\in V} = (\mathcal{D}_{\varLambda,\sigma,\mathbf{v}})_{\mathbf{v}\in V}$. We simply have to verify that:

$$\forall \mathbf{v} \in V, \mathbb{P}_{\mathbf{z}\hookleftarrow\mathcal{D}_{\varLambda,\sigma}}[M\mathcal{D}_{\varLambda,\sigma}(\mathbf{z}-\mathbf{v}) \geq \mathcal{D}_{\varLambda,\sigma}(\mathbf{z})] \geq 1 - 2e^{-\pi t^2}. \tag{17}$$

Let $\mathbf{v} \in V$, and $\mathbf{z} \hookleftarrow \mathcal{D}_{\varLambda,\sigma}$. Then, we have

$$\frac{\mathcal{D}_{\varLambda,\sigma}(\mathbf{z})}{\mathcal{D}_{\varLambda,\sigma}(\mathbf{z}-\mathbf{v})} = \exp\left(\frac{\pi}{\sigma^2}(\|\mathbf{v}\|_2^2 - 2\langle\mathbf{v},\mathbf{z}\rangle)\right).$$

Except with probability at most $2e^{-\pi t^2}$, it holds that $-\langle\mathbf{v},\mathbf{z}\rangle \leq \sigma t\|\mathbf{v}\|_2$ by Lemma 2.3. We now condition on $-\langle\mathbf{v},\mathbf{z}\rangle \leq \sigma t\|\mathbf{v}\|_2$. It yields

$$\begin{aligned}\frac{\mathcal{D}_{\varLambda,\sigma}(\mathbf{z})}{\mathcal{D}_{\varLambda,\sigma}(\mathbf{z}-\mathbf{v})} &\leq \exp\left(\frac{\pi}{\sigma^2}(\|\mathbf{v}\|_2^2 + 2\sigma t\|\mathbf{v}\|_2)\right) \\ &\leq \exp\left(\pi((T/\sigma)^2 + 2t(T/\sigma))\right).\end{aligned}$$

The way we defined $\sigma_{\min}$, we have that $T/\sigma_{\min}$ is the only positive solution to $x^2 + 2tx - \ln(M)/\pi = 0$. Since, we have $\sigma \geq \sigma_{\min}$, we have that T/σ is between the two solution of the equation and as such we have that $(T/\sigma)^2 + 2t(T/\sigma) - \ln(M)/\pi \leq 0$. It can be re-written as

$$\exp\left(\pi((T/\sigma)^2 + 2t(T/\sigma))\right) \leq M,$$

thus proving Equation (17) as required. □

The security properties of the zero-knowledge argument rely on the *Short Integer Solution* (SIS) problem [Ajt96] and the *Learning With Errors* (LWE) problem [Reg05] in Hermite Normal Form (HNF).

Definition E.1 ((HNF) Short Integer Solution). *Let n, m, q be positive integers, and $\beta_2 \geq \beta_\infty \geq 1$. The* Hermite Normal Form Short Integer Solution *problem, denoted by* $\text{HNF-SIS}^{\infty,2}_{n,m,q,\beta_\infty,\beta_2}$*, consists in finding* $\mathbf{x} \in \Lambda_q^\perp([\mathbf{I}_n|\mathbf{A}'])$ *given* $\mathbf{A}' \hookleftarrow U(\mathbb{Z}_q^{n\times(m-n)})$ *such that* $0 < \|\mathbf{x}\|_\infty \leq \beta_\infty$ *and* $0 < \|\mathbf{x}\|_2 \leq \beta_2$.

We say that HNF-SIS is δ-hard if for any probabilistic polynomial-time (PPT) adversary $\mathcal{A}$, the probability of $\mathcal{A}$ finding such a vector is at most δ over the randomness of $\mathbf{A}'$.

Definition E.2 ((HNF) Learning With Errors). *Let n, m, q be positive integers, and ψ a probability distribution over $\mathbb{Z}$. The Hermite Normal Form Learning With Errors problem, denoted by* $\text{HNF-LWE}_{n,m,q,\psi}$*, asks to distinguish between the following two distributions: (1)* $(\mathbf{A}, \mathbf{A}\mathbf{s}+\mathbf{e} \bmod q)$ *with* $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m\times n})$*,* $\mathbf{s} \hookleftarrow \psi^n$ *and* $\mathbf{e} \hookleftarrow \psi^m$*; (2)* $(\mathbf{A}, \mathbf{b})$ *with* $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m\times n})$ *and* $\mathbf{b} \hookleftarrow U(\mathbb{Z}_q^m)$.

We say that HNF-LWE is δ-hard if for any PPT adversary $\mathcal{A}$, the advantage of $\mathcal{A}$ in distinguishing both distributions is at most δ.

Finally, we briefly recall the security properties of a commitment scheme $\mathsf{aCommit}(m; \rho)$ which commits to a message m under randomness ρ. We say that $\mathsf{aCommit}$ is δ-hiding if a PPT adversary $\mathcal{A}$ has advantage at most δ in the following game: $\mathcal{A}$ chooses $m_0 \neq m_1$, receives $\mathsf{aCommit}(m_b; \rho)$ where b is a random bit, and outputs $b' \in \{0,1\}$. $\mathcal{A}$ wins if $b' = b$. We say that $\mathsf{aCommit}$ is δ-binding if a PPT adversary has advantage at most δ in outputting $(m_0, \rho_0), (m_1, \rho_1)$ such that $m_0 \neq m_1$ and $\mathsf{aCommit}(m_0; \rho_0) = \mathsf{aCommit}(m_1; \rho_1)$.

E.2 The Optimized Protocol.

We now present the main protocol with the optimizations we presented. Let ℓ_1, ℓ_2 be two positive integers. We denote by $L_{\mathbf{x}}$ the size of the witness vector, and $L_{\mathcal{M}}$ the size of the quadratic constraints set. We also define $L = \ell_1 + \ell_2 + L_{\mathbf{x}} + L_{\mathcal{M}}$. As is done in [YAZ+19], we employ the homomorphic commitment scheme from [BDL+18] over the integers. More precisely, we define

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_{\ell_1} & & \mathbf{C}_1 \\ \mathbf{0}_{L_{\mathbf{x}}+L_{\mathcal{M}}\times\ell_1} & \mathbf{I}_{L_{\mathbf{x}}+L_{\mathcal{M}}} & \mathbf{C}_2 \end{bmatrix} \in \mathbb{Z}_q^{(\ell_1+L_{\mathbf{x}}+L_{\mathcal{M}})\times L},$$

with $\mathbf{C}_1 \hookleftarrow U(\mathbb{Z}_q^{\ell_1\times(L_{\mathbf{x}}+L_{\mathcal{M}}+\ell_2)})$, $\mathbf{C}_2 \hookleftarrow U(\mathbb{Z}_q^{(L_{\mathbf{x}}+L_{\mathcal{M}})\times\ell_2})$. We then set $\delta = \sqrt{\frac{32}{25\log_2 e}\cdot\frac{\lambda}{L}}$. By Lemma E.1, it holds that for $\mathbf{s} \hookleftarrow \psi_1^L$, $\|\mathbf{s}\|_2 \leq \sqrt{(1+\delta)\frac{5}{8}L}$ except with probability $2^{-\lambda}$. Let $M > 1$ defining the repetition rate of the rejection sampling procedure. Then, let $t = \sqrt{(\lambda+1)/(\pi\log_2 e)}$ so that the tail bound needed in the rejection sampling verifies $2e^{-\pi t^2} = 2^{-\lambda}$. Let $p = 2^\lambda$ be the maximal magnitude of the challenges. We define $s_2 = (-t + \sqrt{t^2 + \ln(M)/\pi})^{-1} \cdot p \cdot \sqrt{5(1+\delta)/(8L)}$. For any $\mathbf{v}, \mathbf{z} \in \mathbb{Z}^L$, we define the rejection sampling probability function by $\mathfrak{p}(\mathbf{v}, \mathbf{z}) = \min(1, \mathcal{D}_{\mathbb{Z}^L, s_2}(\mathbf{z})/(M \cdot \mathcal{D}_{\mathbb{Z}^L, s_2}(\mathbf{z} - \mathbf{v})))$. Finally, let $\mathsf{aCommit}$ be an auxiliary commitment scheme with randomness space $\{0,1\}^\kappa$ and message space $\mathbb{Z}_q^{k+2(\ell_1 + L_{\mathbf{x}} + L_{\mathcal{M}})}$, and that is binding and hiding. The following interactive protocol involves a prover $\mathcal{P}$ with public input $\overline{\mathbf{A}} \in \mathbb{Z}_q^{k \times L_{\mathbf{x}}}$, $\mathbf{y} \in \mathbb{Z}_q^k$, and $\mathcal{M} \subseteq [L_{\mathbf{x}}]^3$ with $|\mathcal{M}| = L_{\mathcal{M}}$ and private input $\mathbf{x} \in \mathbb{Z}_q^{L_{\mathbf{x}}}$. The verifier $\mathcal{V}$ is only given the public input. In the protocol, $\mathcal{P}$ must convince $\mathcal{V}$ in zero-knowledge that they know $\mathbf{x}$ verifying

$$\begin{cases} \overline{\mathbf{A}}\mathbf{x} = \mathbf{y} \bmod q \\ \forall (h,i,j) \in \mathcal{M}, \mathbf{x}[h] = \mathbf{x}[i]\mathbf{x}[j] \bmod q \end{cases} \tag{18}$$

Theorem E.1. *The protocol described in Figure E.1 is complete with completeness error at most* $\delta_c = 1 - 1/M + \mathsf{negl}(\lambda)$*.*

We define $\beta_\infty = 8ps_2 \log_2 L$ *and* $\beta_2 = 8ps_2\sqrt{L}$*. Assume* HNF-SIS$_{\ell_1, L, q, \beta_\infty, \beta_2}$ *is* δ_{SIS}*-hard and that* $\mathsf{aCommit}$ *is* δ_b^a*-binding. Then, there exists an extractor* $\mathcal{E}$ *that for any* $\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}$ *and any* PPT *cheating prover* $\widehat{\mathcal{P}}$*, if* $\widehat{\mathcal{P}}$ *can convince a verifier* $\mathcal{V}$ *without knowing a witness with probability at least* $2/(2p+1) + \varepsilon$ *for a non-negligible* ε*, then* $\mathcal{E}$ *can extract an* $\mathbf{x}$ *that verifies (18) in polynomial time, except with probability* δ_{SIS}*.*

Finally, assume that HNF-LWE$_{\ell_2, \ell_1 + L_{\mathbf{x}} + L_{\mathcal{M}}, q, \psi_1}$ *is* δ_{LWE}*-hard, and that the commitment* $\mathsf{aCommit}$ *is* δ_h^a*-hiding. Then, there exists a simulator* $\mathcal{S}$ *that with input* $\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}$ *outputs a transcript that is* $(\delta_h^a + 2^{-\lambda}/M + \delta_{\mathrm{LWE}})$*-indistinguishable from the transcript of an honest execution of the protocol with a prover knowing a witness* $\mathbf{x}$ *satisfying (18).*

Although the proof of Theorem E.1 follows naturally from that of [YAZ+19], we give it in Section E.3. The above protocol can be turned into a non-interactive zero-knowledge arugument of knowledge via the Fiat-Shamir heuristic in the random oracle model. In this case, the resulting proof does not contain the whole transcript as some elements are uniquely determined by the others for the proof to be correct. More precisely, the proof is $\pi = (\alpha, \rho, \mathbf{c}_1, \mathbf{z}_0, \mathbf{z}_1)$ where the challenge $\alpha = H(\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}, C_{aux}; AUX)$ with AUX an auxiliary input. The verification algorithm then re-computes $\mathbf{t}$ from the verification equation (4), $\mathbf{c}_2$ from equation (5) and C_{aux} from equation (1). We end up with a proof of size

$$\begin{aligned} |\pi| &= \lceil \log_2(2p+1) \rceil + \kappa + (\ell_1 + L_{\mathbf{x}} + L_{\mathcal{M}}) \lceil \log_2 q \rceil + L_{\mathbf{x}} \lceil \log_2 q \rceil \\ &\quad + L \lceil \log_2(s_2 \log_2 L) \rceil \qquad (19) \\ &= \lceil \log_2(2p+1) \rceil + \kappa + (\ell_1 + 2L_{\mathbf{x}} + L_{\mathcal{M}}) \lceil \log_2 q \rceil + L \lceil \log_2(s_2 \log_2 L) \rceil \qquad (20) \end{aligned}$$

The last term in (19) does not appear in the proof size of [YAZ+19] as they treat $\mathbf{z}_1$ (and $\mathbf{z}_2$ in their case) as vectors in $\mathbb{Z}_q$. However, due to the rejection sampling, one has that they are Gaussian vectors and we can therefore reduce the amount of storage needed. Depending on the chosen parameters, this simple observation reduces the proof size by up to 20%.

Prover $\mathcal{P}[\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}; \mathbf{x}]$ Verifier $\mathcal{V}[\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}]$

$\mathbf{r} \hookleftarrow U(\mathbb{Z}_q^{L_\mathbf{x}})$
$\mathbf{t} \leftarrow \overline{\mathbf{A}}\mathbf{r} \bmod q$
$\mathbf{s}_1 \hookleftarrow \psi_1^L$
$\mathbf{s}_2 \hookleftarrow \mathcal{D}_{\mathbb{Z}^L, s_2}$
Let $\mathbf{a}, \mathbf{b}$ be $L_\mathcal{M}$-dimensional vectors
For $e \in [L_\mathcal{M}]$, let (h, i, j) be the e-th element of $\mathcal{M}$
  $\mathbf{a}[e] \leftarrow \mathbf{r}[h] - \mathbf{r}[i]\mathbf{x}[j] - \mathbf{r}[j]\mathbf{x}[i]$
  $\mathbf{b}[e] \leftarrow -\mathbf{r}[i]\mathbf{r}[j]$

$$\mathbf{c}_1 \leftarrow \mathbf{C}\mathbf{s}_1 + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{x} \\ \mathbf{a} \end{bmatrix} \bmod q$$

$$\mathbf{c}_2 \leftarrow \mathbf{C}\mathbf{s}_2 + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{r} \\ \mathbf{b} \end{bmatrix} \bmod q$$

$\rho \hookleftarrow U(\{0,1\}^\kappa)$
$C_{aux} \leftarrow \mathsf{aCommit}(\mathbf{t} \| \mathbf{c}_1 \| \mathbf{c}_2; \rho)$

$$\xrightarrow{\quad C_{aux} \quad}$$

$\alpha \hookleftarrow U([-p, p])$

$$\xleftarrow{\quad \alpha \quad}$$

$\mathbf{z}_0 \leftarrow \alpha\mathbf{x} + \mathbf{r}$
$\mathbf{z}_1 \leftarrow \alpha\mathbf{s}_1 + \mathbf{s}_2$
Abort with probability $1 - \mathfrak{p}(\alpha\mathbf{s}_1, \mathbf{z}_1)$

$$\xrightarrow{\quad \mathbf{t}, \mathbf{c}_1, \mathbf{c}_2, \rho, \mathbf{z}_0, \mathbf{z}_1 \quad}$$

Let $\mathbf{d}$ be an $L_\mathcal{M}$-dimensional vector
For $e \in [L_\mathcal{M}]$, let (h, i, j) be the e-th element of $\mathcal{M}$
  $\mathbf{d}[e] \leftarrow \alpha\mathbf{z}_0[h] - \mathbf{z}_0[i]\mathbf{z}_0[j]$
Accept if:
  (1) $C_{aux} = \mathsf{aCommit}(\mathbf{t} \| \mathbf{c}_1 \| \mathbf{c}_2; \rho)$
  (2) $\|\mathbf{z}_1\|_\infty \le s_2 \log_2 L$
  (3) $\|\mathbf{z}_1\|_2 \le s_2\sqrt{L}$
  (4) $\overline{\mathbf{A}}\mathbf{z}_0 = \alpha\mathbf{y} + \mathbf{t} \bmod q$
  (5) $\mathbf{C}\mathbf{z}_1 + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{z}_0 \\ \mathbf{d} \end{bmatrix} = \alpha\mathbf{c}_1 + \mathbf{c}_2 \bmod q$

Fig. E.1. Zero-knowledge Argument of Knowledge for Equation (18) with compacted commitments.

E.3 Proof of Theorem E.1

Proof. Completeness: Consider an honest execution of the protocol, i.e., between a prover $\mathcal{P}[\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}; \mathbf{x}]$ with $\mathbf{x}$ satisfying (18), and a verifier $\mathcal{V}[\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}]$. Since the execution is honest and since $\mathsf{aCommit}$ does not use any internal randomness other than ρ, (1) is trivially verified. Next, due to the rejection sampling, $\mathcal{P}$ respond in the third move only with probability $\mathfrak{p}(\alpha\mathbf{s}_1, \mathbf{z}_1)$. Yet it holds that $\|\alpha\mathbf{s}_1\|_2 \le p\sqrt{5(1+\delta)/(8L)}$ except with probability at most $2^{-\lambda}$ by Lemma E.1. Because of how we set s_2, Lemma E.2 yields that the prover does not abort with probability at least $(1 - 2^{-\lambda+1})/M$ and that $\mathbf{z}_1$ is within statistical distance $2^{-\lambda}/M$ of $\mathcal{D}_{\mathbb{Z}^L, s_2}$. We further condition on a non-aborting transcript.

Lemma 2.3 combined with the union bound gives

$$\mathbb{P}[\|\mathbf{z}_1\|_\infty \le s_2 \log_2 L \wedge \|\mathbf{z}_1\|_2 \le s_2\sqrt{L}] \ge 1 - (2^{-\lambda}/M + 2^{-2L} + 2Le^{-\pi \log_2^2 L}).$$

Equation (4) is easily verified as $\overline{\mathbf{A}}\mathbf{z}_0 = \overline{\mathbf{A}}(\alpha\mathbf{x}+\mathbf{r}) = \alpha(\overline{\mathbf{A}}\mathbf{x})+\overline{\mathbf{A}}\mathbf{r} = \alpha\mathbf{y}+\mathbf{t} \bmod q$. Now, let $e \in [L_{\mathcal{M}}]$ and let (h, i, j) be the e-th element of $\mathcal{M}$. We have

$$\begin{aligned}\mathbf{d}[e] &= \alpha\mathbf{z}_0[h] - \mathbf{z}_0[i]\mathbf{z}_0[j]\\ &= \alpha(\alpha\mathbf{x}[h] + \mathbf{r}[h]) - (\alpha\mathbf{x}[i] + \mathbf{r}[i])(\alpha\mathbf{x}[j] + \mathbf{r}[j])\\ &= \alpha^2(\mathbf{x}[h] - \mathbf{x}[i]\mathbf{x}[j]) + \alpha(\mathbf{r}[h] - \mathbf{r}[i]\mathbf{x}[j] - \mathbf{r}[j]\mathbf{x}[i]) + (-\mathbf{r}[i]\mathbf{r}[j])\\ &= \alpha\mathbf{a}[e] + \mathbf{b}[e] \bmod q.\end{aligned}$$

As a result, it holds that $\mathbf{d} = \alpha\mathbf{a} + \mathbf{b} \bmod q$. It thus yields

$$\begin{aligned}\mathbf{C}\mathbf{z}_1 + \begin{bmatrix}\mathbf{0}_{\ell_1}\\ \mathbf{z}_0\\ \mathbf{d}\end{bmatrix} &= \mathbf{C}(\alpha\mathbf{s}_1 + \mathbf{s}_2) + \begin{bmatrix}\mathbf{0}_{\ell_1}\\ \alpha\mathbf{x}+\mathbf{r}\\ \alpha\mathbf{a}+\mathbf{b}\end{bmatrix} \bmod q\\ &= \alpha\left(\mathbf{C}\mathbf{s}_1 + \begin{bmatrix}\mathbf{0}_{\ell_1}\\ \mathbf{x}\\ \mathbf{a}\end{bmatrix}\right) + \left(\mathbf{C}\mathbf{s}_2 + \begin{bmatrix}\mathbf{0}_{\ell_1}\\ \mathbf{r}\\ \mathbf{b}\end{bmatrix}\right) \bmod q\\ &= \alpha\mathbf{c}_1 + \mathbf{c}_2 \bmod q,\end{aligned}$$

proving (5). Combining it all yields

$$\mathbb{P}[\langle \mathcal{P}[\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}; \mathbf{x}], \mathcal{V}[\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M}]\rangle \ne 1] \le 1 - 1/M + \mathsf{negl}(\lambda).$$

Extractor: Now, assume that a cheating prover $\widehat{\mathcal{P}}$ can convince the verifier that they possess a witness for $(\overline{\mathbf{A}}, \mathbf{y}, \mathcal{M})$ with probability $2/(2p + 1) + \varepsilon$ for some non-negligible ε. We construct the extractor $\mathcal{E}$ that uses $\widehat{\mathcal{P}}$ via black-box access. First, $\mathcal{E}$ runs $\widehat{\mathcal{P}}$ until it obtains an accepting transcript $(\mathbf{t}, \mathbf{c}_1, \mathbf{c}_2, \alpha, \mathbf{z}_0, \mathbf{z}_1)$. Between each run, $\mathcal{E}$ rewinds the inner randomness of $\widehat{\mathcal{P}}$ to have the same first move response. This first transcript is obtained in expected time $T_1 = (2/(2p + 1) + \varepsilon)^{-1}$. Then, $\mathcal{E}$ re-iterates the same process but sends challenges $\alpha' \ne \alpha$ to $\widehat{\mathcal{P}}$. Assuming $\mathsf{aCommit}$ is δ_b^a-binding and since the first move always uses the same randomness, $\mathcal{E}$ can therefore obtain another accepting transcript $(\mathbf{t}, \mathbf{c}_1, \mathbf{c}_2, \alpha', \mathbf{z}_0', \mathbf{z}_1')$ in expected time $T_2 = (1/(2p+1) + \varepsilon - \delta_b^a)^{-1}$. It then continues running $\widehat{\mathcal{P}}$ with challenges $\alpha'' \notin \{\alpha, \alpha'\}$ to get a third accepting transcript $(\mathbf{t}, \mathbf{c}_1, \mathbf{c}_2, \alpha'', \mathbf{z}_0'', \mathbf{z}_1'')$ in expected time $T_3 = (\varepsilon - \delta_b^a)^{-1}$. The total expected time is therefore $T = T_1 + T_2 + T_3 \le \mathsf{poly}(\lambda)$. Finally, the extractor $\mathcal{E}$ outputs the witness $\overline{\mathbf{x}} = (\alpha' - \alpha)^{-1}(\mathbf{z}_0' - \mathbf{z}_0) \bmod q$. We now analyze the correctness of $\mathcal{E}$. We further define $\varDelta_1 = \alpha' - \alpha$ and $\varDelta_2 = \alpha'' - \alpha$. First, we have

$$\begin{aligned}\overline{\mathbf{A}}\overline{\mathbf{x}} &= \varDelta_1^{-1}(\overline{\mathbf{A}}\mathbf{z}_0' - \overline{\mathbf{A}}\mathbf{z}_0) \bmod q\\ &= \varDelta_1^{-1}(\alpha'\mathbf{y} + \mathbf{t} - (\alpha\mathbf{y} + \mathbf{t})) \bmod q \text{ (by (4))}\\ &= \varDelta_1^{-1}(\alpha' - \alpha)\mathbf{y} \bmod q\\ &= \mathbf{y} \bmod q.\end{aligned}$$

We now prove that $\overline{\mathbf{x}}$ verifies the quadratic constraints except with probability δ_{SIS}. For that, we define $\mathbf{e}' = \mathbf{z}_1' - \mathbf{z}_1$, $\mathbf{e}'' = \mathbf{z}_1'' - \mathbf{z}_1$, $\mathbf{f}' = \mathbf{z}_0' - \mathbf{z}_0$, $\mathbf{f}'' = \mathbf{z}_0'' - \mathbf{z}_0$, and $\mathbf{g}' = \mathbf{d}' - \mathbf{d}$, $\mathbf{g}'' = \mathbf{d}'' - \mathbf{d}$. The verification equation (5) gives

$$\begin{cases} \mathbf{C}\mathbf{e}' + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{f}' \\ \mathbf{g}' \end{bmatrix} = \Delta_1 \mathbf{c}_1 \bmod q \\ \mathbf{C}\mathbf{e}'' + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{f}'' \\ \mathbf{g}'' \end{bmatrix} = \Delta_2 \mathbf{c}_1 \bmod q. \end{cases}$$

Cancelling the right-hand side provides us with

$$\mathbf{C}(\Delta_2 \mathbf{e}' - \Delta_1 \mathbf{e}'') + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ (\Delta_2 \mathbf{f}' - \Delta_1 \mathbf{f}'') \\ (\Delta_2 \mathbf{g}' - \Delta_1 \mathbf{g}'') \end{bmatrix} = \mathbf{0} \bmod q.$$

The first block then gives $[\mathbf{I}_{\ell_1} | \mathbf{C}_1](\Delta_2 \mathbf{e}' - \Delta_1 \mathbf{e}'') = \mathbf{0} \bmod q$. Yet, we can bound the norms of $\Delta_2 \mathbf{e}' - \Delta_1 \mathbf{e}''$ using the verification equations (2) and (3) and get $\|\Delta_2 \mathbf{e}' - \Delta_1 \mathbf{e}''\|_\infty \leq 8ps_2 \log_2 L = \beta_\infty$ and $\|\Delta_2 \mathbf{e}' - \Delta_1 \mathbf{e}''\|_2 \leq 8ps_2\sqrt{L} = \beta_2$. Since we assume that $\text{HNF-SIS}_{\ell_1,L,q,\beta_\infty,\beta_2}$ is δ_{SIS}-hard, then no PPT adversary can solve it with advantage more than δ_{SIS}. Hence, we get that $\Delta_2 \mathbf{e}' - \Delta_1 \mathbf{e}'' = \mathbf{0}$ except with probability at most δ_{SIS}. We now condition on $\Delta_2 \mathbf{e}' - \Delta_1 \mathbf{e}'' = \mathbf{0}$. The second and third blocks in the above yields $\Delta_2 \mathbf{f}' = \Delta_1 \mathbf{f}'' \bmod q$ and $\Delta_2 \mathbf{g}' = \Delta_1 \mathbf{g}'' \bmod q$. We now define $\overline{\mathbf{r}} = \mathbf{z}_0 - \alpha\overline{\mathbf{x}} \bmod q$. Then

$$\mathbf{z}_0' - \alpha'\overline{\mathbf{x}} = \mathbf{z}_0' - \Delta_1\overline{\mathbf{x}} - \alpha\overline{\mathbf{x}} = \mathbf{z}_0' - \mathbf{f}' - \alpha\overline{\mathbf{x}} \bmod q = \overline{\mathbf{r}} \bmod q \tag{21}$$

$$\begin{aligned} \mathbf{z}_0'' - \alpha''\overline{\mathbf{x}} = \mathbf{z}_0'' - \Delta_2\overline{\mathbf{x}} - \alpha\overline{\mathbf{x}} &= \mathbf{z}_0'' - \Delta_2 \Delta_1^{-1}\mathbf{f}' - \alpha\overline{\mathbf{x}} \\ &= \mathbf{z}_0'' - \Delta_2 \Delta_2^{-1}\mathbf{f}'' - \alpha\overline{\mathbf{x}} \bmod q \\ &= \overline{\mathbf{r}} \bmod q. \end{aligned} \tag{22}$$

Now let $e \in [L_{\mathcal{M}}]$ and (h,i,j) be the e-th element of $\mathcal{M}$. We have

$$\begin{aligned} \mathbf{d}[e] &= \alpha\mathbf{z}_0[h] - \mathbf{z}_0[i]\mathbf{z}_0[j] \\ &= \alpha(\alpha\overline{\mathbf{x}}[h] + \overline{\mathbf{r}}[h]) - (\alpha\overline{\mathbf{x}}[i] + \overline{\mathbf{r}}[i])(\alpha\overline{\mathbf{x}}[j] + \overline{\mathbf{r}}[j]) \\ &= \alpha^2(\overline{\mathbf{x}}[h] - \overline{\mathbf{x}}[i]\overline{\mathbf{x}}[j]) + \alpha(\overline{\mathbf{r}}[h] - \overline{\mathbf{r}}[i]\overline{\mathbf{x}}[j] - \overline{\mathbf{r}}[j]\overline{\mathbf{x}}[i]) + (-\overline{\mathbf{r}}[i]\overline{\mathbf{r}}[j]) \\ &= \mathbf{c}[e]\alpha^2 + \mathbf{a}[e]\alpha + \mathbf{b}[e]. \end{aligned}$$

Due to Equations (21) and (22), we also have $\mathbf{d}' = {\alpha'}^2\mathbf{c} + \alpha'\mathbf{a} + \mathbf{b}$ and $\mathbf{d}'' = {\alpha''}^2\mathbf{c} + \alpha''\mathbf{a} + \mathbf{b}$. Hence, since $\Delta_1^{-1}\mathbf{g}' = \Delta_2^{-1}\mathbf{g}'' \bmod q$, we obtain

$$(\alpha' + \alpha)\mathbf{c} + \mathbf{a} = (\alpha'' + \alpha)\mathbf{c} + \mathbf{a} \bmod q$$

which leads to $(\alpha'' - \alpha)\mathbf{c} = \mathbf{0} \bmod q$. Since $\alpha'' \neq \alpha'$ and q is prime, then $\alpha'' - \alpha' \in \mathbb{Z}_q^\times$ and therefore $\mathbf{c} = \mathbf{0} \bmod q$. This proves that for all $(h,i,j) \in \mathcal{M}$, $\overline{\mathbf{x}}[h] =$

$\overline{\mathbf{x}}[i]\overline{\mathbf{x}}[j] \bmod q$. As a result, the output of $\mathcal{E}$ is correct except with probability at most δ_{SIS}.

Simulator: We construct the following simulator $\mathcal{S}$ that simulates the distribution of an honest transcript but only using the public inputs. It proceeds as follows

1. $\alpha \hookleftarrow U([-p, p])$
2. $\mathbf{z}_0 \hookleftarrow U(\mathbb{Z}_q^{L_{\mathbf{x}}})$
3. $\mathbf{t} \leftarrow \overline{\mathbf{A}}\mathbf{z}_0 - \alpha\mathbf{y} \bmod q$
4. For $e \in [L_{\mathcal{M}}]$, let (h, i, j) be the e-the element of $\mathcal{M}$. Then, $\mathbf{d}[e] \leftarrow \alpha\mathbf{z}_0[h] - \mathbf{z}_0[i]\mathbf{z}_0[j]$
5. $\mathbf{z}_1 \hookleftarrow \mathcal{D}_{\mathbb{Z}^L, s_2}$
6. $\mathbf{c}_1 \hookleftarrow U(\mathbb{Z}_q^{\ell_1 + L_{\mathbf{x}} + L_{\mathcal{M}}})$
7. $\mathbf{c}_2 \leftarrow \mathbf{C}\mathbf{z}_1 + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{z}_0 \\ \mathbf{d} \end{bmatrix} - \alpha\mathbf{c}_1 \bmod q$
8. $\rho \hookleftarrow U(\{0,1\}^\kappa)$
9. $C_{\mathsf{aux}} \leftarrow \mathsf{aCommit}(\mathbf{t} \| \mathbf{c}_1 \| \mathbf{c}_2; \rho)$
10. $C'_{\mathsf{aux}} \leftarrow \mathsf{aCommit}(\mathbf{0}; \rho)$
11. Output $(C_{aux}, \alpha, \mathbf{t}, \mathbf{c}_1, \mathbf{c}_2, \rho, \mathbf{z}_0, \mathbf{z}_1)$ with probability $1/M$ and $(C'_{aux}, \alpha, \perp)$ otherwise.

We now prove that the output of $\mathcal{S}$ is computationally indistinguishable from the transcript of an honest execution of the protocol. We proceed by game hopping.
Game G_0: This corresponds to an honest execution.
Game G_1: Here, the prover $\mathcal{P}$ retrieves the challenge α from the honest verifier by sending $\mathsf{aCommit}(\mathbf{0}; \rho)$ for some $\rho \hookleftarrow U(\{0,1\}^\kappa)$. It then rewinds the verifier to its initial state including its inner randomness. It then proceeds as follows:

1. $\mathbf{r} \hookleftarrow U(\mathbb{Z}_q^{L_{\mathbf{x}}})$
2. $\mathbf{t} \leftarrow \overline{\mathbf{A}}\mathbf{r} \bmod q$
3. $\mathbf{s}_1 \hookleftarrow \psi_1^L$
4. $\mathbf{s}_2 \hookleftarrow \mathcal{D}_{\mathbb{Z}^L, s_2}$
5. For $e \in [L_{\mathcal{M}}]$, let (h, i, j) be the e-the element of $\mathcal{M}$. Then, $\mathbf{a}[e] \leftarrow \mathbf{r}[h] - \mathbf{r}[i]\mathbf{x}[j] - \mathbf{r}[j]\mathbf{x}[i]$ and $\mathbf{b}[e] \leftarrow -\mathbf{r}[i]\mathbf{r}[j]$
6. $\mathbf{c}_1 \leftarrow \mathbf{C}\mathbf{s}_1 + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{x} \\ \mathbf{a} \end{bmatrix} \bmod q$
7. $\mathbf{c}_2 \leftarrow \mathbf{C}\mathbf{s}_2 + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{r} \\ \mathbf{b} \end{bmatrix} \bmod q$
8. $\mathbf{z}_0 \leftarrow \alpha\mathbf{x} + \mathbf{r}$
9. $\mathbf{z}_1 \leftarrow \alpha\mathbf{s}_1 + \mathbf{s}_2$
10. Set the binary variable abort to 1 with probability $1 - \mathfrak{p}(\alpha\mathbf{s}_1, \mathbf{z}_1)$
11. $\rho \hookleftarrow U(\{0,1\}^\kappa)$
12. $C_{aux} \leftarrow \mathsf{aCommit}(\mathbf{t} \| \mathbf{c}_1 \| \mathbf{c}_2; \rho)$ and it sends C_{aux} to the verifier

13. When receiving α' from the verifier, the prover aborts if $\mathsf{abort} = 1$ and otherwise sends $(\mathbf{t}, \mathbf{c}_1, \mathbf{c}_2, \rho, \mathbf{z}_0, \mathbf{z}_1)$.

Game G_2: It is identical to G_1 except in the computation of $\mathbf{t}$ and $\mathbf{c}_2$. They are instead computed to verify equations (4) and (5) in the verification:

1. $\mathbf{t} \leftarrow \overline{\mathbf{A}}\mathbf{z}_0 - \alpha\mathbf{y} \bmod q$
2. For $e \in [L_{\mathcal{M}}]$, let (h, i, j) be the e-the element of $\mathcal{M}$. Then, $\mathbf{d}[e] \leftarrow \alpha\mathbf{z}_0[h] - \mathbf{z}_0[i]\mathbf{z}_0[j]$
3. $\mathbf{c}_2 \leftarrow \mathbf{C}\mathbf{z}_1 + \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \mathbf{z}_0 \\ \mathbf{d} \end{bmatrix} - \alpha\mathbf{c}_1 \bmod q$

Game G_3: It is identical to G_2 except for the computation of C_{aux}.

1. $C_{aux} \leftarrow \mathsf{aCommit}(\mathbf{0}; \rho)$ if $\mathsf{abort} = 1$ and $C_{aux} \leftarrow \mathsf{aCommit}(\mathbf{t}\|\mathbf{c}_1\|\mathbf{c}_2; \rho)$ otherwise.

Game G_4: It is identical to G_3 except in the computation of $\mathbf{z}_1$ and abort.

1. $\mathbf{z}_1 \hookleftarrow \mathcal{D}_{\mathbb{Z}^L, s_2}$
2. Set $\mathsf{abort} = 1$ with probability $1 - 1/M$ and 0 otherwise

Game G_5: It is identical to G_4 except in the computation of $\mathbf{c}_1$.

1. $\mathbf{c}_1 \hookleftarrow U(\mathbb{Z}_q^{\ell_1 + L_{\mathbf{x}} + L_{\mathcal{M}}})$

Game G_6: It is identical to G_5 except in the computation of $\mathbf{z}_0$

1. $\mathbf{z}_0 \hookleftarrow U(\mathbb{Z}_q^{L_{\mathbf{x}}})$

We now prove that each game is indistinguishable from the next. First, since the verifier $\mathcal{V}$ is honest, the challenge α' is fully determined by its inner randomness. As it is rewinded, we always have $\alpha' = \alpha$. All other variables are identically distributed, which gives

$$\Delta(\mathsf{View}_{G_0}(\mathcal{V}), \mathsf{View}_{G_1}(\mathcal{V})) = 0. \tag{23}$$

By the completeness of the protocol, $\mathbf{t}$ and $\mathbf{c}_2$ are uniquely determined by the other variables and the verification equations (4) and (5). Thus

$$\Delta(\mathsf{View}_{G_1}(\mathcal{V}), \mathsf{View}_{G_2}(\mathcal{V})) = 0. \tag{24}$$

Since $\mathsf{aCommit}$ is δ_h^a-hiding, it holds that a PPT adversary $\mathcal{A}$ can distinguish between games G_2 and G_3 with advantage at most δ_h^a.

$$|\mathbb{P}[\mathcal{A}(\mathsf{View}_{G_2}(\mathcal{V})) = 1] - \mathbb{P}[\mathcal{A}(\mathsf{View}_{G_3}(\mathcal{V})) = 1]| \leq \delta_h^a. \tag{25}$$

Then, by Lemma E.2, it directly holds that the computation of $\mathbf{z}_1$ and abort in G_4 is within statistical distance $2^{-\lambda}/M$ of that of game G_3. Hence

$$\Delta(\mathsf{View}_{G_3}(\mathcal{V}), \mathsf{View}_{G_4}(\mathcal{V})) \leq 2^{-\lambda}/M. \tag{26}$$

We then use the hiding property of the commitment scheme from [BDL+18] to argue that G_4 and G_5 are indistinguishable under the LWE assumption. The details are already provided in [YAZ+19]. More precisely, since we assume that $\mathrm{HNF\text{-}LWE}_{\ell_2,\ell_1+L_{\mathbf{x}}+L_{\mathcal{M}},q,\psi_1}$ is δ_{LWE}-hard, then for any PPT adversary $\mathcal{A}$ we get

$$|\mathbb{P}[\mathcal{A}(\mathsf{View}_{G_4}(\mathcal{V}))=1]-\mathbb{P}[\mathcal{A}(\mathsf{View}_{G_5}(\mathcal{V}))=1]|\leq\delta_{\mathrm{LWE}}. \tag{27}$$

In G_5, $\mathbf{z}_0=\alpha\mathbf{x}+\mathbf{r}$ where $\mathbf{r}$ is uniform in $\mathbb{Z}_q^{L_{\mathbf{x}}}$ and independent of $\alpha\mathbf{x}$. Hence, $\mathbf{z}_0$ is also uniform in $\mathbb{Z}_q^{L_{\mathbf{x}}}$. Thus:

$$\varDelta(\mathsf{View}_{G_5}(\mathcal{V}),\mathsf{View}_{G_6}(\mathcal{V}))=0. \tag{28}$$

Then, the distribution of the transcript in G_6 no longer depends on the witness $\mathbf{x}$ and is exactly the same as the output of $\mathcal{S}$. Combining Equations (23), (24), (25), (26), (27) and (28) yields

$$\left|\mathbb{P}[\mathcal{A}(\mathsf{View}_{G_0}(\mathcal{V}))=1]-\mathbb{P}[\mathcal{A}(\mathcal{S}(\overline{\mathbf{A}},\mathbf{y},\mathcal{M}))=1]\right|\leq\delta_h^a+2^{-\lambda}/M+\delta_{\mathrm{LWE}},$$

as desired. □

F Parameters and Efficiency

In this section, we instantiate the two versions of our signature scheme with concrete parameters in order to reach $\lambda=128$ bits of quantum security. All the concrete hardness estimates for the SIS, LWE, M-SIS, M-LWE problems are done using the BKZ cost model with sieving SVP oracle. In this model, the classical security is given by $\lambda_c=0.292b$ [BDGL16] and the quantum security by $\lambda_q=0.265b$ [Laa15], where b is the BKZ blocksize. We explain our choice of parameters for both the standard and module version by encompassing the zero-knowledge arguments of message-signature possession. We however note that for a standard use of the signature schemes, one could choose different parameters. We choose to instantiate it for $Q=2^{30}$ signature queries, representing the number of signature issuance. We believe this choice is reasonable for most applications.

F.1 Instantiating the Standard Signature

We provide in Table F.2 an example parameter set along with the size of the keys, signature, and proof of possession for the signature of Section 3. It makes use of the zero-knowledge framework of [YAZ+19] improved with the enhanced fast mode from Section 5.1 and the optimizations of Appendix E (except the compacted commitments, as explained below) that we have introduced.

As explained in Remark 5.2, in order to have as few iterations of the proof system as possible, we need to choose large enough challenges, which in turns require to take a sufficiently large modulus. We then start by choosing the number of iterations N and the challenge size p, which imply we must take $q\geq\mathsf{poly}(\lambda)\cdot p^2$. To avoid an exponential reduction loss, we set $q'\approx Q^2$. We then fix n so that when the other parameters are set using Algorithm 1, we obtain a quantum security of λ. Since the proofs of Lemma 3.1 and 3.2 both have a reduction loss between the advantage of a signature forger ($\delta = 2^{-\lambda}$) and the advantage against SIS ($\mathrm{Adv}[\mathcal{B}]$) which can be substantial, we need to take it into account. More precisely, we compute the required SIS security $\lambda_{\mathrm{I}}, \lambda_{\mathrm{II}}$ so that the SIS problem stays hard even with the relations of Lemma 3.1 and 3.2. For our parameter, we need $\lambda_{\mathrm{I}} = 189$ and $\lambda_{\mathrm{II}} = 181$ for the respective SIS problems which only slightly differ by their bounds. Hence, we must reach for a root Hermite factor of $\delta_0 = 1.0026$. We also account for key recovery attacks, consisting of recovering $\mathbf{R}$ from $\mathbf{A}, \mathbf{B}$. This attack is however much more costly than forgeries as $\mathbf{R}$ is *statistically* hidden in $(\mathbf{A}, \mathbf{B})$ by the leftover hash lemma. We then set the other parameters of the zero-knowledge argument as described in our optimized framework in Appendix E and taking ℓ_1, ℓ_2 to reach 128 bits of quantum security for the HNF-SIS and HNF-LWE problems. The security estimates of HNF-LWE are performed using the estimator of Albrecht et al. [APS15]. We note that although we take the secret and error ternary from distribution ψ_1, we are never in the regime of polynomial algebraic attacks [AG11]. Such attacks for ternary error would require roughly ℓ_2^3 samples. In our cases, we have $\ell_1 + \max(L_{\mathbf{x}}, L_{\mathcal{M}}) \ll \ell_2^2$.

We also instantiate the scheme of [LLM+16]. For a fair comparison, we aim for the same security and make use of the same improvements of the zero-knowledge argument. The relation of [LLM+16] is instantiated in the framework of [YAZ+19] in Appendix D.

For both our scheme and the one from [LLM+16], the ZKAoK are instantiated to be run twice, and thus do not include the compacted commitments discussed in Appendix E. Table 1.1 shows the construction of [LLM+16] leads to intractable parameters and key sizes. We note that one could reduce the value of q at the expense of increasing the number of proof iterations to achieve negligible soundness. However, not only does this approach still leads to intractable key sizes, but it also yields substantially larger proofs. Our results also summarized in Table 1.1 shows the feasibility of signature with efficient protocols based on lattice assumptions, as we gain several orders of magnitude in the size of key materials and proof size, while maintaining the same security. The complete parameter sets used to obtained these results can be found in Tables F.1 and F.2.

Remark F.1. We recall that, although the fast mode reduces the size of the witness vector, it also introduces a soundness gap, which is the object of Lemma 5.1. As a result, the bounds on $\mathbf{v}_1^*, \mathbf{v}_2^*$ used in Lemma 3.1 and 3.2 are larger as discussed in Remark 5.1. We thus take this increase of the SIS bounds into account when estimating the SIS security, which entails an increase of the dimension n.

F.2 Instantiating the Module Signature

We now rely on the framework of [LNP22] for the zero-knowledge argument. The module construction no longer suffers from the requirement of a large modulus. Indeed, in the module case, we can choose an exponentially large challenge space while keeping the size of the challenges constant. The same thing occurs for our tag space. Before, we needed to take $q \geq q'$ where q' was both the bound on the tags and the size of the tag space. In the module case, we can take binary tags while adjusting the value of w in order to have a sufficiently large tag space. Additionally, because the modulus of the signature q is different from the modulus of the proof system $q_\pi = q_1 q$, we can first adjust the parameters of our signature before setting the parameters of the proof system. We proceed as in the previous section, accounting for the reduction loss of Lemma 6.2 and 6.3. To choose the parameters of the proof system, we proceed as prescribed in [LNP22, Sec. 6.1], with the challenge space of [LNP22, Fig. 3]. For simplicity, we choose parameters close to those provided in their group signature instantiation. We give the detailed parameter set in Table F.3 with security and efficiency estimates. To avoid collision between our notations and the proof system parameters, we specify the notations used in [LNP22] in the description column.

This construction based on structured lattices leads to drastic efficiency gains in both key and proof sizes as summarized in Table 1.1, which further reinforce the concrete feasibility of efficient privacy-enhancing post-quantum signatures. In particular, it shows that a proof of knowledge of a signature issued on a committed (secret value), one of the main building blocks of privacy-preserving primitives, can represent less than 700 KB, which is a considerable improvement over [LLM+16] and may have many applications.

F.3 Parameter Sets

Parameters	Description	Exact Proof	Fast Mode
Signature			
λ	Security parameter	128	128
n	SIS dimension	1540	2400
q	Modulus	$2^{155}+15$	$2^{155}+15$
ℓ	Tag bit-size	61	61
m	Trapdoor dimension	480480	748800
N_{msg}	Number of message blocks	1	1
σ	Pre-image sampling width	23354	30918
σ_1	Commitment randomness width	44001388284877	113328266566679
$\lambda_{\mathrm{I}}/\lambda_{\mathrm{I}}^*$	Required/Reached SIS security (I)	164/164	164/164
$\lambda_{\mathrm{II}}/\lambda_{\mathrm{II}}^*$	Required/Reached SIS security (II)	159/161	159/166
$\lambda_{\mathrm{III}}/\lambda_{\mathrm{III}}^*$	Required/Reached SIS security (III)	128/568	128/540
$\lvert\mathsf{pk}\rvert$	Public key size (MB)	$8670 \cdot 10^2$	$2053 \cdot 10^3$
$\lvert\mathsf{sk}\rvert$	Secret key size (MB)	$1376 \cdot 10^1$	$3259 \cdot 10^1$
$\lvert\mathsf{sig}\rvert$	Signature size (KB)	8094	13162
$\lvert\mathsf{pp}\rvert$	Public parameters size (MB)	$1482 \cdot 10^1$	$3510 \cdot 10^1$
Proof			
ℓ_1	HNF-SIS dimension	8350	8000
ℓ_2	HNF-LWE dimension	7900	7900
p	Size of challenges	$2^{\lambda/2}$	$2^{\lambda/2}$
N	Number of proof iterations	2	2
M	Rejection sampling repetition rate	27	27
$L_{\mathbf{x}}$	Witness length	69857541	5483250
$L_{\mathcal{M}}$	Relation set length	69763601	2380920
δ_s	Soundness error	$2^{-\lambda}$	$2^{-\lambda}$
$\lambda^*_{\mathrm{SIS},\pi}$	Reached HNF-SIS security	128	129
$\lambda^*_{\mathrm{LWE},\pi}$	Reached HNF-LWE security	130	130
$\lvert\pi\rvert$	Proof size (KB)	9580555	566318

Table F.1. Selected parameters, security and efficiency estimates of the signature scheme of [LLM+16].

Parameters	Description	Exact Proof	Fast Mode
Signature			
λ	Security parameter	128	128
n	SIS dimension	495	795
q	Modulus	$2^{155}+15$	$2^{155}+15$
q'	Tag bound	2^{61}	2^{61}
m_1	First trapdoor dimension	48732	78070
m_2	Second trapdoor dimension	77220	124020
m_3	Message bit-size	128	128
t	Spectral norm slack	7.5	7.5
σ	Pre-image sampling width	6026.03	7608.76
σ_1	$\sqrt{\sigma^2+\sigma_2^2}$	6026.05	7608.77
σ_2	Commitment randomness width	12.73	12.73
$\lambda_{\mathrm{I}}/\lambda_{\mathrm{I}}^*$	Required/Reached SIS security (I)	189/190	189/189
$\lambda_{\mathrm{II}}/\lambda_{\mathrm{II}}^*$	Required/Reached SIS security (II)	182/190	182/189
$\lvert\mathsf{pk}\rvert$	Public key size (MB)	1160	2988
$\lvert\mathsf{sk}\rvert$	Secret Key size (MB)	898	2308
$\lvert\mathsf{sig}\rvert$	Signature size (KB)	262	420
$\lvert\mathsf{pp}\rvert$	Public parameters size (MB)	1.2	1.9
Proof			
ℓ_1	HNF-SIS dimension	7850	7500
ℓ_2	HNF-LWE dimension	7850	7850
p	Size of challenges	$2^{\lambda/2}$	$2^{\lambda/2}$
N	Number of proof iterations	2	2
M	Rejection sampling repetition rate	27	27
L_{x}	Witness length	2268317	211572
$L_{\mathcal{M}}$	Relation set length	2267821	8686
δ_s	Soundness error	$2^{-\lambda}$	$2^{-\lambda}$
$\lambda^*_{\mathrm{SIS},\pi}$	Reached HNF-SIS security	128	129
$\lambda^*_{\mathrm{LWE},\pi}$	Reached HNF-LWE security	129	129
$\lvert\pi\rvert$	Proof size (KB)	308963	17809

Table F.2. Selected parameters, security and efficiency estimates of the signature scheme of Section 3.

Parameters	Description	Value
Signature		
λ	Security parameter	128
n	Ring degree	128
d	M-SIS module rank	10
q	Modulus	$2^{47}+9$
k	Number of splitting factors	4
w	Tag norm bound	14
$\binom{n}{w}$	Size of tag space	$\approx 2^{60.6}$
κ	Gadget matrix term	$\lceil \log_2 q \rceil$
m_1	First trapdoor rank	620
m_2	Second trapdoor rank	480
m_3	Number of message polynomials	1
t	Spectral norm slack	7.5
σ	Pre-image sampling width	5404
σ_1	$\sqrt{\sigma^2+\sigma_2^2}$	5935
σ_2	Commitment randomness width	2454
$\lambda_{\mathrm{I}}/\lambda_{\mathrm{I}}^*$	Required/Reached M-SIS security (I)	189/192
$\lambda_{\mathrm{II}}/\lambda_{\mathrm{II}}^*$	Required/Reached M-SIS security (II)	182/183
$\lvert\mathsf{pk}\rvert$	Public key size (MB)	8.06
$\lvert\mathsf{sk}\rvert$	Secret Key size (MB)	9.08
$\lvert\mathsf{sig}\rvert$	Signature size (KB)	275
$\lvert\mathsf{pp}\rvert$	Public parameters size (MB)	0.007
Proof		
d'	Height of commitment matrices $\mathbf{A}_1, \mathbf{A}_2$ (n)	12
q_1	Slack Modulus (q_1)	$2^{28}+105$
q_π	Proof modulus (q)	$\approx 2^{75}$
-	Bound on challenges (κ)	2
$\lvert\mathcal{C}\rvert$	Size of challenge space $(\lvert\mathcal{C}\rvert)$	$\approx 2^{147}$
σ_{-1}	Proof automorphism (σ)	σ_{-1}
η	Second bound on challenges (η)	72
ν	Randomness $\mathbf{s}_2$ bound (ν)	1
-	Number of garbage terms (λ)	5
-	Length of $\mathbf{s}_1$ (m_1)	1102
-	Length of $\mathbf{m}$ (ℓ)	0
-	Length of $\mathbf{s}_2$ (m_2)	41
γ_1	Rejection sampling constant for $c\mathbf{s}_1$ (γ_1)	5
γ_2	Rejection sampling constant for $c\mathbf{s}_2$ (γ_2)	3
$\gamma^{(e)}$	Rejection sampling constant for exact ARP $(\gamma^{(e)})$	2
δ_s	Soundness error	$\approx 2^{-131}$
$\lambda^*_{\text{M-SIS},\pi}$	Reached M-SIS security	131
$\lambda^*_{\text{M-LWE},\pi}$	Reached ext-M-LWE security	149
$\lvert\pi\rvert$	Proof size (KB)	638.1

Table F.3. Selected parameters, security and efficiency estimates of the signature scheme of Section 6.