Private Set Operations from Multi-Query Reverse Private Membership Test

Abstract. Private set operations allow two parties to perform secure computation on two private sets, such as intersection or union related functions. In this paper, we identify a framework for performing private set operations. At the technical core of our framework is multi-query reverse private membership test (mqRPMT), which is a natural extension of RPMT recently proposed by Kolesnikov et al. [KRTW19]. In mqRPMT, a client with a vector $X = (x_1, \ldots, x_n)$ interacts with a server holding a set Y. As a result, the server only learns a bit vector $(e_1, \ldots, e_n)$ indicating whether $x_i \in Y$ but without knowing the value of x_i, while the client learns nothing. We present two constructions of mqRPMT from newly introduced cryptographic primitive and protocol. One is based on commutative weak pseudorandom function (cwPRF), the other is based on permuted oblivious pseudorandom function (pOPRF). Both cwPRF and pOPRF can be realized from the decisional Diffie-Hellman like assumptions in the random oracle model. We also introduce a slightly weak version of mqRPMT dubbed mqRPMT*, in which the client additionally learns the cardinality of $X \cap Y$. We show that mqRPMT* can be built from a category of multi-query private membership test (mqPMT) called Sigma-mqPMT, which in turn can be realized from DDH-like assumptions or oblivious polynomial evaluation. This makes the first step towards establishing the relation between mqPMT and mqRPMT.

We demonstrate the practicality of our framework with implementations. By plugging our cwPRF-based mqRPMT to the general framework, we obtain various PSO protocols that are superior or competitive to the state-of-the-art protocols. For intersection functionality, our protocol is faster than the most efficient one for small sets. For cardinality functionality, our protocol achieves a $2.4 - 10.5\times$ speedup in running time and a $10.9 - 14.8\times$ shrinking in communication cost. For cardinality-with-sum functionality, our protocol achieves a $28.5 - 76.3\times$ speedup in running time and 7.4× shrinking in communication cost. For union functionality, our protocol achieves strict linear complexity, and requires the least concrete computation and communication costs in all settings. Specifically, for input set of size 2^{20}, our PSU protocol requires roughly 100 MB of communication, and 16 seconds using 4 threads in the LAN setting. For private-ID functionality, our protocol achieves a $2.7 - 4.9\times$ speedup in running time. Moreover, by plugging our FHE-based mqRPMT* to the general framework, we obtain a PSU* protocol (the sender additionally learns the intersection size) suitable for unbalanced setting, whose communication complexity is linear in the size of the smaller set, and logarithmic in the larger set.

1 Introduction

Consider two parties, each with a private set of items, wanting to compute on their respect sets without revealing any other information to each other. Two-party private set operation (PSO) refers to such family of interactive cryptographic protocols that takes two private sets X and Y as input, computes the desired function, and outputs the result to one or both of the participants. If one party obtains the result, we call this party the receiver and the other party the sender, and refer to the protocol as one-sided. Two-sided protocol in the semi-honest setting can be realized by having the receiver in one-sided protocol forward the result to the sender. In what follows, we briefly review two-party PSO protocols in the semi-honest model in terms of typical functionalities.

Private set intersection. PSI has found many applications including privacy-preserving sharing, private contact discovery, DNA testing, pattern matching and so on. Due to its importance and wide applications, in the past two decades PSI has been extensively studied in a long sequence of works and has become truly practical with extremely fast implementation. The most efficient PSI protocols [KKRT16, PRTY19, CM20, GPR$^+$21, RS21] mainly rely on symmetric-key operations, except a little public-key operations in base OT used in the OT extension protocol. We refer to [PSZ18] for a good survey of different PSI paradigms.

Private computing on set intersection. Certain real-world application scenarios only require partial/aggregated information about the intersection. In this setting fine-grained private computation on set intersection (PCSI) is needed, such as PSI-card for intersection cardinality [HFH99, CGT12], PSI-card-sum for intersection cardinality and sum [IKN$^+$20, GMR$^+$21]. For general-purpose PCSI (also known as circuit-PSI) [HEK12, PSTY19, RS21], the parties learn secret shares of the set intersection, which can be further fed into generic two-party computation (2PC) to compute $g(X \cap Y)$ for arbitrary function g.

Private set union. Like PSI, PSU also has numerous applications in practice, such as cyber risk assessment and management via joint IP blacklists and joint vulnerability data. According to the underlying cryptographic techniques, existing PSU protocols can be roughly divided into two categories. The first is mainly based on public-key techniques [KS05, Fri07, HN10, DC17]. The second is mainly based on symmetric-key techniques [KRTW19, GMR$^+$21, JSZ$^+$22]. We refer to [ZCL$^+$23] for a good survey of existing PSU protocols.

PSO protocols are primarily designed for balanced setting, in which the two sets' sizes are approximately the same. Recently, some works begin to consider unbalanced setting, in which one set is much more larger than the other. Among all PSO protocols, PSI has been extensively studied. In balanced setting, numerous PSI protocols achieve linear complexity, and the current state-of-the-art PSI [RR22] is almost as efficient as the naive insecure hash-based protocol. In unbalanced setting, a series of works [CLR17, CHLR18, CMdG$^+$21] show how to leverage fully homomorphic encryption (FHE) to build PSI protocols with sublinear complexity in the larger set size. In contrast to the affairs of PSI, the studies of PCSI and PSU are less satisfactory. As to PCSI, in balanced setting few protocols [PSTY19, IKN$^+$20] achieve linear complexity, but the practical performance is poor. As pin-pointed by [GMR$^+$21], semi-honest PCSI – even in the simplest case, like PSI-card – is concretely about 20× slower and requires over 30× more communication than PSI. [CHLR18] also propose PSI-card and PSI-card-sum protocols based on generic 2PC in unbalanced setting, but these protocols are more of theoretical interest, and are not accompanied by implementations. As to PSU, no protocol with linear complexity in either balanced or unbalanced setting is known for a long time being. It is until very recently, Zhang et al. [ZCL$^+$23] make a breakthrough by proposing the first PSU with linear complexity. However, their work does not close this issue. Their concrete PSU protocols have large constants in computation and communication complexity, incurring a large efficiency gap compared with PSI: roughly 20× slower and requires 25× more communication than PSI.

It is somewhat surprising that different PSO protocols have significantly different efficiency. Why is this case? Observe that PSI essentially can be viewed as multi-query private membership test (mqPMT), which has very efficient realizations in both balanced and unbalanced settings. However, mqPMT generally does not imply PCSI or PSU. The reason is that mqPMT reveals information about intersection, which should be hidden from the receiver in PCSI and PSU.

1.1 Motivation

Our motivation of this work is threefold. First, the above discussion indicates that the most efficient PSI protocols may not be easily adapted to PCSI and PSU protocols. Therefore, different approaches are employed for different private set operations, requiring much more engineering effort. We are motivated to seek for the minimal common protocol that enables all private set operations via a unified framework. Second, given the huge efficiency gap between PSI and other closely related protocols, we are also motivated to give efficient instantiations of the framework to close the gap. Last but not the least, recall that the seminal PSI protocol known as DH-PSI [Mea86] (related ideas were appeared in [Sha80, HFH99]) is derived from the Diffie-Hellman key-exchange protocol based on the decisional Diffie-Hellman (DDH) assumption. After roughly four decades, DH-PSI is still the most easily understood and implemented one among numerous PSI protocols. Somewhat surprisingly, no counterpart is known in the PSU setting yet. It is curious to know if the DDH assumption can strike back. In summary, we are intrigued to know:

Is there a central building block that enables a unified framework for all private set operations? If so, can we give efficient instantiations with optimal asymptotic complexity and good concrete efficiency? Can the DDH assumption strike back with efficient PSU protocols?

1.2 Our Contribution

In this work, we make positive answers to the aforementioned questions. We summarize our contribution as below.

A framework of PSO. We identify that multi-query reverse private membership test (mqRPMT) is a "Swiss Army Knife" for various private set operations. mqRPMT itself already implies PSI-card; by coupling with OT, mqRPMT implies PSI and PSU; by further coupling with simple secret sharing, mqRPMT implies PSI-card-sum and PSI-card-secret-sharing (further admits general-purpose PCSI with cardinality). Therefore, mqRPMT enables a unified PSO framework, which can perform a variety of private set operations in a flexible manner.

Efficient construction of mqRPMT. We propose two generic constructions of mqRPMT. The first is based on a new cryptographic primitive called commutative weak PRF (cwPRF), while the second is based on a new secure protocol called permuted oblivious PRF (pOPRF). Both of them can be realized from DDH-like assumptions in the random oracle model, yielding incredibly simple mqRPMT constructions with linear communication and computation complexity. Note that the complexity of our PSO framework is dominated by the underlying mqRPMT. Therefore, all resulting PSO protocols inherit optimal linear complexity. Notably, the obtained PSU protocol is arguably the most simple and efficient one among existing protocols.

Connection to mqPMT. mqRPMT is of great theoretical interest since it is the core building block of the PSO framework. It is thus interesting to investigate the relation between mqRPMT and mqPMT. Towards this goal, we put forward a variant of mqRPMT called mqRPMT with cardinality (denoted by mqRPMT* hereafter). Compared to the standard mqRPMT, mqRPMT* additionally reveals the intersection size to the client. We show that mqRPMT* can be built from a broad class of mqPMT called Sigma-mqPMT in a black-box manner via the "permute-then-test" approach. This makes the initial step towards establishing the connection between mqRPMT and mqPMT. We argue that though mqRPMT* deviates from standard mqRPMT in revealing additional information (intersection size) to the client, it could also be a desirable feature in application scenarios where both parties want to learn intersection size, for example, PSI-card-sum [IKN$^+$20]. We leave the general connection between mqPMT and mqRPMT as a challenging open problem.

Evaluations. We give efficient instantiation of our generic framework from cwPRF-based mqRPMT protocol. We provide C++ implementations. The experimental results demonstrate that almost all PSO protocols derived from our generic framework are superior or competitive to the state-of-the-art corresponding protocols.

1.3 Technical Overview

PSO from mqRPMT. As discussed above, mqPMT (a.k.a. PSI) protocol generally is not applicable for computing PCSI and PSU. We examine the reverse direction, i.e., whether the core protocol underlying PSU can be used for computing PSI and PCSI. We identify that the central protocol beneath all the existing PSU protocols is actually mqRPMT, which is a generalization of RPMT formalized in [KRTW19]. Roughly speaking, mqRPMT is a two-party protocol between a server holding a set Y and a client holding a vector $X = (x_1, \dots, x_n)$. After execution, the server learns an indication bit vector $(e_1, \ldots, e_n)$ such that $e_i = 1$ if and only if $x_i \in Y$ but without knowing x_i, while the client learns nothing. Superficially, mqRPMT is similar to mqPMT, except that it is the server but not the client learns the test results instead. This subtle difference turns out to be significant. To see this, note that in mqRPMT the intersection information (a.k.a. x_i and e_i) are shared between the two parties, while in mqPMT the intersection information are entirely known by the client. In light of this difference, mqRPMT is not only particularly suitable for functionalities that have to keep intersection private, but also retains the necessary information to compute the intersection. With mqRPMT in hand, PSI-card protocol is immediate. PSI (resp. PSU) protocol can be done by having the receiver (playing the role of server) and the sender (playing the role of client) invoke a mqRPMT protocol in the first place, then carry out n one-sided OTs with $1 - e_i$ (resp. e_i) and x_i. PSI-card-sum and PSI-card-secret-sharing protocols can be constructed by further coupling with OT and simple secret-sharing trick.

Next, we show two generic constructions of mqRPMT. For convenience of narration, we explicitly parameterize RPMT and PMT with two parameters n_1 and n_2, namely (n_1, n_2)-(R)PMT, where n_1 is the size of server's set Y, n_2 is the length of client's vector X, a.k.a. the number of membership test queries.

mqRPMT from cwPRF. We observe that private equality test (PEQT) protocol [PSZ14] not only can be viewed as an extreme case of mqPMT, but can also be viewed as an extreme case of mqRPMT. Under the terminology introduced above, PEQT is essentially $(1,1)$-PMT and $(1,1)$-RPMT. We choose PEQT as the starting point of our first mqRPMT construction.

The basic idea of building $(1,1)$-RPMT protocol that is amenable to extension is *oblivious joint encoding*, i.e., an element can only be encoded to a codeword by two parties in a joint manner, while the process reveals nothing to the party without the element. To implement this idea, we introduce a new cryptographic primitive called commutative weak PRF (cwPRF). Let $F : K \times D \rightarrow R$ be a family of weak PRF, where $R \subseteq D$. We say F is commutative if for any $k_1, k_2 \in K$ and any $x \in D$, it holds that $F_{k_1}(F_{k_2}(x)) = F_{k_2}(F_{k_1}(x))$. In other words, the two composite functions $F_{k_1} \circ F_{k_2}$ and $F_{k_2} \circ F_{k_1}$ are essentially the same function, say, $\hat{F}$.

Now we are ready to describe the construction of $(1,1)$-RPMT from cwPRF. The server P_1 holding y and client P_2 holding x can conduct PEQT functionality via the following steps: (1) P_1 and P_2 generate cwPRF key k_1 and k_2 respectively, and map their items to domain D of F via a common cryptographic hash function H, which will be modeled as a random oracle; (2) P_1 computes and sends $F_{k_1}(\mathsf{H}(y))$ to P_2; (3) P_2 computes and sends $F_{k_2}(\mathsf{H}(x))$ and $F_{k_2}(F_{k_1}(\mathsf{H}(y)))$ to P_1; (4) P_1 then learns the test result by comparing $F_{k_1}(F_{k_2}(\mathsf{H}(x))) =? F_{k_2}(F_{k_1}(\mathsf{H}(y)))$. The commutative property of F ensures the correctness. The weak pseudorandomness of F guarantees that P_2 learns nothing and P_1 learns nothing beyond the test result. In the above construction, $F_{k_2}(F_{k_1}(\mathsf{H}(\cdot))) = F_{k_1}(F_{k_2}(\mathsf{H}(\cdot))) = \hat{F}_k(\mathsf{H}(\cdot))$ serves as a pseudorandom encoding function in the joint view, while $F_{k_1}(\mathsf{H}(\cdot))$ and $F_{k_2}(\mathsf{H}(\cdot))$ serve as a partial encoding function in the individual views of server and client respectively.

We then extend the above $(1,1)$-RPMT protocol to $(n_1,1)$-RPMT. However, naive repetition by sending back $F_{k_2}(F_{k_1}(\mathsf{H}(y_i)))$ for each $y_i \in Y$ in the same order of server's first move message $F_{k_1}(\mathsf{H}(y_i))$ does not lead to a secure $(n_1,1)$-RPMT. The reason is that $\{\hat{F}_k(\mathsf{H}(y_i))\}_{i\in[n_1]}$ constitutes an order-preserving pseudorandom encoding of $(y_1,\ldots,y_{n_1})$. As a consequence, the server will learn the exact value of x if $x \in Y$. In order to perform the membership test in an oblivious manner, the idea is to make the pseudorandom encoding of $(y_1,\ldots,y_{n_1})$ independent of the order known by the server. A straightforward approach is to shuffle $\{\hat{F}_k(\mathsf{H}(y_i))\}$. In this way, we obtain a $(n_1,1)$-RPMT protocol from cwPRF, and can further batch it to a full-fledged (n_1,n_2)-RPMT protocol by reusing the encoding key k_2. A simple calculation shows that for a (n_1,n_2)-RPMT protocol the computation cost is (n_1+n_2) times mapping, $(2n_1+n_2)$ times evaluation of F and n_2 times look-up, and the communication cost is $(2n_1+n_2)$ elements in the range of F. The resulting mqRPMT protocol is optimal in the sense that both computation and communication complexity are linear to the set size. We can further reduce the communication cost by inserting $\{\hat{F}(\mathsf{H}(y_i))\}$ into an order-hiding data structure such as Bloom filter, instead of shuffling them.

We show that cwPRF can be realized from DDH-like assumptions. Henceforth, DDH strikes back with an incredibly simple PSU protocol. This once again demonstrates that the DDH assumption is truly a golden goose in cryptography.

mqRPMT from permuted OPRF. We choose $(n,1)$-RPMT as the starting point of our second mqRPMT construction. The idea is *oblivious permuted encoding*, i.e., only one party say P_2 is able to encode, and the other party say P_1 can learn the codewords of its elements $(y_1,\ldots,y_{n_1})$ in a permuted order, while P_2 learns nothing. A tempting approach to implement this idea is using multi-point OPRF that underlies many PSI protocols [PRTY19, CM20]. More precisely, having P_1 (acts as receiver) and P_2 (acts as sender) engage in an OPRF protocol. Eventually, P_1 obtains PRF values of $(y_1,\ldots,y_{n_1})$ as encodings, and P_2 obtains a PRF key k. However, OPRF does not readily enable oblivious permuted encoding. The reason is that the standard OPRF functionality always gives the PRF values with the same order of inputs. To remedy this issue, we introduce a new cryptographic protocol called permuted OPRF (pOPRF). pOPRF can be viewed as a generalization of OPRF. The difference is that the sender additionally obtains a random permutation π over $[n_1]$ besides PRF key k, while the receiver obtains PRF values in a permuted order as per π. pOPRF immediately implies a $(n_1,1)$-RPMT protocol: The server with $Y=(y_1,\ldots,y_{n_1})$ and the client with $X=\{x\}$ first engage in a pOPRF protocol. As a result, the server obtains $\{F_k(y_{\pi(i)})\}_{i\in[n_1]}$, while the client learns a PRF key k and a permutation π. The client then computes and sends $F_k(x)$ to the server as RPMT query. Finally, the server learns if $x\in Y$ by testing whether $F_k(x)\in\{F_k(y_{\pi(i)})\}_{i\in[n_1]}$, but learns nothing more since its PRF values are of permuted order. At a high level, $F_k(\cdot)$ serves as an encoding function in client's view, while $F_k(\pi(\cdot))$ serves as a pseudorandom and permuted encoding function in server's view. Extending the above $(n_1, 1)$-RPMT to (n_1, n_2)-RPMT is straightforward by having the client reuse k and send $\{F_k(x_i)\}_{i \in [n_2]}$ as RPMT queries.

The question remains is how to build pOPRF. One common approach to build OPRF is "mask-then-unmask". We choose this category of OPRF as the starting point. The rough idea is exploiting the input homomorphism to mask inputs[1], then unmask the outputs. If the mask procedure is different per input, then the unmask procedure must be carried out accordingly. Therefore, OPRF protocols of this case cannot be easily adapted to pOPRF, since the receiver is unable to perform the unmask procedure over permuted masked outputs correctly, namely, to recover outputs in permuted order. The above analysis indicates us that if the masking procedure can be done via a unifying manner, then the receiver might be able to unmask the permuted masked outputs correctly. Observe that the simplest way to perform unified masking is to apply a weak pseudorandom function F_s to the intermediate input $\mathsf{H}(x)$. To enable efficient unmask procedure, we further require that F_s is a permutation and commutative with respect to F_k. This yields a simple pOPRF construction from commutative weak pseudorandom permutation. More precisely, to build pOPRF, the sender picks a random PRP key k for F, while the receiver with input $X = (x_1, \ldots, x_n)$ picks a random PRP key s for F. The receiver then sends $\{F_s(\mathsf{H}(x_i))\}_{i \in [n]}$ to the sender. Upon receiving the masked intermediate inputs, the sender applies F_k to them, then sends the results in permuted order, a.k.a. $\{F_k(F_s(\mathsf{H}(x_{\pi(i)})))\}_{i \in [n]}$. Finally, the receiver applies F_s^{-1} to the permuted masked outputs, and will obtain $\{F_k(\mathsf{H}(x_{\pi(i)}))\}_{i \in [n]}$ by the commutative property.

Note that many efficient OPRF constructions [CM20] seem not amenable to pOPRF construction due to lack of nice algebra structures. This somehow explains the efficiency gap between the state-of-the-art PSI and PCSI/PSU.

mqRPMT* from Sigma-mqPMT. In Appendix A, we study the connection between mqRPMT and mqPMT. We first abstract a category of mqPMT protocols called Sigma-mqPMT, which is built from Sigma-PMT. Roughly speaking, Sigma-PMT is a three-move protocol, which proceeds as below: (1) in the first move, the server holding a set Y sends a message a to the client, where a is best interpreted as an encoding of Y; (2) in the second move, the client makes a test query q of its item x; (3) in the last move, the server responds with z, and eventually the client can decide if $x \in Y$ by running algorithm $\mathsf{Test}(a, q, x, z)$. To enable efficient parallel composition, we introduce the following two properties for Sigma-PMT: (i) reusable property, which ensures the first move message can be safely reused over multi-instance; (ii) context-independent property, which means the test query only depends on the item in test. With these two properties, one can build mqPMT by running multiple instances of Sigma-PMT in parallel, without increasing round complexity. If the underlying Sigma-PMT additionally satisfies stateless testing, namely, Test algorithm can be done without learning (q, x), we refer to the resulting mqPMT as Sigma-mqPMT, which captures the

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[1] Standard pseudorandomness denies input homomorphism. Rigorously speaking, we utilize the homomorphism over intermediate input.

common form of several PSI protocols [Mea86, FIPR05, CLR17]. By utilizing the stateless property, we can tweak Sigma-mqPMT to permuted mqPMT via the "permute-then-test" approach, without incurring computation and communication overhead, while permuted mqPMT instantly implies mqRPMT*. Therefore, we can expand a series of results in PSI setting to PSO setting, on the premise that revealing intersection size is acceptable. Notably, by applying the conversion to fully homomorphic encryption (FHE) based Sigma-mqPMT, we obtain an efficient mqRPMT* in unbalanced setting, which yields the first PSU* protocol whose communication complexity is sublinear to the size of the large set X.

In Figure 1, we give an overview of the main contribution of this work.

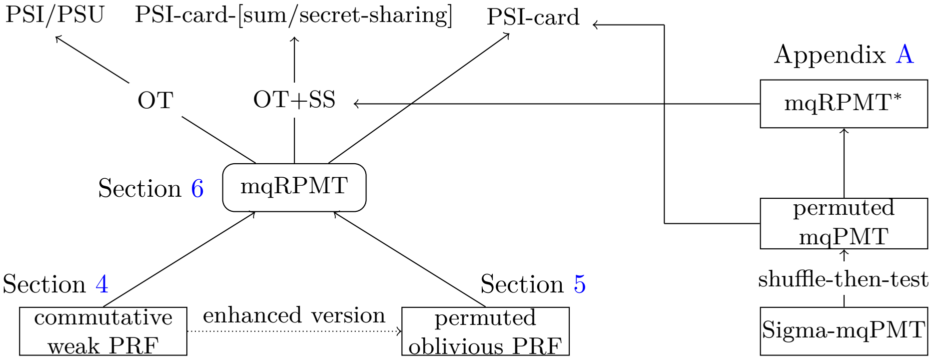

Fig. 1: An overview of our main results. The rectangles denote our contributions. The rounded rectangles denotes notions in previous works.

1.4 Related Works

We review previous PSI-card, PSI-card-sum and PSU protocols that are related to our work. Ion et al. [IKN$^+$20] showed how to transform single-point OPRF-based [PSZ14, KKRT16], garbled Bloom filter-based [DCW13, RR17], and DDH-based [HFH99] PSI protocols into ones for computing PSI-card-sum by leveraging additively homomorphic encryption (AHE). However, their conversions are not efficient due to the usage of AHE, and as noted by the authors, detailed conversions to each category of protocols differ significantly, especially in the way of making use of the underlying AHE. In contrast to their work, we distill a broad class of PSI protocols as Sigma-mqPMT, then show how to tweak it to mqRPMT* in a generic and black-box manner, without relying on any additional cryptographic tools. Our conversion works in a more abstract and lower level, and such generality lends it may find more potential applications.

Miao et al. [MPR+20] put forward shuffled distributed oblivious PRF as a central tool to build PSI-card-sum with malicious security. Compared to shuffled distributed OPRF, our notion of permuted OPRF is much simpler and should be best viewed as a useful extension of standard OPRF. The conceptual simplicity lends it can be easily built from commutative weak pseudorandom permutation and find more potential applications. For example, permuted OPRF immediately implies permuted multi-point private equality test, which is a key tool in building FHE-based PSU [TCLZ22]. Davidson and Cid [DC17] proposed a framework for constructing PSI, PSU, and PSI-card. Their protocols have linear complexity, but both the computation and communication complexities additionally rely on the statistical security parameter λ (a typical concrete choice is 40), resulting in low performance in practice. Kolesnikov et al. [KRTW19] showed that the performance of PSU in [DC17] is four orders of magnitude worse than the state-of-the-art at that time-being. Garimella et al. [GMR+21] proposed a framework for all private set operations. At their technical core is a new protocol called permuted characteristic, which could be viewed as an extension of mqRPMT protocol. Nevertheless, the oblivious shuffle in permuted characteristic functionality is not necessary for PSO, but seems unavoidable due to the use of oblivious switching networks. This incurs superlinear complexity to permuted characteristic protocol and all the enabling PSO protocols. Moreover, we note that the PSI-card-sum functionality defined in [GMR+21] differs from the original functionality defined in [IKN+20]. The distinction is that in the original functionality of PSI-card-sum, both parties are given the cardinality of intersection, and the party initially holding values is also given the intersection sum, while in the functionality described in [GMR+21], the party without holding values is given the cardinality and sum of intersection. To highlight this subtle difference, we prefer to call the functionality presented in [GMR+21] as reverse PSI-card-sum.

Concurrent work. Very recently, Zhang et al. [ZCL+23] propose a generic construction of mqRPMT with linear complexity from oblivious key-value store, set-membership encryption and oblivious vector decryption-then-test functionality. By instantiating their generic construction from public-key and symmetric-key encryption respectively and combining OT, they make the breakthrough by giving the first PSU protocol with optimal linear complexity. However, as noted by the authors, their more efficient PKE-based construction is leaky, failing to satisfy the standard security of mqRPMT. Besides, the communication complexity of their two constructions additionally depends on the statistical security parameter. Compared with their work, our construction of mqRPMT is much simpler. The instantiation meets the standard definition, and achieves strict linear complexity. Moreover, we explore mqRPMT as a central building block for a family of private set operations, while their main focus is limited to PSU.

2 Preliminaries

Notations. We use κ and λ to denote the computational and statistical parameter respectively. Let $\mathbb{Z}_n$ be the set $\{0, 1, \ldots, n-1\}$, $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n \mid \gcd(x, n) =$

$1\}$. We use $[n]$ to denote the set $\{1, \ldots, n\}$, and use $\mathsf{Perm}[n]$ to denote all the permutations over the set $\{1, \ldots, n\}$. We assume that every set X has a default order (e.g. lexicographical order), and write it as $X = \{x_1, \ldots, x_n\}$. For a set X, we use $|X|$ to denote its size and use $x \xleftarrow{\text{R}} X$ to denote sampling x uniformly at random from X. We use $(x_1, \ldots, x_n)$ to denote a vector, and its ith element is x_i. A function is negligible in κ, written $\mathsf{negl}(\kappa)$, if it vanishes faster than the inverse of any polynomial in κ. A probabilistic polynomial time (PPT) algorithm is a randomized algorithm that runs in polynomial time.

2.1 MPC in the Semi-honest Model

We use the standard notion of security in the presence of semi-honest adversaries. Let $\mathit{\Pi}$ be a two-party protocol for computing the function $f(x_1, x_2)$, where party P_i has input x_i. We define security in the following way. For each party P_i where $i \in \{1, 2\}$, let $\text{View}_{P_i}(x_1, x_2)$ denote the view of party P_i during an honest execution of $\mathit{\Pi}$ on inputs x_1 and x_2. The view consists of P_i's input, random tape, and all messages exchanged as part of the $\mathit{\Pi}$ protocol.

Definition 1. *Two-party protocol $\mathit{\Pi}$ securely realizes f in the presence of semi-honest adversaries if there exists a simulator* Sim *such that for all inputs x_1, x_2 and all $i \in \{1, 2\}$:*

$$\mathsf{Sim}(i, x_i, f(x_1, x_2)) \approx_c \mathit{View}_{P_i}(x_1, x_2)$$

Roughly speaking, a protocol is secure if the party P_i with x_i learns no more information other than $f(x_1, x_2)$ and x_i.

2.2 Private Set Operation

PSO is a special case of secure two-party computation. We call the two parties engaging in PSO the *sender* and the *receiver*. The sender holds a set X of size n_1, and the receiver holds a set Y of size n_2 (we write $n_1 = n_2 = n$ in balanced setting). The ideal PSO functionality (depicted in Figure 2) computes the intersection, union, cardinality, intersection sum with cardinality and intersection secret-sharing with cardinality.

3 Protocol Building Blocks

3.1 Oblivious Transfer

Oblivious Transfer (OT) [Rab05] is a central cryptographic primitive in the area of secure computation. 1-out-of-2 OT allows a sender with two input strings (m_0, m_1) and a receiver with an input choice bit $b \in \{0, 1\}$. As a result of the OT protocol, the receiver learns m_b and neither party learns any additional information. In some cases, it suffices to use a "one-sided" version of OT, which

Parameters: P_1's input size n_1 and P_2's input size n_2.
Inputs: The receiver P_1 inputs a set of elements $Y = \{y_1, \ldots, y_{n_1}\}$ where $y_i \in \{0,1\}^\ell$. The sender P_2 inputs a set of elements $X = \{x_1, \ldots, x_{n_2}\}$ where $x_i \in \{0,1\}^\ell$ and possibly a set of values $V = \{v_1, \ldots, v_{n_2}\}$ where $v_i \in \mathbb{Z}_p$ for some integer modular p.
Output:

- **intersection:** The receiver P_1 gets $X \cap Y$.
- **union:** The receiver P_1 gets $X \cup Y$.
- **union*:** The receiver P_1 gets $X \cup Y$. The sender P_2 gets $|X \cap Y|$.
- **card:** The receiver P_1 gets $|X \cap Y|$.
- **card-sum:** The receiver P_1 gets $|X \cap Y|$. The sender P_2 gets $|X \cap Y|$ and $S = \sum_{i: x_i \in Y} v_i$.
- **card-secret-sharing:** The receiver P_1 gets $|X \cap Y|$ and $\{z_i^1\}_{i \in [n_2]}$. The sender P_2 gets $\{z_i^2\}_{i \in [n_2]}$. For each (z_i^1, z_i^2), $z_i^1 \oplus z_i^2 = x_i$ if $x_i \in Y$ and $z_i^1 \oplus z_i^2 = 0$ otherwise.

Fig. 2: Ideal functionality $\mathcal{F}_{\mathsf{PSO}}$ for PSO

conditionally transfers the only item of the sender or nothing to the receiver depending on the choice bit.

Though expensive public-key operations are unavoidable for a single OT, a powerful technique called OT extension [IKNP03, KK13, ALSZ15] allows one to perform n OTs by only performing $O(\kappa)$ public-key operations and $O(n)$ fast symmetric-key operations. In Figure 3 we formally define the ideal functionality for OT that provides n parallel instances of OT.

Parameters: Number of OT instances n and message length ℓ.
Inputs: The sender P_1 inputs $\{(m_{i,0}, m_{i,1})\}_{i \in n}$, where each $m_{i,b} \in \{0,1\}^\ell$. The receiver P_2 inputs a bit vector $(b_1, \ldots, b_n) \in \{0,1\}^n$.
Output: The sender P_1 gets nothing. The receiver P_2 gets $\{m_{i,b_i}\}_{i \in [n]}$.

Fig. 3: Ideal functionality $\mathcal{F}_{\mathsf{OT}}$ for OT

3.2 Multi-Query RPMT

RPMT [KRTW19] refers to a protocol where the client with input x interacts with a server holding a set Y. As a result, the server learns (only) the bit indicating whether $x \in Y$, while the client learns nothing about the set Y. The default notion of RPMT allows the client to query for a single element. While this procedure can be repeated several times, one may seek more efficient solutions allowing the client to make n distinct queries at a reduced cost. It is straightforward to define this generalized notion of n-time RPMT. Hereafter, we refer to n-time RPMT as multi-query RPMT. In Figure 4 we formally define the ideal functionality for mqRPMT. We also define a relaxed version of mqRPMT called mqRPMT*, in which the client is given $|X \cap Y|$.

Parameters: P_1's set size n_1 and number of RPMT queries n_2.
Inputs: The server P_1 inputs a set $Y = (y_1, \dots, y_{n_1})$, where $y_i \in \{0,1\}^\ell$. The client P_2 inputs a set $X = (x_1, \dots, x_{n_2})$ (should be interpreted as a vector), where $x_i \in \{0,1\}^\ell$.
Output: The server P_1 gets a vector $\vec{e} = (e_1, \dots, e_{n_2}) \in \{0,1\}^{n_2}$, where $e_i = 1$ if $x_i \in Y$ and $e_i = 0$ otherwise. The client P_2 gets nothing.

Fig. 4: Ideal functionality $\mathcal{F}_{\mathsf{mqRPMT}}$ for multi-query RPMT

Family of PMT protocols. For completeness and fixing terminology, we are tempting to systematically list the whole family of PMT protocols. We identify two characteristics of PMT protocols. One is direction, which consists of two options, namely forward or reverse. Standard option means the indication bit indicates the membership of the server's elements, while reverse option means the indication bit indicates the membership of the client's elements. The other one is order, which also consists of two options, namely ordered and permuted. The ordered option means the indication bit is of the right order (known by the receiver). The permuted option means the indication bit is of the permuted order unknown by the sender. By mix-match two characteristics, we obtain four types PMT protocols, shown in Table 1.

4 Commutative Weak Pseudorandom Function

In Appendix B we review the notion of PRF, including formal definition as well as simple construction from the DDH assumption. On this basis, we introduce a new notion called commutative weak PRF as below.

4.1 Definition of Commutative Weak PRF

We first formally define two standard properties for keyed functions.

Composable. For a family of keyed functions $F : K \times D \to R$, F is 2-composable if $R \subseteq D$, namely, for any $k_1, k_2 \in K$, the function $F_{k_1}(F_{k_2}(\cdot))$ is well-defined. In this work, we are interested in a special case namely $R = D$.

Table 1: The family of PMT protocols

Protocol	Direction		Order		Direct usage
	forward	reverse	ordered	permuted	
mqPMT	✓		✓		PSI
mqRPMT		✓	✓		PSI-card
permuted mqPMT	✓			✓	PSI-card
permuted mqRPMT		✓		✓	PSI-card

mqPMT and PSI are the same protocol under different names. mqRPMT is formalized in [KRTW19, ZCL$^+$23]. Permuted mqRPMT is introduced in [GMR$^+$21] under the name of permuted characteristic. To the best of our knowledge, the notion of permuted mqPMT is new to this work, which could be viewed as a high-level abstraction of the DH-based PSI-card protocol due to [HFH99].

Commutative. For a family of composable keyed functions, we say it is commutative if:

$$\forall k_1, k_2 \in K, \forall x \in D : F_{k_1}(F_{k_2}(x)) = F_{k_2}(F_{k_1}(x))$$

It is easy to see that the standard pseudorandomness denies commutative property. Consider the following attack against the standard pseudorandomness of F_k as below: the adversary $\mathcal{A}$ picks $k' \overset{\text{R}}{\leftarrow} K$, $x \overset{\text{R}}{\leftarrow} D$, and then queries the real-or-random oracle at point $F_{k'}(x)$ and point x respectively, receiving back responses y' and y. $\mathcal{A}$ then outputs '1' iff $F_{k'}(y) = y'$. Clearly, $\mathcal{A}$ breaks the pseudorandomness with advantage $1/2$. Provided commutative property exists, the best security we can expect is weak pseudorandomness. Looking ahead, weak pseudorandomness and commutative property may co-exist based on some well-studied assumptions.

Definition 2 (Commutative Weak PRF). *Let F be a family of keyed functions $K \times D \to D$. F is called commutative weak PRF if it satisfies weak pseudorandomness and commutative property simultaneously. If F is a permutation, we say F is a commutative weak pseudorandom permutation (cwPRP).*

Further generalization. Instead of sticking to one family of keyed functions, commutative property can be defined over two families of keyed functions. Let F be a family of weak PRFs from $K \times D$ to D, G be a family of weak PRFs $S \times D$ to D. If the following equation holds,

$$\forall k \in K, s \in S, \forall x \in D : F_k(G_s(x)) = G_s(F_k(x))$$

we say (F, G) is a pair of cwPRF.

Remark 1. We note that our notion of cwPRF is similar to but strictly weaker than a previous notion called commutative encryption [AES03]. The difference is that cwPRF neither requires F_k be a permutation nor F_k^{-1} be efficiently computable.

4.2 Construction of Commutative Weak PRF

We observe that the weak PRF construction presented in Section B.1 already satisfies commutative property. This gives us a simple cwPRF construction from the DDH assumption.

4.3 mqRPMT from Commutative Weak PRF

In Figure 5, we show how to build mqRPMT from cwPRF $F : K \times D \to D$ and cryptographic hash function $\mathsf{H} : \{0,1\}^\ell \to D$.

Parameters: P_1's set size n_1 and P_2's set size n_2, cwPRF $F : K \times D \to D$, and hash function $\mathsf{H} : \{0,1\}^\ell \to D$.
Inputs: The server P_1 inputs a set $Y = \{y_1, \dots, y_{n_1}\}$, where $y_i \in \{0,1\}^\ell$. The client P_2 inputs a set $X = \{x_1, \dots, x_{n_1}\}$ (should be interpreted as a vector), where $x_i \in \{0,1\}^\ell$.
Protocol:

1. P_1 picks $k_1 \xleftarrow{\text{R}} K$, then sends $\{F_{k_1}(\mathsf{H}(y_i))\}_{i \in [n_1]}$ to P_2.
2. P_2 picks $k_2 \xleftarrow{\text{R}} K$, then computes and sends $\{F_{k_2}(\mathsf{H}(x_i))\}_{i \in [n_2]}$ to P_1. P_2 also computes $\{F_{k_2}(F_{k_1}(\mathsf{H}(y_i)))\}_{i \in [n_1]}$, picks a random permutation $\pi \xleftarrow{\text{R}} [n_1]$, then sends $\{F_{k_2}(F_{k_1}(\mathsf{H}(y_{\pi(i)})))\}_{i \in [n_1]}$ to P_1. An alternative choice instead of explicit shuffle is inserting $\{F_{k_2}(F_{k_1}(\mathsf{H}(y_i)))\}_{i \in [n_1]}$ to a Bloom filter, then sends the resulting filter to P_1. We slightly abuse the notation, and still use Ω to denote the Bloom filter.
3. P_1 computes $\{F_{k_1}(F_{k_2}(\mathsf{H}(x_i)))\}_{i \in [n_2]}$, then sets $e_i = 1$ iff $F_{k_1}(F_{k_2}(\mathsf{H}(x_i))) \in \Omega$.

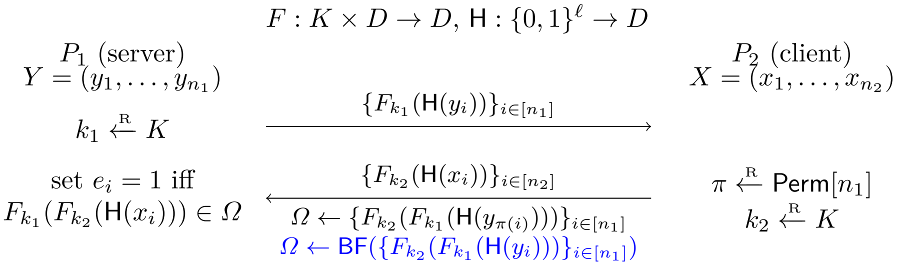

Fig. 5: Multi-query RPMT from commutative weak PRF

Remark 2. We observe that thanks to the nice properties of cwPRF, the same cwPRF-based mqRPMT protocol can also be tweaked to permuted mqPMT by checking if $\hat{F}_k(\mathsf{H}(y_{\pi(i)})) \in \{\hat{F}_k(\mathsf{H}(x_i))\}_{i \in [n_2]}$.

Correctness. The protocol is correct except the event E that $F_{k_1}(F_{k_2}(\mathsf{H}(x))) = F_{k_1}(F_{k_2}(\mathsf{H}(y)))$ for some $x \neq y$ occurs. In what follows, we fix a tuple (x, y) such that $x \neq y$. Let E_0 be the event $\mathsf{H}(x) = \mathsf{H}(y)$. By the collision resistance of H, we have $\Pr[E_0] = 2^{-\kappa}$. Let E_1 be the event that $\mathsf{H}(x) \neq \mathsf{H}(y)$ but $F_{k_1}(F_{k_2}(\mathsf{H}(x))) = F_{k_1}(F_{k_2}(\mathsf{H}(y)))$, which can further be divided into sub-cases E_{10}—$F_{k_2}(\mathsf{H}(x)) = F_{k_2}(\mathsf{H}(y))$ and E_{11}—$F_{k_2}(\mathsf{H}(x)) \neq F_{k_2}(\mathsf{H}(y))$ but $F_{k_1}(F_{k_2}(\mathsf{H}(x))) = F_{k_1}(F_{k_2}(\mathsf{H}(y)))$. By the weak pseudorandomness of F, we have $\Pr[E_{10}] = (1 - \Pr[E_0]) \cdot 1/|D|$, and $\Pr[E_{11}] = (1 - \Pr[E_0]) \cdot (1 - 1/|D|) \cdot 1/|D|$. If $|D| = \omega(\kappa)$, then both $\Pr[E_0]$, $\Pr[E_{10}]$ and $\Pr[E_{11}]$ are negligible in κ. Therefore, by union bound we have $\Pr[E] \leq n_1 n_2 \cdot (\Pr[E_0] + \Pr[E_{10}] + \Pr[E_{11}]) = \mathsf{negl}(\kappa)$.

Theorem 1. *The mqRPMT protocol described in Figure 5 is secure in the semi-honest model assuming* H *is a random oracle and* F *is a family of cwPRFs.*

Due to space limit, we defer the security proof to Appendix D.1.

5 Permuted Oblivious Pseudorandom Function

5.1 Definition of Permuted OPRF

An oblivious pseudorandom function (OPRF) [FIPR05] is a two-party protocol in which the sender learns a PRF key k and the receiver learns $F_k(x_1), \ldots, F_k(x_n)$, where F is a pseudorandom function (PRF) and $(x_1, \ldots, x_n)$ are the receiver's inputs. Nothing about the receiver's inputs is revealed to the sender and nothing more about the key k is revealed to the receiver.

We consider an extension of OPRF which we called permuted OPRF. Roughly speaking, the sender additionally picks a random permutation π over $[n]$, and the receiver learns its PRF values in permuted order, namely, $y_i = F_k(x_{\pi(i)})$. In Figure 6 we formally define the ideal functionality for pOPRF.

Parameters: Number of OPRF queries n.
Inputs: The sender P_1 inputs nothing. The receiver P_2 inputs a set $X = (x_1, \ldots, x_n)$, where $x_i \in \{0,1\}^\ell$.
Output: The sender P_1 gets a random PRF key k and a random permutation π over $[n]$. The client P_2 gets $y_i = F_k(x_{\pi(i)})$.

Fig. 6: Ideal functionality $\mathcal{F}_{\mathsf{pOPRF}}$ for permuted OPRF

5.2 Construction of Permuted OPRF

As we sketched in the introduction part, we can create a permuted OPRF from cwPRP F. At a high level, the unified masking procedure is done by applying a weak PRF $F_s(\cdot)$ to $\mathsf{H}(x)$, and the unmasking process is enabled by the commutative property of F and the fact that $F_s(\cdot)$ is an efficiently invertible permutation. We depict the construction in Figure 7.

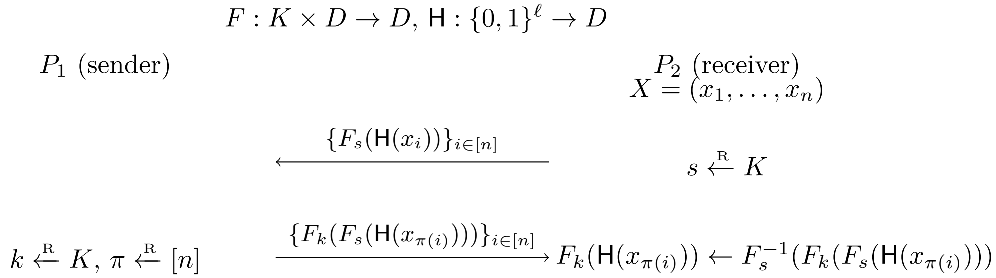

Fig. 7: Permuted OPRF from cwPRP

Remark 3. We note that it suffices to build permuted OPRF from a tuple of cwPRF (F_k, G_s) where G_s is a weak permutation.

Theorem 2. *The above permuted OPRF protocol described in Figure 7 is secure in the semi-honest model assuming* H *is a random oracle and F is a family of cwPRPs.*

Due to space limit, we defer the security proof to Appendix D.2.

5.3 mqRPMT from Permuted OPRF

In Figure 8, we show how to build mqRPMT from permuted OPRF for $F : K \times D \to R$. For simplicity, we assume that $\{0,1\}^\ell \subseteq D$. Otherwise, we can always map $\{0,1\}^\ell$ to D via collision resistant hash function.

Correctness. The above protocol is correct except the case $E = \vee_{i,j} E_{ij}$ occurs, where E_{ij} denotes $F_k(x_i) = F_k(y_j)$ but $x_i \neq y_j$. By pseudorandomness of F, we have $\Pr[E_{ij}] = 2^{-\ell}$. Apply the union bound, we have $\Pr[E] \leq n_1 n_2 \cdot \Pr[E_{ij}] = n_1 n_2 / 2^\ell = \mathsf{negl}(\lambda)$.

Theorem 3. *The above mqRPMT protocol described in Figure 8 is secure in the semi-honest model assuming the security of permuted OPRF F.*

Due to space limit, we defer the security proof to Appendix D.3.

Parameters: P_1's set size n_1 and P_2's set size n_2, a permuted OPRF for $F : K \times D \to R$.
Inputs: The server P_1 inputs a set $Y = \{y_1, \ldots, y_{n_1}\}$, where $y_i \in \{0,1\}^\ell$. The client P_2 inputs a set $X = \{x_1, \ldots, x_{n_2}\}$, where $x_i \in \{0,1\}^\ell$.
Protocol:

1. P_1 with inputs $Y = \{y_1, \ldots, y_{n_1}\}$ and P_2 invoke the permuted OPRF protocol. At the end of the protocol, P_1 obtains $\{F_k(y_{\pi(i)})\}_{i \in [n_1]}$, P_2 obtains k and a permutation π over $[n_1]$.
2. P_2 computes and sends $(F_k(x_1), \ldots, F_k(x_{n_1}))$ to P_1.
3. P_1 sets $e_i = 1$ iff $F_k(x_i) \in \{F_k(y_{\pi(i)})\}_{i \in [n_1]}$.

Fig. 8: mqRPMT from permuted OPRF

Parameters: P_1's set size n_1 and P_2's set size n_2.
Inputs: The receiver P_1 inputs a set $Y = \{y_1, \ldots, y_{n_1}\}$, where $y_i \in \{0,1\}^\ell$. The sender P_2 inputs a set $X = \{x_1, \ldots, x_{n_2}\}$ and $V = \{v_1, \ldots, v_{n_2}\}$, where $x_i \in \{0,1\}^\ell$ and $v_i \in \mathbb{Z}_p$. Let q be a big integer greater than $n_2 \cdot p$.
Protocol:

0. P_2 shuffles the set $(x_1, \ldots, x_{n_2})$ and $(v_1, \ldots, v_{n_2})$ according to the same random permutation over $[n_2]$. For simplicity, we still use the original notation to denote the vector after permutation.
1. P_1 (playing the role of server) with Y and P_2 (playing the role of client) with $X = \{x_1, \ldots, x_{n_2}\}$ invoke $\mathcal{F}_{\mathsf{mqRPMT}}$. P_1 obtains an indication bit vector $\vec{e} = (e_1, \ldots, e_{n_2})$. P_2 obtains nothing.
    - **cardinality:** P_1 learns the cardinality by calculating the Hamming weight of $\vec{e}$.
2. P_1 and P_2 invoke n_2 instances of OT via $\mathcal{F}_{\mathsf{OT}}$. P_1 uses $\vec{e}$ as the choice bits.
    - **intersection:** P_1 holding e_i and P_2 holding $(\perp, x_i)$ invoke one-sided OT n_2 times. P_1 learns $\{x_i \mid e_i = 1\}_{i \in [n_2]} = X \cap Y$.
    - **union:** P_1 holding e_i and P_2 holding $(x_i, \perp)$ invoke one-sided OT n_2 times. P_1 learns $\{x_i \mid e_i = 0\}_{i \in [n_2]} = X \backslash Y$, and outputs $\{X \backslash Y\} \cup Y = X \cup Y$.
    - **card-sum:** P_2 randomly picks $r_i \in \mathbb{Z}_q$ and computes $r' = \sum_{i=1}^{n_2} r_i \bmod q$. Subsequently, P_1 holding e_i and P_2 holding $(r_i, r_i + v_i)$ invoke 1-out-of-2 OT n_2 times. P_1 learns $S' = \{\sum_{i=1}^{n_2} v_i \mid e_i = 1\}_{i \in [n_2]} + \{\sum_{i=1}^{n_2} r_i\}_{i \in [n_2]} \bmod q$, then sends S' and the Hamming weight of $\vec{e}$ to P_2. P_2 computes $S = (S' - r') \bmod q$.
    - **card-secret-sharing:** P_2 randomly picks $r_i \in \mathbb{Z}_q$. Subsequently, P_1 holding e_i and P_2 holding $(r_i, r_i + x_i)$ invoke 1-out-of-2 OT n_2 times. P_1 learns $\{z_i\}_{i \in [n_2]}$. $\{(z_i, r_i)\}_{e_i = 1}$ constitutes the shares of intersection.

Fig. 9: PSO from mqRPMT

6 Applications of mqRPMT

6.1 PSO Framework from mqRPMT

We show how to build a PSO framework centering around mqRPMT in Figure 9.

We prove the security of the above PSO framework by the case of PSU. The security proof of other functionality is similar.

Theorem 4. *The PSU derived from the above framework described in Figure 9 is secure by assuming the semi-honest security of mqRPMT and OT.*

Due to space limit, we defer the security proof to Appendix D.4.

We compare our PSI-card-sum protocol with protocols [IKN+20, GMR+21] as below. As mentioned in the introduction part, the PSI-card-sum protocols presented in [IKN+20] are built from concrete primitives (e.g. DH-protocol, ROT-protocol, Phasing+OPPRF etc.) with general 2PC techniques or AHE schemes. This renders their protocols less general and efficient. The protocol presented in [GMR+21] is built from permuted characteristic (permuted mqRPMT under our terminology) and secret sharing. Our protocol is similar to their protocol but with the following differences. First, mqRPMT underlying our protocol is conceptually simpler than its permuted version. More importantly, mqRPMT admits instantiations with optimal linear complexity, while the current best instantiation of permuted mqRPMT requires superlinear complexity. Second, as we pointed out in the introduction part, the protocol due to [GMR+21] deviates from the standard functionality of PSI-card-sum. In contrast, our protocol meets the standard functionality of PSI-card-sum as defined in [IKN+20]. We do so by simply removing the constraint $\sum_{i=1}^{n} r_i = 0$ on the receiver side (as did in [GMR+21]), and having the sender send back the masked sum value to the receiver, and the receiver finally recovers the intersection sum by unmasking.

We also briefly discuss the differences between our card-secret-sharing protocol with related work. The most related functionality is circuit-PSI [HEK12, PSTY19, RS21]. The only difference between our card-secret-sharing and circuit-PSI is that our protocol additionally leaks the cardinality to the receiver. However, as pointed out by Garimella et al. [GMR+21], in many applications of interest, the functions that need to be computed indeed imply such leakage. Garimella et al. [GMR+21] also proposed a similar functionality named secret-shared intersection, in which the parties only get the sharing of intersection elements. As a result, their protocol leaks the cardinality to both the sender and the receiver.

6.2 Private-ID

Recently, Buddhavarapu et al. [BKM+20] proposed a two-party functionality called private-ID, which assigns two parties, each holding a set of items, a truly random identifier per item (where identical items receive the same identifier).

As a result, each party obtains identifiers to his own set, as well as identifiers associated with the union of their input sets. With private-ID, two parties can sort their private set with respect to a global set of identifiers, and then can proceed any desired private computation item by item, being assured that identical items are aligned. Buddhavarapu et al. [BKM$^+$20] also gave a concrete DDH-based private-ID protocol. Garimella et al. [GMR$^+$21] showed how to build private-ID from oblivious PRF and PSU. Roughly speaking, their approach proceeds in two phases. In phase 1, P_1 holding X and P_2 holding Y run an OPRF twice by switching the roles, so that first P_1 learns k_1 and P_2 learns $F_{k_1}(y_i)$, and second P_2 learns k_2 and P_1 learns $F_{k_2}(x_i)$. The random identifier of an item z is thus defined as $id_z = F_{k_1}(z) \oplus F_{k_2}(z)$. After phase 1, both parties can compute identifiers for their own items. In phase 2, they simply engage a PSU protocol on their sets $id(X)$ and $id(Y)$ to finish private-ID.

Our method is largely inspired by the approach presented in [GMR$^+$21]. We first observe that in phase 1, two parties essentially need to engage a distributed OPRF protocol, as we formally depict in Figure 10. The random identifier of an item z is defined as $G_{k_1,k_2}(z)$, where G is a PRF determined by key (k_1, k_2). Furthermore, note that $id(X)$ and $id(Y)$ are pseudorandom, which means in phase 2 a distributional PSU protocol suffices, whose semi-honest security is additionally defined on the input distribution. Looking ahead, such relaxation may lead to nice efficiency improvement.

In this work, we instantiate the generic private-ID construction as below: (1) realize the distributed OPRF protocol by running the multi-point OPRF [CM20] twice in reverse order; (2) run the PSU protocol from cwPRF-based mqRPMT with the obtained two sets of pseudorandom identifiers as inputs to fulfill the private-ID functionality.

Parameters: PRF $G : K \times D \rightarrow R$, where $K = K_1 \times K_2$.
Inputs: P_1 inputs a set $X = \{x_1, \ldots, x_{n_1}\}$, where $x_i \in D$. P_2 inputs a set $Y = \{y_1, \ldots, y_{n_2}\}$, where $y_i \in D$.
Output: P_1 gets $\{G_{k_1,k_2}(x_i)\}_{i\in[n_1]}$ and k_1. P_2 gets $\{G_{k_1,k_2}(y_i)\}_{i\in[n_2]}$ and k_2,

Fig. 10: Ideal functionality for distributed OPRF

Distributional PSU. Standard security notions for MPC are defined w.r.t. any private inputs. This treatment facilitates secure composition of different protocols. We find that in certain settings it is meaningful to consider a weaker security notion by allowing the real-ideal indistinguishability to also base on the distribution of private inputs. This is because such relaxed security suffices if the protocol's input is another protocol's output which obeys some distribution, and the relaxation may admit efficiency improvement. Suppose choosing the DDH-based distributed OPRF and DDH-based PSU in the same elliptic curve

(EC) group as ingredients, faithful implementation according to the above recipe requires $4n$ hash-to-point operations. Observe that the output of distributed DDH-based OPRF are already pseudorandom EC points. In this case, it suffices to use distributional DDH-based PSU instead, and thus can save $2n$ hash-to-point operations, which are costly in the real-world implementation.

7 Performance

We describe details of our implementation and report the performance of the following set operations: (1) **psi**: intersection of the sets; (2) **psi-card**: cardinality of the intersection; (3) **psi-card-sum**: sum of the associated values for every item in the intersection with cardinality; (4) **psu**: union of the sets; (5) **private-ID**: a universal identifier for every item in the union. We compare our work with the current fastest known protocol implementation for each functionality.

7.1 Implementation Details

Our protocols are written in C++ with detailed documentations, which is available upon request. In consistency with our paper, our implementation is organized in a modular and unified fashion: we first implement the core mqRPMT protocol, then build various PSO protocols upon it. Besides, it is only built upon the OpenSSL library [Opea], and can smoothly run on both Linux and x86_64 MacOS platforms.

7.2 Experimental Setup

We run all our protocols and related protocols on Ubuntu 20.04 with a single Intel i7-11700 2.50 GHz CPU (8 physical cores) and 16 GB RAM. We simulate the network connection using Linux `tc` command. For the WAN setting, we set the average RTT to be 80 ms and bandwidth to be 50 Mbps. We use `iptables` command to calculate the communication cost, and use running time to compute the computation complexity, which is the maximal time from the beginning to the end of protocol, including the messages transmission time.

For a fair comparison, we stick to the following setting for all protocols:

- We set the computational security parameter $\kappa = 128$ and the statistical security parameter $\lambda = 40$.
- We test the balanced scenario by setting the input set size $n_1 = n_2$ (our implementation supports unbalanced scenario as well), and randomly generate two input sets with 128 bits length item conditioned on the intersection size being roughly $0.5n$. The exception is the protocol in [GMR$^+$21], whose item length is set as 61 bits in default and cannot exceed 64 bits.
- The PSI-card-sum protocol [IKN$^+$20] and the private-id protocol [BKM$^+$20] are two of the related works we are going to compare. The former implementation is built upon `nist P-256` (also known as `secp256r1` and `prime256v1`),

while the latter implementation relies on special elliptic curve `Curve25519` realized in the highly-optimized Dalek library. For a fair and comprehensive comparison, we implement our protocols under both standard elliptic curve `nist P-256` and special elliptic curve `Curve25519`. For protocols based on standard elliptic curve, we denote the one not using or using point compression technique with $\blacklozenge$ and $\blacktriangledown$ respectively. For protocols based on `Curve25519`, we denote with $\bigstar$.

7.3 Evaluation of Our Core Protocol

We first report the performances of our core protocol cwPRF-based mqRPMT described in Section 4.3, which dominates the communication and computation overheads of its enabling PSO protocols. We test our protocol up to 4 threads, since both the server and the client run on a single CPU with 8 physical cores. Our cwPRF-based mqRPMT achieves optimal linear complexity, and thus is scalable, which is demonstrated by the experimental results in Table 2. Moreover, the computation tasks on both sides in our cwPRF-based mqRPMT are highly parallelable, thus we can effortlessly using OpenMP [Opeb] to make the program multi-threaded.

Table 2: The computation and communication complexity of mqRPMT.

Protocol	T	Running time (s)						Commu. (MB)		
		LAN			WAN			total		
		2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}
mqRPMT$^{\blacklozenge}$	1	0.50	7.20	114.16	1.39	9.68	136.27	0.52	8.35	133.6
	2	0.31	3.89	62.09	1.14	6.54	86.60			
	4	0.22	2.37	40.41	1.11	5.08	62.77			
Speedup		1.6-2.3×	1.9-3.0×	1.8-2.8×	1.2-1.3×	1.5-1.9×	1.6-2.2×	–	–	–
mqRPMT$^{\blacktriangledown}$	1	0.50	8.00	128.00	1.35	10.15	141.52	0.27	4.35	69.6
	2	0.32	5.05	80.69	1.18	7.11	94.19			
	4	0.23	3.54	58.40	1.08	5.54	71.26			
Speedup		1.6-2.2×	1.6-2.3×	1.6-2.2×	1.1-1.3×	1.4-1.8×	1.5-2×	–	–	–
mqRPMT$^{\bigstar}$	1	0.26	3.51	54.85	0.81	5.41	68.68	0.26	4.23	67.66
	2	0.15	1.79	28.24	0.75	3.83	41.38			
	4	0.10	1.07	15.32	0.72	3.09	28.31			
Speedup		1.7-2.6×	2.0-3.3×	1.9-3.6×	1.1-1.1×	1.4-1.8×	1.7-2.4×	–	–	–

7.4 Benchmark Comparison

We derive all kinds of PSO protocols from cwPRF-based mqRPMT protocol, and compare them with the state-of-the-art related protocols. We report the performances of all protocols on 3 input sizes $n = \{2^{12}, 2^{16}, 2^{20}\}$, executed over a single thread for LAN and WAN configurations. When testing the PSI-card, PSI-card-sum and PSU protocols in [GMR$^+$21], we set the number of mega-bins as $\{1305, 16130, 210255\}$ and the number of items in each mega-bin as $\{51, 62, 72\}$ for set sizes $n = \{2^{12}, 2^{16}, 2^{20}\}$ respectively. These parameter choices have been tested to be much more optimal than their default ones.

PSI. We compare our mqRPMT-based PSI protocol to the classical DH-PSI protocol implemented by [PRTY19] and re-implemented by ourselves. We remark that all PSI protocols in comparison are not competitive with the state-of-the-art PSI protocol. We include them merely for illustrative purposes. PSI protocols from public-key techniques are always thought to be very inefficient, but our experiment demonstrates that by carefully choosing modern crypto library with optimized parameters they could be pretty practical. Our mqRPMT-based PSI protocol is more than an order of magnitude faster than the DH-PSI protocol[2] implemented in [PRTY19]. By leveraging the features of `Curve25519` in important ways (see Section 7.5 in details), our re-implemented DH-PSI protocol (denoted by DH-PSI$^\bigstar$) achieves 26× speedup, which is arguably the most efficient implementation known to date.

Table 3: Communication cost and running time of PSI protocol.

PSI	Running time (s)						Comm. (MB)		
	LAN			WAN			total		
	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}
[PRTY19]$^\bigstar$	5.51	88.64	1418.20	5.82	90.79	1498.67	0.30	4.74	76.60
Our PSI$^\blacklozenge$	**0.50**	**7.24**	**114.66**	**1.71**	**10.50**	**142.45**	**0.67**	**10.38**	**165.77**
Our PSI$^\blacktriangledown$	**0.55**	**8.04**	**128.18**	**1.73**	**11.02**	**148.18**	**0.41**	**6.38**	**101.63**
Our PSI$^\bigstar$	**0.29**	**3.56**	**55.11**	**1.19**	**6.38**	**75.56**	**0.40**	**6.25**	**99.71**
DH-PSI$^\bigstar$	**0.22**	**3.39**	**54.79**	**0.92**	**5.57**	**69.31**	**0.28**	**4.57**	**74.1**

Recently, Rosulek and Trieu [RT21] propose a PSI protocol based on Diffie-Hellman key agreement, which requires the least time and communication of any known PSI protocols for small sets. Somewhat surprisingly, Table 4 shows that for small sets our mqRPMT-based PSI protocol is faster than their protocol in LAN setting, and our re-implemented DH-PSI is faster than their protocol in all settings.

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[2] We remark that except inefficiency, their implementation also has a severe security issue. More precisely, they realize the hash-to-point function $\{0,1\}^* \to \mathbb{G}$ as $x \mapsto g^{\mathsf{H}(x)}$, where H is some cryptographic hash function. However, such hash-to-point function cannot be modeled as random oracle anymore since it exposes the algebra structure of output in the clear, and hence totally compromise security. Similar issue also appears in `libPSI`.

Table 4: Communication cost and running time of PSI protocol on small sets.

PSI	Running time (ms)						Comm. (KB)		
	LAN			WAN			total		
	2^8	2^9	2^{10}	2^8	2^9	2^{10}	2^8	2^9	2^{10}
[RT21]$^{\bigstar}$	50.0	71.0	147.3	224.1	260.2	457.9	17.9	34.1	66.3
Our PSI$^{\bigstar}$	**41.9**	**69.5**	**99.3**	**577.0**	**582.9**	**646.1**	**38.6**	**63.5**	**113.3**
DH-PSI$^{\bigstar}$	**16.49**	**31.80**	**56.91**	**210.42**	**227.33**	**252.32**	**18.48**	**36.68**	**72.8**

PSI-card. We compare our mqRPMT-based PSI-card protocol to the PSI-card protocol in [GMR$^+$21]. Table 5 shows that our protocol achieves a $2.4 - 10.5\times$ speedup in running time, and reduces the communication cost by a factor $10.9 - 14.8\times$.

Table 5: Communication cost and running time of PSI-card protocol.

PSI-card	Running time (s)						Comm. (MB)		
	LAN			WAN			total		
	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}
[GMR$^+$21]	1.00	8.41	126.01	8.60	27.46	323.52	2.93	55.49	1030
Our PSI-card$^{\blacklozenge}$	**0.49**	**7.20**	**114.31**	**1.30**	**9.68**	**136.06**	**0.52**	**8.36**	**133.71**
Our PSI-card$^{\blacktriangledown}$	**0.53**	**8.00**	**128.00**	**1.35**	**10.16**	**141.31**	**0.27**	**4.35**	**69.6**
Our PSI-card$^{\bigstar}$	**0.27**	**3.51**	**54.89**	**0.82**	**5.42**	**68.31**	**0.26**	**4.23**	**67.70**

PSI-card-sum. We compare our mqRPMT-based PSI-card-sum protocol to the PSI-card-sum protocol (the most efficient and the deployed one based on DH-protocol+Paillier) in [IKN$^+$20]. We do not compare the protocol described in [GMR$^+$21] since its functionality is not the standard one, as we discussed in the introduction. Our protocol is more advantageous than the protocol of [GMR$^+$21] due to our random masking trick is much simpler and efficient than the AHE-based technique. Particularly, the upper bound of intersection sum in [GMR$^+$21] is closely tied to the AHE scheme in use, which requires sophisticated parameter tuning and ciphertext packing techniques. In our protocol, the upper bound of intersection sum can be flexibly adjusted according to applications. Table 6 shows that, compared with the protocol in [IKN$^+$20], our protocol is roughly $28.5 - 76.3\times$ faster and reduces the communication cost by a factor $7.4\times$.

PSU. We compare our PSU protocol to recently emerging PSU protocols in [GMR$^+$21, JSZ$^+$22, ZCL$^+$23]. The work [JSZ$^+$22] provides two PSU protocols

Table 6: Communication cost and running time of PSI-card-sum protocol.

PSI-card-sum	Running time (s)						Comm. (MB)		
	LAN			WAN			total		
	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}
[IKN+20]▼ (deployed)	23.64	176.34	–	30.10	186.29	–	2.72	43.24	–
Our PSI-card-sum♦	**0.51**	**7.22**	**113.66**	**1.46**	**9.68**	**136.27**	**0.64**	**9.89**	**157.80**
Our PSI-card-sum▼	**0.57**	**8.12**	**129.66**	**1.94**	**11.83**	**157.66**	**0.38**	**5.87**	**93.74**
Our PSI-card-sum★	**0.31**	**3.73**	**57.44**	**1.36**	**6.53**	**76.16**	**0.37**	**5.75**	**91.70**

Communication cost and running time of PSI-card-sum protocol. We assume each associated value is a non-negative integer in $[0, 2^{32})$ conditioned on the upper bound of intersection sum being 2^{32}. We note that the implementation of [IKN+20] only works in our environment at set sizes 2^{12} and 2^{16}. For set size 2^{20}, we encounter a run time error reported in [Pri], which has not been fixed yet. The corresponding cells are marked with "–".

called PSU-R and PSU-S. The work [ZCL+23] also provides two PSU protocols from public-key and symmetric-key respectively. We choose the most efficient PKE-PSU [ZCL+23] and PSU-S [JSZ+22] for comparison. Among all the mentioned PSU protocols, only the PSU protocols in [ZCL+23] and our PSU protocol achieve linear communication and computation complexity. The experimental results in Table 7 indicate that our PSU protocol is the most superior one. Comparing to the state-of-the-art PSU protocol of [ZCL+23], our protocol is 2× smaller in terms of communication cost, and thus achieves a $2.7 - 17\times$ speedup.

Table 7: Communication cost and running time of PSU protocol.

PSU	Running time (s)						Comm. (MB)		
	LAN			WAN			total		
	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}
[GMR+21]	1.16	10.06	151.34	10.34	38.52	349.43	3.85	67.38	1155
[ZCL+23]♦	4.87	12.19	141.38	5.78	15.75	182.88	1.35	21.41	342.38
[ZCL+23]▼	5.10	15.13	187.29	5.82	17.37	210.06	0.77	12.20	195.17
[JSZ+22]	2.29	8.50	516.04	5.33	27.00	736.30	3.59	70.37	1341.55
Our PSU♦	**0.52**	**7.27**	**114.44**	**1.70**	**10.56**	**143.29**	**0.68**	**10.38**	**165.77**
Our PSU▼	**0.57**	**8.04**	**128.20**	**1.76**	**10.92**	**148.15**	**0.41**	**6.38**	**101.63**
Our PSU★	**0.30**	**3.55**	**55.48**	**1.19**	**6.38**	**74.96**	**0.40**	**6.25**	**99.71**

Private-ID. We compare our concrete private-ID protocol described in Section 6.2 to the state-of-the-art protocols in [BKM+20, GMR+21]. The experimental results in Table 8 show that our private-ID protocol achieves a $2.7-4.9\times$ speedup comparing to the existing most computation efficient private-ID protocol of [GMR$^+$21], while its bandwidth is only marginally larger than the most communication efficient private-ID protocol [BKM$^+$20]. Thereby, our protocol is arguably the most efficient one in terms of monetary cost. By instantiating the distributed OPRF from the current best OPRF due to [RR22], we will obtain a private-ID protocol with better performance.

Table 8: Communication cost and running time of private-ID protocol.

Private-ID	Running time (s)						Comm. (MB)		
	LAN			WAN			total		
	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}
[GMR$^+$21]	1.65	11.023	158.76	13.82	43.00	385.12	4.43	76.57	1293
[BKM$^+$20]$^{\bigstar}$	2.21	37.56	671.75	7.98	46.97	710.94	1.00	15.97	226.70
Our Private-ID$^{\blacklozenge}$	**0.77**	**8.40**	**114.45**	**2.91**	**13.62**	**148.48**	**1.46**	**20.52**	**330.40**
Our Private-ID$^{\blacktriangledown}$	**0.89**	**9.57**	**146.73**	**3.07**	**14.53**	**186.98**	**1.13**	**16.43**	**266.31**
Our Private-ID$^{\bigstar}$	**0.61**	**5.11**	**71.57**	**2.83**	**10.06**	**114.66**	**1.13**	**16.31**	**265.15**

7.5 Tips For ECC-based Implementations

In what follows, we summarize the lessons we learned during the implementation of ECC-based protocols, with the hope to uncover some dark details and correct imprecise impressions.

We first highlight the following two caveats when implementing with standard elliptic curves:

Pros and cons of point compression technique. Point compression is a standard trick in elliptic-curve cryptography (ECC), which can roughly reduce the storage cost of EC point by half, at the cost of performing decompression when needed. Point decompression was empirically thought to be cheap, but experiment indicates that it could be as expensive as point multiplication. Our perspective is that point compression offers a natural trade-offs between communication and computation. The above experimental results demonstrate that the total running time gives a large weight to communication cost in bandwidth constrained scenarios. Therefore, in the LAN setting (involving parties are co-located), we recommend not to apply point compression trick, while in the WAN setting (involving parties cannot be co-located), we recommend to apply point compression trick. A quick take-away is that point compression trick pays off in setting where communication is much more expensive than computation.

Tricky hash-to-point operation. The hash to point operation is very tricky in ECC. So far, there is no universal method to securely map arbitrary bit strings to points on elliptic curves. Here, the vague term "securely" indicates the hash function could be modeled as a random oracle. A folklore method is the "try-and-increment" algorithm [BLS01], which is also the method adopted in this work. Nevertheless, even such simple hash-to-point operation could be as expensive as point multiplication, which should be avoid if possible.

Regarding to the two caveats discussed above, the following questions arise: (1) is it possible to get the best of two worlds of point compression? (2) could the hash-to-point operation be cheaper? Luckily, the answers are both yes under some circumstance.

With the aim to avoid many potential implementation pitfalls, in 2005 Bernstein [Ber06] designed an elliptic curve dedicated to ECDH function known as `Curve25519`. Due to its many efficiency/security advantages, it has been widely deployed in numerous applications and has become the *de facto* alternative to `NIST P-256`. Here, we highlight its two nice features that are particularly beneficial for our cwPRF-based mqRPMT protocol: (i) it allows efficient scalar multiplication in compressed form (only X coordinates); (ii) by design, any 32-byte bit string (interpreting as X coordinate) can be ambiguously identified as a valid point on curve. Feature (i) brings us the best of two worlds of point compression, without making trade-off anymore, while feature (ii) makes the hash-to-point operation almost free, by simply hashing the input to a 32-byte bit string via cryptographic hash function. Naturally, `Curve25519` has deficiencies coming with its nice features. All the known implementations of `Curve25519` that support efficient scalar multiplication in X-coordinate compressed form do not provide interfaces for point addition, subtraction, and scalar inverse multiplication. The reason is that (a) point addition and subtraction operations cannot be performed using only X coordinates, thus in turn requiring expensive decompression operation; (b) giving any 32-byte integer value as the scalar, existing implementations would automatically "clamp" it before scalar multiplication, thus requiring complicated treatment to support scalar inverse multiplication.

Luckily, our cwPRF-based mqRPMT protocol only requires scalar multiplication and hash-to-point operations, and thus can enjoy the nice features without being affected by the deficiencies. This explains the advantages our cwPRF-based mqRPMT protocol based on `Curve25519` over that based on `NIST P-256`. To the best of our knowledge, this is also the first time that `Curve25519` fully unleashes its advantages in the area of private set operations. Prior to this work, Rosulek and Tireu [RT21] employed `Curve25519` to build a PSI protocol from Diffie-Hellman key agreement (DHKA) with strongly uniform property [FMV19], whose instantiation inherently requires the elligator encoding/decoding mechanism [BHKL13]. The optimizations originated from feature (i) and (ii) does not apply to their construction because it requires encoding/decoding EC points to bit strings (thus points cannot only be represented by X coordinates), rather than hashing elements to EC points. In summary, for protocols that are not involved with point addition/subtraction and scalar inverse multiplication, `Curve25519` would be a good choice.

Public-key operations are always rashly thought to be much expensive than symmetric-key operations, and thus the design philosophy of many practical protocols opts to avoid public-key operations. Our experimental results demonstrates this impression is not precise anymore after rapid advances on ECC-based cryptography in recent years. By leveraging optimized implementation, public-key operations could be as efficient as symmetric-key operations. As a concrete example, in EC group with 128 bit security level, one EC point scalar operation takes 0.026 ms and one EC point addition takes 0.00028 ms on a laptop.

8 Summary

In this work, we show that mqRPMT protocol is complete for most private set operations. By coupling with OT, we create a unified PSO framework from mqRPMT, which can greatly reduce the deployment and maintaining costs of PSO in the real world. We build the core mqRPMT protocol from two newly introduced cryptographic primitives, namely cwPRF and pOPRF respectively. By instantiating cwPRF and pOPRF from DDH-like assumptions, we obtain mqRPMT protocols with linear complexity. The significance of this result is two folds. The first is of practical interest, namely providing a simple PSO framework. Particularly, we view the simplicity as a great merit since it yields a family of PSO protocols that are competitive or superior to existing ones. The second is of more theoretical interest, namely introducing cwPRF and pOPRF. The notion of cwPRF can be viewed as the right cryptographic abstraction of the celebrated DH functions, which not only demonstrates that the DDH assumption is complete for PSO, but also opens the door for possible new instantiations beyond DDH-like assumptions. The notion of pOPRF is of independent interest. It enriches the OPRF family, and help us to understand which OPRF-based PSI protocols can (or cannot) be adapted to PCSI/PSU protocols. We left more applications and efficient constructions of pOPRF as an interesting problem.

In addition, we present a semi-generic conversion from a category mqPMT protocols called Sigma-mqPMT to mqRPMT, making the first step towards investigating relations between the two core protocols. As an application of such conversion, we obtain a mqRPMT protocol from FHE which is suitable in the unbalanced setting. However, the resulting mqRPMT is a slightly weak version in the sense that the intersection size is leaked to the sender. We left the construction of standard mqRPMT in the unbalanced setting as an open problem.

To demonstrate the efficiency of our framework, we opensource our C++ implementation with detailed documentations. When conducting performance comparison, we find that quite a few PSO implementations suffer from one or more of the following deficiencies: (i) rely on multiple libraries, but configurations are not well documented; (ii) require sophisticated parameters tuning, but optimized parameters are not explicitly given; (iii) codes are not faithful to protocols described in paper, such as insecure random oracle instantiation, incorrect thread number counting etc. Sometimes, even making these programs successfully running would require tremendous efforts. We are thus expect a high-quality MPC platform that admits easy and fair benchmarking of all PSO protocols.

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Supplementary Materials

A mqRPMT from Sigma-mqPMT

A.1 Sigma-mqPMT

Private membership test (PMT) protocol [PSZ14] is a two-party protocol in which the client with input x learns whether or not its item is in the input set Y of the server. PMT can be viewed as a special case of private keyword search protocol [FIPR05] by setting the payload as any indication string. We consider three-move PMT, which we refer to Sigma-PMT hereafter.

Sigma-PMT proceeds via the following pattern.

1. Server P_1 sends the first round message a to client P_2, which is best interpreted as an encoding of Y.
2. Client P_2 sends query q w.r.t. to his item x.
3. Server P_1 responds with z.

After receiving z, client P_2 can decide if $x \in Y$ by running $\mathsf{Test}(a, x, q, z)$. The basic notion of Sigma-PMT allows the client P_2 to test for a single item. While this procedure can be repeated several times, one may seek for more efficient protocol allowing the client to test n items at reduced communication cost and round complexity. To this end, we introduce the following two properties for Sigma-PMT:

- **Reusable:** The first round message is performed by the server P_1 once and for all.
- **Context-independent:** Each test query q_i is only related to the element x_i under test and the randomness of P_2.

The first property helps to reduce communication cost, while the second property admits parallelization, hence the round complexity is unchanged even when handling multiple items. Sigma-PMT may enjoy an additional property:

- **Stateless:** For any x_i and associated (q_i, z_i), $\mathsf{Test}(a, x_i, q_i, z_i)$ can work in a memoryless way, namely, without looking at (x_i, q_i). In this case, the test algorithm can be simplified as $\mathsf{Test}(a, z_i)$.

By running Sigma-PMT with reusable, context-independent, and stateless properties in parallel, we obtain mqPMT with three-move pattern (depicted in Figure 11), which we refer to as Sigma-mqPMT.

To reduce the semi-honest security of mqRPMT* to that of Sigma-mqPMT, we assume the simulator $\mathsf{Sim}(X, \vec{e})$ for client P_2 is composed of two sub-routines $(\mathsf{Sim}', \mathsf{Sim}'')$, and satisfies the following properties:

- **Locality:** $z_i \approx \mathsf{Sim}'(e_i; r_i)$, a.k.a. the i-th response can be emulated via invoking a sub-routine $\mathsf{Sim}'(e_i)$ with independent random coins r_i;
- **Order invariance:** $a \approx \mathsf{Sim}''(\{e_{\pi(i)}, r_{\pi(i)}\}_{i \in [n_2]}; s)$, where π could be an arbitrary permutation over $[n_2]$, s is the random coins.

$$\begin{array}{ccc} P_1 \text{ (server)} & & P_2 \text{ (client)} \\ Y = (y_1, \ldots, y_{n_1}) & & X = (x_1, \ldots, x_{n_2}) \\ a \leftarrow \mathsf{Encode}(Y) & \xrightarrow{\qquad\qquad a \qquad\qquad} & \\ & \xleftarrow{\quad \vec{q} = \{q_1, \ldots, q_{n_2}\} \quad} & q_i \leftarrow \mathsf{GenQuery}(a, x_i) \\ z_i \leftarrow \mathsf{Response}(q_i) & \xrightarrow{\quad \vec{z} = \{z_1, \ldots, z_{n_2}\} \quad} & e_i \leftarrow \mathsf{Test}(a, z_i) \end{array}$$

Fig. 11: Sigma-mqPMT

A.2 Instantiations of Sigma-mqPMT

A.2.1 Sigma-mqPMT from DDH

We first present an instantiation of Sigma-mqPMT based on the DDH assumption, which is obtained by plugging DDH-based OPRF to the above generic construction.

$$\begin{array}{ccc} P_1 \text{ (server)} & & P_2 \text{ (client)} \\ Y = (y_1, \ldots, y_n) & & X = (x_1, \ldots, x_n) \\ k \overset{\text{R}}{\leftarrow} \mathbb{Z}_p & \xrightarrow{\quad a \leftarrow \{\mathsf{H}(y_1)^k, \ldots, \mathsf{H}(y_{n_1})^k\} \quad} & \\ & \xleftarrow{\quad \vec{q} = \{q_1, \ldots, q_{n_2}\} \quad} & r \overset{\text{R}}{\leftarrow} \mathbb{Z}_p,\ q_i \leftarrow \mathsf{H}(x_i)^r \\ z_i \leftarrow (\mathsf{H}(x_i)^r)^k & \xrightarrow{\quad \vec{z} = \{z_1, \ldots, z_{n_2}\} \quad} & e_i := z_i \in a \end{array}$$

A.2.2 Sigma-mqPMT from FHE

We then present an instantiation of Sigma-mqPMT based on oblivious polynomial evaluation (OPE). By instantiating OPE from FHE, we obtain the following mqPMT protocol, which is the backbone of [CLR17].

$$\begin{array}{ccc}
\begin{array}{c} P_1 \text{ (server)} \\ Y = (y_1, \ldots, y_{n_1}) \end{array} & & \begin{array}{c} P_2 \text{ (client)} \\ X = (x_1, \ldots, x_{n_2}) \end{array} \\
a \leftarrow \perp & \xrightarrow[]{\perp} & \\
& \xleftarrow{\vec{q} = \{q_1, \ldots, q_{n_2}\}} & q_i \leftarrow \mathsf{FHE.Enc}(pk, x_i) \\
\begin{array}{c} r_i \overset{\text{R}}{\leftarrow} \mathbb{F} \\ f_i \leftarrow r_i \prod_{y \in Y}(y_i - x) \\ z_i \leftarrow \mathsf{FHE.Eval}(pk, f_i, q_i) \end{array} & \xrightarrow{\vec{z} = \{z_1, \ldots, z_{n_2}\}} & e_i := \mathsf{FHE.Dec}(dk, z_i) \overset{?}{=} 0
\end{array}$$

Alternatively, we can realize OPE from additively homomorphic encryption. The change is that each q_i now consists of n_1 ciphertexts of the following form: $\{\mathsf{AHE.Enc}(pk, x_i^1), \ldots, \mathsf{AHE.Enc}(pk, x_i^{n_1})\}$.

Remark 4. As noted in [CLR17], the above protocol only serves as a toy example to illustrate the idea of how to using FHE to build PSI, which is impractical. They also show how to make the basic protocol efficient. However, the optimizing techniques destroy structure and properties of Sigma-mqPMT. As a consequence, so far the transformation from Sigma-mqPMT to mqRPMT* does not have efficient instantiation in the unbalanced setting, and only serves as a proof of concept.

A.3 Connection to Sigma-mqPMT

Next, we show a generic construction of mqRPMT* from Sigma-mqPMT. With the nice properties of Sigma-mqPMT, the construction is pretty simple, a.k.a. having the server P_1 shuffle the last move message in Sigma-mqPMT (yielding permuted mqPMT upon this step), then having the client P_2 send the test results back to P_1, and finally P_1 recovers the indication bits in the right order. We formally describe the construction in Figure 12.

Theorem 5. *The above mqRPMT* protocol depicted in Figure 12 is secure in the semi-honest model assuming the semi-honest security of the starting Sigma-mqPMT protocol.*

Proof. We exhibit simulators Sim_{P_1} and Sim_{P_2} for simulating corrupt server P_1 and corrupt client P_2 respectively. Let $|X \cap Y| = m$.

Security against corrupt client. Sim_{P_2} simulates the view of corrupt client P_2, which consists of P_2's randomness, input, output and received messages.

We argue that the output of Sim_{P_2} is indistinguishable from the real execution. We formally show Sim_{P_2}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid_0: P_2's view in the real protocol.

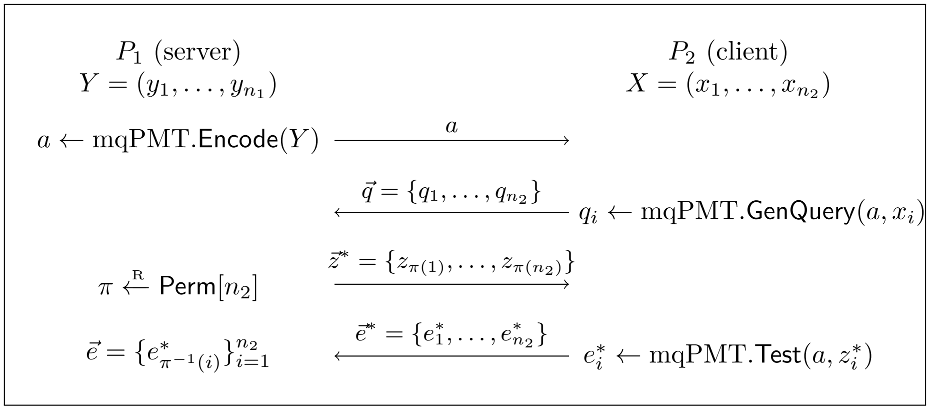

Fig. 12: mqRPMT* from Sigma-mqPMT

Hybrid$_1$: Sim_{P_2} chooses the randomness for P_1, and simulates with the knowledge of Y. Clearly, Sim_{P_2}'s simulation is identical to the real view of P_2.
Hybrid$_2$: Sim_{P_2} does not choose the randomness for P_1, and simulates without the knowledge of Y. Instead, it invokes the Sigma-mqPMT's simulator for P_2 on his private input X and output $\vec{e}^*$ to emulate the view $(a, \vec{z}^*)$ in the following manner:

- for $1 \leq i \leq n_2$, run $\mathsf{Sim}'(e^*_i; r_i) \to z^*_i$, obtaining $\vec{z}^* = (z^*_1, \ldots, z^*_n)$.
- run $\mathsf{Sim}''(\{(e^*_i, r_i)\}_{i \in [n_2]}; s) \to a$.

By the *locality* and *order invariance* properties, the simulated view in Hybrid$_2$ and Hybrid$_1$ are computationally indistinguishable based on semi-honest security of mqPMT on P_2 side.
Security against corrupt server. Sim_{P_1} simulates the view of corrupt server P_1, which consists of P_1's randomness, input, output and received messages. We formally show Sim_{P_1}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid$_0$: P_1's view in the real protocol.
Hybrid$_1$: Sim_{P_1} chooses the randomness for P_2, and simulates with the knowledge of X. Clearly, Sim_{P_1}'s simulation is identical to the real view of P_1.
Hybrid$_2$: Sim_{P_1} does not choose the randomness for P_2, and simulates without the knowledge of X. Instead, given $(Y, \vec{e})$ it first invokes the Sigma-mqPMT's simulator for P_1 on input Y to generate $\vec{q}$, then picks a random permutation π over $[n_2]$ and computes $\vec{e}^* = \pi^{-1}(\vec{e})$, outputs $(\vec{q}, \vec{e}^*)$.
Clearly, the view in Hybrid$_1$ and Hybrid$_2$ are computationally indistinguishable based on the semi-honest security of Sigma-mqPMT on P_1's side.

This proves the theorem. $\square$

Remark 5. As a byproduct, we note that if P_1 only permutes and sends the last move message in Sigma-mqPMT, then we obtain a standard PSI-card protocol.

From this perspective, it is fair to say Sigma-mqPMT distills sufficient characteristics of what kind of PSI protocols can be converted to PSI-card with no extra overhead.

B Pseudorandom Function

In this section, we recap the standard notions of PRF, as well as the canonical construction from the DDH like assumption. Looking ahead, we will build more advanced variants of PRF with richer properties on these basis. We first recall the notion of standard pseudorandom functions (PRFs) [GGM86].

Definition 3 (PRF). *A family of PRFs consists of three polynomial-time algorithms as follows:*

- $\mathsf{Setup}(1^\kappa)$*: on input a security parameter* κ*, outputs public parameter* pp*.* pp *specifies a family of keyed functions* $F : K \times D \to R$*, where* K *is the key space,* D *is domain, and* R *is range.*
- $\mathsf{KeyGen}(pp)$*: on input* pp*, outputs a secret key* $k \xleftarrow{\text{R}} K$*.*
- $\mathsf{Eval}(k, x)$*: on input* $k \in K$ *and* $x \in D$*, outputs* $y \leftarrow F(k, x)$*. For notation convenience, we will write* $F(k, x)$ *as* $F_k(x)$ *interchangeably.*

The standard security requirement for PRF is pseudorandomness.

Pseudorandomness. *Let* $\mathcal{A}$ *be an adversary against PRF and define its advantage as:*

$$\mathsf{Adv}_{\mathcal{A}}(\kappa) = \Pr\left[\beta' = \beta : \begin{array}{l} pp \leftarrow \mathsf{Setup}(1^\kappa); \\ k \leftarrow \mathsf{KeyGen}(pp); \\ \beta \leftarrow \{0,1\}; \\ \beta' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{ror}}(\beta,\cdot)}(\kappa); \end{array}\right] - \frac{1}{2},$$

where $\mathcal{O}_{\mathsf{ror}}(\beta, \cdot)$ *denotes the real-or-random oracle controlled by* β*, i.e.,* $\mathcal{O}_{\mathsf{ror}}(0, x) = F_k(x)$*,* $\mathcal{O}_{\mathsf{ror}}(1, x) = \mathsf{H}(x)$ *(here* H *is chosen uniformly at random from all the functions from* D *to* R[3]*).* $\mathcal{A}$ *can adaptively access the oracle* $\mathcal{O}_{\mathsf{ror}}(\beta, \cdot)$ *polynomial many times. We say that* F *is pseudorandom if for any PPT adversary* $\mathsf{Adv}_{\mathcal{A}}(\kappa)$ *is negligible in* κ*. We refer to such security as full PRF security.*

Sometimes the full PRF security is not needed and it is sufficient if the function cannot be distinguished from a uniform random one when challenged on random inputs. The formalization of such relaxed requirement is *weak pseudorandomness*, which is defined the same way as pseudorandomness except that the inputs of oracle $\mathcal{O}_{\mathsf{ror}}(b, \cdot)$ are uniformly chosen from D by the challenger instead of adversarially chosen by $\mathcal{A}$. PRF that satisfy weak pseudorandomness are referred to as *weak PRF*.

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[3] To efficiently simulate access to a uniformly random function H from D to R, one may think of a process in which the adversary's queries to $\mathcal{O}_{\mathsf{ror}}(1, \cdot)$ are "lazily" answered with independently and randomly chosen elements in R, while keeping track of the answers so that queries made repeatedly are answered consistently.

B.1 Weak PRF from the DDH Assumption

We recall the folklore weak PRF from the DDH assumption as below.

- $\mathsf{Setup}(1^\kappa)$: runs $\mathsf{GroupGen}(1^\kappa) \to (\mathbb{G}, g, p)$, outputs $pp = (\mathbb{G}, g, p)$. pp defines a family of functions from $\mathbb{Z}_p \times \mathbb{G}$ to $\mathbb{G}$, a.k.a. on input $k \in \mathbb{Z}_p$ and $x \in \mathbb{G}$ outputs $F_k(x) = x^k$.
- $\mathsf{KeyGen}(pp)$: outputs $k \xleftarrow{\text{R}} \mathbb{Z}_p$.
- $\mathsf{Eval}(k, x)$: on input $k \in \mathbb{Z}_p$ and $x \in D$, outputs $y \leftarrow x^k$.

The following theorem establishes its pseudorandomness based on the DDH assumption.

Theorem 6. *$F_k(x)$ is a family of weak pseudorandom functions assuming the hardness the DDH assumption holds w.r.t.* $\mathsf{GroupGen}(1^\kappa) \to (\mathbb{G}, g, p)$.

Proof. DDH assumption states that DDH tuple (g^a, g^b, g^{ab}) and random tuple (g^a, g^b, g^c) are computationally indistinguishable. By exploiting the random self-reducibility of the DDH problem [NR95], the standard DDH assumption implies that $(g^a, g^{b_1}, \ldots, g^{b_n}, g^{ab_1}, \ldots, g^{ab_n})$ and $(g^a, g^{b_1}, \ldots, g^{b_n}, g^{c_1}, \ldots, g^{c_n})$ are computationally indistinguishable, where $a, b_i, c_i \xleftarrow{\text{R}} \mathbb{Z}_p$. We are now ready to reduce the weak pseudorandomness of $F_k(\cdot)$ based on the DDH assumption. Let $\mathcal{B}$ be an adversary against the DDH assumption. Given a DDH challenge instance $(g^a, g^{b_1}, \ldots, g^{b_n}, g^{c_1}, \ldots, g^{c_n})$, $\mathcal{B}$ interacts with an adversary $\mathcal{A}$ in the weak pseudorandomness experiment, with the aim to determine if $c_i = ab_i$ or c_i is a random value.

Setup: $\mathcal{B}$ sends $pp = (\mathbb{G}, g, p)$ to $\mathcal{A}$. $\mathcal{B}$ implicitly sets a as the key of PRF.
Real-or-random query: Upon receiving the i-th query to oracle $\mathcal{O}_{\mathsf{ror}}$, $\mathcal{B}$ sets the i-th random input $x_i := g^{b_i}$, computes $y_i = g^{c_i}$, then sends (x_i, y_i) to $\mathcal{A}$.
Guess: $\mathcal{A}$ makes a guess $\beta' \in \{0, 1\}$ for β, where '0' indicates real mode and '1' indicates random mode. $\mathcal{B}$ forwards β' to its own challenger.

Clearly, if $c_i = ab_i$ for all $i \in [n]$, then $\mathcal{A}$ simulates the real oracle. If c_i are random values, then $\mathcal{A}$ simulates the random oracle. Thereby, $\mathcal{B}$ breaks the DDH assumption with the same advantage as $\mathcal{A}$ breaks the pseduorandomness of $F_k(\cdot)$.

Remark 6. We note that $F_k(x) = x^k$ is actually a permutation over $\mathbb{G}$, and it is efficiently invertible.

B.2 PRF from the DDH Assumption

We next recall the standard PRF from the DDH assumption known as HashDH presented in [NPR99]. The construction is very similar to the weak PRF construction. The only modification is to map the input to $\mathbb{G}$ via a cryptographic hash function H first, then apply F_k in a cascade way, yielding a composite function $F_k \circ \mathsf{H} : D \to \mathbb{G}$. By leveraging the programmability of H, we reduce to pseudorandomness of the composite function $F_k \circ \mathsf{H}$ to the weak pseudorandomness of F_k. In other words, random oracle amplifies weak pseudorandomness to standard pseudorandomness.

For completeness, we provide the details as below.

- $\mathsf{Setup}(1^\kappa)$: runs $\mathsf{GroupGen}(1^\kappa) \to (\mathbb{G}, g, p)$, picks a cryptographic hash function H from domain D to $\mathbb{G}$, outputs $pp = (\mathbb{G}, g, p, \mathsf{H})$. pp defines a family of functions from $\mathbb{Z}_p \times D$ to $\mathbb{G}$, which takes $k \in \mathbb{Z}_p$ and $x \in D$ as inputs and outputs $F_k(\mathsf{H}(x)) = \mathsf{H}(x)^k$.
- $\mathsf{KeyGen}(pp)$: outputs $k \xleftarrow{\text{R}} \mathbb{Z}_p$.
- $\mathsf{Eval}(k, x)$: on input $k \in \mathbb{Z}_p$ and $x \in D$, outputs $\mathsf{H}(x)^k$.

The following theorem establishes its pseudorandomness based on the DDH assumption.

Theorem 7. *$F_k(\mathsf{H}(x))$ is a family of PRF assuming H is a random oracle and the DDH assumption holds w.r.t. $\mathsf{GroupGen}(1^\kappa) \to (\mathbb{G}, g, p)$.*

Proof. We now reduce the pseudorandomness of $F_k(\mathsf{H}(\cdot))$ to the hardness of DDH problem. Let $\mathcal{B}$ be an adversary against the DDH problem. Given a DDH challenge instance $(g^a, g^{b_1}, \dots, g^{b_n}, g^{c_1}, \dots, g^{c_n})$, $\mathcal{B}$ interacts with an adversary $\mathcal{A}$ in the pseudorandomness experiment, with the aim to determine if $c_i = ab_i$ or c_i is a random value. $\mathcal{B}$ simulates the random oracle H and real-or-random oracle as below:

- Setup: $\mathcal{B}$ sends $pp = (\mathbb{G}, g, p, \mathsf{H})$ to $\mathcal{A}$, and implicitly sets a as the key of PRF.
- Random oracle query: for random oracle query $\langle x_i \rangle$, $\mathcal{B}$ programs $\mathsf{H}(x_i) := g^{b_i}$.
- Real-or-random query: without loss of generality, it is safe to assume adversary has already made the corresponding random oracle (RO) queries before making the evaluation queries. For evaluation query $\langle x_i \rangle$, $\mathcal{B}$ returns $y_i := g^{c_i}$ to $\mathcal{A}$.
- Guess: $\mathcal{A}$ makes a guess $\beta \in \{0, 1\}$, where '0' indicates real mode and '1' indicates random mode. $\mathcal{B}$ forwards β to its own challenger.

Clearly, if $c_i = ab_i$ for all $i \in [n]$, then $\mathcal{A}$ simulates the real oracle. If c_i's are random values, then $\mathcal{A}$ simulates the random oracle. Thereby, $\mathcal{B}$ breaks the DDH assumption with the same advantage as $\mathcal{A}$ breaks the pseduorandomness of $F_k(\mathsf{H}(\cdot))$.

Remark 7. (Weak) PRF can be built from weak pseudorandom group action (c.f. Definition in Appendix C) in a similar way.

C Weak Pseudorandom EGA

We begin by recalling the definition of a group action.

Definition 4 (Group Actions). *A group $\mathbb{G}$ is said to* act on *a set X if there is a map $\star : \mathbb{G} \times X \to X$ that satisfies the following two properties:*

1. *Identity: if e is the identity element of $\mathbb{G}$, then for any $x \in X$, we have $e \star x = x$.*
2. *Compatibility: for any $g, h \in \mathbb{G}$ and any $x \in X$, we have $(gh) \star x = g \star (h \star x)$.*

From now on, we use the abbreviated notation $(\mathbb{G}, X, \star)$ to denote a group action. If $(\mathbb{G}, X, \star)$ is a group action, for any $g \in \mathbb{G}$ the map $\phi_g : x \mapsto g \star x$ defines a permutation of X.

We then define an effective group action (EGA) [AFMP20] as follows.

Definition 5 (Effective Group Actions). *A group action $(\mathbb{G}, X, \star)$ is* effective *if the following properties are satisfied:*

1. *The group $\mathbb{G}$ is finite and there exist PPT algorithms for:*
   - *(a) Membership testing, i.e., to decide if a given bit string represents a valid group element in $\mathbb{G}$.*
   - *(b) Equality testing, i.e., to decide if two bit strings represent the same group element in $\mathbb{G}$.*
   - *(c) Sampling, i.e., to sample an element g from a uniform (or statistically close to) distribution on $\mathbb{G}$.*
   - *(d) Operation, i.e., to compute gh for any $g, h \in \mathbb{G}$.*
   - *(e) Inversion, i.e., to compute g^{-1} for any $g \in \mathbb{G}$.*
2. *The set X is finite and there exist PPT algorithms for:*
   - *(a) Membership testing, i.e., to decide if a bit string represents a valid set element.*
   - *(b) Unique representation, i.e., given any set element $x \in X$, compute a string $\hat{x}$ that canonically represents x.*
3. *There exists a distinguished element $x_0 \in X$, called the origin, such that its bit-string representation is known.*
4. *There exists an efficient algorithm that given (some bit-string representations of) any $g \in \mathbb{G}$ and any $x \in X$, outputs $g \star x$.*

Definition 6 (Weak Pseudorandom EGA). *A group action $(G, X, \star)$ is weakly pseudorandom if the family of efficiently commutable permutation $\{\phi_g : X \to X\}_{g \in G}$ is weakly pseudorandom, i.e., there is no PPT adversary that can distinguish tuples of the form $(x_i, g \star x_i)$ from (x_i, u_i) where $g \xleftarrow{\text{R}} \mathbb{G}$ and each $x_i, u_i \xleftarrow{\text{R}} X$.*

D Missing Security Proofs

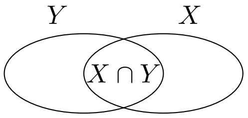
D.1 Proof of mqRPMT from cwPRF

Theorem 8. *The mqRPMT protocol described in Figure 5 is secure in the semi-honest model assuming H is a random oracle and F is a family of cwPRFs.*

Proof. We exhibit simulators Sim_{P_1} and Sim_{P_2} for simulating corrupt P_1 and P_2 respectively, and argue the indistinguishability of the simulated transcript from the real execution. Let $|X \cap Y| = m$.

Security against corrupt client. Sim_{P_2} simulates the view of corrupt client P_2, which consists of P_2's randomness, input, output and received messages. We formally show Sim_{P_2}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid$_0$: P_2's view in the real protocol.
Hybrid$_1$: Given P_2's input X, Sim_{P_2} chooses the randomness for P_1 (i.e., picks $k_1 \overset{\mathrm{R}}{\leftarrow} K$), and simulates with the knowledge of Y.

- RO query: Sim_{P_2} emulates the random oracle H honestly. For each query $\langle z_i \rangle$, Sim_{P_2} picks $\alpha_i \overset{\mathrm{R}}{\leftarrow} D$, and assigns $\mathsf{H}(z_i) := \alpha_i$.
- Sim_{P_2} outputs $(F_{k_1}(\mathsf{H}(y_1)), \ldots, F_{k_1}(\mathsf{H}(y_{n_1})))$.



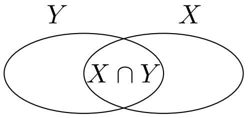
for $z_i \in D$, $\mathsf{H}(z_i) := \alpha_i \overset{\mathrm{R}}{\leftarrow} D$

Clearly, Sim_{P_2}'s simulated view in Hybrid$_1$ is identical to P_2's real view.
Hybrid$_2$: Sim_{P_2} does not choose the randomness for P_1 (i.e., picks $k_1 \overset{\mathrm{R}}{\leftarrow} K$), and simulates without the knowledge of Y. It emulates the random oracle H honestly as before, and only changes the simulation of P_1's message.

- Sim_{P_2} outputs $(\eta_1, \ldots, \eta_{n_1})$ where $\eta_i \overset{\mathrm{R}}{\leftarrow} D$.

We argue that the simulated view in Hybrid$_1$ and Hybrid$_2$ are computationally indistinguishable. More precisely, a PPT adversary $\mathcal{A}$ (with knowledge of X and Y) against cwPRF (with secret key k) is given n tuples (γ_i, η_i) where $\gamma_i \overset{\mathrm{R}}{\leftarrow} D$, and is asked to distinguish if $\eta_i = F_k(\gamma_i)$ or η_i are random values. $\mathcal{A}$ implicitly sets P_1's randomness $k_1 := k$, and simulates as below.

- RO query: for each random oracle query $\langle z_i \rangle$, if $z_i \notin Y$, picks $\alpha_i \overset{\mathrm{R}}{\leftarrow} D$ and sets $\mathsf{H}(z_i) := \alpha_i$; if $z_i \in Y$, sets $\mathsf{H}(z_i) := \gamma_i$.
- outputs $(\eta_1, \ldots, \eta_{n_1})$.



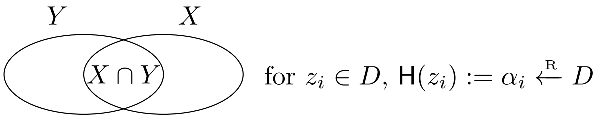
for $z_i \notin Y$, $\mathsf{H}(z_i) := \alpha_i \overset{\mathrm{R}}{\leftarrow} D$
for $z_i \in Y$, $\mathsf{H}(z_i) := \gamma_i \overset{\mathrm{R}}{\leftarrow} D$

If $\eta_i = F_k(\gamma_i)$ for $i \in [n_1]$, then $\mathcal{A}$'s simulation is identical to Hybrid$_1$. If η_i are random values, then $\mathcal{A}$'s simulation is identical to Hybrid$_2$.

Security against corrupt server. Sim_{P_1} simulates the view of corrupt server P_1, which consists of P_1's randomness, input, output and received messages. We formally show Sim_{P_1}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.

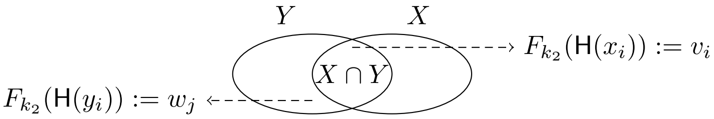
Hybrid$_0$: P_1's view in the real protocol.
Hybrid$_1$: Given P_1's input Y and output $(e_1, \ldots, e_{n_1})$, Sim_{P_1} chooses the randomness for P_2 (i.e., picks $k_2 \overset{\text{R}}{\leftarrow} K$ and a random permutation π over $[n_1]$), and simulates with the knowledge of X.

- RO queries: Sim_{P_1} emulates the random oracle H honestly. For each query $\langle z_i \rangle$, Sim_{P_1} picks $\alpha_i \overset{\text{R}}{\leftarrow} D$ and assigns $\mathsf{H}(z_i) := \alpha_i$.
- Sim_{P_1} outputs $\{F_{k_2}(\mathsf{H}(x_i))\}_{i \in [n_1]}$ and $\Omega \leftarrow \{F_{k_2}(F_{k_1}(\mathsf{H}(y_{\pi(i)})\}_{i \in [n_1]}$.



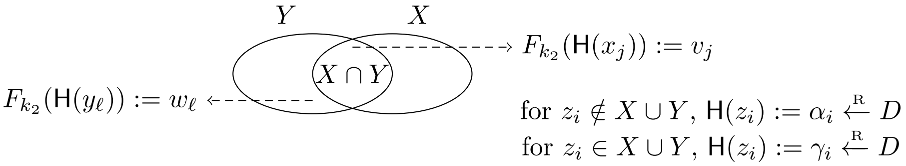
Clearly, Sim_{P_1}'s simulation in Hybrid$_1$ is identical to the real view of P_1.
Hybrid$_2$: Sim_{P_1} does not choose randomness for P_2, and simulates without the knowledge of X. It simulates the random oracle H honestly as before, and changes its simulation of P_2's message. Let m be the Hamming weight of $(e_1, \ldots, e_{n_1})$.

- Sim_{P_1} picks $v_i \overset{\text{R}}{\leftarrow} D$ for $i \in [n_2]$ (associated with $F_{k_2}(\mathsf{H}(x_i))$ where $x_i \in X$), outputs $\{v_i\}_{i \in [n_2]}$; picks $w_j \overset{\text{R}}{\leftarrow} D$ for $j \in [n_1 - m]$ (associated with $F_{k_2}(\mathsf{H}(y_j))$ where $y_j \in Y - X \cap Y$), outputs a random permutation of $(\{F_{k_1}(v_i)\}_{e_i=1}, \{F_{k_1}(w_j)\}_{j \in [n_1 - m]})$.



We argue that the view in Hybrid$_1$ and Hybrid$_2$ are computationally indistinguishable. More precisely, a PPT adversary $\mathcal{A}$ (with knowledge of X and Y) against cwPRF are given $n_1 + n_2 - m$ tuples (γ_i, η_i) where $\gamma_i \overset{\text{R}}{\leftarrow} D$, and is asked to determine if $\eta_i = F_k(\gamma_i)$ or random values. $\mathcal{A}$ implicitly sets P_2's randomness $k_2 := k$, picks $k_1 \overset{\text{R}}{\leftarrow} K$.

- RO queries: for $z_i \notin X \cup Y$, picks $\alpha_i \overset{\text{R}}{\leftarrow} D$ and assigns $\mathsf{H}(z_i) := \alpha_i$; for $z_i \in X \cup Y$, assigns $\mathsf{H}(z_i) := \gamma_i$.
- For each $z_i \in X$, $\mathcal{A}$ picks out the associated η_i to form $\{v_j\}_{j \in [n_2]}$; for each $z_i \in Y - X \cap Y$, $\mathcal{A}$ picks out the associated η_i to form $\{w_\ell\}_{\ell \in [n_1 - m]}$. $\mathcal{A}$ outputs $\{v_j\}_{j \in [n_2]}$ and a random permutation of $(\{F_{k_1}(v_j)\}_{x_j \in X \cap Y}, \{F_{k_1}(w_\ell)\}_{\ell \in [n_1 - m]})$.



If $\eta_i = F_k(\gamma_i)$, then $\mathcal{A}$'s simulation is identical to Hybrid_1. If η_i are random values, then $\mathcal{A}$'s simulation is identical to Hybrid_2.

This proves the theorem. □

D.2 Proof of Permuted OPRF from cwPRP

Theorem 9. *The above permuted OPRF protocol described in Figure 7 is secure in the semi-honest model assuming* H *is a random oracle and* F *is a family of cwPRPs.*

Proof. We exhibit simulators Sim_{P_1} and Sim_{P_2} for simulating corrupt P_1 and P_2 respectively, and argue the indistinguishability of produced transcript from the real execution.

Security against corrupt sender. Sim_{P_1} simulates the view of corrupt sender P_1, which consists of P_1's randomness, input, output and received messages. We formally show Sim_{P_1}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid_0: P_1's view in the real protocol.
Hybrid_1: Given P_1's output k and π, Sim_{P_1} chooses the randomness s for P_2, and simulates with the knowledge of $X = (x_1, \ldots, x_n)$:

- RO queries: Sim_{P_1} honestly emulates random oracle H. For every query $\langle z_i \rangle$, picks $\alpha_i \xleftarrow{\text{R}} D$ and assigns $\mathsf{H}(z_i) := \alpha_i$.
- Sim_{P_1} outputs $(F_s(\beta_1), \ldots, F_s(\beta_n))$, where $\mathsf{H}(x_i) = \beta_i$.

Clearly, Sim_{P_1}'s simulated view in Hybrid_1 is identical to P_1's real view.
Hybrid_2: Sim_{P_1} does not choose the randomness for P_2, and simulates without the knowledge of X. It honestly emulates random oracle H as in Hybrid_1, and only changes the simulation of P_2's message.

- Sim_{P_1} outputs $(\eta_1, \ldots, \eta_n)$ where $\eta_i \xleftarrow{\text{R}} D$.

We argue that the view in Hybrid_1 and Hybrid_2 are computationally indistinguishable. Let $\mathcal{A}$ be a PPT adversary against the weak pseudorandom of F_s. Given a real-or-random oracle $\mathcal{O}_{\mathsf{ror}}(\cdot)$, $\mathcal{A}$ is asked to distinguish which mode he is in. $\mathcal{A}$ queries the $\mathcal{O}_{\mathsf{ror}}(\cdot)$ n times, and obtains (γ_i, η_i) in return. $\mathcal{A}$ then simulates (with the knowledge of X) as below:

- RO queries: for each query $\langle z_i \rangle$, if $z_i \notin X$, picks $\alpha_i \xleftarrow{\text{R}} D$ and assigns $\mathsf{H}(z_i) := \alpha_i$; if $z_i \in X$, assigns $\mathsf{H}(x_i) := \gamma_i$.
- Outputs $(\eta_1, \ldots, \eta_n)$.

Clearly, if $\eta_i = F_s(\gamma_i)$, $\mathcal{A}$ simulates Hybrid_1. Else, it simulates Hybrid_2. Thereby, Sim_{P_1}'s simulated view is computationally indistinguishable to P_1's real view.
Security against corrupt receiver. Sim_{P_2} simulates the view of corrupt receiver P_2, which consists of P_2's randomness, input, output and received messages. We formally show Sim_{P_2}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.

Hybrid$_0$: P_2's view in the real protocol.
Hybrid$_1$: Given P_2's input $X = (x_1, \ldots, x_n)$ and output $\{F_k(\mathsf{H}(x_{\pi(i)}))\}_{i \in [n]}$, Sim_{P_2} emulates the random oracle H honestly, picks $s \xleftarrow{\text{R}} \mathbb{Z}_p$, simulates message from P_1 as $\{F_s(F_k(\mathsf{H}(x_{\pi(i)})))\}_{i \in [n]}$.

According to the commutative property of cwPRF, Sim_{P_2}'s simulated view is identical to the real view.

This proves the theorem. □

Observe that the cwPRF construction presented in Section 4.2 is actually a family of cwPRPs. Plugging it to the above generic construction, we obtain a concrete pOPRF protocol as described in Figure 7.

Parameters: hash function $\mathsf{H} : \{0,1\}^\ell \to \mathbb{G}$.
Inputs: The receiver P_2 inputs a set $X = \{x_1, \ldots, x_n\}$, where $x_i \in \{0,1\}^\ell$.
Protocol:

1. P_2 picks $s \xleftarrow{\text{R}} \mathbb{Z}_p$, then sends $(\mathsf{H}(x_1)^s, \ldots, \mathsf{H}(x_n)^s)$ to the sender P_1.
2. P_1 picks $k \xleftarrow{\text{R}} \mathbb{Z}_p$ and computes $(\mathsf{H}(x_1)^{sk}, \ldots, \mathsf{H}(x_n)^{sk})$, then picks a random permutation π over $[n]$ and sends $y'_i = \mathsf{H}(x_{\pi(i)})^{sk}$ for $i \in [n]$ to P_2.
3. P_1 outputs k and π.
4. P_2 outputs $y_i = (y'_i)^{s^{-1}}$ for each $i \in [n]$.

Fig. 13: Permuted OPRF based on the DDH assumption

The security of the above pOPRF protocol is guaranteed by Theorem 2 and the security of the underlying cwPRP, which is in turn based on the DDH assumption. For completeness, we provide a direct security proof based on the DDH assumption as below.

Theorem 10. *The permuted OPRF protocol described in Figure 7 is secure in the semi-honest model assuming* H *is a random oracle and the DDH assumption holds.*

Proof. We exhibit simulators Sim_{P_1} and Sim_{P_2} for simulating corrupt P_1 and P_2 respectively, and argue the indistinguishability of produced transcript from the real execution.

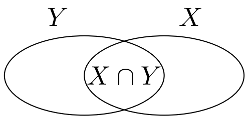
Security against corrupt receiver. Sim_{P_2} simulates the view of corrupt receiver P_2, which consists of P_2's randomness, input, output and received messages. We formally show Sim_{P_2}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid$_0$: P_2's view in the real protocol.

Hybrid$_1$: Given P_2's input $X = (x_1, \ldots, x_n)$ and output $\{y_{\pi(1)}, \ldots, y_{\pi(n)}\}$, Sim_{P_2} emulates the random oracle H honestly, picks $s \xleftarrow{R} \mathbb{Z}_p$, simulates message from P_1 as $\{y^s_{\pi(1)}, \ldots, y^s_{\pi(n)}\}$.

Clearly, Sim_{P_2}'s simulated view is identical to the real view.

Security against corrupt sender. Sim_{P_1} simulates the view of corrupt sender P_1, which consists of P_1's randomness, input, output and received messages. We formally show Sim_{P_1}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts,
Hybrid$_0$: P_1's view in the real protocol.
Hybrid$_1$: Given P_1's output k and π, Sim_{P_1} chooses the randomness s for P_2, and simulates with the knowledge of $X = (x_1, \ldots, x_n)$:

- RO queries: Sim_{P_1} honestly emulates random oracle H. For every query $\langle z_i \rangle$, picks $\alpha_i \xleftarrow{R} \mathbb{G}$ and assigns $\mathsf{H}(z_i) := \alpha_i$.
- Sim_{P_1} outputs $(\beta^s_1, \ldots, \beta^s_n)$, where $\mathsf{H}(x_i) = \beta_i$.



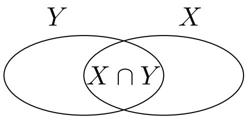
for $z_i \in \{0,1\}^\ell$, $\mathsf{H}(z_i) := \alpha_i \xleftarrow{R} \mathbb{G}$

Clearly, Sim_{P_1}'s simulated view in Hybrid$_1$ is identical to P_1's real view.
Hybrid$_2$: Sim_{P_1} does not choose the randomness for P_2, and simulates without the knowledge of X. It honestly emulates random oracle H as in Hybrid$_1$, and only changes the simulation of P_2's message.

- Sim_{P_1} outputs $(g^{c_1}, \ldots, g^{c_n})$ where $c_i \xleftarrow{R} \mathbb{Z}_p$.

We argue that the view in Hybrid$_1$ and Hybrid$_2$ are computationally indistinguishable. Let $\mathcal{A}$ be a PPT adversary against the DDH assumption. Given the DDH challenge $g^a, g^{b_1}, \ldots, g^{b_n}, g^{c_1}, \ldots, g^{c_n})$ where $a, b_i \xleftarrow{R} \mathbb{Z}_p$, $\mathcal{A}$ is asked to distinguish if $c_i = ab_i$ or random values. $\mathcal{A}$ implicitly sets P_2's randomness $s := a$, and simulates (with the knowledge of X) as below:

- RO queries: for each query $\langle z_i \rangle$, if $z_i \notin X$, picks $\alpha_i \xleftarrow{R} \mathbb{G}$ and assigns $\mathsf{H}(z_i) := \alpha_i$; if $z_i \in X$, assigns $\mathsf{H}(x_i) := g^{b_i}$.
- Outputs $(g^{c_1}, \ldots, g^{c_n})$.



for $z_i \notin X$, $\mathsf{H}(z_i) := \alpha_i \xleftarrow{R} \mathbb{G}$
for $z_i \in X$, $\mathsf{H}(z_i) := g^{b_i}$

Clearly, if $c_i = ab_i$, $\mathcal{A}$ simulates Hybrid$_1$. Else, it simulates Hybrid$_2$. Thereby, Sim_{P_1}'s simulated view is computationally indistinguishable to P_1's real view.

This proves the theorem. $\square$

Remark 8. In the above security proof, when establishing the security against corrupt sender, we can obtain a more modular proof by reducing the indistinguishability of simulated views in Hybrid_1 and Hybrid_2 to the pseudorandomness of $F_k(\mathsf{H}(\cdot))$, which is in turn based on the DDH assumption.

D.3 Proof of mqRPMT from pOPRF

Theorem 11. *The above mqRPMT protocol described in Figure 8 is secure in the semi-honest model assuming the security of permuted OPRF F.*

Proof. We exhibit simulators Sim_{P_1} and Sim_{P_2} for simulating corrupt P_1 and P_2 respectively, and argue the indistinguishability of the produced transcript from the real execution. Let $|X \cap Y| = m$.

Security against corrupt client. Sim_{P_2} simulates the view of corrupt client P_2, which consists of P_2's randomness, input, output and received messages. We formally show Sim_{P_2}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid_0: P_2's view in the real protocol.
Hybrid_1: Sim_{P_2} simply picks k and π, then invokes the simulator for P_2 in the permuted OPRF with (k, π) as output. By the semi-honest security of permuted OPRF on P_2's side, the simulation is indistinguishable to the real view.

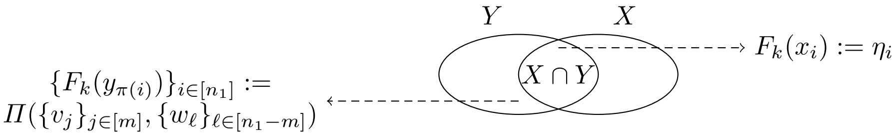
Security against corrupt server. Sim_{P_1} simulates the view of corrupt server P_1, which consists of P_1's randomness, input, output and received messages. We formally show Sim_{P_1}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid_0: P_1's view in the real protocol. Note that P_1's view consists of its view in stage 1 (the permuted OPRF part) and its view in stage 2.
Hybrid_1: Given P_1's input $Y = (y_1, \ldots, y_{n_1})$ and output $(e_1, \ldots, e_{n_2})$, Sim_{P_1} creates the simulated view as below:

- pick a random PRF key k and a random permutation π;
- compute $(F_k(y_{\pi(1)}), \ldots, F_k(y_{\pi(n_1)}))$, then generate its stage 1's view by invoking the simulator for P_1 of permuted OPRF with input $(y_1, \ldots, y_{n_1})$ and output $(F_k(y_{\pi(1)}), \ldots, F_k(y_{\pi(n_1)}))$;
- generate stage 2's view $(F_k(x_1), \ldots, F_k(x_{n_2}))$ using k with the knowledge of P_2's input X.

The simulated stage 2's view is identical to the real one. By the semi-honest security of permuted OPRF on P_1' side, the stage 1's simulated view is computationally indistinguishable to the real one. Thereby, the simulated view in Hybrid_1 is computationally indistinguishable to the real one.
Hybrid_2: Sim_{P_1} creates the simulated view without the knowledge of X, and it neither picks k nor explicitly picks π:

- generate stage 2's view by outputting $(\eta_1, \ldots, \eta_{n_2})$, where $\eta_i \xleftarrow{\text{R}} R$; this implicitly sets $F_k(x_i) := \eta_i$.

- for each $e_i = 1$, pick out the associated η_i to form $\{v_j\}_{j\in[m]}$; for each $e_i = 0$, pick random values to form $\{w_\ell\}_{\ell\in[n_1-m]}$; apply a random permutation Π of $(\{v_j\}_{j\in[m]}, \{w_\ell\}_{\ell\in[n_1-m]})$, treat the result as $(F_k(y_{\pi(1)}), \ldots, F_k(y_{\pi(n_1)}))$ (note that the real permutation π is unknown to the simulator since it does not know $X \cap Y$); then generate its stage 1's view by invoking the simulator for P_1 of permuted OPRF with input $(y_1, \ldots, y_{n_1})$ and output $(F_k(y_{\pi(1)}), \ldots, F_k(y_{\pi(n_1)}))$.



We argue that the simulated views in Hybrid_1 and Hybrid_2 are computationally indistinguishable based on the pseudorandomness of F. Let $\mathcal{A}$ be an adversary against F. Given X and Y, $\mathcal{A}$ simulates as below:

- query the real-or-random oracle $\mathcal{O}_{\mathsf{ror}}(\cdot)$ with $(x_1, \ldots, x_{n_1})$, output the response $(\eta_1, \ldots, \eta_{n_1})$.
- pick a random permutation π;
- query the real-or-random oracle with $(y_{\pi(1)}, \ldots, y_{\pi(n_1)})$ and obtain $(\zeta_1, \ldots, \zeta_{n_1})$ in return; then generate its stage 1's view by invoking the simulator for P_1 of permuted OPRF with input $(y_1, \ldots, y_{n_1})$ and output $(\zeta_1, \ldots, \zeta_{n_1})$.

Clearly, if $\mathcal{A}$ queries the real oracle, then its simulation is identical to Hybrid_1. Else, its simulation is identical to that Hybrid_2. This reduces the computational indistinguishability of views in Hybrid_1 and Hybrid_2 to the pseudorandomness of $F_k(\cdot)$. Therefore, Sim_{P_1}'s simulation is indistinguishable to the real one.

This proves the theorem. □

D.4 Proof of PSU from mqRPMT

Theorem 12. *The PSU derived from the above framework described in Figure 9 is semi-honest secure by assuming the semi-honest security of mqRPMT and OT.*

Proof. We exhibit simulators Sim_{P_1} and Sim_{P_2} for simulating corrupt P_1 and P_2 respectively, and argue the indistinguishability of the produced transcript from the real execution. Let $|X \cap Y| = m$.

Security against corrupt sender. Sim_{P_2} simulates the view of corrupt sender P_2, which consists of P_2's randomness, input, output and received messages. We formally show Sim_{P_2}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid_0: P_2's view in the real protocol. Note that P_2's view consists of two parts, i.e., the mqRPMT part of view (stage 1) and the OT part of view (stage 2).

Hybrid$_1$: Sim_{P_2} first invokes the simulator for client in the mqRPMT with X as input to generate the stage 1's part of view, then invokes the simulator for sender in the OT with $\{(x_i, \perp)\}_{i\in[n_2]}$ as input to generate stage 2's part of view. By the semi-honest security of mqRPMT on the client side and the semi-honest security for OT on the sender side, the simulation is indistinguishable to the real view via standard hybrid argument.

Security against corrupt receiver. Sim_{P_1} simulates the view of corrupt receiver P_1, which consists of P_1's randomness, input, output and received messages. We formally show Sim_{P_1}'s simulation is indistinguishable from the real execution via a sequence of hybrid transcripts.
Hybrid$_0$: P_1's view in the real protocol. Note that P_1's view also consists of two parts, i.e., the mqRPMT part of view (stage 1) of and the OT part of view (stage 2).
Hybrid$_1$: Given P_1's input $Y = (y_1, \dots, y_{n_1})$ and output $X \cup Y$, Sim_{P_1} creates the simulated view as below:

- pick a random indication vector $\vec{e} = (e_1, \dots, e_{n_2})$ with Hamming weight $m = |X \cap Y|$, then generate the output vector $\vec{z} = (z_1, \dots, z_{n_2})$ from $\vec{e}$ and $X \cup Y$ in the following manner: randomly shuffle the $(n_2 - m)$ elements in $X \backslash Y$, and assign them to z_i if $e_i = 0$, then assign $z_i = \perp$ iff $e_i = 1$; then invoke the simulator for OT receiver with input $\vec{e}$ and output $\vec{z}$ and generate stage 2's view.
- invoke the simulator for mqRPMT server with input Y and output $\vec{e} = (e_1, \dots, e_{n_2})$ to generate stage 1's view.

It is easy to check that the distribution of $\vec{e}$ and $\vec{z}$ is identical to that (induced by the distribution of mqRPMT's input vector $(x_1, \dots, x_{n_2})$) in the real protocol. By the semi-honest security of mqRPMT on the server side and the semi-honest security for OT on the receiver side, the simulation is indistinguishable to the real view via standard hybrid argument.

This proves the theorem. $\square$

Table of Contents