Breaking the Quadratic Barrier: Quantum Cryptanalysis of Milenage, Telecommunications' Cryptographic Backbone

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Abstract. The potential advent of large-scale quantum computers in the near future poses a threat to contemporary cryptography. One ubiquitous usage of cryptography is currently present in the vibrant field of cellular networks. The cryptography of cellular networks is centered around seven secret-key algorithms $f_1, \ldots, f_5, f_1^*, f_5^*$, aggregated into an authentication and key agreement algorithm set. Still, to the best of our knowledge, these secret key algorithms have not yet been subject to quantum cryptanalysis. Instead, many quantum security considerations for telecommunication networks argue that the threat posed by quantum computers is restricted to public-key cryptography. However, various recent works have presented quantum attacks on secret key cryptography that exploit quantum period finding to achieve more than a quadratic speedup compared to the best known classical attacks. Motivated by this quantum threat to symmetric cryptography, this paper presents a quantum cryptanalysis for the Milenage algorithm set, the prevalent instantiation of the seven secret-key $f_1, \ldots, f_5, f_1^*, f_5^*$ algorithms that underpin cellular security. Building upon recent quantum cryptanalytic results, we show attacks that go beyond a quadratic speedup. Concretely, we provide quantum attack scenarios for all Milenage algorithms, including exponential speedups distinguishable by different quantum attack models. Our results do not constitute an immediate quantum break of the Milenage algorithms, but they do show that Milenage suffers from structural weaknesses making it susceptible to quantum attacks.

Keywords: Quantum cryptanalysis · Simon's Algorithm · Quantum Security · Milenage · Cellular network · AKA protocol · Post-quantum cryptography

1 Introduction

Telecommunication operators are evidently expecting the advent of general purpose quantum computers, as indicated by their funding of various research projects investigating the new technologies' potential [33]. As part of these efforts, telecommunication standardization bodies also pay increasing attention

to post-quantum security in telecommunication networks. As a result, the sixth generation of telecommunication networks (6G) is intended to be post-quantum secure, and proposals for extensions of the fifth generation (5G) already integrate quantum security considerations, cf. [15, 29, 40]. . These security considerations are often focused on the threat quantum computers pose to asymmetric cryptography. To mitigate this threat, telecommunication protocols can replace the vulnerable cryptographic primitives with post-quantum secure cryptography, which does not rely on the hardness of factoring or the discrete logarithm problem. The National Institute of Standards and Technology (NIST), a standardization body, leads an ongoing process to evaluate and standardize asymmetric post-quantum primitives [30]. This process is now in its final stages, with four candidate algorithms already selected for standardization and an ongoing fourth round to analyze additional alternative constructions [31]. Multiple works have already demonstrated how the now standardized post-quantum secure public key cryptographic schemes can replace the (quantum) vulnerable public-key cryptography in present telecommunication protocols [13, 38].

In contrast to public key cryptography, quantum security considerations for cellular networks often do not consider the aspect of quantum attacks against symmetric cryptography. Instead, they assume symmetric cryptography to be unaffected by quantum cryptananalysis, except for a quadratic speed-up of exhaustive search due to Grover's algorithm . Hence, so the argument goes, increasing the key size of symmetric cryptography used in 6G to 256-bit would provide sufficient protection against quantum adversaries [29, 40].

In light of recent quantum cryptanalytic results however, this common belief can no longer be assumed to be trivially true. Starting with the seminal works of Kuwakado and Morii [25, 26], various works have shown that *quantum period finding* – through Simon's algorithm [37] – can speed up attacks on symmetric-key cryptography schemes beyond best known classical bounds [9, 10, 14, 24, 35]. The attacks demonstrate that, depending on the assumed attacker capabilities, quantum computers can be used to either efficiently break certain symmetric-key cryptography schemes or reduce the time needed to attack them. The distinguishing feature in the attacker capabilities for quantum cryptanalysis is the kind of oracle access that is provided to the attacker, commonly referred to as Q_2 and Q_1. In the Q_2 setting, also called the quantum known plaintext attack, the attacker can make superposition queries to an encryption oracle. This model enables quantum attackers to significantly reduce the security of classically secure symmetric ciphers. For example, in the Q_2 setting, Simon's algorithm enables attackers to execute forgery attacks against an otherwise *classically* secure CBC-MACs in polynomial time [24]. However, due to its powerful attacker model, the Q_2 model remains of mainly theoretical interest. In the Q_1 model the attacker has access to a general purpose quantum computer, but can only make classical queries to an encryption oracle. Attacks in this model can be executed as soon as general-purpose quantum computers come into existence. In the Q_1 model, symmetric cryptography can be attacked as well. Bonnetain et al. [11] demonstrated Q_1 attacks on symmetric cryptogra-

phy that improve upon the best-known classical bounds. Their attacks extend quantum-cryptanalysis of symmetric ciphers that was rooted in the Q_2 model, i.e., relied on superposition queries to an encryption oracle.The cornerstone of these attacks is the offline Simon's algorithm [9], which combines quantum search and quantum period finding to transfer the Q_2 attacks to the Q_1 model.

These results call for a careful re-evaluation of the truism that has guided quantum security considerations for 6G so far. Doubling the key-size might not be sufficient to ensure long-term security of telecommunication protocols. Instead, symmetric-key cryptographic schemes used in telecommunications protocols must be evaluated towards their resilience against quantum enabled adversaries as well.

1.1 Contributions

We conduct such a quantum cryptanalysis for the Milenage algorithm set, a set of symmetric-key cryptographic algorithms ubiquitously used in the cellular world. The Milenage algorithm set's main usage is the Authentication and Key Agreement (AKA) protocol, used for authentication and session establishment in cellular networks. All Milenage algorithms make use of the network authentication key K, a secret key shared between the subscriber (stored in his network provider's SIM card) and the network. The algorithm set consists of the functions $f1, \ldots, f5, , f1^*, f5^*$, and makes use of the AES block cipher.

In summary, when a user wants to authenticate to the network, the operator generates a random challenge and calculates the output of one of the Milenage algorithms, keyed with the network authentication key K. If the user, upon receiving the challenge, replies with the correct response, thus demonstrating knowledge of K, the authentication request is accepted. Other functions of the Milenage algorithm set are used to calculate a Message Authentication Code (MAC) or derive keys for later usage. Breaking the Milenage algorithm set would therefore allow attackers to perform account takeover attacks. Thus, the security of Milenage algorithms is crucial for the security of pervasive cellular networks in general. As such, the algorithms underpin the security of the worldwide cellular networks and provide a great starting point for the required quantum cryptanalysis of symmetric ciphers.

In conducting the quantum cryptanalysis, we take a gentle approach that can be followed by researchers who are not familiar with the internals of quantum computing. First, in Section 3, we distill a *quantum toolbox* from the various works on quantum cryptanalysis and quantum algorithms, i.e., a minimum set of quantum algorithms and results about their complexity that have proven to be useful in quantum cryptanalysis. For each algorithm in the toolbox, we explain the requirements that an attacker needs to meet in order to use the respective algorithm. For example, whether a quantum algorithm requires superposition access or can also be executed with only classical oracle access to the encryption under attack. Once equipped with this quantum toolbox, no more detailed understanding of quantum computing is required. The attacker then only needs

to construct a function that meets the respective requirements, after which the algorithms can be applied as a black-box.

Leveraging this minimum quantum toolbox, we develop multiple attacks on the Milenage algorithm set inspired by various prior works. The quantum cryptanalysis of Milenage is the main contribution of this paper and can be found in Section 4. We analyze the Milenage set from several dimensions. In the two different query models Q_1 and Q_2, considering different attacker goals such as full key recovery or existential forgery and considering more powerful attacker models such as the related key model. Our results show that the quantum toolbox can be utilized to provide speedups over classical attacks in all dimensions, and even leads to polynomial time attacks in the Q_2 model. As a helpful overview, Table 1 summarizes the breadth of our results. The complexity analysis is additionally parameterized by three circuit complexities. First, $\mathtt{T}_{\mathtt{QAES}}$ refers to the depth of a quantum circuit computing an AES encryption, as presented for example in Ref. [21]. Second, $\mathtt{T}_{\mathtt{AES}}$ refers to runtime-complexity of a classical circuit computing an AES encryption. Finally, $\mathtt{T}_O$ refers to the time required for an oracle query. We find that a Q_2 attacker can execute existential forgery attacks against Milenage in polynomial time. It is an explicit design goal of Milenage to resist existential forgery attacks of classical adversaries (the design document does not consider a quantum adversary model) [4]. Less powerful adversaries are still able to significantly speed up their attacks, albeit not to an extent that fully breaks the algorithm set.

In summary, the attacks show that the Milenage algorithms exhibit a structure that can be exploited by quantum computers to obtain attacks that are more efficient than Grover's search. In the Q_2 model, Milenage must be considered broken, as it is vulnerable to a polynomial-time existential forgery attack. We emphasize that the Q_2 attacks remain of mainly theoretical interest for now, and *do not* imply that Milenage is broken once general purpose quantum computers come into existence. Notably however, the Q_2 attacker model encompasses all potential Q_1 attacks. An absence of Q_2 attacks would have implied an absence of Q_1 attacks as well. Given its vulnerability in the Q_2 model, further Q_1 attacks against Milenage cannot be ruled out. We encourage further quantum cryptanalysis and security proofs for Milenage, and its alternative TUAK, based on the Keccack-f-permutation [2], to lay the ground for post-quantum secure cellular networks.

2 Background

2.1 Notation

Throughout this paper, we will make use of a block cipher encryption function E, which takes as input an m-bit message, an n-bit key and returns an m-bit output. We denote by $E_K[m]$ the encryption of bit-string m under block cipher E with secret k. Similarly, if a function f takes as input a secret key k and a message m, we denote by $f_k(m)$ the invocation of that function with k and

Attack	Model	Classical Queries	Superposition Queries	Circuit Depth Complexity	Best Known Classical Attack	Sec.
Grover's attack for key recovery, OP known	Q_1	$O(1)$	0	$O\left(2^{\lvert K\rvert/2} \cdot \mathtt{T}_{\mathtt{QAES}}\right)$	$O\left(2^{\lvert K\rvert} \cdot \mathtt{T}_{\mathtt{AES}}\right)$	Sec. 4.1
Grover's attack for key recovery, OP unknown	Q_1	$O(1)$	0	$O\left(2^{(\lvert K\rvert+\lvert OP_c\rvert)/2} \cdot \mathtt{T}_{\mathtt{QAES}}\right)$	$O\left(2^{\lvert K\rvert+\lvert OP_c\rvert} \cdot \mathtt{T}_{\mathtt{AES}}\right)$	Sec. 4.1
Key Recovery $f2$, OP unknown	Q_2	0	$O\left(\lvert M\rvert\right)$	$\tilde{O}\left((\lvert M\rvert \cdot \mathtt{T}_{\mathtt{QAES}}) \cdot 2^{\lvert K\rvert/2}\right)$	$O\left(2^{\frac{M}{2}} \cdot \mathtt{T}_O + 2^{\lvert K\rvert+\frac{\lvert M\rvert}{2}} \cdot \mathtt{T}_{\mathtt{AES}}\right)$	Sec. 4.2
Offline Key Recovery $f2$, OP unknown	Q_1	$O\left(2^{\lvert M\rvert}\right)$	0	$\tilde{O}\left(2^{\lvert M\rvert} \cdot \mathtt{T}_O + (\lvert M\rvert \cdot \mathtt{T}_{\mathtt{QAES}}) \cdot 2^{\frac{\lvert K\rvert}{2}}\right)$	$O\left(2^{\frac{M}{2}} \cdot \mathtt{T}_O + 2^{\lvert K\rvert+\frac{\lvert M\rvert}{2}} \cdot \mathtt{T}_{\mathtt{AES}}\right)$	Sec. 4.2
Existential Forgery $f1$	Q_2	$O\left(1\right)$	$O\left(\lvert M\rvert\right)$	$O\left(\lvert M\rvert \cdot \mathtt{T}_O\right)$	$O\left(2^{\lvert M\rvert/2} \cdot \mathtt{T}_O\right)$	Sec. 4.3
Related Key Attack $f1, \ldots, f_5$	Q_2	0	$O\left(\lvert K\rvert + \lvert OP_c\rvert\right)$	$\tilde{O}\left((\lvert K\rvert + \lvert OP_c\rvert) \cdot \mathtt{T}_O\right)$	$O\left(S \cdot \mathtt{T}_O + S \cdot \mathtt{T}_{\mathtt{AES}}\right)$ where $S = 2^{\frac{\lvert K\rvert+\lvert OP_c\rvert}{2}}$	Sec. 4.4
Offline Related Key Attack $f1, \ldots, f5$	Q_1	$O\left(2^{\frac{\lvert K\rvert+\lvert OP_c\rvert}{3}}\right)$	0	$\tilde{O}\left(S \cdot \mathtt{T}_O + S \cdot \mathtt{T}_{\mathtt{QAES}}\right)$ where $S = 2^{\frac{\lvert K\rvert+OP_c\rvert}{3}}$	$O\left(S \cdot \mathtt{T}_O + S \cdot \mathtt{T}_{\mathtt{AES}}\right)$ where $S = 2^{\frac{\lvert K\rvert+\lvert OP_c\rvert}{2}}$	Sec. 4.4

Table 1. Summary of the results. $|K|$ is the length of the message authentication key, $|OP_C|$ is the length of the OP_c bitstring and $|M|$ is the block length of the underlying block cipher. In the case of Milenage, $|K| = |OP_C| = |M| = 128$. For all complexity estimates, the big-O notation hides only a very small multiplicative constant.

message m. For a bit-string $x \in \{0,1\}^*$, we denote by $|x|$ the length of the bit-string. We write 0^n to denote the bit-string of n zeros.

Additionally, we define the function $rot_r(x)$ and $rot_r^{-1}(x)$ which are the results of cyclically rotating the 128-bit value x by r bit positions towards the most significant or least significant bit, respectively. If $x = x[0]||x[1]||\dots x[127]$, and $y = rot_r(x)$, then $y = x[r]||x[r+1]||\dots x[127]||x[0]||x[1]||x[r-1]$. Of course, it holds that $rot_r(rot_r^{-1}(x)) = x$ and $rot_r^{-1}(rot_r(x)) = x$.

To state complexities, we use the big-O notation, where we use $O(f(n))$ to hide constant factors and $\tilde{O}(f(n))$ to hide polynomial factors.

2.2 The AKA Protocol and Milenage Algorithms

Cellular protocols base their security on seven secret-key cryptographic functions, referred to as a authentication and key generation algorithm set. Upon session establishment between the home network and the subscriber, these algorithms are used to authenticate the subscriber to the network and derive keys that are in turn used protect subsequent communication. To this end, telecommunication operators assign each subscriber a secret key, the network authentication key, denoted as K. The operator provisions each subscriber's SIM card with their individual network authentication key. To authenticate itself to the network, the subscriber then takes part in a challenge-response protocol, the so-called AKA protocol. The use of the AKA protocol is mandated through standardization bodies — all cellular networks follow this protocol.

The AKA protocol is built around a set of cryptographic functions $f1, \dots, f5$ and $f1^*, f5^*$, keyed with the network authentication key K. In summary, the protocol follows a challenge-response structure. The subscriber sends the telecommunication operator an authentication request, containing the subscriber's identity. The operator then generates a random challenge $RAND$ and uses one of the provided cryptographic functions to calculate a corresponding response [5]. The operator then sends the challenge $RAND$ to the subscriber's device, which derives the response using the same cryptographic function and sends the derived response to the network. If the derived response and the expected response match, the subscriber has successfully authenticated themselves to the operator. In addition, the cryptographic functions $f1, \dots, f5^*$ serve to a derive a MAC and additional key material used for encryption and integrity protection of subsequent messages as well as transferred user data. Figure 1 describes the authentication towards the network as implemented in the 4th generation of cellular networks (LTE), using the AKA protocol and the functions $f1, \dots, f5$.

The exact details of the AKA protocol are not required to understand the present analysis — however, it is important to note that the results of the functions $f1$ and $f2$ are sent in cleartext over the network upon authentication. The AKA protocol itself has been subject to formal security analysis [5, 17], proving AKA's security under the assumption that the function $f1, \dots, f5$ and $f1^*, f5^*$ are pseudorandom. The analysis resulted in improvement suggestions to harden the protocol's privacy guarantees. A more detailed protocol description is given in Appendix B.

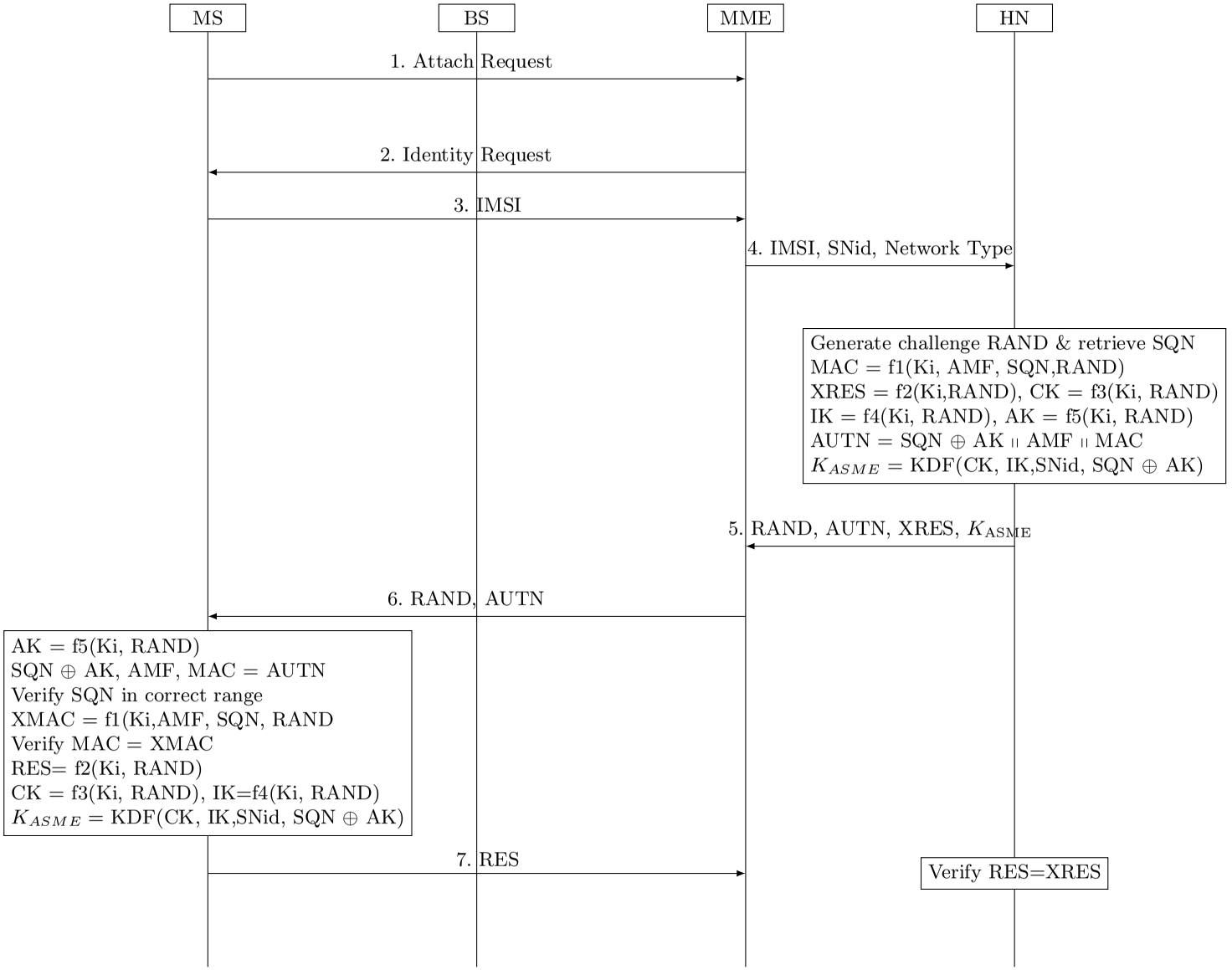

Fig. 1. The Authentication and Key Agreement (AKA) protocol as used in Long-Term Evolution (LTE). The user's device, referred to as Mobile Station (MS), communicates with the Base Station (BS) to authenticate towards the network. The BS forwards the request to the Mobility Management Entity (MME), which in turn forwards it to the Home Network (HN). The home network uses the function $f1, \ldots, f5$ to calculate session information and secret key material and forwards the necessary information back to the Mobility Management Entity (MME).

Note that if an attacker obtains a subscriber's secret key K, the attacker can impersonate the respective subscriber towards the home network. This amounts to a complete account takeover. In addition, an attacker can derive all keys used for encryption and integrity protection and thus eavesdrop on all communication between the subscriber and the home network. Therefore, the security of cellular networks is completely contingent on the security of the cryptographic functions used in the AKA protocol.

The most commonly used set of functions for the AKA protocol is the Milenage authentication and key generation algorithm set. The Milenage algorithm set consists of five basis functions, $h1, \ldots, h5$, [1] whose outputs are mapped to the seven required outputs for the functions $f1, \ldots, f5^*$. Figure 2 describes the Milenage algorithm set, standardized through the 3rd Generation Partnership Project (3GPP) [3]. All five functions take as input the random 128-bit challenge $RAND$, generated by the operator upon registration of the subscriber's device towards the network. The second to fifth basis function, $h2, \ldots, h5$, take this random challenge as an input and output:

$$hi_{K,OP_c}(RAND) = E_K[c_i \oplus rot_{r_i}(OP_c \oplus E_K[RAND \oplus OP_c])] \oplus OP_c,$$

where the function E_K, also referred to as the kernel, is a block cipher with block lenght of 128-bit.

The first basis function $h1$ takes as an additional input a 128 bit-string $IN1$, that is composed of the concatenation of a sequence number SQN and a fixed authentication management field AMF. The Sequence Number (SQN) acts as sequential counter to prevent replay attacks. The Authentication Management Field (AMF) specifies the type of authentication to be used and is usually fixed [1]. The function $h1$ is then defined as:

$$h1_{K,OP_c}(RAND, IN1) = E_K[TEMP \oplus rot_{r_1}(IN1 \oplus OP_c) \oplus c_1] \oplus OP_c,$$

where $TEMP = E_K[RAND \oplus OP_C]$.

The output of the basis functions is mapped to the seven required outputs $f1, \ldots, f5^*$ as follows. The first 64 bits of the $h1$ output are mapped to represent the output of $f1$, the last 64 bits of $h1$'s output are used as the output of $f1^*$. The output of $h2$ is split in the same vein, to obtain the outputs for $f5$ and $f2$. The basis function $h3, h4, h5$ are used as-is for the output of $f3, f4, f5^*$. To highlight this almost one-to-one relation between the basis functions and their respective AKA counterparts and to support an intuitive understanding of the implications of our attacks, we will simply refer to the basis function $h1, \ldots, h5$ as the functions $f1, \ldots, f5$ for the remainder of this paper. This is also done to emphasize that vulnerabilities in the basis functions translate into immediate insecurities of their respective AKA counterparts.

All functions in the Milenage algorithm use AES as the underlying block cipher E_K. The cipher is keyed with the network authentication key K, a 128-bit-string shared between the operator and the subscriber. The bit-strings $c_1, \ldots, c_5$

[1] The standard denotes the basis functions as OUT1, ..., OUT5

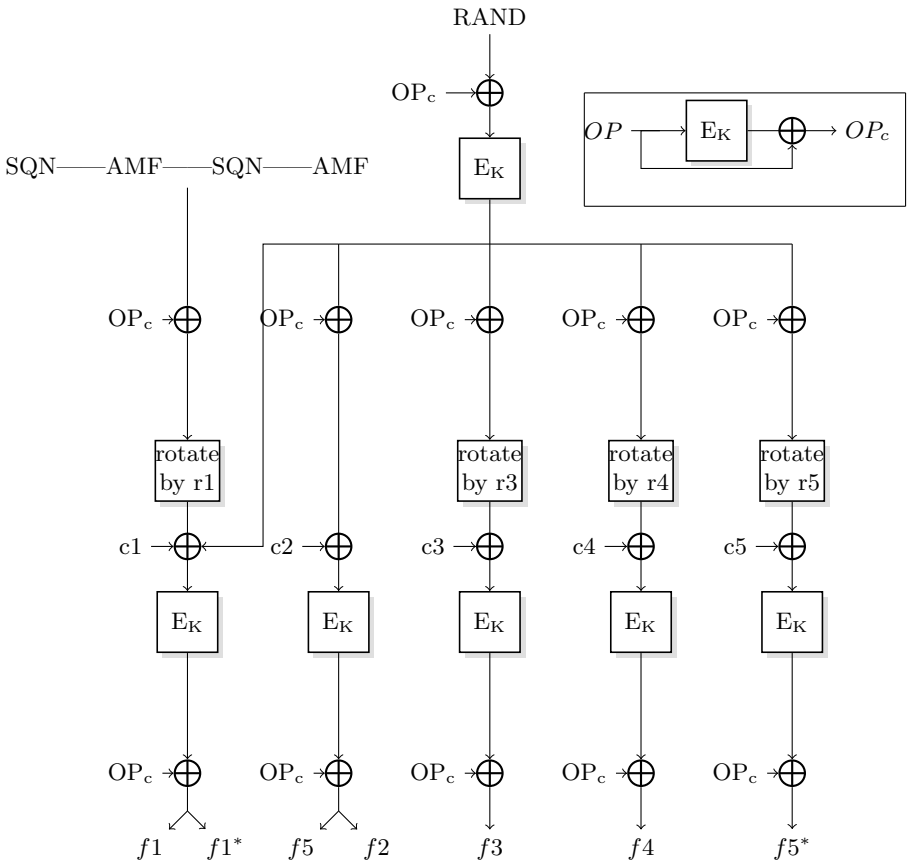

Fig. 2. The Milenage algorithm set as standardized by 3GPP [3]. The outputs of the five Milenage basis functions are mapped almost one-to-one to the seven required outputs.

and $r_1, \ldots, r_5$ are public constants which are defined in the standard. Notably, $r_2 = 0$ and $c_1 = 0$. As additional key material, the OP_c bit-string is derived from a (potentially secret) constant OP, defined by the operator. The operator provides the additional 128-bit string OP, which was intended to provide separation between different operators [3]. The per-subscriber secret OP_c is then derived as $OP_c = E_K[OP] \oplus OP$. Note that the OP-bit string is never used directly in the Milenage algorithm set, only the derived value OP_c. As such, it suffices to store the OP_c bit-string on a subscriber's SIM card, without ever revealing the operator constant OP.

There are no requirements on how the operators generate and manage the OP-bit string. It is conceivable that each operator uses the same OP bit-string for all handed-out SIM cards, but the operator could also rotate the OP for every batch of produced SIM cards. Although the Milenage algorithm set is designed to be secure even if the OP is public, in practice, operators do not reveal the value of OP. Instead of the OP, they store the OP_c bit-string on the SIM card. In the present analysis, we will show attacks for both the case when the OP bit-string is known and when it is secret.

2.3 Classical Cryptanalysis of Milenage algorithms

The Milenage algorithm set was designed to fulfill the following security requirements, as specified in [4]:

1. *Without knowledge of secret keys, the functions f1, f1*, f2, f3, f4, f5 and f5* should be practically indistinguishable from independent random functions of their inputs (RAND——SQN——AMF) and RAND. Examples: Knowledge of the values of one function on a fairly large number of given inputs should not enable its values to be predicted on other inputs. [...]*
2. *It should be infeasible to determine any part of the secret key K, or the operator variant configuration field, OP, by manipulation of the inputs and examination of the outputs to the algorithm.*
3. *Events tending to violate criteria 1 and 2 should be regarded as insignificant if they occur with probability approximately 2^{-128} or less (or require approximately 2^{128} operations).*
4. *Events tending to violate criteria 1 and 2 should be examined if they occur with probability approximately 2^{-64} (or require approximately 2^{64} operations) to ensure that they do not have serious consequences. Serious consequences would include recovery of a secret key, or ability to emulate the algorithm on a large number of future inputs.*

So far, no attack violating this criteria has been identified. Simplified versions (not using the constant OP_c) of the Milenage algorithm set have been proven to be pseudorandom under the assumption that the kernel function E_K is a random permutation [4, 18]. The proof gives rise to a lower bound of 2^{64} queries for attacks on the Milenage algorithms. This lower bound is tight, i.e., 2^{64} queries

suffice to identify collisions between the functions $f1$ and $f2$ or in the function $f1$ itself. Once identified, a collision allows an attacker to perform existential forgery [4]. For a full key recovery however, no attacks that perform better than exhaustive search are known. The brute-force attacks amount to a complexity of $O\left(2^{|K|}\right)$ if the OP bit-string is known, and $O\left(2^{|K|+|OP_c|}\right)$ if OP is unknown.

2.4 Quantum Computation

For a thorough introduction to quantum computing, we refer to the accessible exposition of [34]. Briefly, quantum computation can be described as follows. Quantum computation is usually modelled in the quantum circuit model. A quantum circuit consists of a sequence of quantum gates, acting on logical qubits. A *qubit* is encoded in the state of a system, which is described by a vector in a 2-dimensional Hilbert space. This vector describes a complex linear superposition of two computational basis state vectors $|0\rangle$ and $|1\rangle$, i.e, $\alpha_0 |0\rangle + \alpha_1 |1\rangle$, where α_0, α_1 are called the complex amplitudes of the basis states and adhere to the normalization constraint $|\alpha_0|^2 + |\alpha_1|^2 = 1$. An n-qubit state $|\psi\rangle$ is described by the complex linear superposition over all 2^n computational basis states $|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x_1, \ldots, x_n\rangle$, where again it must hold that $\sum_x |\alpha_x|^2 = 1$. *Measuring* a state $|\psi\rangle$ will output the label x with probability $|\alpha_x|^2$ and leave the system in state $|x\rangle$. *Quantum gates* that act on n qubits are unitary operators U that transform a quantum state $|\psi\rangle$ into a quantum state $U|\psi\rangle$.

Quantum Oracles and Quantum Complexity When acting on a function $f : \{0,1\}^n \to \{0,1\}^n$, quantum computation requires some kind of oracle access to this function. The oracle access is usually given through a unitary operator $\mathcal{O}_f$, that performs the following calculation $\mathcal{O}_f : |x\rangle \otimes |y\rangle \to |x\rangle |y \oplus f(x)\rangle$, where $x, y \in \{0,1\}^n$ and $|x\rangle, |y\rangle$ are the corresponding quantum states.

There are multiple ways to measure the complexity of quantum algorithms. We will focus here on two fundamental dimensions. The *query* complexity and the *time* complexity. Query complexity measures the number of accesses to the oracle $\mathcal{O}_f$, while *time* complexity is measured by the depth of the respective quantum circuit consisting of elementary gate operators from a universal quantum gate set, cf. [34]. We will use the terms *time* complexity and *depth* of a quantum circuit interchangeably.

We note here that this model abstracts away constraints that arise when actually implementing physical systems for quantum computation. For example, instead of measuring just the depth of the circuit, it has been proposed to include also the number of qubits (the width of the circuit) [22], to account for the fact that ensuring coherence of idle qubits might be costly. Unless otherwise mentioned, our work will focus on the time and query complexity of the described attacks. Accounting for other metrics would require to model the designed circuits in more detail, which we leave as future work.

2.5 Attacker Model

Almost all attacks described in this paper assume access to an encryption oracle which can be queried with arbitrary plaintexts. This follows the security model of a chosen plaintext attack. In quantum cryptanalysis, the attacker's capabilities are additionally determined by the kind of queries that are allowed to this oracle, namely whether only classical or also superposition queries are allowed.

In more detail, let $F = \{f_k : \{0,1\}^n \to \{0,1\}^n\}_{k\in\{0,1\}^n}$ be a family of functions indexed by k and assume that for any given $k, x \in \{0,1\}^n$, there exists a polynomial-time algorithm to compute $f_k(x)$. Intuitively, each function f_k defines encryption under key k. For a given function f_k sampled from F, the attacker is given oracle access to f_k, denoted by $\mathcal{O}_{f_k}$. Following other quantum cryptanalytic works [23, 41], we will consider two quantum adversary models, distinguished by the capabilities of their oracle access.

In the *standard security* model, or Q_1 model, the attacker can only make classical queries to the function f_k. In this case, the oracle $\mathcal{O}_{f_k}$ is a classical function $\mathcal{O}_{f_k} : \{0,1\}^n \mapsto \{0,1\}^n$.

In the *quantum security model*, or Q_2 model, the attacker is allowed to query the oracle in superposition. That is, the attacker can provide as input to the oracle $\mathcal{O}_{f_k}$ a superposition $\sum_{x,y} \lambda_{x,y} |x\rangle |y\rangle$ and the oracle will return the output $\sum_{x,y} \lambda_{x,y} |x\rangle |y \oplus f_k(x)\rangle$. Note that quantum security implies standard security.

We stress that even in the Q_1 model, the attacker can still guess the key k and then construct (and access) a quantum circuit that, given any $k, x \in \{0,1\}^n$, efficiently evaluates $f_k(x)$. This quantum circuit can receive as input any superposition of k and x. We will make use of this *offline* computation later on.

Note that all Milenage functions $f1, \ldots, f5$ can be viewed as a function family F, where generating a random secret key k amounts to sampling a function from the family F. The attacker is given access to an oracle $\mathcal{O}_{f_k}$, which evaluates a function f_k with a fixed key k, where k is not known by the attacker.

3 The Quantum Cryptanalysis Toolbox

In recent years, symmetric cryptography has received increasing scrutiny with respect to resilience against quantum attacks. This quantum cryptanalysis of symmetric cryptography has mostly uncovered new attacks in the Q_2 model, but also yielded more than quadratic speedups (over classical attacks) in the Q_1 model [9, 11, 26]. Most of the cryptanalytic works present quantum algorithms that equip quantum attackers with powerful attack primitives that can be used as a black box. We follow this approach and present in this section a *quantum toolbox*. I.e., a set of algorithms that facilitate cryptanalytic attacks on symmetric key cryptography. To keep our work accessible to researchers outside of the quantum community, we will hereafter use these algorithms only as a black box.

The quantum cryptanalysis presented in this paper is based on three algorithms. Grover's algorithm to speed up exhaustive search, Simon's algorithm to

identify a hidden period, and the offline version of Simon's algorithm, which combines the two former algorithms to speed up attacks in the Q_1 model. In this section, we will briefly describe the intuition of the relevant algorithms, the problems they solve, the requirements for their usage and their respective complexity. For the remainder of this work, we will then use these algorithms as a black box and focus our analysis on classical constructions that will then allow us to employ quantum algorithms in a simple fashion.

3.1 Grover's Algorithm: Fast unstructured search

In his seminal work, Grover [19] described an algorithm that achieves a quadratic speedup when performing an unstructured, brute-force search. We state the main result as relevant for this paper as follows, where we ignore small constants in Grover's time and query complexity and also the extremely high success probability for better readability.

Theorem 1 (Grover's Algorithm). *Consider a function $f : \{0,1\}^n \to \{0,1\}$, such that 2^t inputs map to 1 and the rest maps to 0. Given quantum oracle access to the function f, Grover's algorithm finds a preimage of 1, i.e., a $k \in \{0,1\}^n$ satisfying $f(k) = 1$, in $O\left(\sqrt{2^n/2^t}\right)$ time and oracle queries. If there is exactly one preimage of 1, i.e. only one k such that $f(k) = 1$, then Grover's algorithm finds this k in $O\left(\sqrt{2^n}\right)$ time.*

Intuitively, Grover's algorithm "cooks" a solution k_0, such that $f(k_0) = 1$, by constructing an equal superposition over all inputs in the domain of f and repeating a sub-procedure that increases the amplitude of k_0 while decreasing all other amplitudes. For a detailed explanation, we refer the reader to the standard literature [19, 34]. Note that Grover's algorithm requires quantum oracle access to f.

In quantum cryptanalysis, Grover's algorithm is typically used to speed up the exhaustive search (bruteforce) of a key. To this end, an attacker can construct a quantum circuit for a given cipher, e.g., AES. This circuit will take as input a message and a key guess k^* and will return the encryption of the message under the key k^*. To then bruteforce the key for a fixed but unknown key k, the attacker first captures enough plaintext-ciphertext pairs so that the secret key is uniquely determined by those pairs. An attacker can then easily construct a quantum circuit for a function f that, on input of a key guess k^* returns 1 if k^* is equal to the correct k and zero otherwise. The construction works as follows. The quantum circuit encrypts the collected plaintexts under the key guess k^* and compares the resulting ciphertexts with the captured ciphertexts. If they match, f returns 1, otherwise f returns 0. Thus, an attacker can construct a quantum cirucit for f and then leverage Grover's algorithm to find the key k with $2^{|k|/2}$ queries to the quantum circuit implementing f.

The effectiveness of Grover attacks are limited by two factors. First, the search cannot be parallelized [6, 16]. Second, by the complexity of the circuit actually implementing the oracle f. For example, Jang et al. [21] present a circuit

for AES-128 encryption which results in a circuit depth of roughly 2^{80} gates in an end-to-end key recovery attack using Grover's search. To the best of our knowledge, this is the most efficient quantum circuit for AES presented so far.

3.2 Simon's Algorithm: Quantum Period Finding

Simon's algorithm can identify hidden period in a function f in polynomial time, given quantum oracle access to this function. This powerful primitive has been successfully used in various quantum attacks on symmetric cryptography [9, 24, 27] and to show *quantum separation*, i.e., the existence of functions that are learnable in the quantum setting, but not in the classical setting (under standard cryptographic assumptions) [36]. Formally, Simon's algorithm solves the following problem:

Definition 1 (Simon's problem). *Let $f : \{0,1\}^n \to \{0,1\}^n$ be a function that is either injective, or there exists a single period $s \neq 0^n$ such that*

$$\forall x \neq x' \colon f(x) = f(x') \iff x' = x \oplus s;$$

determine s.

Given quantum oracle access to f through an oracle $\mathcal{O}_f$, this problem can be solved with $O(n)$ quantum queries to f and $O(n^3)$ time using Simon's algorithm [37]. In summary, Simon's algorithm relies on a quantum subroutine which queries the function f with a superposition query and returns a random value y, s.t. $y \oplus s = 0$ or a random y if f is injective. After $c \cdot n$ invocations of Simon's quantum subroutine (for a small constant $c \geq 1$), we obtain n linear independent vectors $y_1, \ldots, y_n$, such that $y_i \oplus s = 0$. This gives rise to an equation system and allows us to recover s via Gaussian elimination.

Note that for cryptanalytic purposes, where f represents some sort of cryptographic construction, f does not necessarily fulfill the requirement of Simon's problem perfectly. Instead, there might be unwanted collisions in f. Kaplan et al. [24] showed that Simon's algorithm can still recover the period s efficiently, provided that the probability of an unwanted collision is bounded away from 1. They prove the following theorem.

Theorem 2 (Simon's algorithm with approximate promise). *Let $f : \{0,1\}^n \to X$ be a function with period s. Define the probability of an unwanted collision as*

$$\varepsilon(f, s) = \max_{t \in \{0,1\}^n \setminus \{0,s\}} Pr_x[f(x) = f(x \oplus t)].$$

If $\varepsilon(f, s) \leq p_0 < 1$, then with $c \cdot n$ calls of the quantum subroutine, Simon's algorithm returns s with probability at least

$$1 - \left(2 \cdot \left(\frac{1 + p_0}{2}\right)^c\right)^n.$$

Note that the theorem also holds for cases where the codomain of the function is smaller than the domain, i.e., $|X| < 2^n$. It follows from Theorem 2 that as long as $c \geq 3/(1-p_0)$ the error probability decreases exponentially in n. Thus, given a constant bound on p_0 on the probability of unwanted collision for a function f, we can recover that function's period s with $O(n)$ quantum queries and polynomial time. Throughout this paper, we will make implicit use of a related theorem. For almost all functions with large enough outputs (in terms of bit length), the impact of unwanted collisions on the query cost is negligible, c.f. [8]. This allows us to ignore the issue of unwanted collisions for the remainder of this paper at all, since we will only deal with functions that have large enough outputs.

3.3 Offline Simon's algorithm: Attacks without superposition queries

In the Q_1 model, superposition queries to an oracle $\mathcal{O}_f$ are not possible. Instead, the attacker can only query $\mathcal{O}_f$ classically. Many quantum cryptanalytic attacks on symmetric ciphers thus are not applicable in the Q_1 setting, since the attacks require superposition queries to the attacked cipher. However, even in the Q_1 setting, quantum computers can speed up attacks. Indeed, Bonnetain et al. [9] introduced a new algorithm, called the "Offline Simon's Algorithm", which leverages structural properties of cryptographic schemes to execute quantum attacks which are ways faster than their known classical counterparts [9, 12]. The "Offline Simon's Algorithm" can be divided into two phases. An online phase, in which the attacker makes classical queries to the oracle. The results of the classical queries are then used to assemble a database of function inputs/outputs in superposition. Once this database is established, an offline phase follows. In the offline phase the attacker uses the database to run a quantum search and period finding algorithms. The key idea of the offline Simon's algorithm is that the database can be reused throughout the whole offline phase, without any further additional oracle queries. Reusing the database leads can be exploited to reduce query complexity, speedup existing algorithms, or reduce memory requirements. [9].

In more detail, the offline Simon's algorithm is applicable in the following situation. Consider a function $g : \{0,1\}^n \to \{0,1\}^l$ to which an attacker has only classical oracle access and a family of functions $F = \{f_i : \{0,1\}^n \to \{0,1\}^l, i \in \{0,1\}^m\}$. Assume that given any $(i,x) \in \{0,1\}^m \times \{0,1\}^n$, there exists a polynomial-time quantum circuit to compute $F(i,x) = f_i(x)$. For example, g might be an encryption oracle for an encryption under a fixed (and unknown) key k with a cipher E, while the function $F(i,x)$ is an encryption through the cipher E under a key i that is provided as input to the circuit. Further assume that there exists an $i_0 \in \{0,1\}^m$ such that $f_{i_0} \oplus g$ has a hidden period, i.e., $f_{i_0}(x) \oplus g(x) = f_{i_0}(x \oplus s) \oplus g(x \oplus s)$ for some $s \in \{0,1\}^n$.

The following result due to Bonnetain et al. [9] shows that in this setting, the strategy described above can be used to achieve a substantial speed up over classical algorithms when searching for the value i_0 and the period s.

Theorem 3 (Asymmetric Search of a Period). *Let $F = \{f_i : \{0,1\}^n \to \{0,1\}^l, i \in \{0,1\}^m\}$ be a family of functions, define $F(i,\cdot) = f_i(\cdot)$ and let g be a function $g : \{0,1\}^n \to \{0,1\}^l$. Assume that we are given quantum oracle access to F. Further, assume that there exists exactly one $i_0 \in \{0,1\}^m$ such that $f_{i_0} \oplus g$ has a hidden period, i.e., for all $x \in \{0,1\}^n$ it holds that $f_{i_0}(x) \oplus g(x) = f_{i_0}(x \oplus s) \oplus g(x \oplus s)$ for some s. Moreover, let the probability of unwanted collisions for all $f_i \oplus g$ be bounded from above by $1/2$, i.e.,*

$$\max_{\substack{i \in \{0,1\}^m \setminus \{i_0\} \\ t \in \{0,1\}^n \setminus \{0^n\}}} Pr_x[f_i(x) \oplus g(x) = f_i(x \oplus t) \oplus g(x)] \leq \frac{1}{2}.$$

Then, offline Simon's algorithm can identify i_0 with the following complexities:

1. *If we are given classical oracle access to g, then we can identify i_0 with extremely high success probability using $O(2^n)$ classical queries to g and additional computations with a time complexity of $O((n^3 + nT_F) \cdot 2^{m/2})$, where T_F is the time required to evaluate F once.*
2. *If we are given quantum oracle access to g, then we can identify i_0 with extremely high success probability, using $O(n)$ quantum queries to g and additional computations with time complexity $O((n^3 + nT_F) \cdot 2^{m/2})$.*

The offline version of Simon's algorithm leverages Grover's algorithm to search for the i_0 such that $f_{i_0} \oplus g$ has a period, and uses Simon's algorithm as a sub-procedure in that search to verify that a given guess i^* indeed results in a period for the function $f_{i^*} \oplus g$.

In the case where only classical access to g is provided, Bonnetain et al. [9] first build up a database of all $O(2^n)$ input-outputs pairs of g to obtain a superposition

$$|\phi_g\rangle = \bigotimes^{c \cdot n} \left(\sum_{x \in \{0,1\}^n} |x\rangle \, |g(x)\rangle \right),$$

where $\bigotimes$ is the usual tensor product, cf. [34]. This database can then be used to run the above-mentioned combination of Grover and Simon without any additional classical or quantum queries to g. In the case where quantum access to g is provided, this database can be built faster by querying g in superposition directly. Note that once that i_0 such that $f_{i_0} \oplus g$ has a period s is identified, we can recover the actual period s in polynomial time using Simon's algorithm — again reusing the g-database $|\phi_g\rangle$.

Throughout this paper, we will make use of the fact that the offline Simon's algorithm is also applicable in a more generalized setting, where the attacker combines the function g with a quantum circuit through means other than xoring the results [8, 9].

Theorem 4 (Generalized Offline Simon's Algorithm). *Consider a family of functions $F_i : \{0,1\}^n \times \{0,1\}^l \to \{0,1\}^l$, indexed by $i \in \{0,1\}^m$. Let g be a function $g : \{0,1\}^n \to \{0,1\}^l$ to which the attacker has classical or quantum oracle access and $p_i : \{0,1\}^n \to \{0,1\}^n$ be a permutation. Assume that for the*

index value i_0, the function $F_{i_0}(x, g(p_{i_0}(x)))$ has some period s. The Offline Simon's algorithm can identify i_0 with extremely high success probability, with the following complexities:

1. *If we are given classical oracle access to g, then we can identify i_0 using $O(2^n)$ classical queries to g and additional computations with time complexity $O((n^3 + nT_F) \cdot 2^{m/2})$, where T_F is the time required to evaluate F once.*
2. *If we are given quantum oracle access to g, then we can identify i_0 using $O(n)$ quantum queries to g and additional computations with time complexity $O((n^3 + nT_F) \cdot 2^{m/2})$.*

In the same vein as Simon's algorithm, the offline Simon's algorithm can deal with unwanted collisions; again, for functions with large enough output the impact of unwanted collisions can be neglected [8].

4 Quantum Cryptanalysis of the Milenage algorithms

The main idea of this paper is to leverage the above described quantum toolbox to perform a quantum cryptanalysis of the Milenage algorithm set. To this end, we extend existing attacks on symmetric ciphers to perform forgery attacks or recover the secret key K and the bit-string OP_c.

To describe the complexities of the presented attacks, we will consider three parameters:

- the length of the secret key K,
- the length of the OP_c bit-string, and
- the block length of the underlying block-cipher E_K, which we denote by $|M|$.

Note that for the current Milenage configuration it holds that $|K| = 128, |OP_c| = 128$ and $|M| = 128$. Quantum security considerations for 5G have proposed to increase the key-size $|K|$ to 256 bits [29, 40]. With this we can summarize our four different attacks as follows.

1. For reasons of (exposition) completeness, we include the trivial Grover attack that results in a quadratic reduction of the query complexity of exhaustive key search.
2. A quantum slide attack against the $f2$ function, which reduces the complexity of recovering the secret key material in case the OP bit-string is not known. If quantum superposition access to $f2$ is granted, the attacker can acquire the OP_c and the key K with only $O(|M|)$ superposition queries and $\tilde{O}\left(2^{|K|/2} \cdot \mathtt{T}_{\mathtt{QAES}}\right)$ time. If the attacker is given only classical access to $f2$, then we require $O(2^{|M|})$ online classical queries, and the attack has a time complexity of $\tilde{O}\left(2^{|M|} \cdot \mathtt{T}_O + \mathtt{T}_{\mathtt{QAES}} \cdot 2^{\frac{|K|}{2}}\right)$. To the best of our knowledge, recovering the network authentication key K as well as the OP_c bit-string via a classical slide attack requires $O\left(2^{\frac{M}{2}}\right)$ oracle quries and $O\left(2^{|K|+\frac{|M|}{2}}\right)$ operations.

3. A quantum polynomial time existential forgery attack on the MAC function $f1$, assuming quantum superposition access to $f1$. Classical attacks that achieve existential forgery on the $f1$ cipher require $O(2^{|M|/2})$ operations and queries.
4. A quantum related key attack against Milenage, which can recover the secret key in polynomial time in the Q_2 model, and in $\tilde{O}\left(2^{(|K|+|OP_c|)/3}\right)$ time and queries in the Q_1 model.

4.1 The Grover Key Recovery for $f1, \ldots, f5$

We first describe the most obvious attack on the Milenage algorithms, that gives an upper bound on the complexity of quantum attacks. Note that the Milenage algorithms only rely on AES encryption and the xor operation — both of these operations can be fully simulated by a quantum computer [42]. We can thus use Grover to execute the following attack:

1. Using classical oracle access to one of the functions $f1, \ldots, f5$, obtain enough function input/outputs pairs $(c_1, m_1), \ldots, (c_r, m_r)$ to uniquely determine the network authentication key K and — if required — the bitstring OP_c.
2. Given these plaintext/ciphertext pairs, we can construct a quantum circuit for the following function f: on input of a key guess K^*, OP_C^*, return 1 if $K^* = K$ and $OP_c^* = OP_c$ and zero otherwise. This circuit can be constructed as described in Section 3.1.
3. By this quantum circuit, we now have quantum oracle access to the function f. This allows us to apply Grover's algorithm to search for the key K and the bit-string OP_c.

With Theorem 1, the attack can recover the key with a circuit of depth $O\left(2^{|K|/2} \cdot \mathtt{T}_{\mathtt{QAES}}\right)$ or $O\left(2^{\frac{|K|+|OP_c|}{2}} \cdot \mathtt{T}_{\mathtt{QAES}}\right)$ if the bit-string OP_c is not known.

4.2 Quantum Slide Attacks Against $f2$

Bonnetain et al. [9] describe that the offline Simon algorithm can be used to execute a quantum slide attack against a 2-round self-similar cipher. A self-similar cipher builds upon a block cipher E to encrypt a message m, using two keys k_1, k_2 in the following way:

$$iFX(m) = E_{k_2}[E_{k_2}[m \oplus k_1] \oplus k_1] \oplus k_1.$$

The attack described by Bonnetain et al. [9] yields a speedup compared to classical attacks. This *quantum slide attack* can be adapted to work on the $f2$ function as well.

To this end, we first show how the $f2$ function can be transformed into a 2-round self-similar cipher and then describe how the attack described by Bonnetain et al. [9] can be applied to our construction. This leads to an attack that reduces the additional security provided by the OP_c bit-string, a value which is unknown in practice.

In more detail, recall that function f_2 is defined as

$$f2(m) = E_K[rot_{r_2}(E_K[m \oplus OP_C] \oplus OP_C) \oplus c_2] \oplus OP_c.$$

Now, the standard defines r_2 as $r_2 = 0$, which simplifies $f2$ to

$$f2(m) = E_K[E_K[m \oplus OP_C] \oplus OP_C \oplus c_2] \oplus OP_c$$

To transform $f2$ into a self-similar cipher, we define the function f_2', which for each input m instead queries f_2 for $m \oplus c_2$ and then xors the result with c_2. I.e.,

$$\begin{aligned} f2'(m) &\stackrel{\text{def}}{=} f2(m \oplus c_2) \oplus c_2 \\ &= E_K[E_K[m \oplus c_2 \oplus OP_C] \oplus OP_C \oplus c_2] \oplus OP_c \oplus c_2. \end{aligned}$$

Note that c_2 is public. As a result, if the attacker has (quantum) oracle access to $f2$, the attacker can easily construct a quantum circuit to also have (quantum) oracle access $f2'$. Clearly, $f2'$ follows the description of a self-similar cipher, as visualized in Figure 3.

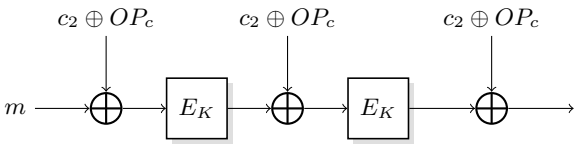

Fig. 3. The $f2'$ function, which now resembles an iterated FX cipher.

This enables us to execute the attack presented in [9], which we now describe in the following. Define the functions p_i, F_i, g as follows:

$$F_i((b,x),y) \stackrel{\text{def}}{=} \begin{cases} y \oplus x & \text{if } b = 0 \\ E_i(y) \oplus x & \text{if } b = 1 \end{cases} \qquad p_i((b,x)) \stackrel{\text{def}}{=} \begin{cases} E_i(x) & \text{if } b = 0 \\ x & \text{if } b = 1 \end{cases}$$

$$g(x) \stackrel{\text{def}}{=} f2'(x).$$

We combine now the above functions into a function F_i^*, indexed by i, which will have the desired hidden period,

$$F_i^*(b,x) \stackrel{\text{def}}{=} F_i((b,x), g(p_i(b,x))).$$

Note that for a given i, an attacker can easily construct an efficient quantum circuit for $F_i((b,x),y)$ and $F_i^*(b,x)$.

The function $F_k^*(b,x) = F_k((b,x), g(p_k(b,x)))$ has a hidden period $(1, OP_c \oplus c_2)$, as shown in Appendix C. This is sufficient to apply the offline Simon's algorithm. Armed with Theorem 4 and the above definitions, we arrive at the following complexities.

- In the Q_2 setting, the attack requires $O(|M|)$ superposition queries to $f2$ and $\tilde{O}\left((|M| \cdot \mathtt{T}_{\mathtt{QAES}}) \cdot 2^{|K|/2}\right)$ time.
- In the Q_1 setting, the attack requires more time and queries to prepare the database of g's input-output pairs. To this end, the attacker needs to query $f2'(x)$ for all possible $2^{|M|}$ inputs. Once the database is prepared, the attacker can recover the key K as well as the OP_c bit-string via the offline Simon's algorithm. As such, the attack requires $O(2^{|M|})$ online classical queries, and has an *additional* time complexity of $\tilde{O}\left(|M| \cdot \mathtt{T}_{\mathtt{QAES}}) \cdot 2^{\frac{|K|}{2}}\right)$. For the current Milenage configuration, this results in $c \cdot 2^{128}$ superposition queries and altogether $c \cdot (2^{128} \cdot \mathtt{T}_O + \mathtt{T}_{\mathtt{QAES}} \cdot 2^{64})$ operations for a small constant c. Note that while this is no improvement over the trivial Grover attack if $|K| = 128$, the advantage of the quantum slide attack shows when increasing the AES key length to 256 bit. Then, the quantum slide attack requires $c \cdot (2^{128} + 2^{128} \cdot \mathtt{T}_{\mathtt{QAES}})$ operations, while the Grover attack requires $c \cdot (2^{384/2}) \cdot \mathtt{T}_{\mathtt{QAES}} = c \cdot 2^{192} \cdot \mathtt{T}_{\mathtt{QAES}}$ operations, for a small constant c.

To the best of our knowledge, the best classical attack against the $f2$ construction — when both the OP bit-string as well the network authentication key K are unknown — is a slide attack as well. The attacker guesses a key $i \in \mathcal{K}$ and tries to find a collision in the function $F_i^*(b, x)$ to recover the period $(1, OP_c \oplus c_2)$. The attack requires $O\left(2^{\frac{M}{2}}\right)$ classical queries to the encryption oracle and approximately $O\left(2^{|K|+\frac{|M|}{2}}\right)$ time.

Therefore, the presented quantum slide attack reduces the additional security provided by the OP_c bit-string significantly.

4.3 Existential forgery of $f1$

Our third attack is based on the seminal work of Kaplan et al. [24], who describe a polynomial time existential forgery attack against a CBC-MAC construction in the Q_2 model. As a result, if superposition queries against the CBC-MAC oracle are allowed, CBC-MACs must be considered insecure. The attack can be extended to an attack that allows for polynomial time existential forgery against the $f1$ function from the Milenage algorithm set. In the following, we provide the details of our novel quantum attack.

In summary, the attack assumes superposition access to an oracle $\mathcal{O}_{f1_{K,OP_C}}(x, y) = f1_{K,OP_c}(x, y)$, invoking the function $f1$ on input (x, y) with a fixed network authentication key k and fixed value OP_c. Given this access, the attacker can efficiently construct $q + 1$ outputs of the function $f1_{K,OP_c}$ after issuing a total of q quantum and classical queries to the function $f1_{K,OP_c}$.

Before we provide the details of the attack, recall that the function $f1$ is defined as

$$\begin{aligned} & f1_{K,OP_c}(RAND, IN1) \\ \overset{\text{def}}{=}\ & E_K[E_K[RAND \oplus OP_C] \oplus rot_{r_1}(IN1 \oplus OP_c) \oplus c_1] \oplus OP_c. \end{aligned}$$

Also, for the sake of brevity, we will set $x = RAND$, and $y = IN1$, where $x, y \in \{0,1\}^{|M|}$. Then, the function $f1$ can be a bit "shortened" to

$$f1_{K,OP_c}(x,y) = E_K[E_K[x \oplus OP_C] \oplus rot_{r_1}(y \oplus OP_c) \oplus c_1] \oplus OP_c.$$

To now perform an existential forgery attack, pick two arbitrary bit-strings $\alpha_0, \alpha_1 \in \{0,1\}^{|M|}$ with $\alpha_0 \neq \alpha_1$. We then define the following function $f' : \{0,1\} \times \{0,1\}^{|M|} \to \{0,1\}^{|M|}$ by

$$\begin{aligned} & f'(b,y) \\ \stackrel{\text{def}}{=}\ & f1_{K,OP_c}(\alpha_b, y) \\ =\ & E_K[E_K[\alpha_b \oplus OP_C] \oplus rot_{r1}(y) \oplus rot_{r1}(OP_c) \oplus c_1] \oplus OP_c. \end{aligned}$$

Clearly, if an attacker has access to a quantum oracle for $f1_{K,OP_C}$, the attacker can construct an efficient quantum circuit for f' as well. As shown in Appendix D, the function f' has the hidden period $(1, rot_{r1}^{-1}(\alpha_0^* \oplus \alpha_1^*))$, where $\alpha_b^* = E_k[\alpha_b \oplus OP_c]$. This hidden period can be recovered in polynomial time using Simon's algorithm. Once an attacker obtained the period $(1, rot_{r_1}^{-1}(\alpha_0^* \oplus \alpha_1^*))$, the attacker can easily perform an existential forgery. Assume the attacker knows the value $t = f1_{K,OP_c}(\alpha_0, x)$, where $x \in \{0,1\}^{|M|}$. Then he also knows the output of the function call $f1_{K,OP_c}(\alpha_1, x \oplus rot_r^{-1}(\alpha_0^* \oplus \alpha_1^*)) = f1_{K,OP_c}(\alpha_0, x) = t$. Since the $f1$ function is intended to be used as a MAC, this amounts to an existential forgery attack.

The attacks proceeds then as follows.

1. Recover the hidden period $(1, rot_{r1}^{-1}(\alpha_0^* \oplus \alpha_1^*))$ using Simon's algorithm. Let q' denote the number of quantum queries made through running Simon's algorithm.
2. Repeat the following steps $q' + 1$ times:
   (a) Pick an arbitrary bit-string $y \in \{0,1\}^{|M|}$.
   (b) Query the function $f1_{K,OP_c}$ on input (α_0, y) to obtain $t = f1_{K,OP_c}(\alpha_0, y)$.
   (c) The same value t is also a value output/MAC tag for the input $(\alpha_1, y \oplus rot_r^{-1}(\alpha_0^* \oplus \alpha_1^*))$

This will produce a total of $2q' + 2$ tags after issuing only $2q' + 1$ queries. Overall the attack has a query complexity of $O(|M|)$ quantum queries to $f1_{K,OP_c}$ and $O(|M|^3)$ classical computation time. For the Milenage key lengths, this translates to $c \cdot 128$ quantum queries for a small constant c and a negligible amount of computation.

Resistance against classical existential forgery attacks is a design goal of the $f1$ function [4] – our *quantum existential forgery* attack demonstrates that this resistance does not transfer to the quantum security setting.

4.4 Quantum Related Key Attacks against $f1, \ldots, f5$

Related key attacks, as introduced by Biham [7], consider attackers that can request encryption under multiple related keys. The exact values of the keys are

unknown, but the way in which the keys are related is known to the attacker. The attacks can be modelled through a related key oracle, which provides the attacker access to encryption of a chosen-plaintext under related keys. Related key attacks are of interest because they have practical implications, for example when conducting fault-injection attacks. Recent works have shown that related key attacks on block ciphers can be sped up through quantum computers, both in the Q_2 as well as the Q_1 model. In the Q_2 model, with quantum superposition queries to the related key oracle, related key attacks can break any block cipher in polynomial time [35]. Using the offline Simon algorithm, the attack from [35] can be adapted to yield a super-quadratic speedup in the Q_1 model as well. Both attacks assume the following attacker model. For a given block-cipher E with a fixed secret K, the attacker has access to a related key oracle $\mathcal{O}_{E,K}$ defined as follows. The oracle $\mathcal{O}_{E_K}$ takes as input a bitmask L and a bit string x and outputs $E_{K\oplus L}(x)$.

Considering this attacker model, classical related key attacks on an ideal block cipher require at least $2^{n/2}$ operations, where n is the key length and the bound is tight, cf. [39].

In this section, we will describe the attacks in detail and show how to apply these attacks to the Milenage algorithm set, yielding a polynomial time attack in the Q_2 model, and a super-quadratic speedup over classical attacks in the Q_1 model. The described attacks can be mounted on all Milenage functions $f1, \ldots, f5$, regardless of whether the OP bit string is known or unknown. To focus on an intuitive understanding, we will assume that the OP bitstring is public and thus the functions $f1, \ldots, f5$ take only the network authentication K as key material. The analysis for the case when OP is unknown follows in an analogue fashion.

In the following, we denote by f the Milenage function under attack. Then, for a given function f_K, we assume that the attacker has access to an $\mathcal{O}_{f_K}$ that takes as input a bitmask $L \in \{0,1\}^n$ and a bit string $x \in \{0,1\}^n$ and outputs $f_{K\oplus L}(x)$, i.e., $\mathcal{O}_{f_k}(L, x) = f_{K\oplus L}(x)$. In the Q_2 model, the attacker has superposition access to this oracle, while in the Q_1 model, the attacker only has classical access.

Quantum Related Key Attacks with Superposition Access The quantum related key attacks described by Roetteler and Steinwandt [35] can be transferred in a one-to-one fashion to attack the Milenage algorithm set in the attacker model described above. Their attack works as follows.

Let $c = (c_1, \ldots, c_l)$ and $m = (m_1, \ldots, m_l)$ be a set of output-inputs pairs $c = (f_K(m_1), \ldots, f_K(m_l))$ such that (c, m) uniquely determines K. Assume an attacker has superposition access to a related key oracle for

$$\mathcal{O}_{f_K}(s, m) = f_{K\oplus s}(m) = (f_{K\oplus s}(m_1), \ldots, f_{K\oplus s}(m_l)).$$

Then, define the following mapping

$$f'(s) \stackrel{\text{def}}{=} \{f_{K\oplus s}(m), f_s(m)\}.$$

Given quantum access to a related key oracle oracle $\mathcal{O}_{f_K}(s, m)$ for f_K, one can construct an efficient quantum circuit for f'. To be efficiently encodable, f' outputs can be encoded as integers [35].

The mapping f' is two-to-one with period K, as shown below. Using Simon's algorithm, we can recover this period efficiently with only a linear number of queries to the related key oracle.

To see why f' is 2-to-1 with period K, let s, s' be two different bit-strings such that $f'(s) = f'(s')$ and assume $K \neq 0^n$. We consider two cases.

1. Assume $f_s(m) = f_{s'}(m)$. As we choose the plaintexts $m = (m_1, \ldots, m_l)$ so that they uniquely determine the key, this would imply $s = s'$, which contradicts our assumption.
2. Now let $f_s(m) \neq f_{s'}(m)$. Thus, if $f'(s) = f'(s')$, then $f_{K \oplus s}(m) = f_{s'}(m)$. The choice of plaintexts implies $K \oplus s = s'$.

Quantum Related Key Attacks without Superposition Access In the Q_1 setting, the attacker only has classical access to the related key oracle $\mathcal{O}_{f_K}(s, m)$. However, leveraging the offline Simon's algorithm, the attacker can still achieve a super-quadratic speedup over classical attacks [9]. We now show how to apply the offline Simon related key attack as stated by Bonnetain et al. [9] to the Milenage algorithm set.

Intuitively, the attack works by dividing the key k and the bitmask l into two parts, i.e., $k = k_1||k_2$, $l = l_1||l_2$ where $l_1, k_1 \in \{0,1\}^{|M|/3}$. We then query the oracle $\mathcal{O}_{f_K}$ for each possible l_1 and construct a quantum circuit F so that $F_{k_2}(l) \oplus g(l)$ has period k_1, where g is a function derived from the related key oracle. This allows us to employ the offline Simon algorithm.

Let $l = l_1||l_2$, where $l_1 \in \{0,1\}^{|M|/3}$, $l_2 \in \{0,1\}^{|M| \cdot 2/3}$ and define the following function $g : \{0,1\}^{|M|/3} \to \{0,1\}^{l \cdot |M|}$ by

$$g(l_1) \stackrel{\text{def}}{=} \mathcal{O}(l_1||0^{n\frac{2}{3}}) = f_{(k_1||k_2) \oplus (l_1||0^{2/3 \cdot |M|})}(m).$$

Moreover let F be a family of functions indexed by h so that

$$F_h(j) = f_{j||h}(m).$$

Clearly F can be efficiently represented as a quantum circuit, while querying g requires oracle access. The function $F_{k_2}(l) \oplus g(l)$ has period k_1. Thus, we have a family of functions F such that there exists a k_2 so that $f_{k_2} \oplus g$ has a hidden period. This suffices to apply the offline Simon's algorithm to recover the key part k_2. Once we obtain the k_2, we can efficiently recover k_1 as well.

Applying now Theorem 3, the attack requires $O(2^{|K|/3})$ classical queries to the related key oracle and a has a time complexity of $\tilde{O}\left(2^{\frac{|K|}{3}} \cdot \mathtt{T}_{\mathtt{QAES}}\right)$. If the OP bit-string is known, this translates to approximately 2^{43} oracle queries and encryption operations. If the OP bit-string is not known, then the attack requires approximately $2^{85.3}$ oracle queries and encryption operations.

To see why the function $F_{k_2}(l) \oplus g(l)$ has period k_1 note that

$$\begin{aligned} F_{k_2}(l \oplus k_1) \oplus g(l \oplus k_1) &= f_{l \oplus k_1 || k_2}(m) \oplus f_{(k_1 \oplus l \oplus k_1) || k_2}(m) \\ &= f_{l \oplus k_1 || k_2}(m) \oplus f_{l || k_2}(m) \\ &= g(l) \oplus F_{k_2}(l). \end{aligned}$$

5 Discussion

The presented attacks expose a structural weakness in the Milenage algorithm set, namely that it exhibits a structure that makes it susceptible to quantum period finding attacks. The attacks *do not* imply the Milenage is broken once general quantum computer come into existence, since the required superposition oracle is not given to the attacker in Milenage's typical use-cases.

However, they do show that Milenage cannot be considered secure in the quantum security (Q_2) setting. This result has merit in and of itself, as an absence of Q_2 attacks would have implied an absence of Q_1 attacks as well. Further research is thus required to assess whether the vulnerability in the Q_2 model transfers to further attacks in the Q_1 model or security proofs for Milenage can be established. For other ciphers, Q_2 attacks have already been elevated to the Q_1 model [11]. In addition, the Q_1 attacks we presented already improve on best-known classical attacks, as well as the trivial Grover, "quantum bruteforce" attack (depending on Milenage's configuration). On the other hand, other works have managed to established security proofs for FX-constructions in the Q_1 model [20].

The 3GPP has also standardized an alternative instantiation of the secret key functions $f1, \ldots, f5$, the TUAK algorithm set [2]. The TUAK algorithm set is based on the Keccak-f-permutation, which so far withstood quantum cryptanalysis and seemingly does not exhibit the structural properties that enabled the presented attacks. We thus conjecture it be secure against the "quantum period finding" attacks presented in this paper. In addition, the TUAK algorithm set was found to provide sufficient performance to be executed on a SIM card [28], and thus poses a (great) alternative to the Milenage algorithm set.

6 Conclusion

Given that experts increasingly view large-scale quantum computers as likely [32] and faced with the slow nature of standardization bodies, quantum security considerations for cellular networks and infrastructure need to start now.

Bringing together research results from recent quantum cryptanalytic work and synthesizing their results into a quantum toolbox, we took a step in this direction. We present various novel attacks against the Milenage algorithm set. Against the strongest (but purely theoretical) quantum adversary, Milenage must be considered insecure. We see the following research directions as necessary to ensure the security of telecommunication networks against quantum adversaries.

1. Symmetric cryptography that is used in telecommunication networks needs to be subject to scrutiny, investigating the resilience against quantum-enabled attacks. With the synthesized quantum toolbox, we hope to make this work accessible to non-quantum experts in the research community as well. This scrutiny should also encompass the investigation whether the results of our attacks can be improved.
2. It is necessary to clarify what security guarantees suffice and what kind of quantum adversary models can be ignored in quantum security considerations for cellular networks. The answer to this question can then guide the choice for appropriate cryptographic algorithms.

Standardizing an algorithm which later turns out to be vulnerable to quantum adversaries would be a disaster in a post-quantum world and should be prevented under any circumstances. To this end, this work should serve as a starting point to spark further investigations into the above-mentioned questions now, to ensure a smooth transition into quantum-resistant telecommunication networks in the future.

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A List of Abbreviations

3GPP Third Generation Partnership Project 8
AK Anomity Key 28
AKA Authentication and Key Agreement 3
AMF Authentication Management Field 8
SQN Sequence Number 8
MAC Message Authentication Code 3
HN Home Network
MME Mobility Management Entity
BS Base Station
MS Mobile Station
LTE Long-Term Evolution
EAP Extensible Authentication Protocol 28
3GPP 3rd Generation Partnership Project 8

B The AKA Protocol

The Milenage algorithm set's main usage is the AKA protocol, used for authentication and session establishment in cellular networks as well as other cellular related applications, e.g., as a variant of the Extensible Authentication Protocol (EAP), the EAP-AKA.

In summary, the LTE-AKA protocol is a challenge-response protocol that allows the subscriber to authenticate themselves to the network. The AKA protocol also derives a session key K_{ASME} that is used for encryption and integrity protection of communication at later points. The functions $f1, \ldots, f4$ from the Milenage algorithm set serve to derive a MAC, an expected response to a challenge, and the confidentiality and integrity keys (commonly denoted as CK and IK), which are in turn used to derive session keys. The function $f5$ is used to derive an Anomity Key (AK). The AK serves to mask the SQN, where the purpose of the SQN itself is to prevent replay attacks.

The authentication procedure in the fifth generation (5G) of cellular networks networks add various security and privacy enhancements to the LTE-AKA protocol, but uses the functions $f1, \ldots, f5$ in the same way. Given that the functions provide authentication and serve as a basis for later encryption and integrity protection, the security of cellular networks is completely contigent on the security of the functions $f1, \ldots, f5$.

C Proof of the Hidden Period Required for the Quantum Slide Attack

To see why $F_k^*(b,x) = F_k((b,x), g(p_k(b,x)))$ indeed has the hidden period $(1, OP_c \oplus c_2)$, first observe that

$$f2'(E_K(x \oplus OP_c^*)) \oplus (x \oplus OP_c^*) = E_K(f2'(x)) \oplus x, \tag{1}$$

where we write $OP_c^* = OP_c \oplus c_2$ for the sake of brevity. To see why Equation 1 holds, note that:

$$\begin{aligned} & f2'(E_K[x \oplus OP_c^*]) \oplus (x \oplus OP_c^*) \\ = {} & E_K[E_K[E_K[x \oplus OP_c^*] \oplus OP_c^*] \oplus OP_c^*] \oplus OP_c^* \oplus (x \oplus OP_c^*) \\ = {} & E_K[E_K[E_K[x \oplus OP_c^*] \oplus OP_c^*] \oplus OP_c^*] \oplus x \end{aligned}$$

and

$$\begin{aligned} & E_K(f2'(x)) \oplus x \\ = {} & E_K[E_K[E_K[x \oplus OP_c^*] \oplus OP_c^*] \oplus OP_c^*] \oplus x \\ = {} & f2'(E_K[x \oplus OP_c^*]) \oplus (x \oplus OP_c^*). \end{aligned}$$

Thus, it follows that $F_k^*(1,x) = F_k^*(0, x \oplus OP_c \oplus c_2)$ because

$$\begin{aligned} F_k^*(1,x) &= F_k((1,x), g(p_k(1,x))) \\ &= F_k((1,x), g(x)) \\ &= F_k((1,x), f_2'((x))) \\ &= E_k(f_2'(x)) \oplus x \end{aligned}$$

and

$$\begin{aligned} & F_k^*(0, x \oplus OP_c \oplus c_2) \\ = {} & F_k((0, x \oplus OP_c^*), g(p_k(0, x \oplus OP_c^*))) \\ = {} & F_k((0, x \oplus OP_c^*), g(E_k(x \oplus OP_c^*))) \\ = {} & f2'(E_k(x \oplus OP_c^*)) \oplus x \oplus OP_c^* \\ = {} & E_k(f2'(x)) \oplus x, \end{aligned}$$

where the last step follows from equation 1.

D Proof of the Hidden Period Required for the Existential Forgery Attack

It remains to be shown that f' as defined in Section 4.3 indeed has the hidden period $(1, rot_{r1}^{-1}(\alpha_0^* \oplus \alpha_1^*))$. To this end, we need to show that

$$f'(0,y) = f'(1, y \oplus rot_{r1}^{-1}(E_k[\alpha_0 \oplus OP_c] \oplus E_k[\alpha_1 \oplus OP_c])).$$

First, observe that by linearity of rotation it holds that

$$\begin{aligned} & f1_{K,OP_c}(x,y) \\ &= E_K[E_K[x \oplus OP_C] \oplus rot_{r_1}(y \oplus OP_c) \oplus c_1] \oplus OP_c \\ &= E_K[E_K[x \oplus OP_C] \oplus rot_{r_1}(y) \oplus rot_{r_1}(OP_c) \oplus c_1] \oplus OP_c. \end{aligned}$$

Thus, we have

$$f'(0,y) = E_K[\alpha_0^* \oplus rot_{r1}(y) \oplus rot_{r1}(OP_c) \oplus c_1] \oplus OP_c,$$

and

$$\begin{aligned} & f'(1, y \oplus rot_r^{-1}(\alpha_0^* \oplus \alpha_1^*)) \\ &= E_K[\alpha_1^* \oplus rot_{r_1}(y \oplus rot_{r_1}^{-1}(\alpha_0^* \oplus \alpha_1^*)) \oplus rot_{r_1}(OP_c) \oplus c_1] \oplus \\ & \quad OP_c \\ &= E_K[\alpha_1^* \oplus rot_{r_1}(y) \oplus rot_{r_1}(rot_{r_1}^{-1}(\alpha_0^* \oplus \alpha_1^*)) \oplus \\ & \quad rot_{r_1}(OP_c) \oplus c_1] \oplus OP_c. \end{aligned}$$

Now, using $rot_{r_1}(rot_{r_1}^{-1}(x)) = x$ we can continue as

$$\begin{aligned} &= E_K[\alpha_1^* \oplus rot_{r_1}(y) \oplus \alpha_0^* \oplus \alpha_1^* \oplus rot_{r_1}(OP_c) \oplus c_1] \oplus OP_c \\ &= E_K[rot_{r_1}(y) \oplus \alpha_0^* \oplus rot_{r_1}(OP_c) \oplus c_1] \oplus OP_c \\ &= f'(0,y), \end{aligned}$$

which indeed yields $f'(0,y) = f'(1, y \oplus rot_r^{-1}(\alpha_0^* \oplus \alpha_1^*))$.