Multi-key and Multi-input Predicate Encryption from Learning with Errors

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October 7, 2022

Abstract

We put forward two natural generalizations of predicate encryption (PE), dubbed *multi-key* and *multi-input* PE. More in details, our contributions are threefold.

- **Definitions.** We formalize security of multi-key PE and multi-input PE following the standard indistinguishability paradigm, and modeling security both against malicious senders (i.e., corruption of encryption keys) and malicious receivers (i.e., collusions).

- **Constructions.** We construct multi-key and multi-input PE supporting the conjunction of poly-many arbitrary single-input predicates, assuming the hardness of the standard learning with errors (LWE) problem.

- **Applications.** We show that multi-key and multi-input PE for expressive enough predicates suffices for interesting cryptographic applications, including non-interactive multi-party computation (NI-MPC) and matchmaking encryption (ME).

In particular, plugging in our constructions of multi-key and multi-input PE, under the standard LWE assumption, we obtain the first ME supporting *arbitrary policies* with unbounded collusions, as well as robust (resp. non-robust) NI-MPC for so-called *all-or-nothing* functions satisfying a non-trivial notion of reusability and supporting a constant (resp. polynomial) number of parties. Prior to our work, both of these applications required much heavier tools such as indistinguishability obfuscation or compact functional encryption.

Keywords: predicate encryption, non-interactive MPC, matchmaking encryption, LWE.

Contents

1 Introduction

Predicate encryption (PE) [BW07, KSW08, GVW15] is a powerful cryptographic primitive that enriches standard encryption with fine-grained access control to the encrypted data. In PE, the ciphertext is associated to both a message m and an attribute[1] x, whereas the secret key is associated to a predicate $\mathbb{P}$, in such a way that the decryption process reveals the message if and only if the attribute x satisfies the predicate $\mathbb{P}$ (i.e., $\mathbb{P}(x) = 1$). Typically, security of PE requires indistinguishability in the presence of *collusion attacks*, namely, for any pair of attributes (x^0, x^1) and for any pair of messages (m^0, m^1), ciphertexts corresponding to (x^0, m^0) and to (x^1, m^1) are computationally indistinguishable, even for an adversary possessing poly-many decryption keys $\mathsf{dk}_\mathbb{P}$, so long as $\mathbb{P}(x^0) = \mathbb{P}(x^1) = 0$ (otherwise it is easy to distinguish).

Recently, there has been a lot of progress in constructing PE supporting expressive predicates under standard assumptions [BW07, KSW08, LOS+10, OT10, OT12, AFV11, Wat12, Wee14, Att14, GVW15]. In particular, Gourbunov, Vaikuntanathan and Wee [GVW15] give a construction of PE (with unbounded collusions) for arbitrary predicates under the learning with errors (LWE) assumption.

1.1 Our Contributions

In this paper, we put forward two natural generalizations of PE which we dub *multi-key* PE and *multi-input* PE. Furthermore, we construct both multi-key PE and multi-input PE for a particular class of predicates, under the LWE assumption. As we show, the class of predicates our schemes can handle is powerful enough to yield interesting cryptographic applications, including matchmaking encryption (ME) [AFNV19, AFNV21] for arbitrary policies and non-interactive multi-party computation (NI-MPC) [HLP11] satisfying a weaker (but still non-trivial) notion of reusability. We elaborate on these contributions in Section 1.3.

Prior to our work, all of the above applications required much stronger tools such as indistinguishability obfuscation (iO) [BGI+01]. While recent work made significant progress towards basing iO on standard assumptions [JLS21, JLS22], these constructions are fairly complex and still require a careful combination of multiple assumptions (i.e., learning parity with noise, the SXDH assumption on bilinear groups, and the existence of pseudorandom generators computable in constant depth). Furthermore, such constructions are not secure in the presence of a quantum attacker. Candidate constructions of *post-quantum* iO also exist [GP21, WW21, BDGM22], but they are based on problems whose hardness is less understood.

Multi-key PE. In multi-key PE, we consider an ensemble of predicates $\mathcal{P} = \{\mathbb{P}_v\}$ indexed by a value $v \in \mathcal{V}$ which is uniquely represented as a sequence $v = (v_1, \ldots, v_n) \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$. A sender can encrypt a message under an input x using the public-key encryption algorithm $\mathsf{Enc}(\mathsf{mpk}, x, m)$. A trusted authority generates decryption keys dk_{v_i} (using the corresponding master secret key msk_i) for each $i \in [n]$, with the guarantee that, given the decryption keys $\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n}$, the receiver can decrypt successfully the ciphertext c (associated to plaintext m and attributes x), so long as $\mathbb{P}_v(x) = \mathbb{P}_{v_1, \ldots, v_n}(x) = 1$.

Security of multi-key PE says that, for any pair of attributes (x^0, x^1) and for any pair of messages (m^0, m^1), ciphertexts c associated to (x^0, m^0) and (x^1, m^1) should be computationally indistinguishable even under unbounded collusions, where the latter essentially means that the adversary can obtain decryption keys for (poly-many) arbitrary values $v_1, \ldots, v_n$ which correspond to predicates indexed by any value $v = (v_1, \ldots, v_n)$ such that $\mathbb{P}_v(x^0) = \mathbb{P}_v(x^1) =$

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[1]Sometimes, we also refer to x as the predicate input. Throughout the paper, we use the terms attribute and input interchangeably.

0. This yields so-called CPA-1-sided security. The stronger notion of CPA-2-sided security additionally allows for predicates indexed by values v such that $\mathbb{P}_v(x^0) = \mathbb{P}_v(x^1) = 1$, so long as $m^0 = m^1$. These notions mimic the corresponding notions that are already established for standard PE.

Our first result is a construction of multi-key PE, from the standard LWE assumption, supporting conjunctions of arbitrary predicates, i.e. for predicates of the form $\mathbb{P}_v(x) = \mathbb{P}_{v_1}(x_1) \wedge \ldots \wedge \mathbb{P}_{v_n}(x_n)$, where $x = (x_1, \ldots, x_n)$ and $v = (v_1, \ldots, v_n)$.

Theorem 1 (Informal). *Assuming the hardness of LWE, there exists a CPA-1-sided secure multi-key PE scheme supporting conjunctions of $n = \mathsf{poly}(\lambda)$ arbitrary predicates with unbounded collusions.*

Multi-input PE. In multi-input PE, we consider predicates $\mathbb{P}$ with n inputs, i.e. predicates of the form $\mathbb{P}(x_1, \ldots, x_n)$. A trusted authority produces encryption keys ek_i which are associated to the i-th slot of an input for $\mathbb{P}$; namely, given a (possibly secret)[2] encryption key ek_i, a sender can generate a ciphertext c_i which is an encryption of message m_i under attribute x_i. At the same time, the authority can produce a decryption key $\mathsf{dk}_\mathbb{P}$ associated to an n-input predicate $\mathbb{P}$, with the guarantee that the receiver can successfully decrypt $c_1, \ldots, c_n$, and thus obtain $m_1, \ldots, m_n$, so long as $\mathbb{P}(x_1, \ldots, x_n) = 1$.

As for security, we consider similar flavors as CPA-1-sided and CPA-2-sided security for standard PE. Namely, for any pair of sequences of attributes $(x_1^0, \ldots, x_n^0)$ and $(x_1^1, \ldots, x_n^1)$ and for any pair of sequences of messages $(m_1^0, \ldots, m_n^0)$ and $(m_1^1, \ldots, m_n^1)$, ciphertexts $c_1, \ldots, c_n$ corresponding to either $(x_1^0, m_1^0), \ldots, (x_n^0, m_n^0)$ or $(x_1^1, m_1^1), \ldots, (x_n^1, m_n^1)$ should be computationally indistinguishable. Here, we additionally consider two cases:

- In the setting with no corruptions (a.k.a. the secret-key setting), all of the encryption keys ek_i are secret and cannot be corrupted (and thus all the senders are honest).

- In the setting with adaptive corruptions, the attacker can adaptively reveal some of the encryption keys ek_i (and thus corrupt a subset of the senders).

Naturally, for both of these flavors, one can define CPA-1-sided and CPA-2-sided security with or without collusions.

Our second result is a construction of multi-input PE, from the standard LWE assumption, supporting conjunctions of $n = \mathsf{poly}(\lambda)$ arbitrary predicates *with wildcards*, i.e. for predicates of the form $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$ such that, for each $i \in [n]$, there exists a (public) wildcard input $x_i^\star$ for which $\mathbb{P}_i(x_i^\star) = 1$ for every i-th predicate $\mathbb{P}_i$.[3] Our multi-input PE construction retains its security only in the setting of no corruptions (i.e., the encryption keys ek_i are kept secret) and no collusions (i.e., the adversary only knows a single decryption key $\mathsf{dk}_\mathbb{P}$ for an adversarially chosen predicate $\mathbb{P}$).

Theorem 2 (Informal). *Assuming the hardness of LWE, there exists a CPA-1-sided secure multi-input PE scheme supporting conjunctions of $n = \mathsf{poly}(\lambda)$ arbitrary predicates with wildcards, without corruptions and without collusions.*

Our third result is a construction of multi-input PE, from the standard LWE assumption, supporting the same class of predicates as above but tolerating adaptive corruptions of up to $n-1$ parties. However, this particular scheme only supports predicates with constant arity.

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[2]This is one of the differences between multi-key PE and multi-input PE: the former has a public-key encryption algorithm, whereas the latter could have a secret-key encryption algorithm.

[3]Note that, in the setting with no corruptions, assuming the presence of a (single) wildcard $x_i^\star$ for each $\mathbb{P}_i$ does not affect the expressiveness and the security guarantees of multi-input PE. This is because the i-th sender can simply choose not to encrypt $x_i^\star$, which will not permit the receiver to evaluate $\mathbb{P}_i$ over $x_i^\star$.

Theorem 3 (Informal). *Assuming the hardness of LWE, there exists a CPA-1-sided secure multi-input PE scheme supporting conjunctions of $n = O(1)$ arbitrary predicates with wildcards, under $n - 1$ adaptive corruptions and without collusions.*

Finally, we highlight that we can extend our results to the CPA-2-sided setting. In particular, our techniques allow us to construct: (i) CPA-2-sided secure multi-key PE with unbounded collusions for $n = \text{poly}(\lambda)$, (ii) CPA-2-sided secure multi-input PE without corruptions and without collusions for $n = O(\log(\lambda))$[4], and (iii) CPA-2-sided secure multi-input PE under $n-1$ corruptions and without collusions for $n = O(1)$. However, the LWE assumption is not enough for CPA-2-sided security. Indeed, even for single-input PE for arbitrary predicates, CPA-2-sided security implies iO [BV15].

1.2 Technical Overview

We now give a high level overview of our constructions. As explained above, both our multi-key and multi-input PE constructions handle conjunctions of arbitrary predicates, i.e., predicates of the form:

$$\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n). \tag{1}$$

We start by explaining how to build multi-key PE for the above class of predicates by combining single-input PE and so-called *lockable obfuscation* [WZ17, GKW17]. Informally, a lockable obfuscation scheme allows to obfuscate a circuit $\mathbb{C}$ under a lock y together with a message m, in such a way that evaluating the obfuscated circuit, on input x, returns m if $\mathbb{C}(x) = y$. As for security, an obfuscated circuit can be simulated in a virtual black-box (VBB) fashion whenever the lock is random and unknown to the adversary. Lockable obfuscation exists under the standard LWE assumption.

Then, we explain how to build multi-input PE (for the same class of predicates) by additionally using SKE and PKE. Here, we consider two settings: without corruptions (a.k.a. the secret-key setting) and with corruptions. The former assumes that all the encryption keys (each corresponding to an input) are secret. The latter is a stronger model that allows the adversary to leak one or more encryption keys (i.e., corruption of the senders). We achieve security in each setting by changing the way lockable obfuscation is used. In particular, part of the contribution of this paper is a new technique based on nested (lockable obfuscated) circuits that execute each other. This technique allows us to construct a multi-input PE that can handle adaptive corruptions. We provide a high-level overview in the remaining part of this section. For more details, we refer the reader to Section 4 and Section 5.

Multi-key Predicate Encryption. An n-key PE allows a sender to encrypt a message m under an attribute x, by running $c \leftarrow_{\$} \mathsf{Enc}(\mathsf{mpk}, x, m)$. Similarly to single-input PE, a receiver can correctly decrypt c if it has a decryption key for a predicate $\mathbb{P}_v$, within a family $\mathcal{P}$ of predicates indexed by values $v \in \mathcal{V}$, such that $\mathbb{P}_v(x) = 1$. The main difference between single-input PE and n-key PE is that in the latter the receiver must have n independent decryption keys $(\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n})$ that uniquely represent the predicate $\mathbb{P}_v(\cdot) = \mathbb{P}_{v_1, \ldots, v_n}(\cdot)$, i.e., the decryption key associated to a particular predicate is decomposed into n decryption keys. Each decryption key dk_{v_i} is generated by the authority via $\mathsf{KGen}(\mathsf{msk}_i, v_i)$ where $(\mathsf{msk}_1, \ldots, \mathsf{msk}_n)$ are the master secret keys generated during the setup. Hence, once obtained $(\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n})$ from the authority, the receiver can decrypt the ciphertext c (encrypted under attribute x) by executing $\mathsf{Dec}(\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n}, c)$. The message is returned if the predicate $\mathbb{P}_{v_1, \ldots, v_n}(x) = 1$, where

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[4]Note that, in case of no corruptions, our CPA-1-sided construction supports $n = \text{poly}(\lambda)$. However, to achieve CPA-2-sided security we use complexity leveraging and this reduces n from $\text{poly}(\lambda)$ to $O(\log(\lambda))$.

$\mathbb{P}_{v_1,\ldots,v_n}(\cdot)$ is the predicate represented by the combination of the n decryptions keys $\mathsf{dk}_{v_1},\ldots,$ dk_{v_n}. The security of n-key PE is analogous to that of single-input PE, where the validity of the adversary A is defined with respect to the (poly-many) tuples $(\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_n})$ of n decryption keys that the adversary has access to. In particular, we consider the well-known notion of CPA-1-sided security, i.e., the attacker cannot distinguish between $\mathsf{Enc}(\mathsf{mpk}, x^0, m^0)$ and $\mathsf{Enc}(\mathsf{mpk}, x^1, m^1)$ so long as it only holds combinations of n decryption keys $(\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_n})$ such that $\mathbb{P}_{v_1,\ldots,v_n}(x^0) = \mathbb{P}_{v_1,\ldots,v_n}(x^1) = 0$ (i.e., the adversary cannot decrypt the challenge ciphertext).[5]

As explained above, we focus on conjunctions of arbitrary predicates $\mathbb{P}_{v_1,\ldots,v_n}(x) = \mathbb{P}_{v_1,\ldots,v_n}(x_1,\ldots,x_n) = \mathbb{P}_{v_1}(x_1) \wedge \cdots \wedge \mathbb{P}_{v_n}(x_n)$ as defined in Equation (1); hence, $x = (x_1,\ldots,x_n)$ and each dk_{v_i} identifies the i-th predicate of the conjunction (and, in turn, any tuple of n decryption keys uniquely identifies the global predicate). We build an n-key PE handling this class of predicates by extending the technique of Goyal et al. [GKW17], that uses lockable obfuscation to transform any CPA secure attribute-based encryption (ABE) (recall that ABE schemes only guarantee the secrecy of the message) into a CPA-1-sided secure PE (i.e., secrecy of both message and attribute). Let $\mathsf{PE}_i = (\mathsf{Setup}_i, \mathsf{KGen}_i, \mathsf{Enc}_i, \mathsf{Dec}_i)$ for $i \in [n]$ be n single-input PE schemes. In a nutshell, our n-key PE scheme $\mathsf{kPE} = (\mathsf{Setup}, \mathsf{KGen}, \mathsf{Enc}, \mathsf{Dec})$ works as follows. The setup algorithm Setup simply executes Setup_i of each PE_i and outputs the master public key $\mathsf{mpk} = (\mathsf{mpk}_1,\ldots,\mathsf{mpk}_n)$ and n master secret keys $(\mathsf{msk}_1,\ldots,\mathsf{msk}_n)$. To generate a decryption key $\mathsf{dk}_{v_i} \leftarrow_\$ \mathsf{KGen}(\mathsf{msk}_i, v_i)$ (representing the i-th predicate $\mathbb{P}_{v_i}(\cdot)$ of the conjunction), the authority can use the key generation algorithm of the i-th PE, i.e., $\mathsf{dk}_{v_i} \leftarrow_\$ \mathsf{KGen}_i(\mathsf{msk}_i, \mathbb{P}_{v_i})$. To encrypt a message m under an input $x = (x_1,\ldots,x_n)$, a sender samples a random lock y and encrypts it n times using $\mathsf{PE}_1,\ldots,\mathsf{PE}_n$, i.e.,

$$c \leftarrow_\$ \mathsf{Enc}_n(\mathsf{mpk}_n, x_n, \mathsf{Enc}_{n-1}(\mathsf{mpk}_{n-1}, x_{n-1}, \cdots, \mathsf{Enc}_1(\mathsf{mpk}_1, x_1, y))).$$

The final ciphertext of the n-key PE kPE will be the obfuscation of the circuit $\mathbb{C}_c$ under the lock y together with the message m (i.e., $\widetilde{\mathbb{C}} \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{C}_c, y, m)$), where $\mathbb{C}_c$, on input $(\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_n})$, iteratively decrypts c and returns the last decrypted value, i.e., $y = \mathbb{C}_c(\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_n}) = \mathsf{Dec}_1(\mathsf{dk}_{v_1},\cdots,\mathsf{Dec}_n(\mathsf{dk}_{v_n}, c))$. Decryption is straightforward: the receiver simply executes $\widetilde{\mathbb{C}}$ using its n decryption keys.

The CPA-1-sided security of our construction follows by the CPA security (i.e., secrecy of the message) of $\mathsf{PE}_1,\ldots,\mathsf{PE}_n$ and by the security of lockable obfuscation.[6] Intuitively, the proof works as follows. In order to be valid, an adversary A cannot hold a tuple of decryption keys $(\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_n})$ such that $\mathbb{P}_{v_1,\ldots,v_n}(x^b) = \mathbb{P}_{v_1,\ldots,v_n}(x_1^b,\ldots,x_n^b) = 1$, where $x^b = (x_1^b,\ldots,x_n^b)$ is the input chosen by A during the challenge phase, and b is the challenge bit. Since $\mathbb{P}_{v_1,\ldots,v_n}(x_1^b,\ldots,x_n^b)$ is a conjunction of arbitrary predicates (see Equation (1)), this implies that there exists an $i \in [n]$ such that $\mathbb{P}_{v_i}(x_i^b) = 0$ for every i-th decryption key dk_{v_i} obtained by A. We can leverage this observation together with the CPA security of PE_i to do a first hybrid in which the challenger computes the i-th layer of the challenge ciphertext as $\mathsf{Enc}_i(\mathsf{mpk}_i, x_i^b, 0\ldots 0)$. Now, since the lock y is not encrypted anymore, we can use the security of lockable obfuscation to do a second hybrid in which the challenge ciphertext $\widetilde{\mathbb{C}}$ is simulated by using the simulator of lockable obfuscation. In this last hybrid, the challenge ciphertext does not depend on the bit b sampled by the challenger.

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[5]Observe that the decryption keys can be interleaved. For example, starting from $(\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_i},\ldots\mathsf{dk}_{v_n})$ representing the predicate $\mathbb{P}_{v_1,\ldots,v_i,\ldots,v_n}$, the adversary can ask for an additional i-th decryption key $\mathsf{dk}_{v_i'}$ and rearrange the decryption keys as $(\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_i'},\ldots\mathsf{dk}_{v_n})$ in order to obtain the tuple representing a different predicate $\mathbb{P}_{v_1,\ldots,v_i',\ldots,v_n} \neq \mathbb{P}_{v_1,\ldots,v_i,\ldots,v_n}$.

[6]When we write CPA secure PE, without specifying 1-sided or 2-sided security, we refer to a PE scheme that guarantees only the secrecy of the message. CPA secure PE is the same as CPA secure ABE.

Despite we focused the discussion on CPA-1-sided security, we stress that the same construction achieves CPA-2-sided security if the underlying n single-input PE schemes $\mathsf{PE}_1, \ldots, \mathsf{PE}_n$ are CPA-2-sided secure, i.e., $\mathsf{Enc}(\mathsf{mpk}, x^0, m^0)$ and $\mathsf{Enc}(\mathsf{mpk}, x^1, m^1)$ are indistinguishable even when $\mathbb{P}_{v_1, \ldots, v_n}(x^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^1) = 1$ and $m^0 = m^1$.

Multi-input Predicate Encryption. We now turn to the more challenging setting of multi-input PE.[7] Here, each of the n senders can use its corresponding encryption key to independently encrypt messages under different inputs for the predicate. For this reason, the setup algorithm of n-input PE outputs n encryption keys $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n)$ and a master secret key msk. Each encryption key ek_i is given to the i-th sender and allows the latter to handle the i-th slot of a multi-input predicate. The i-th party encrypts a message m_i under an input x_i by using its encryption key ek_i, i.e., $c_i \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_i, x_i, m_i)$. On the other hand, a receiver can use the decryption key $\mathsf{dk}_\mathbb{P}$ associated to an n-input predicate $\mathbb{P}$ (recall that $\mathsf{dk}_\mathbb{P}$ is generated by the authority via $\mathsf{KGen}(\mathsf{msk}, \mathbb{P})$) to execute $\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, c_1, \ldots, c_n)$. Intuitively, the decryption algorithm returns $(m_1, \ldots, m_n)$ when $\mathbb{P}(x_1, \ldots, x_n) = 1$ where (m_i, x_i) are the message and the input associated to the i-th ciphertext c_i.

The CPA-1-sided security of n-input PE is similar to that of n-key PE, but adapted to the multi-input setting. Informally, an adversary A must not be able to distinguish between ciphertexts $(\mathsf{Enc}(\mathsf{ek}_i, x_i^0, m_i^0))_{i \in [n]}$ and $(\mathsf{Enc}(\mathsf{ek}_i, x_i^1, m_i^1))_{i \in [n]}$ where $(x_1^0, \ldots, x_n^0)$, $(x_1^1, \ldots, x_n^1)$ and $(m_1^0, \ldots, m_n^0)$, $(m_1^1, \ldots, m_n^1)$ are chosen by A. Naturally, this is subject to the usual validity condition, informally saying that A should not be able to decrypt (part of) the challenge ciphertext. This condition can assume different meanings depending on whether the encryption keys are all secret or some of them are public (or can be leaked). Because of this, we formalize security with and without corruptions. Throughout the rest of this section, we describe how CPA-1-sided security of n-input PE changes in these two settings, and give some intuition on our constructions for each setting.

Security in the secret-key setting. Here, no corruptions are allowed and thus the encryption keys are all secrets. Hence, an adversary A playing the CPA-1-sided security game has adaptive oracle access to both the key generation oracle $\mathsf{KGen}(\mathsf{msk}, \cdot)$ and to n encryption oracles $\{\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)\}_{i \in [n]}$. The latter oracles allow A to generate ciphertexts (associated to the i-th input/sender) on adversarially chosen predicate inputs and messages. Since these ciphertexts are created independently, the adversary has the power to interleave part of the challenge ciphertext $(c_1^*, \ldots, c_n^*)$ with the ciphertexts obtained trough the encryption oracles. This has a huge impact on the security of the a n-input PE scheme and on the validity condition that A must satisfy. For example, during the challenge phase, A could choose two vectors of messages $(m_1^0, \ldots, m_n^0)$ and $(m_1^1, \ldots, m_n^1)$ and two vectors of predicate inputs $(x_1^0, \ldots, x_n^0)$ and $(x_1^1, \ldots, x_n^1)$ such that for every predicate $\mathbb{P}$ (submitted to oracle $\mathsf{KGen}(m, \cdot)$) we have $\mathbb{P}(x_1^0, \ldots, x_n^0) = \mathbb{P}(x_1^1, \ldots, x_n^1) = 0$. Although the vector $(c_1^*, \ldots, c_n^*)$ can not be directly decrypted, A could still be able to decrypt part of it by leveraging the encryption oracles. In more details, A could: (i) adversarially choose x_i' such that $\mathbb{P}(x_1^0, \ldots, x_i', \ldots x_n^0) = 1$ and $\mathbb{P}(x_1^1, \ldots, x_i', \ldots x_n^1) = 0$; (ii) submit (x_i', m_i') to oracle $\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)$ and obtain c_i'; and (iii) simply decrypt the vector $(c_1^*, \ldots, c_i', \ldots, c_n^*)$. When $b = 0$ (resp. $b = 1$), the adversary knows that the challenge ciphertext must (resp. must not) decrypt successfully. This allows it to easily win the CPA-1-sided security experiment of n-input PE. As a consequence, the condition defining when A is valid depends on both the queries submitted to $\mathsf{KGen}(\mathsf{msk}, \cdot)$ and to the oracles $\{\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)\}_{i \in [n]}$. More precisely, for every decryption key $\mathsf{dk}_\mathbb{P}$ corresponding to a predicate $\mathbb{P}$, for every vector of ciphertexts obtained by interleaving the

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[7]Indeed, as we discuss in Remark 2, CPA-1-sided (resp. CPA-2-sided) secure multi-input PE for arbitrary predicates implies CPA-1-sided (resp. CPA-2-sided) secure multi-key PE.

challenge ciphertext $(c_1^*, \ldots, c_n^*)$ with the ciphertexts generated trough any of the n encryption oracles, we must have that $\mathbb{P}$ is not satisfied. This is formalized by the following condition: $\forall \mathbb{P} \in \mathcal{Q}_{\mathsf{KGen}}, \forall j \in [n], \forall i_1 \in [k_1 + 1], \ldots, \forall i_n \in [k_n + 1]$, it holds that

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\begin{aligned}
&\mathbb{P}(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) = \\
&\mathbb{P}(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 0,
\end{aligned} \tag{2}
$$

where $\mathcal{Q}_{\mathsf{KGen}}$ are the queries submitted to oracle $\mathsf{KGen}(\mathsf{msk}, \cdot)$, $(x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1)$ are the predicate inputs chosen by A during the challenge phase, and $\mathcal{Q}_i^b = \{x_i^{(1,b)}, \ldots, x_i^{(k_i,b)}, x_i^{(k_i+1,b)} = x_i^b\}$ is the ordered list composed of the k_i predicate inputs submitted to oracle $\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)$ and the challenge input x_i^b for $b \in \{0, 1\}, i \in [n]$ (observe that $\mathcal{Q}_i^0$ and $\mathcal{Q}_i^1$ are identical except for the last element). The formal security definition appears in Section 4.2.

Construction in the secret-key setting. We propose a construction of n-input PE for conjunctions of arbitrary predicates (see Equation (1)) *with wildcards* from single-input PE, lockable obfuscation, and SKE. In particular, we start from single-input PE for arbitrary predicates. Actually, it will suffice that the underlying PE itself supports the predicates $\mathbb{P}(x_1, \ldots, x_n)$ as defined in Equation (1), where we view $(x_1, \ldots, x_n)$ as a single input chosen by the sender. In addition, the predicate must have a (possibly unique) wildcard input $(x_1^\star, \ldots, x_n^\star)$ such that $x_i^\star$ satisfies every i-th predicate of the conjunction, i.e., $\mathbb{P}_i(x_i^\star) = 1$. As we will describe next, the $n-1$ subset of wildcards $(x_1^\star, \ldots, x_{i-1}^\star, x_{i+1}^\star, \ldots, x_n^\star)$ will permit the i-th sender to put a "don't care" placeholder on the slots of the other senders. This will allow the construction to deal with multiple inputs without compromising the evaluation of the predicate.

The main intuition behind our construction is to evaluate the conjunction of the predicates inside lockable obfuscation in such a way that, as soon as one of the predicates (of the conjunction) is not satisfied, both the messages and the predicate inputs remain hidden (even if another predicate $\mathbb{P}_i$ is satisfied). To accomplish that, we need to create a link between the independently generated ciphertexts (each produced by different senders). This is done by leveraging an SKE scheme as follows.

In a nutshell, the i-th secret encryption key has the form $\mathsf{ek}_i = (\mathsf{mpk}, \mathsf{k}_i, \mathsf{k}_{i+1})$ where mpk is the master public key of the single-input PE, and k_i for $i \in [n]$ is a secret key for the SKE (we also let $\mathsf{ek}_{n+1} = \mathsf{k}_1$). In order to encrypt a message m_i under an input x_i, the i-th sender samples a random lock y_i and encrypts (y_i, k_{i+1}) via the single-input PE, using the input made by all the wildcards $x_j^\star$ except for the position $j = i$, where, instead, the sender places its real input x_i, i.e., $c_i^{(1)} \leftarrow_{\$} \mathsf{Enc}(\mathsf{mpk}, (x_1^\star, \ldots, x_{i-1}^\star, x_i, x_{i+1}^\star, \ldots, x_n^\star), (y_i, \mathsf{k}_{i+1}))$. The final ciphertext c_i will be $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$, where $c_i^{(2)} \leftarrow_{\$} \mathsf{Enc}(\mathsf{k}_i, c_i^{(1)})$ and $\widetilde{\mathbb{C}}_i$ is the obfuscation of the circuit $\mathbb{C}_{c_i^{(2)}, \mathsf{k}_{i+1}}$ under the lock y_i and message m_i. Similarly to the case of multi-key PE, the latter circuit is responsible for the decryption. In particular, upon input the ciphertexts $(c_{i+1}^{(2)}, \ldots, c_n^{(2)}, c_1^{(2)}, \ldots, c_{i-1}^{(2)})$—note the order of the ciphertexts—and the decryption key $\mathsf{dk}_{\mathbb{P}}$ for $\mathbb{P}(x_1, \ldots, x_n)$, the circuit $\mathbb{C}_{c_i^{(2)}, \mathsf{k}_{i+1}}$ acts as follows:

1. Set $\mathsf{k} = \mathsf{k}_{i+1}$ where k_{i+1} is the secret key hardcoded into the circuit.

2. For $c_j^{(2)} \in \{c_{i+1}^{(2)}, \ldots, c_n^{(2)}, c_1^{(2)}, \ldots, c_{i-1}^{(2)}\}$ do:

   (a) Decrypt $c_j^{(2)}$ using the secret key k, i.e., $c_j^{(1)} \leftarrow_{\$} \mathsf{Dec}(\mathsf{k}, c_j^{(2)})$.

   (b) Decrypt $c_j^{(1)}$ using $\mathsf{dk}_{\mathbb{P}}$ in order to get (y_j, k_{j+1}). If $c_j^{(1)}$ decrypts correctly, k_{j+1} is the secret key used to encrypt the next ciphertext $c_{j+1}^{(2)}$.

(c) Set $\mathsf{k} = \mathsf{k}_{j+1}$.

3. Compute $(y_i, \mathsf{k}_{i+1}) = \mathsf{Dec}(\mathsf{dk}_{\mathbb{P}}, \mathsf{Dec}(\mathsf{k}, c_i^{(2)}))$, where $c_i^{(2)}$ is the ciphertext hardcoded into the circuit.

4. Return y_i (note that if none of the decryptions fails then y_i is the lock used to obfuscate the circuit).

By the above description, decryption is immediate: Upon input $(c_i)_{i \in [n]}$, the receiver computes $m_i = \widetilde{\mathbb{C}}_i(c_{i+1}^{(2)}, \ldots, c_n^{(2)}, c_1^{(2)}, \ldots, c_{i-1}^{(2)}, \mathsf{dk}_{\mathbb{P}})$ where $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$ and $\mathsf{dk}_{\mathbb{P}}$ is the decryption key of the underlying single-input PE for a predicate $\mathbb{P}(x_1, \ldots, x_n)$. We highlight that the combination of the SKE with the PE wildcards is what allows our construction to correctly implement the predicates of Equation (1). This is because, when $c_i^{(1)}$ correctly decrypts under the key $\mathsf{dk}_{\mathbb{P}}$ (Item 2b), we are guaranteed that $\mathbb{P}_i(x_i) = 1$ (recall that x_i is the input of the i-th sender). In particular, the latter holds as, in any other slot, the i-th sender has used the wildcards. By repeating this argument, we can conclude that $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$ is satisfied if the execution of each $\mathbb{C}_{c_i^{(2)}, \mathsf{k}_{i+1}}$ goes as expected. The formal construction is described in Section 5.2.

As for security, we show that our construction satisfies CPA-1-sided security in the presence of *no collusions* (i.e., the adversary can submit a single query to the oracle KGen) if the underlying PE is CPA-1-sided secure, SKE is CPA secure, and the lockable obfuscation is secure. Roughly, the proof works as follows. Let $\mathbb{P}^*$ be the *only* predicate submitted to KGen by the adversary. Starting from A's validity condition, we infer that, for any choice of the challenge bit $b \in \{0, 1\}$, then attacker A must maintain one of the following two conditions:

(i) either $\mathbb{P}_1^*(x_1^b) = \ldots = \mathbb{P}_n^*(x_n^b) = 0$ (i.e., all the predicates of the conjunctions are false);

(ii) or (if at least one predicate $\mathbb{P}_i^*$ is satisfied, i.e., $\mathbb{P}_i^*(x_i^b) = 1$) there exists $j \neq i$ such that, for every $x_j \in \mathcal{Q}_j^b$, it holds that $\mathbb{P}_j^*(x_j) = 0$ where $\mathcal{Q}_j^b$ is the ordered list composed of predicate inputs submitted to the oracle $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ and the challenge input x_j^b (see Equation (2)).[8]

When the first condition is satisfied, we can leverage the CPA-1-sided security of the single-input PE to show that the every lock y_i (encrypted using the PE), and every input x_i (encrypted in $c_i^{(2)}$), is completely hidden to the adversary. The latter allows us to use the security of lockable obfuscation to move to a hybrid experiment in which all the (obfuscated) circuits are simulated (including the messages).

On the other hand, when the second condition is satisfied, we can transition to a hybrid experiment (this time by leveraging the security of the underlying PE scheme) in which $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ computes $c_j^{(1)}$ by encrypting the all-zero string (instead of (y_j, k_{j+1})). Thus, we can use the security of lockable obfuscation to move to another hybrid in which $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ simulates all the obfuscations. At this point, the symmetric key k_{j+1} is not used anymore. Hence, we can use the security of SKE to transition to another hybrid in which $\mathsf{Enc}(\mathsf{ek}_{j+1}, \cdot, \cdot)$ computes $c_{j+1}^{(2)}$ by encrypting the all-zero string (instead of $c_{j+1}^{(1)}$ that, in turn, contains the lock y_{j+1} and the symmetric key k_{j+2}). After this hybrid, we can again use the security of lockable obfuscation to simulate all the obfuscations computed by $\mathsf{Enc}(\mathsf{ek}_{j+1}, \cdot, \cdot)$, and so on. By repeating these last two hybrids, we reach an experiment whose distribution does not depend on the challenge bit. We present the formal construction in Section 5.2.

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[8]If this condition is not satisfied, the adversary has obtained through the encryption oracles a set of ciphertexts that can be interleaved with one (or more) parts of the challenge ciphertext in order to satisfy the predicate $\mathbb{P}^*$.

We highlight that our scheme is not secure in the presence of collusions. In particular, the fact that the adversary can obtain a single decryption key $\mathsf{dk}_\mathbb{P}$ is crucial in order to get the validity condition (ii), i.e., for every $b \in \{0, 1\}$ there exists a j such that for every predicate (submitted to $\mathsf{KGen}(\mathsf{msk}, \cdot)$) we have $\mathbb{P}_j(x_j^b) = 0$. In fact, in the case of collusions, the adversary can ask for two decryption keys $\mathsf{dk}_\mathbb{P}$ and $\mathsf{dk}_{\mathbb{P}'}$ such that for every $b \in \{0, 1\}$:

$$\mathbb{P}_1(x_1^b) = 0 \text{ and } \mathbb{P}_2(x_2^b) = \ldots = \mathbb{P}_n(x_n^b) = 1$$
$$\mathbb{P}'_1(x_1^b) = 1 \text{ and } \mathbb{P}'_2(x_2^b) = \ldots = \mathbb{P}'_n(x_n^b) = 0.$$

Note that these are valid queries for the CPA-1-sided security experiment of n-input PE (the ciphertext cannot be decrypted). However, such a *unique j* for every predicate (as per condition (ii)) does not exist. When this happens, we are not able to conclude the proof by making a reduction to the security of single-input PE (the reduction will make an invalid set of queries to the KGen oracle of the single-input PE, making it invalid for the CPA-1-sided security of the single-input PE).[9]

Lastly, we stress that since we start from a single-input PE supporting conjunctions of arbitrary predicates *with wildcards*, we end up with an n-input PE for conjunctions of arbitrary predicates (see Equation (1)) *with wildcards*. However, we note that, in the secret-key setting (i.e., no corruptions), the presence of wildcards does not affect the scheme's expressiveness. This is because we can transform any secure multi-input PE for $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$ with a *single* wildcard $(x_1^\star, \ldots, x_n^\star)$ into a secure multi-input PE for the same class of predicates $\mathbb{P}(x_1, \ldots, x_n)$ *without the wildcard*. This can be done by requiring the senders not to encrypt the corresponding wildcard, i.e., for each $i \in [n]$, $\mathsf{Enc}(\mathsf{ek}_i, x_i^\star, m_i)$ outputs $\bot$ whenever $x_i = x_i^\star$. The latter only works in the case of no corruptions (otherwise, an adversary can encrypt wildcards on its own using the leaked encryption keys).

Security under corruptions. Next, let us explain how to define security of multi-input PE in the presence of corruptions. Here, the adversary has the possibility to corrupt a subset of the senders and leak their encryption keys ek_i. We model this by introducing an additional corruption oracle $\mathsf{Corr}(\cdot)$ that, upon input an index $i \in [n]$, returns ek_i. Note that, once obtained ek_i, the adversary A has the possibility to produce arbitrary ciphertexts on any message and predicate input, without interacting with the challenger during the CPA-1-sided security game. As usual, the validity condition heavily depends on the queries submitted to both the encryption oracles and the corruption oracle. More precisely, the validity condition now says that, for every decryption key $\mathsf{dk}_\mathbb{P}$, for every vector of ciphertexts that can be obtained by interleaving the challenge ciphertext $(c_1^*, \ldots, c_n^*)$ with both the ciphertexts obtain trough any of the (uncorrupted) encryption oracles and the ones that A may autonomously produce by using the leaked encryption keys (trough oracle $\mathsf{Corr}(\cdot)$), we have that $\mathbb{P}$ is not satisfied. Hence, the validity condition is identical to that of the secret-key setting (see Equation (2)), except that:

- If the i-th encryption key ek_i has been corrupted/leaked, then $\mathcal{Q}_i^b$ of Equation (2) corresponds to the i-th predicate input space. This is because the adversary can produce a valid ciphertext on any input x_i.

- Else (i.e., the i-th encryption key ek_i is still secret), $\mathcal{Q}_i^b$ is defined as usual, i.e., it is the ordered list of predicate inputs submitted to oracle $\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)$ and challenge input x_i^b.

See Section 4.2 for the formal definition.

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[9]As we discuss in Remark 4, our construction remains secure if we consider a weaker form of collusion in which the adversary can only obtain multiple decryption keys for predicates $\mathbb{P}$ such that there is a unique j *for all predicates* (submitted to KGen) that satisfies the validity condition (ii).

A simple attack. Before explaining our construction in details, let us show why the previous construction is not secure under corruptions. For simplicity, we focus on the 2-input setting. Suppose an adversary A has a single decryption key $dk_\mathbb{P}$ for $\mathbb{P}(x_1, x_2) = \mathbb{P}_1(x_1) \wedge \mathbb{P}_2(x_2)$ and a vector of ciphertexts $(c_1^*, c_2^*) = ((\widetilde{\mathbb{C}}_1, c_1^{(2)}), (\widetilde{\mathbb{C}}_2, c_2^{(2)}))$ encrypted under the predicate input (x_1, x_2) such that $\mathbb{P}_1(x_1) = 0$ and $\mathbb{P}_2(x_2) = 1$. Note that this ciphertext should not decrypt under $dk_\mathbb{P}$, since the conjunction of $\mathbb{P}_1$ and $\mathbb{P}_2$ evaluates to 0. If A can obtain ek_2, then it can easily determine the message m_2 (and thus the bit b). Indeed, once A gets $ek_2 = (mpk, k_2, k_1)$, it can compute a malicious ciphertext $\widetilde{c}_1^{(1)}$ (using the single-input PE) by encrypting $(\widetilde{y}, k_2)$ (where $\widetilde{y}$ is a random lock) under the predicate input composed by (x_1', x_2') such that $\mathbb{P}_1(x_1') = 1$ and $\mathbb{P}_2(x_2') = 1$. Then, it can compute $\widetilde{c}_1^{(2)} \leftarrow_\$ \mathsf{Enc}(k_1, \widetilde{c}_1^{(1)})$ and execute $\widetilde{\mathbb{C}}_2(\widetilde{c}_1^{(2)}, dk_\mathbb{P})$ to get m_2. Note that by definition the execution of $\widetilde{\mathbb{C}}_2$ outputs the correct message, since $\mathbb{P}_1(x_1^\star) \wedge \mathbb{P}_2(x_2) = 1$ and $\widetilde{c}_1^{(2)}$ contains the correct secret encryption key k_2, allowing the circuit to correctly end the computation. Also, note that this attack does not violate the validity condition. This is because $\mathbb{P}_1(x_1) = 0$, and A does not use the oracle $\mathsf{Enc}(ek_1, \cdot, \cdot)$ at all. Hence, any interleaving of the ciphertexts will involve the predicate input x_1 that, in turn, will make the conjunction $\mathbb{P}(x_1, x_2') = \mathbb{P}_1(x_1) \wedge \mathbb{P}_2(x_2')$ unsatisfied for every choice of the input predicate x_2'.

In light of the above attack, we can identify what we need to do in order to extend our techniques to handle corruptions:

- First, following the proof of the previous construction, it is important to hide the (plain) single-input PE ciphertext that a particular sender produces (e.g., in the secret-key setting we re-encrypt $c_i^{(1)}$ using SKE). As we have described for the secret-key setting, this allows us to claim that everything remains hidden whenever one of the predicate $\mathbb{P}_i$ of the conjunction is not satisfied (even if a different $\mathbb{P}_j$ is satisfied).[10]

- Second, the leakage of one (or more) encryption keys should not allow to produce a malicious ciphertext on behalf of the uncorrupted senders (or simply decrypt the ciphertexts of other parties). Otherwise, the attacker can follow a strategy similar to the one above to break security.

Construction under corruptions. In order to achieve the above properties, we propose a new technique based on nested (lockable obfuscated) circuits that can be executed one inside the other. This technique permits to make available secret information (e.g., secret keys) *only* during nested execution. For the sake of clarity, we first present our approach for the case of two inputs. As an initial attempt to deal with corruptions, we replace the SKE in our previous construction with a PKE, so that the encryption key ek_1 (resp. ek_2) is now composed of (mpk, sk_1, pk_1, pk_2) (resp. (mpk, sk_2, pk_2, pk_1)) where (sk_i, pk_i) is a secret/public key pair. Each (sk_i, pk_i) is associated to the i-th sender (indeed, note that ek_i contains also the secret key sk_i). From the perspective of the first sender, in order to encrypt a message m_1 under the input x_1, it samples two random locks (y_1^{in}, y_1^{out}) and encrypts them (using the single-input PE) as before using the wildcard $x_2^\star$, i.e., $c_1^{(0)} \leftarrow_\$ \mathsf{Enc}(mpk, (x_1, x_2^\star), (y_1^{in}, y_1^{out}))$. At this point, the PE ciphertext $c_1^{(0)}$ is re-encrypted twice using pk_1 and pk_2, i.e., $c_1^{(i)} \leftarrow_\$ \mathsf{Enc}(pk_i, c_1^{(i-1)})$ for $i \in [2]$. Intuitively, the two layers of PKE have the role of hiding the PE ciphertexts (that in turn contain the locks) even when the adversary leaks all encryption keys except one. The final ciphertext is composed by the two obfuscations $\widetilde{\mathbb{C}}_1^{out}, \widetilde{\mathbb{C}}_1^{in}$ of the circuits $\mathbb{C}_{sk_1, c_1^{(2)}}^{out}, \mathbb{C}_{sk_1, c_1^{(2)}}^{in}$, respectively. The former is obfuscated under the lock y_1^{out} and message m_1, whereas the latter is

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[10] The secret-key construction achieves this by linking multiple PE ciphertexts via SKE, and including the secret key k_{i+1} into the PE ciphertext.

obfuscated under the lock y_1^{in} and message sk_1. The ciphertext produced by the second sender, is identical, except that it uses sk_2 (instead of sk_1) and that $c_2^{(0)}$ is computed using the predicate input $(x_1^\star, x_2)$ (instead of $(x_1, x_2^\star)$).

The crux of our nesting technique comes from the definition of the circuits $\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_i, c_i^{(2)}}$. More precisely, the outer circuit $\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_1, c_1^{(2)}}$ will take as input the obfuscation $\widetilde{\mathbb{C}}^{\mathsf{in}}_2$ of the inner circuit $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_2, c_2^{(2)}}$ and a decryption key $\mathsf{dk}_{\mathbb{P}}$. Then, in order to securely check the conjunction inside the lockable obfuscation, $\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_1, c_1^{(2)}}$ will execute $\widetilde{\mathbb{C}}^{\mathsf{in}}_2(\mathsf{sk}_1, \mathsf{dk}_{\mathbb{P}})$. At this point, $\widetilde{\mathbb{C}}^{\mathsf{in}}_2$ has everything it needs to check the satisfiability of $\mathbb{P}_2(\cdot)$. It removes the PKE layers from $c_2^{(2)}$ by computing $c_2^{(0)} = \mathsf{Dec}(\mathsf{sk}_2, \mathsf{Dec}(\mathsf{sk}_1, c_2^{(2)}))$. Then, it decrypts the PE ciphertext $(y_2^{\mathsf{in}}, y_2^{\mathsf{out}}) = \mathsf{Dec}(\mathsf{dk}_{\mathbb{P}}, c_2^{(0)})$—observe that the decryption succeeds if $\mathbb{P}_2(x_2) = 1$—and returns y_2^{in}. By correctness of lockable obfuscation, if the computation of $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_2, c_2^{(2)}}(\mathsf{sk}_1, \mathsf{dk}_{\mathbb{P}})$ goes as intended, then $\widetilde{\mathbb{C}}^{\mathsf{in}}_2(\mathsf{sk}_1, \mathsf{dk}_{\mathbb{P}})$ will output sk_2 (the message attached to the obfuscation). Once obtained sk_2, the computation of $\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_1, c_1^{(2)}}$ can continue and perform a similar computation to check the satisfiability of $\mathbb{P}_1(\cdot)$ except that, if the PE ciphertext $c_1^{(0)}$ decrypts correctly, it returns y_1^{out}. If all the decryptions (performed by $\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_1, c_1^{(2)}}$ and $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_2, c_2^{(2)}}$) succeed, the execution of the obfuscation $\widetilde{\mathbb{C}}^{\mathsf{out}}_1$ of $\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_1, c_1^{(2)}}$ will output m_1. A symmetrical argument holds for $\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_2, c_2^{(2)}}$ and $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_1, c_1^{(2)}}$, releasing m_2.

We show that the above 2-input PE construction is CPA-1-sided secure under 1 corruption (i.e., one encryption key remains secret) and no collusions if the underlying single-input PE is CPA secure, PKE is CPA secure, and the lockable obfuscation is secure. The high level intuition is that sk_i remains unknown to the adversary if $\mathbb{P}_i(\cdot) = 0$ (unless the adversary invokes the oracle $\mathsf{Corr}(i)$). This is reflected by the proof technique that is sketched below.

Let $\mathsf{dk}_{\mathbb{P}^*}$ be the decryption key obtained by A for the predicate $\mathbb{P}^*(\cdot, \cdot) = \mathbb{P}_1^*(\cdot) \wedge \mathbb{P}_2^*(\cdot)$ (recall the presence of wildcards), and let $\mathcal{Q}_{\mathsf{Corr}}$ be the queries submitted to the corruption oracle. Starting from the validity condition, we can infer that for any choice of the challenge bit $b \in \{0, 1\}$ we have:

(i) either $\mathbb{P}_1^*(x_1^b) = \mathbb{P}_2^*(x_2^b) = 0$;

(ii) or (i.e., there exists an $i \in [2]$ such that predicate $\mathbb{P}_i$ is satisfied) $j \notin \mathcal{Q}_{\mathsf{Corr}}$ such that $j \neq i$ and, for every $x_j \in \mathcal{Q}_j^b$, $\mathbb{P}_j^*(x_j) = 0$ (recall that $x_j^b \in \mathcal{Q}_j^b$). Observe that this second condition holds because of the following:

 - If there is $x_j \in \mathcal{Q}_j^b$ such that $\mathbb{P}_j^*(x_j) = 1$, A can use the corresponding ciphertext to decrypt the i-th part of the challenge ciphertext since $\mathbb{P}_i^*(x_i^b) = 1$.
 - If $j \in \mathcal{Q}_{\mathsf{Corr}}$, A can simply use ek_j to encrypt a random message under the wildcard $x_j^\star$ (that always exists by design of our construction) and, again, decrypt the i-th part of the challenge ciphertext.[11]

By leveraging the above two conditions, the security of our scheme follows by using a similar argument to that of the secret-key setting. In particular, when the first condition is satisfied, we can show that the locks $(y_1^{\mathsf{in}}, y_1^{\mathsf{out}})$ and $(y_2^{\mathsf{in}}, y_2^{\mathsf{out}})$ (used to encrypt the challenge) are completely hidden. This, in turn, allows us to use the security of lockable obfuscation and simulate the obfuscations of $(\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_1, c_1^{(2)}}, \mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_1, c_1^{(2)}})$, $(\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_2, c_2^{(2)}}, \mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_2, c_2^{(2)}})$, and the corresponding messages.

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[11]Contrarily from the secret-key setting, wildcards affect the security of the multi-input PE when corruptions are in place. Indeed, if an encryption key ek_j gets leaked then a malicious adversary can always encrypt itself the j-th wildcards $x_j^\star$, satisfying the j-th predicate $\mathbb{P}_j$.

On the other hand, when the second condition is satisfied, we can move to a hybrid (by leveraging the security of single-input PE) in which $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ computes $c_j^{(0)}$ by encrypting the all-zero string (instead of $(y_j^{\mathsf{in}}, y_j^{\mathsf{out}})$). Then, we can use the security of lockable obfuscation to transition to another hybrid in which $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ simulates all the obfuscations. At this point, the secret key sk_j of the uncorrupted j-th sender is not used anymore (recall that $j \notin \mathcal{Q}_{\mathsf{Corr}}$). Hence, we can leverage the security of the PKE to remove the locks $(y_i^{\mathsf{in}}, y_i^{\mathsf{out}})$ chosen by the i-th sender (recall $i \neq j$). In more details, we do another hybrid in which the j-th PKE layer $c_i^{(j)}$ of the challenge ciphertext is an encryption of zeroes (instead of $c_i^{(j-1)}$ that, in turn, encrypts the locks $(y_i^{\mathsf{in}}, y_i^{\mathsf{out}})$). After this hybrid, we can again use the security of lockable obfuscation to simulate all the obfuscations (and the corresponding attached messages) that compose the i-th component of the ciphertext. The distribution of this last hybrid does not depend on the challenge bit b since all the ciphertexts are simulated by the simulator of the lockable obfuscation scheme.

To sum up, we can observe that encrypting $c_i^{(0)}$ (the PE ciphertext that contains the locks) with the public keys $(\mathsf{pk}_1, \mathsf{pk}_2)$ of both senders is crucial in order for our proof to work independently of which encryption key the adversary decides to leak. So long as at least one encryption key ek_i remains hidden, then there is a PKE layer that cannot be decrypted by the adversary. This allows the proof to go through.

Generalizing the nesting technique to $(n > 2)$ inputs. By carefully modifying the definition of the outer and inner circuits, we can generalize the above technique to the case of $n > 2$. The structure of the encryption keys and of the encryption algorithm is similar to the case $n = 2$:

- Each encryption key ek_i is of the form $(\mathsf{mpk}, \mathsf{sk}_i, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$.

- To compute the i-th encryption of (x_i, m_i), the sender computes the initial PE ciphertext as $c_i^{(0)} \leftarrow_\$ \mathsf{Enc}(\mathsf{mpk}, (x_1^\star, \ldots, x_i, \ldots, x_n^\star), (y_i^{\mathsf{in}}, y_i^{\mathsf{out}}))$. Then, it re-encrypts n times the ciphertext $c_i^{(0)}$ using $(\mathsf{pk}_1, \ldots, \mathsf{pk}_n)$, i.e., $c_i^{(v)} \leftarrow_\$ \mathsf{Enc}(\mathsf{pk}_v, c_i^{(v-1)})$ for $v \in [n]$. As usual, the final ciphertext $c_i = (\widetilde{\mathbb{C}}_i^{\mathsf{out}}, \widetilde{\mathbb{C}}_i^{\mathsf{in}})$ is composed of the obfuscations of $\mathbb{C}_{\mathsf{sk}_i, c_i^{(n)}}^{\mathsf{out}}$ and $\mathbb{C}_{\mathsf{sk}_i, c_i^{(n)}}^{\mathsf{in}}$.

We now turn on the crucial point: the definition of the outer and inner circuits. Again, for the sake of clarity, we only describe the outer circuit $\mathbb{C}_{\mathsf{sk}_1, c_1^{(n)}}^{\mathsf{out}}$ and of the inner circuits $(\mathbb{C}_{\mathsf{sk}_2, c_2^{(n)}}^{\mathsf{in}}, \ldots, \mathbb{C}_{\mathsf{sk}_n, c_n^{(n)}}^{\mathsf{in}})$ generated by the corresponding senders. The remaining circuits are defined similarly. First off, the input space of these circuits is a follows:

- $\mathbb{C}_{\mathsf{sk}_1, c_1^{(n)}}^{\mathsf{out}}$ takes as input the $n-1$ obfuscations of the circuits $(\mathbb{C}_{\mathsf{sk}_2, c_2^{(n)}}^{\mathsf{in}}, \ldots, \mathbb{C}_{\mathsf{sk}_n, c_n^{(n)}}^{\mathsf{in}})$ and a decryption $\mathsf{dk}_\mathbb{P}$. These obfuscations are the inner circuits that needs to be executed in order to return the message m_1 attached to the obfuscation of $\mathbb{C}_{\mathsf{sk}_1, c_1^{(n)}}^{\mathsf{out}}$.

- On the other hand, $\mathbb{C}_{\mathsf{sk}_i, c_i^{(n)}}^{\mathsf{in}}$, for $i \in [n] \setminus \{1\}$, takes as input a tuple of n secret keys $(\mathsf{sk}_1, \ldots, \mathsf{sk}_n)$ (where some can be set to $\perp$), a decryption key $\mathsf{dk}_\mathbb{P}$, and the obfuscations of $(\mathbb{C}_{\mathsf{sk}_{i+1}, c_{i+1}^{(n)}}^{\mathsf{in}}, \ldots, \mathbb{C}_{\mathsf{sk}_n, c_n^{(n)}}^{\mathsf{in}})$. Intuitively, these obfuscations are the remaining inner circuits that we need to still execute in order to complete the nested execution.

Intuitively, the decryption of m_1 requires the nested execution of these circuits (starting from the outer one) in order to get all the secret keys required to decrypt the PE ciphertext. This is achieved as follows.

The outer circuit $\mathbb{C}^{\mathsf{out}}_{\mathsf{sk}_1,c_1^{(n)}}$ starts the nested execution by invoking the obfuscation of $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_2,c_2^{(n)}}$ upon input $(\mathsf{sk}_1, \bot, \dots, \bot)$, $\mathsf{dk}_\mathbb{P}$, and the remaining obfuscations of $(\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_3,c_3^{(n)}}, \dots, \mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_n,c_n^{(n)}})$. In turn, $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_2,c_2^{(n)}}$ will do a similar thing: It executes the next obfuscated circuit $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_3,c_3^{(n)}}$ upon input $(\mathsf{sk}_1, \mathsf{sk}_2, \bot, \dots, \bot)$, $\mathsf{dk}_\mathbb{P}$, and the remaining obfuscations $(\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_4,c_4^{(n)}}, \dots, \mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_n,c_n^{(n)}})$. This process is repeated until $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_n,c_n^{(n)}}$ is executed upon input $(\mathsf{sk}_1, \dots, \mathsf{sk}_{n-1}, \bot)$ and $\mathsf{dk}_\mathbb{P}$. At this point, all the secret keys are know (observe that sk_n is hardcoded). From $c_n^{(n)}$, we can remove the n PKE layers, decrypt the PE ciphertext and, in turn, return y_n^{in} if the PE ciphertext decrypts correctly (i.e., $\mathbb{P}_n(\cdot)$ is satisfied). Once $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_n,c_n^{(n)}}$ terminates, the secret key sk_n is released and $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_{n-1},c_{n-1}^{(n)}}$ performs the computation required to check if $\mathbb{P}_{n-1}(\cdot)$ is satisfied. Indeed, $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_{n-1},c_{n-1}^{(n)}}$ has been executed on input $(\mathsf{sk}_1, \dots, \mathsf{sk}_{n-2}, \bot, \bot)$, it has sk_{n-1} harcoded, and the execution of $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_n,c_n^{(n)}}$ has released sk_n. Hence, after the correct termination of $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_n,c_n^{(n)}}$, all secret keys are known.

It may seems that this argument can be iterated. However, there is a problem. Even if $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_{n-1},c_{n-1}^{(n)}}$ correctly terminates, the circuit $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_{n-2},c_{n-2}^{(n)}}$ that invokes it does not have access to the secret key sk_n. This is because the latter circuit receives as input $(\mathsf{sk}_1, \dots, \mathsf{sk}_{n-3}, \bot, \bot, \bot)$, it has sk_{n-2} hardcoded, and the circuit $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_{n-1},c_n^{(n)}}$ has returned sk_{n-1}. As a consequence, $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_{n-2},c_{n-2}^{(n)}}$ must re-run $\mathbb{C}^{\mathsf{in}}_{\mathsf{sk}_n,c_n^{(n)}}$ on input $(\mathsf{sk}_1, \dots, \mathsf{sk}_{n-1}, \bot)$ in order to get sk_n and decrypt every PKE layer. This needs to be done at any level of the nested execution, yielding an asymptotic running time of $O(n^n)$. Hence, this technique only works assuming $n = O(1)$, i.e. for $O(1)$-input predicates. The formal construction is described in Section 5.2.

On achieving CPA-2-sided secure multi-input PE. Until now, we only focused the discussion on achieving CPA-1-sided security. Our multi-input constructions achieve CPA-2-sided security if the underlying single-input PE is CPA-2-sided secure (we highlight that, in our secret-key multi-input PE construction, we need to reduce the n-arity from $\mathsf{poly}(\lambda)$ to $O(\log(\lambda))$ since we use complexity leveraging). We just recall here that, already for the simple notion of single-input PE for arbitrary predicates, CPA-2-sided security implies iO [BV15].

1.3 Applications

Finally, we explore applications of multi-key and multi-input PE. This question is particularly relevant given the fact that we are only able to obtain multi-key and multi-input PE supporting conjunctions of arbitrary predicates (with wildcards). Luckily, we can show that this class of predicates is already expressive enough to yield interesting cryptographic applications which previously required much stronger assumptions.

Matchmaking Encryption. Matchmaking encryption (ME) [AFNV19, AFNV21] allows a sender to publicly encrypt a message m under some attributes σ and a policy $\mathbb{R}$. On the other hand, the receive can use the decryption keys dk_ρ and $\mathsf{dk}_\mathbb{S}$ (encoding the receiver's attributes and policy, respectively) to decrypt the message (i.e., $\mathsf{Dec}(\mathsf{dk}_\rho, \mathsf{dk}_\mathbb{S}, c) = m$) if there is a mutual match $\mathbb{S}(\sigma) = 1 \land \mathbb{R}(\rho) = 1$. The main security guarantee of ME is that:

- In case of a mismatch, nothing is leaked except the fact that a match did not occur (this is reminiscent to the definition of CPA-1-sided security for PE).

- Additionally, in case of a match, nothing is leaked except for the message and the fact that a match occurred (this is reminiscent to the definition of CPA-2-sided security for PE).

Multi-key PE is a direct generalization of ME: 2-key PE for conjunctions $\mathbb{P}_{v_1, v_2}(\cdot, \cdot) = \mathbb{P}_{v_1}(\cdot) \wedge \mathbb{P}_{v_2}(\cdot)$ (i.e., the class of predicates studied in this work) implies ME for arbitrary policies. In a nutshell, the construction works as follows. To encrypt a message m under the sender's attributes σ and the sender's policy $\mathbb{R}$, the ME encryption algorithm corresponds to the public-key encryption algorithm of the 2-key PE scheme, i.e., $c \leftarrow_\$ \mathsf{Enc}(\mathsf{mpk}, (x_1, x_2), m)$ where $x_1 = \sigma$ and $x_2 = \mathbb{R}$. Analogously, the ME decryption keys dk_ρ and $\mathsf{dk}_\mathbb{S}$ correspond to the decryption keys dk_{v_2} and dk_{v_1} of the 2-key PE scheme where $v_1 = \mathbb{S}$ and $v_2 = \rho$. By setting $\mathbb{P}_{v_1, v_2}(x_1, x_2) = \mathbb{P}_{\mathbb{S}, \rho}(\sigma, \mathbb{R}) = \mathbb{P}_\sigma(\mathbb{S}) \wedge \mathbb{P}_\mathbb{R}(\rho) = \mathbb{S}(\sigma) \wedge \mathbb{R}(\rho)$, we obtain the desired ME functionality during decryption. The security analysis is intuitive: if the 2-key PE is CPA-1-sided secure then the ME scheme is secure only in case of mismatch. In addition, if the 2-key PE is CPA-2-sided secure, then the ME security holds also in case of a match. Hence, as a corollary of our results, we achieve the first construction of ME for arbitrary policies with CPA-1-sided security (i.e., mismatch security) and unbounded collusions from LWE. We provide more details in Appendix A.

Previous works construct ME with unbounded collusions for either very restricted policies (i.e., for identity matching) using bilinear maps [FGRV21, CLWW22] (and ROM [AFNV19]), or for arbitrary policies from much stronger assumptions such as 2-input FE with one secret key and one public key (this notion of 2-input FE implies iO) [AFNV19, AFNV21].

For completeness (see Section 4.1 and Appendix C), we highlight that we can build n-key PE from $(n + 1)$-input PE supporting arbitrary predicates and tolerating 1 corruption (this is required to implement the public-key encryption algorithm of n-key PE). As a consequence, multi-input PE implies ME as well. However, recall that our multi-input PE constructions do not support arbitrary predicates but only conjunctions of arbitrary predicates with wildcards.

Non-interactive MPC. Non-interactive MPC (NI-MPC) [BGI+14, HLP11] allows n parties to evaluate a function $f(v_1, \ldots, v_n)$ on their inputs using a single round of communication (i.e., each party sends a single message $c_i \leftarrow_\$ \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, v_i)$). This is achieved by assuming a trusted setup (that may depend on the function itself) that generates (possibly correlated) strings (e.g., common reference string crs and encryption keys ek_i) that can be later used by the parties to perform function evaluation. Security of NI-MPC can be formulated in two different settings, named *non-reusable* and *reusable* NI-MPC. The former retains security only if the setup is executed after every round. The latter retains security even if parties evaluate f on different inputs using the same setup (full-fledged reusability makes use of session identifiers in order to avoid that an adversary can interleave messages from different rounds [HLP11]). Both non-reusable and reusable NI-MPC provide the same security guarantee, formalized using an indistinguishability-based definition: an adversary A cannot distinguish between $(\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, v_i^0))_{i \in [n]}$ and $(\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, v_i^1))_{i \in [n]}$, so long as any combination of the messages known by the adversary (including the ones it can compute using the encryption key ek_i of a corrupted party) yields the same function's evaluation.[12]

As mentioned by several works [BGI+14, GGG+14, HIJ+16, HIJ+17], NI-MPC achieving indistinguishability-based security implies iO even in very restricted settings. In particular, a non-reusable 1-robust (i.e., one malicious party) NI-MPC for two parties implies iO. Intuitively,

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[12]Note that security of NI-MPC for general functions is formalized by an indistinguishability-based definition [HIJ+17, BGI+14]. This is because simulation-based NI-MPC implies virtual black-box (VBB) obfuscation that is known to be impossible for certain classes of functions [BGI+01].

by fixing the NI-MPC function to $f(\mathbb{C}, x) = \mathbb{C}(x)$, we can obfuscate a circuit by simply setting the input of the first (honest) party to $\mathbb{C}$, compute $c_1 \leftarrow_\$ \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_1, \mathbb{C})$, and outputting $\widetilde{\mathbb{C}} = (\mathsf{crs}, c_1, \mathsf{ek}_2)$ where $\mathsf{ek}_1, \mathsf{ek}_2$ are the key material required to encode the inputs of the NI-MPC (note that 1-robustness is necessary since we reveal ek_2). To evaluate the obfuscated circuit, the evaluator only needs to compute $c_2 \leftarrow_\$ \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_2, x)$ and evaluate the NI-MPC function f that will yield $\mathbb{C}(x)$. The security of this iO obfuscator follows from the security of NI-MPC since the residual functions $f(\mathbb{C}_0, \cdot)$ and $f(\mathbb{C}_1, \cdot)$ are identical, as $\mathbb{C}_0(x) = \mathbb{C}_1(x)$ for every input x. Additionally, reusable, 0-robust (i.e., no malicious parties) NI-MPC for $n = \mathsf{poly}(\lambda)$ parties implies iO. In this case, iO can be built using a similar construction to that of iO from secret-key multi-input functional encryption (FE) [GGG$^+$14].

Due to the similarities between multi-input PE and multi-input FE, we observe that multi-input PE is enough to construct NI-MPC for *all-or-nothing* functions defined over the predicates supported by the multi-input PE scheme. In more details, by leveraging our CPA-1-sided n-input PE (for $n = O(1)$) secure under $n - 1$ corruptions and without collusions, we can build an $(n-1)$-robust NI-MPC for a constant number of parties for the following class of functions:

$$f_\mathbb{P}((x_1, m_1), \ldots, (x_n, m_n)) = \begin{cases} (m_1, \ldots, m_n) & \text{if } \mathbb{P}(x_1, \ldots, x_n) = 1 \\ \bot & \text{otherwise} \end{cases}$$

where $\mathbb{P}(x_1, \ldots, x_n)$ is a conjunctions of arbitrary independent predicates (with wildcards) as defined in Equation (1). The resulting NI-MPC satisfies a weaker notion of reusability without session identifiers (i.e., messages produced in different rounds can be interleaved by design) specifically tailored for all-or-nothing functions, which we name *CPA-1-sided reusability*. In a nutshell, CPA-1-sided reusable NI-MPC guarantees the usual indistinguishability-based security only if $f_\mathbb{P}$ outputs $\bot$ (i.e., $\mathbb{P}(\cdot)$ is not satisfied) for any combination of the honest messages and the ones the adversary can maliciously compute using the encryption key ek_i of a corrupted party.

The construction is intuitive. At setup, simply publish $\mathsf{crs} = \mathsf{dk}_\mathbb{P}$ and distribute ek_i to the i-th party where $(\mathsf{msk}, \mathsf{ek}_1, \ldots \mathsf{ek}_n) \leftarrow_\$ \mathsf{Setup}(1^\lambda)$ and $\mathsf{dk}_\mathbb{P} \leftarrow_\$ \mathsf{KGen}(\mathsf{msk}, \mathbb{P})$. During evaluation, each party can send the message $c_i \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_i, x_i, m_i)$ and compute $\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, c_1, \ldots, c_n)$ to evaluate the function $f_\mathbb{P}((x_1, m_1), \ldots, (x_n, m_n))$. The CPA-1-sided reusable security of k-robust NI-MPC for $f_\mathbb{P}$ follows readily from CPA-1-sided security of n-input PE under k corruptions and without collusions.

By plugging in our results, we obtain either CPA-1-sided reusable $(n-1)$-robust NI-MPC with $n = O(1)$, or CPA-1-sided reusable 0-robust NI-MPC with $n = \mathsf{poly}(\lambda)$ where the predicate $\mathbb{P}$ of the function $f_\mathbb{P}$ is a conjunctions of arbitrary predicates (i.e., $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$) with wildcards under the LWE assumption. Note that a CPA-1-sided reusable NI-MPC for $f_\mathbb{P}$ where $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$ can be used to implement a voting protocol with message passing, i.e., only when each parties' vote x_i satisfies its dedicated set of requirements $\mathbb{P}_i(\cdot)$ (i.e., $\mathbb{P}_i(x_i) = 1$ for every $i \in [n]$) the messages are revealed to all the participants. Until this condition is not satisfied, everything remains secret. We provide the formal definition of CPA-1-sided reusability and the construction of NI-MPC from multi-input PE in Appendix B.

We stress that, nonetheless CPA-1-sided reusability is a weakening of the standard reusability definition, our flavor of reusability is still non-trivial to achieve in the setting of general functions. This is because we can build null iO (and, in turn, witness encryption) [GKW17, WZ17, BJK$^+$18] from CPA-1-sided reusable NI-MPC using the same constructions of iO from (standard) reusable NI-MPC, i.e., CPA-1-sided reusable (resp. CPA-1-sided non-reusable) 0-robust (resp. 1-robust) NI-MPC for $n = \mathsf{poly}(\lambda)$ parties (resp. $n = 2$ parties) and general functions implies null iO.

1.4 Relation with Witness Encryption

In the following we recall the notion of witness encryption (WE) [GGSW13], and we discuss its relation with both multi-input and multi-key PE (we anticipate that such relations hold even if we consider weaker versions of multi-input and multi-key PE where only the secrecy of the message is guaranteed).

A WE scheme for a relation $\mathcal{R}$, defined over a language $\mathcal{L}$, allows a sender to encrypt a message m using a statement x. A receiver, holding a witness w, can decrypt the message m if $(x, w) \in \mathcal{R}$. As for security, WE guarantees that the message remains hidden whenever $x \notin \mathcal{L}$, i.e., the corresponding ciphertext can not be decrypted. WE has several disrupting applications such as encrypting messages that can be decrypted in the future (i.e., whenever w will be known). Moreover, WE does not require setup and is fully non-interactive.

As shown by Brakerski et al. [BJK$^+$18], an n-input ABE (and thus an n-input PE) for arbitrary predicates, secure in the secret-key setting and without collusions, implies WE for NP and n-size witnesses. The construction is reminiscent to the one of iO from secret-key multi-input functional encryption [GGG$^+$14] (see also Section 1.3). Unfortunately, we cannot use here our construction of n-input PE since it only supports conjunctions of arbitrary predicates (see Equation (1)). Currently, it is not know how to build n-input PE, with $n > 2$, for arbitrary predicates without iO (the only known construction is for $n = 2$ and it is due to the concurrent work of Agrawal et al. [AYY22]. See Section 2 for a detailed discussion.

Also, we stress that multi-key PE (or simply a multi-key ABE) for arbitrary predicates implies WE. The construction is similar to that of Brakerski et al. [BJK$^+$18], for obtaining WE from multi-input ABE/PE. The only difference is that we substitute the multiple inputs (provided by multi-input PE) with the multiple decryption keys of multi-key PE. For completeness, we describe the construction below. Let $\mathbb{P}_{v_1,\ldots,v_n}(x) = 1$ if and only if $(x, w) \in \mathcal{R}$, where $w = v_1||\ldots||v_n$ defines the class of predicates supported by the multi-key PE. To encrypt a message m under a statement $x \in \mathcal{L}$, the sender computes $(\mathsf{mpk}, \mathsf{msk}_1, \ldots, \mathsf{msk}_n) \leftarrow_{\$} \mathsf{Setup}(1^\lambda)$ and sends to the receiver $(c, (\mathsf{dk}_{v_i}, \mathsf{dk}_{\overline{v_i}})_{i \in [n]})$ where $c \leftarrow_{\$} \mathsf{Enc}(\mathsf{mpk}, x, m)$ and $\mathsf{dk}_{v_i} \leftarrow_{\$} \mathsf{KGen}(\mathsf{msk}_i, 1)$ (resp. $\mathsf{dk}_{\overline{v_i}} \leftarrow_{\$} \mathsf{KGen}(\mathsf{msk}_i, 0)$) for $i \in [n]$. To decrypt the ciphertext under a witness $w = v_1||\ldots||v_n$, the receiver simply executes $\mathsf{Dec}(\mathsf{dk}'_{v_1}, \ldots, \mathsf{dk}'_{v_n}, c)$ where $\mathsf{dk}'_{v_i} = \mathsf{dk}_{v_i}$ if $v_i = 1$, and $\mathsf{dk}'_{v_i} = \mathsf{dk}_{\overline{v_i}}$ if $v_i = 0$.[13] Similarly to the case of multi-input PE, our multi-key PE construction fails to imply WE since it does not support arbitrary predicates.

It may seem that arbitrary predicates are a necessary condition in order to build WE from multi-input PE. However, we highlight that this is not necessarily the case if we consider security under corruptions. In particular, a 2-input PE for conjunctions under 1 corruption and no collusions, implies WE for any relation. This can be accomplished by considering the predicate $\mathbb{P}_{x,\mathcal{R}}(\cdot, \cdot) = \mathbb{P}_1(\cdot) \wedge \mathbb{P}_{x,\mathcal{R}}(\cdot)$ such that $\mathbb{P}_1(x_1^\star) = 1$ (for some wildcard $x_1^\star$) and $\mathbb{P}_{x,\mathcal{R}}(w) = 1$ if and only if $(x, w) \in \mathcal{R}$. Intuitively, to encrypt m using a statement x, the sender can simply output $(c_1, \mathsf{ek}_2, \mathsf{dk}_{\mathbb{P}_{x,\mathcal{R}}})$ such that $c_1 \leftarrow_{\$} \mathsf{Enc}(\mathsf{ek}_1, x_1^\star, m)$, $\mathsf{dk}_{\mathbb{P}_{x,\mathcal{R}}} \leftarrow_{\$} \mathsf{KGen}(m, \mathbb{P}_{x,\mathcal{R}})$, and $(\mathsf{msk}, \mathsf{ek}_1, \mathsf{ek}_2) \leftarrow_{\$} \mathsf{Setup}(1^\lambda)$. Then, the receiver uses w to retrieve m by computing $\mathsf{Dec}(\mathsf{dk}_{x,\mathcal{R}}, c_1, \mathsf{Enc}(\mathsf{ek}_2, w))$.[14] Here, it is crucial the the underlying 2-input PE can handle corruptions, since the latter allows the sender to disclose ek_2 to the (possibly malicious) receiver and give him the opportunity to try different witnesses.

Unfortunately, even in this case, our construction of $O(1)$-input PE under corruptions fails to imply WE. This is because our construction supports conjunctions of arbitrary predicates *each one* having a wildcard. In other words, we do not support predicates that are unsatisfiable.

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[13]Observe that the same construction works if we start from a multi-key PE whose encryption algorithm is secret-key, i.e., the mpk (required to execute Enc) is replaced with an encryption key ek that is kept secret.

[14]A similar construction can be used to build iO from 2-input FE with security under 1 corruption and no collusions.

This is a problem, because supporting unsatisfiable predicates is crucial to prove security of WE when $x \notin \mathcal{L}$.[15]

Given the above discussion, we identify two plausible approaches that could lead to a construction of WE from standard assumptions:

- Enlarging the class of predicates of our secret-key n-input PE or n-key PE constructions: From conjunction of arbitrary predicates (see Equation (1)) to arbitrary predicates (or any restricted class of predicates that permits to implement a specific non-trivial WE relation $\mathcal{R}$).

- Supporting conjunctions of arbitrary predicates (without wildcards) in the setting of 2-input PE with security under 1 corruption.

2 Related Work

Multi-input PE is a special case of multi-input FE [GGG+14]. It is well known that so-called compact FE (supporting arbitrary functions) implies multi-input FE [AJ15, BV15], which in turn implies iO. Constructions of multi-input FE from standard assumptions, in turn, exist for restricted functions [BLR+15, AGRW17, DOT18, ACF+18, CDG+18, ABKW19, LT19, Tom19, ABG19, AGT21, CSW21, AGT22]. Multi-input PE can also be seen as stronger form of multi-input ABE [BJK+18], the difference being that the attributes are not private in the case of ABE. Previously to our work, all (provably secure) constructions of n-input ABE with $n > 2$ required iO (the only exception is the concurrent work of Agrawal et al. [AYY22], which we discuss in the next paragraph).

The multi-input and multi-key settings have also been considered in the context of fully-homomorphic encryption [LTV12, CM15, MW16].

Concurrent and independent work. The independent and concurrent work of Agrawal, Yadav, and Yamada [AYY22] proposes two constructions of secret-key (i.e., no corruptions) 2-input key-policy ABE for NC^1 with unbounded collusions (recall that, in the ABE setting, only the secrecy of the messages is guaranteed, i.e., inputs can be public). The first construction is based on LWE and pairings, and it is provably secure in the generic group model. The second construction is based on function-hiding inner-product FE, a variant of the non-falsifiable KOALA knowledge assumption (which is proven to hold under the bilinear generic group model), and LWE. However, this second construction achieves a weaker selective flavor of security in which the adversary has to submit both the challenge and the decryption key queries before the setup phase. Additionally, they propose two heuristic constructions. The first is a 2-input ABE for P from lattices, and the second is a 3-input ABE for NC^1 from pairings and lattices. However, the security of these heuristic constructions remains unclear.

In comparison, our work directly focuses on the PE setting (i.e., CPA-1-sided security) and provides the first secret-key n-input PE that supports $n = \mathsf{poly}(\lambda)$ inputs, with (adaptive) CPA-1-sided security (i.e., secrecy of both inputs and messages) based solely on LWE. However, our construction only supports a restricted class of predicates (i.e., conjunctions of arbitrary predicates with wildcards) and it is secure only in the case of no collusions. Furthermore, differently from [AYY22], we move away from the secret-key setting and propose a second construction of n-input PE (still for conjunctions of arbitrary predicates) that supports $n = O(1)$ inputs and can tolerate $n - 1$ corruptions (i.e., up to $n - 1$ encryption keys can be adaptively revealed by

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[15]If wildcards exist, a malicious receiver can always decrypt the message by evaluating the predicate over the wildcards.

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the adversary). Finally, we propose the notion of multi-key PE (not covered in [AYY22]), and give the first construction of CPA-1-sided secure n-key PE for $n = \mathsf{poly}(\lambda)$, with unbounded collusions and still supporting conjunctions of arbitrary predicates, based on LWE.

Regarding the techniques, we highlight that both our work and that of [AYY22] introduce (albeit different) nesting techniques based on lockable obfuscation. In particular, the nesting technique of [AYY22] permits to transform any secret-key n-input ABE into a secret-key n-input PE (achieving CPA-1-sided security). We stress that their approach only works in the secret-key setting. In contrast, we propose a different nesting technique which yields n-input PE for $n = O(1)$ while tolerating $n - 1$ corruptions. It is important to note that our nesting technique is not generic, but it is specifically tailored to work with the class of predicates considered in this work.

Turning to applications, we highlight that the multi-input schemes of [AYY22] fail to imply ME, since their constructions are all in the secret-key setting (whereas ME requires a public-key encryption algorithm). As for NI-MPC, the constructions in [AYY22] can be used to obtain a CPA-1-sided 0-robust reusable NI-MPC for all-or-nothing functions defined over arbitrary predicates, but only in the case of 2 parties (3 parties if we consider also the heuristic constructions).

3 Preliminaries

3.1 Notation

We use the notation $[n] = \{1, 2, \ldots, n\}$. Capital bold-face letters (such as $\mathbf{X}$) are used to denote random variables, small letters (such as x) to denote concrete values, calligraphic letters (such as $\mathcal{X}$) to denote sets, serif letters (such as A) to denote algorithms, and bold typeface letters (such as $\mathbb{C}$) to denote circuits. All of our algorithms are modeled as (possibly interactive) Turing machines; if algorithm A has oracle access to some oracle O, we often implicitly write $\mathcal{Q}_{\mathsf{O}}$ for the set of queries asked by A to O.

For a string $x \in \{0, 1\}^*$, we let $|x|$ be its length; if $\mathcal{X}$ is a set, $|\mathcal{X}|$ represents the cardinality of $\mathcal{X}$. When x is chosen uniformly in $\mathcal{X}$, we write $x \leftarrow_\$ \mathcal{X}$. If A is an algorithm, we write $y \leftarrow_\$ \mathsf{A}(x)$ to denote a run of A on input x and output y; if A is randomized, y is a random variable and $\mathsf{A}(x; r)$ denotes a run of A on input x and (uniform) randomness r. An algorithm A is *probabilistic polynomial-time* (PPT) if A is randomized and for any input $x, r \in \{0, 1\}^*$ the computation of $\mathsf{A}(x; r)$ terminates in a polynomial number of steps (in the input size). We write $\mathbb{C}(x) = y$ to denote the evaluation of the circuit $\mathbb{C}$ on input x and output y.

Negligible functions. Throughout the paper, we denote by $\lambda \in \mathbb{N}$ the security parameter and we implicitly assume that every algorithm takes as input the security parameter. A function $\nu(\cdot)$ is called negligible in the security parameter $\lambda \in \mathbb{N}$ if it vanishes faster than the inverse of any polynomial in λ, i.e. $\nu(\lambda) \in O(1/p(\lambda))$ for all positive polynomials $p(\lambda)$. We sometimes write $\mathsf{negl}(\lambda)$ (resp. $\mathsf{poly}(\lambda)$) to denote an unspecified negligible function (resp. polynomial function) in the security parameter.

3.2 Lockable Obfuscation

A lockable obfuscator [GKW17, WZ17] permits to obfuscate a circuit $\mathbb{C}$ together with a "lock" y and a message m. As a result, the obfuscator will output an obfuscated circuit $\widetilde{\mathbb{C}}$ that will

$$\frac{\mathbf{G}^{\text{lock-sim}}_{\Pi,\mathsf{A},\mathsf{S}}(\lambda)}{}$$

$(\mathbb{C}, m, \alpha) \leftarrow\!\!{\scriptstyle\$}\ \mathsf{A}_0(1^\lambda)$

$b \leftarrow\!\!{\scriptstyle\$}\ \{0,1\}, y \leftarrow\!\!{\scriptstyle\$}\ \{0,1\}^{s(\lambda)}$

$\widetilde{\mathbb{C}}_0 \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Obf}(1^\lambda, \mathbb{C}, y, m), \widetilde{\mathbb{C}}_1 \leftarrow\!\!{\scriptstyle\$}\ \mathsf{S}(1^\lambda, 1^{|\mathbb{C}|}, 1^{|m|})$

$b' \leftarrow\!\!{\scriptstyle\$}\ \mathsf{A}_1(1^\lambda, \widetilde{\mathbb{C}}_b, \alpha)$

If $(b' = b)$: **return** 1

Else: return 0
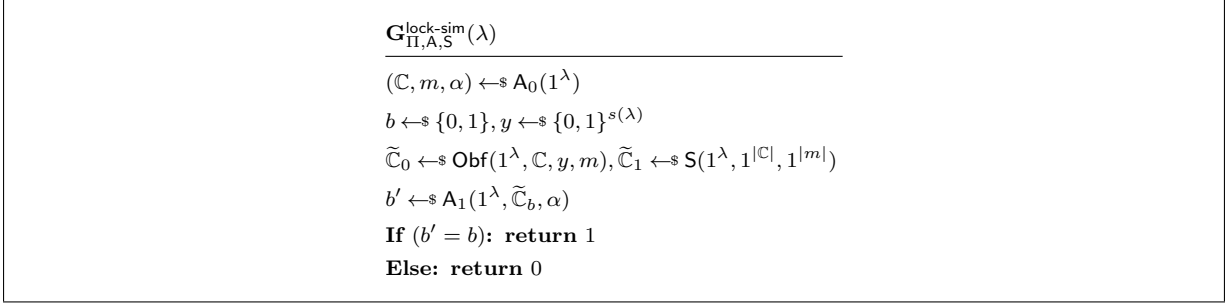
Figure 1: Game defining security of lockable obfuscation.

behave as follows:

$$\widetilde{\mathbb{C}}(x) = \begin{cases} m & \text{if } \mathbb{C}(x) = y \\ \bot & \text{otherwise.} \end{cases}$$

More formally, let $n(\cdot), s(\cdot), d(\cdot)$ be polynomials, and $\mathcal{C}_{n,s,d}(\lambda)$ be the family of circuits of depth $d(\lambda)$ with input size $n(\lambda)$ and output size $s(\lambda)$. A lockable obfuscator for the circuit family $\mathcal{C}_{n,s,d}(\lambda)$ and message space $\mathcal{M}$ is composed of the following polynomial-time algorithms:

$\mathsf{Obf}(1^\lambda, \mathbb{C}, y, m)$**:** Upon input the security parameter 1^λ, a circuit $\mathbb{C} \in \mathcal{C}_{n,s,d}(\lambda)$, a lock $y \in \{0,1\}^{s(\lambda)}$, and a message $m \in \mathcal{M}$, the randomized lockable obfuscator algorithm outputs a circuit $\widetilde{\mathbb{C}}$.

$\mathsf{Eval}(\widetilde{\mathbb{C}}, x)$**:** Upon input an obuscated circuit $\widetilde{\mathbb{C}}$ and an input $x \in \{0,1\}^{n(\lambda)}$, the deterministic evaluation algorithm outputs a message $m \in \mathcal{M} \cup \{\bot\}$.

Definition 1 (Semi-statistical correctness of lockable obfuscation [GKW17])**.** A lockable obfuscator $\Pi = (\mathsf{Obf}, \mathsf{Eval})$ for the circuit family $\mathcal{C}_{n,s,d}(\lambda)$ and message space $\mathcal{M}$ satisfies *semi-statistical correctness* if:

1. $\forall \lambda \in \mathbb{N}, \forall x \in \{0,1\}^{n(\lambda)}, m \in \mathcal{M}, \forall \mathbb{C} \in \mathcal{C}_{n,s,d}(\lambda)$ such that $\mathbb{C}(x) = y$, we have

$$\mathbb{P}\Big[\mathsf{Eval}(\mathsf{Obf}(1^\lambda, \mathbb{C}, y, m), x) = m\Big] = 1.$$

2. $\forall \lambda \in \mathbb{N}, \forall x \in \{0,1\}^{n(\lambda)}, \forall m \in \mathcal{M}, \forall \mathbb{C} \in \mathcal{C}_{n,s,d}(\lambda)$ such that $\mathbb{C}(x) \neq y$, we have

$$\mathbb{P}\Big[\mathsf{Eval}(\mathsf{Obf}(1^\lambda, \mathbb{C}, y, m), x) = m\Big] \leq \mathsf{negl}(\lambda).$$

As for security, lockable obfuscation must hide any information about the circuit $\mathbb{C}$, the message m and the lock y when the lock is randomly chosen. This is defined by requiring that there exists a simulator S that simulates the obfuscated circuit $\widetilde{\mathbb{C}}$.

Definition 2 (Security of lockable obfuscation)**.** A lockable obfuscator $\Pi = (\mathsf{Obf}, \mathsf{Eval})$ for the circuit family $\mathcal{C}_{n,s,d}(\lambda)$ and message space $\mathcal{M}$ is secure if there exists a PPT simulator S such that for every PPT adversary $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$ we have:

$$\left| \mathbb{P}\Big[\mathbf{G}^{\text{lock-sim}}_{\Pi,\mathsf{A},\mathsf{S}}(\lambda) = 1\Big] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda),$$

where $\mathbf{G}^{\text{lock-sim}}_{\Pi,\mathsf{A},\mathsf{S}}(\lambda)$ is depicted in Figure 1.

Remark 1. The definitions above are taken from [GKW17]. Wichs and Zirdelis [WZ17] proposed a slightly more general notion of obfuscation for multi-bit compute-and-compare circuits in which the lock is only required to be unpredictable. They also give an obfuscator for multi-bit compute-and-compare circuits from the LWE assumption.

$$
\begin{array}{ll}
\underline{\mathbf{G}^{\mathsf{CPA\text{-}ske}}_{\Pi,\mathsf{A}}(\lambda)} & \underline{\mathbf{G}^{\mathsf{CPA\text{-}pke}}_{\Pi,\mathsf{A}}(\lambda)} \\
\mathsf{k} \leftarrow_{\$} \mathsf{KGen}(1^\lambda) & (\mathsf{pk},\mathsf{sk}) \leftarrow_{\$} \mathsf{KGen}(1^\lambda) \\
(m^0, m^1, \alpha) \leftarrow_{\$} \mathsf{A}_0^{\mathsf{Enc}(\mathsf{k},\cdot)}(1^\lambda) & (m^0, m^1, \alpha) \leftarrow_{\$} \mathsf{A}_0(1^\lambda, \mathsf{pk}) \\
b \leftarrow_{\$} \{0,1\}, c \leftarrow_{\$} \mathsf{Enc}(\mathsf{k}, m^b) & b \leftarrow_{\$} \{0,1\}, c \leftarrow_{\$} \mathsf{Enc}(\mathsf{pk}, m^b) \\
b' \leftarrow_{\$} \mathsf{A}_1^{\mathsf{Enc}(\mathsf{k},\cdot)}(1^\lambda, c, \alpha) & b' \leftarrow_{\$} \mathsf{A}_1(1^\lambda, c, \alpha) \\
\mathbf{If}\ (b' = b)\colon \mathbf{return}\ 1 & \mathbf{If}\ (b' = b)\colon \mathbf{return}\ 1 \\
\mathbf{Else}\colon \mathbf{return}\ 0 & \mathbf{Else}\colon \mathbf{return}\ 0
\end{array}
$$

Figure 2: Game defining CPA security of SKE and PKE.

3.3 Symmetric and Public Key Encryption

3.3.1 Symmetric key encryption

A symmetric-key encryption (SKE) scheme with message space $\mathcal{M}$ is composed of the following polynomial-time algorithms:

$\mathsf{KGen}(1^\lambda)$**:** The randomized key generator takes as input the security parameter 1^λ and outputs a symmetric key k.

$\mathsf{Enc}(\mathsf{k}, m)$**:** The randomized encryption algorithm takes as input a symmetric key k and a message $m \in \mathcal{M}$, and outputs a ciphertext c.

$\mathsf{Dec}(\mathsf{k}, c)$**:** The deterministic decryption algorithm takes as input a symmetric key k and a ciphertext c, and outputs a message m.

We require a SKE to be correct and secure against chosen-plaintext attacks (CPA).

Definition 3 (Correctness of SKE)**.** A SKE Π with message space $\mathcal{M}$ is correct if $\forall \lambda \in \mathbb{N}$, $\forall \mathsf{k}$ output by $\mathsf{KGen}(1^\lambda)$, $\forall m \in \mathcal{M}$, we have

$$\mathbb{P}[\mathsf{Dec}(\mathsf{k}, \mathsf{Enc}(\mathsf{k}, m)) = m] \geq 1 - \mathsf{negl}(\lambda).$$

Definition 4 (CPA security of SKE)**.** We say that a SKE Π is CPA secure if for all PPT adversaries $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$:

$$\left| \mathbb{P}\Big[\mathbf{G}^{\mathsf{CPA\text{-}ske}}_{\Pi,\mathsf{A}}(\lambda) = 1 \Big] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda),$$

where game $\mathbf{G}^{\mathsf{CPA\text{-}ske}}_{\Pi,\mathsf{A}}(\lambda)$ is depicted in Figure 2.

3.3.2 Public key encryption

A public-key encryption (PKE) scheme with message space $\mathcal{M}$ is composed of the following polynomial-time algorithms:

$\mathsf{KGen}(1^\lambda)$**:** The randomized key generator takes as input the security parameter 1^λ and outputs a public and a secret key pair $(\mathsf{pk}, \mathsf{sk})$.

$\mathsf{Enc}(\mathsf{pk}, m)$**:** The randomized encryption algorithm takes as input a public key pk and a message $m \in \mathcal{M}$, and outputs a ciphertext c.

$\mathsf{Dec}(\mathsf{sk}, c)$**:** The deterministic decryption algorithm takes as input a secret key sk and a ciphertext c, and outputs a message m.

We consider the usual definition of correctness and CPA security of PKE.

Definition 5 (Correctness of PKE). A PKE Π with message space $\mathcal{M}$ is correct if $\forall \lambda \in \mathbb{N}$, $\forall(\mathsf{pk}, \mathsf{sk})$ output by $\mathsf{KGen}(1^\lambda)$, $\forall m \in \mathcal{M}$, we have

$$\mathbb{P}[\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{pk}, m)) = m] \geq 1 - \mathsf{negl}(\lambda).$$

Definition 6 (CPA security of PKE). We say that a SKE Π is CPA secure if for all PPT adversaries $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$:

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}pke}}(\lambda) = 1 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda),$$

where game $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}pke}}(\lambda)$ is depicted in Figure 2.

3.4 Predicate Encryption

In PE, a trusted authority generates a decryption key for the receiver associated to an arbitrary predicate of his choice. The receiver is able to decrypt a ciphertext if and only if the predicate $\mathbb{P}$ (corresponding to its decryption key) is satisfied when evaluated with the predicate input x used for encrypting the plaintext, i.e. $\mathbb{P}(x) = 1$. Formally, a PE with message space $\mathcal{M}$, input space $\mathcal{X}$, and predicate space $\mathcal{P}$, is composed of the following polynomial-time algorithms:

$\mathsf{Setup}(1^\lambda)$: Upon input the security parameter 1^λ the randomized setup algorithm outputs the master public key mpk and the master secret key msk.

$\mathsf{KGen}(\mathsf{msk}, \mathbb{P})$: The randomized key generator takes as input the master secret key msk and a predicate $\mathbb{P} \in \mathcal{P}$. The algorithm outputs a secret decryption key $\mathsf{dk}_\mathbb{P}$ for predicate $\mathbb{P}$.

$\mathsf{Enc}(\mathsf{mpk}, x, m)$: The randomized encryption algorithm takes as the master public key mpk, an input $x \in \mathcal{X}$, and a message $m \in \mathcal{M}$. The algorithm produces a ciphertext c linked to both x and m.

$\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, c)$: The deterministic decryption algorithm takes as input a secret decryption key $\mathsf{dk}_\mathbb{P}$ for predicate $\mathbb{P} \in \mathcal{P}$ and a ciphertext c. The algorithm outputs either a message m or an error $\perp$.

Correctness of PE states that the receiver obtains the message with overwhelming probability if $\mathbb{P}(x) = 1$. On the other hand, if $\mathbb{P}(x) = 0$, the decryption outputs $\perp$ with overwhelming probability.

Definition 7 (Correctness of PE). A PE with message space $\mathcal{M}$, input space $\mathcal{X}$, predicate space $\mathcal{P}$, is correct if $\forall \lambda \in \mathbb{N}$, $\forall(\mathsf{mpk}, \mathsf{msk})$ output by $\mathsf{Setup}(1^\lambda)$, $\forall m \in \mathcal{M}$, $\forall x \in \mathcal{X}, \forall \mathbb{P} \in \mathcal{P}$, $\forall \mathsf{dk}_\mathbb{P}$ output by $\mathsf{KGen}(\mathsf{msk}, \mathbb{P})$ the following two conditions hold:

1. If $\mathbb{P}(x) = 1$, then $\mathbb{P}[\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, \mathsf{Enc}(\mathsf{mpk}, x, m)) = m] \geq 1 - \mathsf{negl}(\lambda)$.

2. If $\mathbb{P}(x) = 0$, then $\mathbb{P}[\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, \mathsf{Enc}(\mathsf{mpk}, x, m)) = \perp] \geq 1 - \mathsf{negl}(\lambda)$.

Security of PE comes in different flavors. The standard CPA security requires the adversary to distinguish between the encryption of two messages for the same predicate input. More formally, the adversary is allowed to perform a polynomial number of queries to the key generation oracle. Then, the adversary chooses two messages m^0 and m^1 and an input x, and wins the CPA security game if it can distinguish between an encryption of $\mathsf{Enc}(\mathsf{mpk}, x, m^0)$ and $\mathsf{Enc}(\mathsf{mpk}, x, m^1)$ with non-negligible probability (a PE scheme that satisfy CPA security is also called attribute-based encryption (ABE)).

$$\mathbf{G}^{\mathsf{CPA\text{-}PE}}_{\Pi,\mathsf{A}}(\lambda)$$

$(\mathsf{mpk}, \mathsf{msk}) \leftarrow\!\!\$\ \mathsf{Setup}(1^\lambda)$

$(m^0, m^1, x, \alpha) \leftarrow\!\!\$\ \mathsf{A}_0^{\mathsf{KGen}(\mathsf{msk},\cdot)}(1^\lambda, \mathsf{mpk})$

$b \leftarrow\!\!\$\ \{0,1\}, c \leftarrow\!\!\$\ \mathsf{Enc}(\mathsf{mpk}, x, m^b)$

$b' \leftarrow\!\!\$\ \mathsf{A}_1^{\mathsf{KGen}(\mathsf{msk},\cdot)}(1^\lambda, c, \alpha)$

If $(b' = b)$**: return** 1

Else: return 0

$$\mathbf{G}^{\mathsf{CPA\text{-}}t\text{-}\mathsf{PE}}_{\Pi,\mathsf{A}}(\lambda)$$

$(\mathsf{mpk}, \mathsf{msk}) \leftarrow\!\!\$\ \mathsf{Setup}(1^\lambda)$

$(m^0, m^1, x^0, x^1, \alpha) \leftarrow\!\!\$\ \mathsf{A}_0^{\mathsf{KGen}(\mathsf{msk},\cdot)}(1^\lambda, \mathsf{mpk})$

$b \leftarrow\!\!\$\ \{0,1\}, c \leftarrow\!\!\$\ \mathsf{Enc}(\mathsf{mpk}, x^b, m^b)$

$b' \leftarrow\!\!\$\ \mathsf{A}_1^{\mathsf{KGen}(\mathsf{msk},\cdot)}(1^\lambda, c, \alpha)$

If $(b' = b)$**: return** 1

Else: return 0

Figure 3: Game defining CPA, CPA-1-sided, and CPA-2-sided security of PE.

Definition 8 (CPA security of PE)**.** We say that a PE Π is CPA secure if for all valid PPT adversaries $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$:

$$\left| \mathbb{P}\left[\mathbf{G}^{\mathsf{CPA\text{-}PE}}_{\Pi,\mathsf{A}}(\lambda) = 1 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda),$$

where game $\mathbf{G}^{\mathsf{CPA\text{-}PE}}_{\Pi,\mathsf{A}}(\lambda)$ is depicted in Figure 3. Adversary A is called valid if $\forall \mathbb{P} \in \mathcal{Q}_{\mathsf{KGen}}$ it holds that $\mathbb{P}(x) = 0$.

We also consider two stronger definitions of security, namely CPA-1-sided and CPA-2-sided security, guaranteeing also the secrecy of the predicate input used during the encryption of a message. In this security games, the adversary is allowed to choose two different inputs x^0 and x^1 and the usual messages m^0 and m^1. CPA-1-sided security guarantees the privacy of the input only when the predicates for which the adversary knows a decryption key (i.e. the ones he received from the key generation oracle) are not satisfied, i.e. the receiver cannot decrypt the message. On the other hand, CPA-2-sided security considers the same property also when the predicate is satisfied, i.e., the receiver can decrypt the challenge ciphertexts.

Definition 9 (CPA-1-sided and CPA-2-sided security of PE)**.** Let $t \in [2]$. We say that a PE Π is CPA-t-sided secure if for all valid PPT adversaries $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$:

$$\left| \mathbb{P}\left[\mathbf{G}^{\mathsf{CPA\text{-}}t\text{-}\mathsf{PE}}_{\Pi,\mathsf{A}}(\lambda) = 1 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda),$$

where game $\mathbf{G}^{\mathsf{CPA\text{-}}t\text{-}\mathsf{PE}}_{\Pi,\mathsf{A}}(\lambda)$ is depicted in Figure 3. Adversary A is called valid if $\forall \mathbb{P} \in \mathcal{Q}_{\mathsf{KGen}}$,

Case $t = 1$**:** $\mathbb{P}(x^0) = \mathbb{P}(x^1) = 0$.

Case $t = 2$**:** Either $\mathbb{P}(x^0) = \mathbb{P}(x^1) = 0$ or $\mathbb{P}(x^0) = \mathbb{P}(x^1) \wedge m^0 = m^1$.

Through the paper, we say Π is CPA-1-sided (resp. CPA-2-sided) secure *without collusions* if $|\mathcal{Q}_{\mathsf{KGen}}| = 1$, i.e., the adversary can not get more than one decryption key.

4 Multi-key and Multi-input Predicate Encryption

We provide the formal definitions of multi-key PE and multi-input PE in the following Sections 4.1 and 4.2, respectively. In Appendix A, we build ME from multi-key PE. In Appendix B, we build CPA-1-sided reusable robust NI-MPC for all-or-nothing functions from multi-input PE.

4.1 Multi-key PE

Formally, an n-key PE with message space $\mathcal{M}$, input space $\mathcal{X}$, and predicate space $\mathcal{P} = \{\mathbb{P}_{v_1,\dots,v_n}(x)\}_{(v_1,\dots,v_n)\in\mathcal{V}}$ indexed by $\mathcal{V} = \mathcal{V}_1 \times \dots \times \mathcal{V}_n$, is composed of the following polynomial-time algorithms:
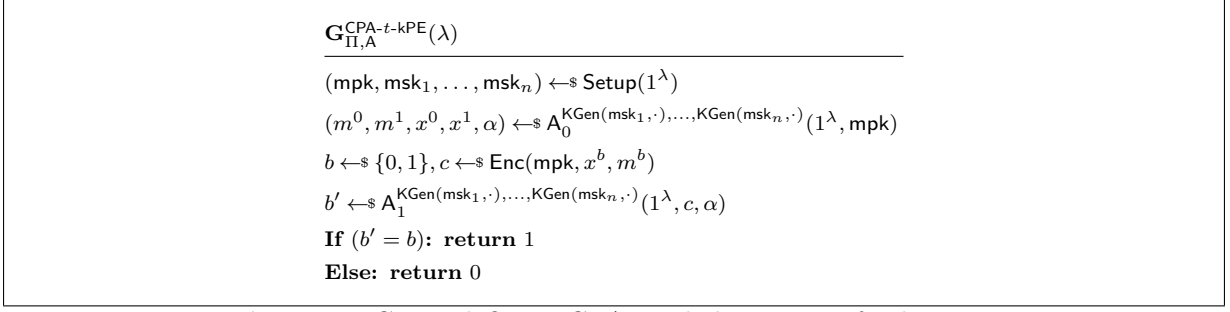
$$\boxed{\begin{array}{l} \mathbf{G}^{\mathsf{CPA}\text{-}t\text{-}\mathsf{kPE}}_{\Pi,\mathsf{A}}(\lambda) \\ \hline (\mathsf{mpk}, \mathsf{msk}_1, \ldots, \mathsf{msk}_n) \leftarrow\!\!\!{}_\$\ \mathsf{Setup}(1^\lambda) \\ (m^0, m^1, x^0, x^1, \alpha) \leftarrow\!\!\!{}_\$\ \mathsf{A}_0^{\mathsf{KGen}(\mathsf{msk}_1, \cdot), \ldots, \mathsf{KGen}(\mathsf{msk}_n, \cdot)}(1^\lambda, \mathsf{mpk}) \\ b \leftarrow\!\!\!{}_\$\ \{0, 1\}, c \leftarrow\!\!\!{}_\$\ \mathsf{Enc}(\mathsf{mpk}, x^b, m^b) \\ b' \leftarrow\!\!\!{}_\$\ \mathsf{A}_1^{\mathsf{KGen}(\mathsf{msk}_1, \cdot), \ldots, \mathsf{KGen}(\mathsf{msk}_n, \cdot)}(1^\lambda, c, \alpha) \\ \textbf{If } (b' = b)\textbf{: return } 1 \\ \textbf{Else: return } 0 \end{array}}$$

Figure 4: Game defining CPA-t-sided security of n-key PE.

$\mathsf{Setup}(1^\lambda)$: Upon input the security parameter 1^λ the setup algorithm outputs the master public key mpk and the n master secret key $(\mathsf{msk}_1, \ldots, \mathsf{msk}_n)$.

$\mathsf{KGen}(\mathsf{msk}_i, v_i)$: Let $i \in [n]$. The randomized key generator takes as input the i-th master secret key msk_i and the i-th index $v_i \in \mathcal{V}_i$. The algorithm outputs the i-th secret decryption key dk_{v_i} for the predicate index v_i.

$\mathsf{Enc}(\mathsf{mpk}, x, m)$: The randomized encryption algorithm takes as the master public key mpk, an input $x \in \mathcal{X}$, and a message $m \in \mathcal{M}$. The algorithm produces a ciphertext c.

$\mathsf{Dec}(\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n}, c)$: The deterministic decryption algorithm takes as input n secret decryption keys $(\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n})$ for the n indexes $(v_1, \ldots, v_n) \in \mathcal{V}$ and a ciphertext c. The algorithm outputs a message m.

Correctness is intuitive: given the decryption keys $(\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n})$ for $(v_1, \ldots, v_n) \in \mathcal{V}$, the decryption algorithm returns the message m (encrypted under the input x) with overwhelming probability, whenever $\mathbb{P}_{v_1, \ldots, v_n}(x) = 1$.

Definition 10 (Correctness of n-key PE). A n-key PE with message space $\mathcal{M}$, input space $\mathcal{X}$, predicate space $\mathcal{P} = \{\mathbb{P}_{v_1, \ldots, v_n}\}_{v_1, \ldots, v_n \in \mathcal{V}}$ indexed by $\mathcal{V} = \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, is correct if $\forall \lambda \in \mathbb{N}$, $\forall (\mathsf{mpk}, \mathsf{msk}_1, \ldots, \mathsf{msk}_n)$ output by $\mathsf{Setup}(1^\lambda)$, $\forall m \in \mathcal{M}$, $\forall x \in \mathcal{X}$, $\forall (v_1, \ldots, v_n) \in \mathcal{V}$ such that $\mathbb{P}_{v_1, \ldots, v_n}(x) = 1$, we have:

$$\mathbb{P}[\mathsf{Dec}(\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n}, \mathsf{Enc}(\mathsf{mpk}, x, m)) = m] \geq 1 - \mathsf{negl}(\lambda),$$

where $\mathsf{dk}_{v_i} \leftarrow\!\!\!{}_\$\ \mathsf{KGen}(\mathsf{msk}_i, v_i)$ for $i \in [n]$.

As for security, we adapt the standard CPA-1-sided and CPA-2-sided security of PE to the n-key setting. In particular, an adversary (with oracle access to $\mathsf{KGen}(\mathsf{msk}_i, \cdot)$ for $i \in [n]$) cannot distinguish between $\mathsf{Enc}(\mathsf{mpk}, x^0, m^0)$ and $\mathsf{Enc}(\mathsf{mpk}, x^1, m^1)$ except with non-negligible probability. When considering CPA-1-sided security, the adversary is valid only if it cannot decrypt the challenge ciphertext, i.e., it asks to the n key generation oracles indexes $(v_1, \ldots, v_n)$ such that $\mathbb{P}_{v_1, \ldots, v_n}(x^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^1) = 0$. Analogously, the CPA-2-sided security captures the indistinguishability of $\mathsf{Enc}(\mathsf{mpk}, x^0, m^0)$ and $\mathsf{Enc}(\mathsf{mpk}, x^1, m^1)$ even when the adversary can decrypt the challenge ciphertext, i.e., $\mathbb{P}_{v_1, \ldots, v_n}(x^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^1) = 1$ and $m^0 = m^1$. These security definitions are formalized below.

Definition 11 (CPA-1-sided and CPA-2-sided security of n-key PE). Let $t \in [2]$. We say that a n-key PE Π is CPA-t-sided secure if for all valid PPT adversaries $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$:

$$\left| \mathbb{P}\left[\mathbf{G}^{\mathsf{CPA}\text{-}t\text{-}\mathsf{kPE}}_{\Pi,\mathsf{A}}(\lambda) = 1 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda),$$

where game $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA}\text{-}t\text{-}\mathsf{kPE}}(\lambda)$ is depicted in Figure 4. Adversary A is called valid if $\forall v_1 \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_1,\cdot)}, \ldots, \forall v_n \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_n,\cdot)}$, we have

$$\textbf{Case } t = 1\text{: } \mathbb{P}_{v_1,\ldots,v_n}(x^0) = \mathbb{P}_{v_1,\ldots,v_n}(x^1) = 0.$$
$$\textbf{Case } t = 2\text{: Either } \mathbb{P}_{v_1,\ldots,v_n}(x^0) = \mathbb{P}_{v_1,\ldots,v_n}(x^1) = 0$$
$$\text{or } \mathbb{P}_{v_1,\ldots,v_n}(x^0) = \mathbb{P}_{v_1,\ldots,v_n}(x^1) \wedge m^0 = m^1.$$

4.2 Multi-input PE

Formally, an n-input PE with message space $\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_n$, input space $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$, and predicate space $\mathcal{P}$, is composed of the following polynomial-time algorithms:

$\mathsf{Setup}(1^\lambda)$**:** Upon input the security parameter 1^λ the setup algorithm outputs the encryption keys $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n)$ and the master secret key msk.

$\mathsf{KGen}(\mathsf{msk}, \mathbb{P})$**:** The randomized key generator takes as input the master secret key msk and a predicate $\mathbb{P} \in \mathcal{P}$. The algorithm outputs a secret decryption key $\mathsf{dk}_\mathbb{P}$ for predicate $\mathbb{P}$.

$\mathsf{Enc}(\mathsf{ek}_i, x_i, m_i)$**:** Let $i \in [n]$. The randomized encryption algorithm takes as input an encryption key ek_i, an input $x_i \in \mathcal{X}_i$, and a message $m_i \in \mathcal{M}_i$. The algorithm produces a ciphertext c_i linked to x_i.

$\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, c_1, \ldots, c_n)$**:** The deterministic decryption algorithm takes as input a secret decryption key $\mathsf{dk}_\mathbb{P}$ for predicate $\mathbb{P} \in \mathcal{P}$ and n ciphertexts $(c_1, \ldots, c_n)$. The algorithm outputs n messages $(m_1, \ldots, m_n)$.

Correctness states that ciphertexts $(c_1, \ldots, c_n)$, each linked to an input x_i, correctly decrypt with overwhelming probability if $\mathbb{P}(x_1, \ldots, x_n) = 1$.

Definition 12 (Correctness of n-input PE)**.** An n-input PE with message space $\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_n$, input space $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$, predicate space $\mathcal{P}$, is correct if $\forall \lambda \in \mathbb{N}$, $\forall (\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk})$ output by $\mathsf{Setup}(1^\lambda)$, $\forall (m_1, \ldots, m_n) \in \mathcal{M}$, $\forall (x_1, \ldots, x_n) \in \mathcal{X}, \forall \mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}(x_1, \ldots, x_n) = 1$, we have:
$$\mathbb{P}[\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, c_1, \ldots, c_n) = (m_1, \ldots, m_n)] \geq 1 - \mathsf{negl}(\lambda),$$
where $\mathsf{dk}_\mathbb{P} \leftarrow_\$ \mathsf{KGen}(\mathsf{msk}, \mathbb{P})$ and $c_i \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_i, x_i, m_i)$ for $i \in [n]$.

Security with and without corruptions. The CPA-1-sided and CPA-2-sided security of n-input PE capture the infeasibility in distinguishing between ciphertexts $(\mathsf{Enc}(\mathsf{ek}_1, x_1^0, m_1^0), \ldots, \mathsf{Enc}(\mathsf{ek}_n, x_n^0, m_n^0))$ and $(\mathsf{Enc}(\mathsf{ek}_1, x_1^1, m_1^1), \ldots, \mathsf{Enc}(\mathsf{ek}_n, x_n^1, m_n^1))$. This is modeled by an adversary having oracle access to a key generation oracle $\mathsf{KGen}(\mathsf{msk}, \cdot)$ (allowing it to get decryption keys $\mathsf{dk}_\mathbb{P}$ on predicates of its choice) and n encryption oracles $\mathsf{Enc}(\mathsf{ek}_1, \cdot, \cdot), \ldots, \mathsf{Enc}(\mathsf{ek}_n, \cdot, \cdot)$ (allowing it to get encryptions of arbitrary messages and inputs). Differently from the n-key setting, we consider different models of security with respect to whether the encryption keys are secret (i.e., no corruptions) or public/leaked (i.e., the adversary has the possibility to get one or more encryption keys of its choice). The corruption of an encryption key is captured by giving access to a corruption oracle $\mathsf{Corr}(\cdot)$ to the adversary that, on input $i \in [n]$, it returns ek_i. Intuitively, the knowledge of ek_i impacts the validity condition that the adversary must satisfy (e.g., the challenge ciphertext cannot be decrypted). Indeed, ek_i would allow the adversary to produce arbitrary i-th ciphertexts on arbitrary i-th inputs x_i and potentially decrypt part of the challenge ciphertext. Concretely, as for CPA-1-sided security, the validity of the adversary can be defined as follows:
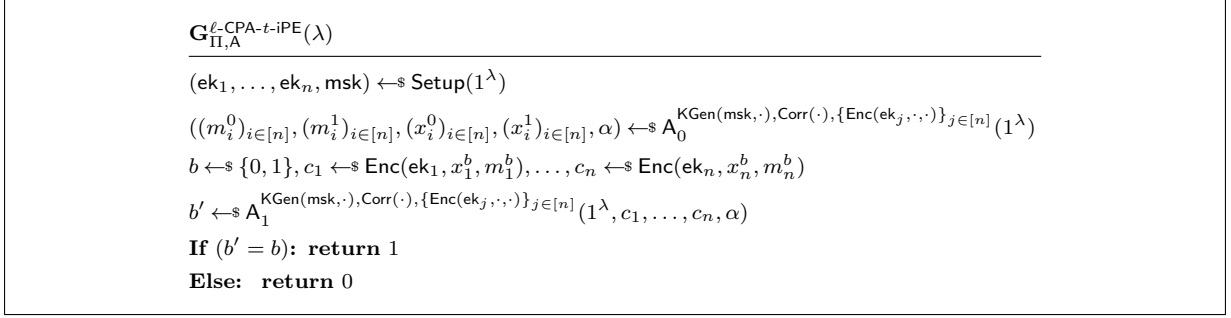
$(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow\!\!{\scriptstyle\$}\; \mathsf{Setup}(1^\lambda)$

$((m_i^0)_{i\in[n]}, (m_i^1)_{i\in[n]}, (x_i^0)_{i\in[n]}, (x_i^1)_{i\in[n]}, \alpha) \leftarrow\!\!{\scriptstyle\$}\; \mathsf{A}_0^{\mathsf{KGen}(\mathsf{msk},\cdot),\mathsf{Corr}(\cdot),\{\mathsf{Enc}(\mathsf{ek}_j,\cdot,\cdot)\}_{j\in[n]}}(1^\lambda)$

$b \leftarrow\!\!{\scriptstyle\$}\; \{0,1\}, c_1 \leftarrow\!\!{\scriptstyle\$}\; \mathsf{Enc}(\mathsf{ek}_1, x_1^b, m_1^b), \ldots, c_n \leftarrow\!\!{\scriptstyle\$}\; \mathsf{Enc}(\mathsf{ek}_n, x_n^b, m_n^b)$

$b' \leftarrow\!\!{\scriptstyle\$}\; \mathsf{A}_1^{\mathsf{KGen}(\mathsf{msk},\cdot),\mathsf{Corr}(\cdot),\{\mathsf{Enc}(\mathsf{ek}_j,\cdot,\cdot)\}_{j\in[n]}}(1^\lambda, c_1, \ldots, c_n, \alpha)$

If $(b' = b)$**: return** 1

Else: return 0

Figure 5: Game defining CPA-t-sided security of n-input PE in the ℓ-corruptions setting. Oracle $\mathsf{Corr}(j)$ returns ek_j for $j \in [n]$.

- *No corruptions (a.k.a. the secret-key setting).* If all the encryption keys $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n)$ are secret the validity conditions of CPA-1-sided security is straightforward. It intuitively states that for every $\mathsf{dk}_\mathbb{P}$ (obtained through oracle $\mathsf{KGen}(\mathsf{msk}, \cdot)$) and any tuple of ciphertexts $(c_1, \ldots, c_n)$ (each linked to an input x_i) obtained through the interleaving of part of the challenge ciphertext with the ciphertexts generated by invoking oracles $\{\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)\}_{i\in[n]}$, we must have that $\mathbb{P}(x_1, \ldots, x_n) = 0$ (otherwise part of the challenge ciphertext can be decrypted).

- *With corruptions.* If some of the encryption keys are known by the adversary (i.e., obtained through the corruption oracle $\mathsf{Corr}(\cdot)$) then the validity condition now changes according to which keys have been obtained. This is because the adversary can now autonomously compute arbitrary ciphertext (for a particular slot i) using the leaked i-th encryption key ek_i. Taking into account this observation, the validity of CPA-1-sided security *with corruptions* says that any tuple of ciphertexts $(c_1, \ldots, c_n)$ that can be obtained by interleaving part of the challenge ciphertexts with both the ones generated through oracles $\{\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)\}_{i\in[n]}$ and the ones that can be autonomously generated using the leaked encryption keys, we must have that $\mathbb{P}(x_1, \ldots, x_n) = 0$.

The validity of CPA-2-sided security (with and without corruptions) can be easily obtained by adapting the above discussion. Below, we provide the formal definition.

Definition 13 (ℓ-Corruptions CPA-1-sided and CPA-2-sided security of n-input PE)**.** Let $t \in [2]$. We say that an n-input PE Π is CPA-t-sided secure in the ℓ-corruptions setting if for all valid PPT adversaries $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$:

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\ell\text{-CPA-}t\text{-iPE}}(\lambda) = 1 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda),$$

where game $\mathbf{G}_{\Pi,\mathsf{A}}^{\ell\text{-CPA-}t\text{-iPE}}(\lambda)$ is depicted in Figure 5. Let $\mathcal{Q}_i = \{x | \exists (x, m) \in \mathcal{Q}_{\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)}\}$ for $i \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$ and $\mathcal{Q}_i = \mathcal{X}_i$ for $i \in \mathcal{Q}_{\mathsf{Corr}}$. Moreover, let $\mathcal{Q}_i^d$ (for $d \in \{0,1\}$) be the ordered list composed of the predicate inputs $\mathcal{Q}_i$ and the challenge input x_i^d, i.e., $\mathcal{Q}_i^d = \{x_i^{(1,d)}, \ldots, x_i^{(k_i,d)}, x_i^{(k_i+1,d)} = x_i^d\}$ where $k_i = |\mathcal{Q}_i|$ and $x_i^{(j,d)} \in \mathcal{Q}_i$ for $j \in [k_i]$.[16] Adversary A is called valid if $|\mathcal{Q}_{\mathsf{Corr}}| \leq \ell$ and $\forall \mathbb{P} \in \mathcal{Q}_{\mathsf{KGen}}, \forall j \in [n], \forall i_1 \in [k_1 + 1], \ldots, \forall i_n \in [k_n + 1]$, we

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[16]Observe that $\mathcal{Q}_i^0$ and $\mathcal{Q}_i^1$ are identical except for the last element.

have

Case $t = 1$: $\mathbb{P}(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$
$$\mathbb{P}(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 0.$$

Case $t = 2$: Either

$$\mathbb{P}(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$$
$$\mathbb{P}(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 0$$

or

$$\mathbb{P}(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$$
$$\mathbb{P}(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) \wedge m_j^0 = m_j^1.$$

Through the paper, for $t \in [2]$, we say that Π is CPA-t-sided secure in the ℓ-corruptions setting and *without collusions* if $|\mathcal{Q}_{\mathsf{KGen}}| = 1$ (i.e., the adversary asks for a single decryption key). If $|\mathcal{Q}_{\mathsf{Corr}}| = 0$ (i.e., no corruptions), we say that Π is CPA-t-sided secure in the *secret-key setting*. In case of both restrictions, we say that Π is CPA-t-sided secure in the *secret-key setting* and *without collusions* (i.e., $|\mathcal{Q}_{\mathsf{Corr}}| = 0$ and $|\mathcal{Q}_{\mathsf{KGen}}| = 1$).

Remark 2 (Relation with multi-key PE). We notice that CPA-t-sided secure $(n+1)$-input PE, supporting arbitrary predicates and tolerating 1 corruption, naturally implies CPA-t-sided secure n-key PE.[17] The idea is to use the first n inputs of the predicate $\mathbb{P}(x_1, \ldots, x_{n+1})$ (of $(n+1)$-input PE) to determine the indexes $(v_1, \ldots, v_n) \in \mathcal{V}$ that define the predicate $\mathbb{P}_{v_1,\ldots,v_n}(x)$ of the n-key PE, i.e., $\mathbb{P}(x_1, \ldots, x_{n+1}) = \mathbb{P}(v_1, \ldots, v_n, x) = \mathbb{P}_{v_1,\ldots,v_n}(x)$ where $x_i = v_i$ for $i \in [n]$ and $x_{n+1} = x$. In a nutshell, the authority uses the Setup algorithm of the $(n+1)$-input PE to generate $n+1$ encryption keys $(\mathsf{ek}_1, \ldots, \mathsf{ek}_{n+1})$ together with the master secret key msk. Then, it uses the latter to compute $\mathsf{dk}_{\mathbb{P}} \leftarrow_\$ \mathsf{KGen}(\mathsf{msk}, \mathbb{P})$ and sets $\mathsf{mpk} = \mathsf{ek}_{n+1}$ and $\mathsf{msk}_i = (\mathsf{dk}_{\mathbb{P}}, \mathsf{ek}_i)$ for $i \in [n]$. To generate dk_{v_i} (for an index v_i) the authority computes $c_{v_i} \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_i, v_i, \bot)$ (using the encryption algorithm of $(n+1)$-input PE) and outputs $\mathsf{dk}_{v_i} = (\mathsf{dk}_{\mathbb{P}}, c_{v_i})$. At this point, the n-key PE encryption and decryption algorithms are straightforward. To encrypt a message m under an input x, the sender computes $c \leftarrow_\$ \mathsf{Enc}(\mathsf{mpk}, x, m)$ where $\mathsf{mpk} = \mathsf{ek}_{n+1}$. On the other hand, the receiver decrypts the ciphertext c by executing $\mathsf{Dec}(\mathsf{dk}_{\mathbb{P}}, c_{v_1}, \ldots, c_{v_n}, c)$ where $\mathsf{dk}_{v_i} = (\mathsf{dk}_{\mathbb{P}}, c_{v_i})$ for $i \in [n]$. Intuitively, the CPA-t-sided security of this n-key PE follows from the CPA-t-sided security in the 1-corruption setting of the $(n+1)$-input PE. Observe that the $(n+1)$-input PE must tolerate 1 corruption since ek_{n+1} (i.e., the mpk of the n-key PE) is made public.[18] We refer the reader to Appendix C for more details.

On the other hand, if we consider restricted classes of predicates (as studied in this work), the above implication does not to hold anymore, making multi-input and multi-key PE incomparable. This is also reflected by the results achieved in this paper. For example, our multi-key PE construction for conjunctions of arbitrary predicates tolerates unbounded collusions whereas our

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[17]If we restrict the n-key PE's encryption algorithm to be secret-key (i.e., $\mathsf{Enc}(\mathsf{ek}, \cdot, \cdot)$ where ek is kept secret) then we can start from a secret-key $(n+1)$-input PE, i.e., 0 corruptions.

[18]We highlight that the same construction works if we use an $(n+1)$-input PE that satisfies a weaker flavor of security under corruptions, named the ℓ-*hybrid setting* (see Appendix C). Here, the Setup algorithm takes an additional parameter 1^ℓ that determines the number of encryption keys that we want to make public (this is equivalent to saying that the corruptions are known ahead of time and the construction can depend on this information). Hence, in this setting the adversary does not have access to the corruption oracle $\mathsf{Corr}(\cdot)$ but, instead, it directly receives as input the public encryption keys generated on setup. Clearly, security in the ℓ-hybrid setting is stronger than security in the secret-key setting but weaker than security in the $(\ell > 0)$-corruptions setting).

$\mathbb{C}_c(\mathsf{dk}_1, \ldots, \mathsf{dk}_n)$	$\mathbb{C}_{c,\mathsf{k}}(c_1, \ldots, c_{n-1}, \mathsf{dk}_{\mathbb{P}})$
Initialize: $c_n = c$	**Initialize:** $\mathsf{k}_1 = \mathsf{k}, c_n = c$
For i **from** n **to** 1 **do:**	**For** i **from** n **to** 1 **do:**
$\quad \mathsf{Dec}_i(\mathsf{dk}_i, c_i) = c_{i-1}$	$\quad \mathsf{Dec}_1(\mathsf{dk}_{\mathbb{P}}, \mathsf{Dec}_2(\mathsf{k}_i, c_i)) = v_i$
end for.	$\quad$ **If** $v_i = \perp$: **return** $\perp$
return c_0	$\quad$ **Else: let** $v_i = (y_i, \mathsf{k}_{i+1})$
	end for.
	return y_n **where** $v_n = (y_n, \mathsf{k}_1)$

Figure 6: On the left, the definition of the circuit $\mathbb{C}_c$ of Construction 1. On the right, the definition of the circuit $\mathbb{C}_{c,\mathsf{k}}$ of Construction 2.

multi-input PE constructions (for the same class of predicates with wildcards) are significantly more complex and are secure only in the case of no collusions (see Section 5).

Lastly, in Appendix D, we discuss the relation between the multi-key and multi-input settings when considering a weaker definition of security. In particular, if we drop the secrecy of the predicate inputs, i.e., only the the messages remain secret (which is equivalent to ABE), then we can show that multi-key ABE implies multi-input ABE only in the presence of no corruptions.

5 Constructions

In this section, we give different constructions of multi-key and multi-input PE (see also Section 1.2) for predicates $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$.

In particular, in Section 5.1 we give a construction of n-key PE from single-input PE and lockable obfuscation for $n = \mathsf{poly}(\lambda)$. This construction is secure against unbounded collusions.

In Section 5.2, we give two constructions of n-input PE from single-input PE, lockable obfuscation, and SKE/PKE. The first handles $\mathsf{poly}(\lambda)$-arity and it is CPA-1-side secure without collusions and in the secret-key setting. The second handles $O(1)$-airity and it is CPA-1-side secure without collusions and in the $(n-1)$-corruptions setting. This second construction leverages a new nesting execution technique of (lockable obfuscated) circuits.

Both multi-input constructions support conjunctions of arbitrary predicates with wildcards, i.e., for every $i \in [n]$, there exists (possibly unique) a wildcard $x_i^\star$ such that for every i-th predicate $\mathbb{P}_i$ we have $\mathbb{P}_i(x_i^\star) = 1$ (in Remark 3 we discuss how to remove the wildcard when no corruptions are in place).

Also, our constructions are generic and achieve CPA-2-sided security if the underlying single-input PE is CPA-2-sided secure (in case of no corruptions, our CPA-2-sided secure multi-input Construction 2 supports $n = O(\log(\lambda))$).

5.1 Multi-key PE from PE and Lockable Obfuscation

Construction 1. *Consider the following primitives:*

1. *For $i \in [n]$, a PE scheme $\mathsf{PE}_i = (\mathsf{Setup}_i, \mathsf{KGen}_i, \mathsf{Enc}_i, \mathsf{Dec}_i)$ with message space $\mathcal{M}_i$, input space $\mathcal{X}_i$, and predicate space $\mathcal{P}_i = \{\mathbb{P}_v(x)\}_{v \in \mathcal{V}_i}$ indexed by $\mathcal{V}_i$. Without loss of generality, we assume that PE_i has ciphertext space $\mathcal{Y}_i$, $\mathcal{M}_1 = \{0,1\}^{m(\lambda)}$, and $\mathcal{M}_i = \mathcal{Y}_{i-1}$ for every $i \in [n] \setminus \{1\}$.*

2. *A lockable obfuscation scheme $\mathsf{LOBF} = (\mathsf{Obf}, \mathsf{Eval})$ with message space $\mathcal{M}$ for the family of circuits $\mathcal{C}_{n,s,d}(\lambda) = \{\mathbb{C}_c\}$ as defined in Figure 6, where $n(\lambda)$, $s(\lambda)$, $d(\lambda)$ depends on the schemes $\mathsf{PE}_1, \ldots, \mathsf{PE}_n$ used, and the circuits $\mathcal{C}_{n,s,d}(\lambda)$.*

We build a n-key PE scheme Π with message space $\mathcal{M}$, input space $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$, and predicate space $\mathcal{P} = \{\mathbb{P}_{v_1,\ldots,v_n}(x_1,\ldots,x_n) = \mathbb{P}_{v_1}(x_1) \wedge \ldots \wedge \mathbb{P}_{v_n}(x_n)\}_{(v_1,\ldots,v_n)\in\mathcal{V}}$ indexed by $\mathcal{V} = \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$ (and $\mathbb{P}_{v_i} \in \mathcal{P}_i$ for $i \in [n]$), as follows:

$\mathsf{Setup}(1^\lambda)$: *Upon input the security parameter 1^λ the randomized setup algorithm outputs* $\mathsf{mpk} = (\mathsf{mpk}_1,\ldots,\mathsf{mpk}_n)$ *and* $\mathsf{msk}_1,\ldots,\mathsf{msk}_n$ *where* $(\mathsf{mpk}_i,\mathsf{msk}_i) \leftarrow_\$ \mathsf{Setup}_i(1^\lambda)$ *for $i \in [n]$.*

$\mathsf{KGen}(\mathsf{msk}_i, v_i)$: *Let $i \in [n]$. Upon input the i-th master secret key msk_i and the i-th predicate index $v_i \in \mathcal{V}_i$, the randomized key generator outputs* $\mathsf{dk}_{v_i} \leftarrow_\$ \mathsf{KGen}_i(\mathsf{msk}_1, \mathbb{P}_{v_i})$ *where* $\mathbb{P}_{v_i} \in \mathcal{P}_i$.

$\mathsf{Enc}(\mathsf{mpk}, x, m)$: *Upon input the master public key $\mathsf{mpk} = (\mathsf{mpk}_1,\ldots,\mathsf{mpk}_n)$, an input $x = (x_1,\ldots,x_n) \in \mathcal{X}$, and a message $m \in \mathcal{M}$, the randomized encryption proceeds as follows:*

    1. Sample $y \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and let $c_0 = y$.

    2. For $i \in [n]$, compute $c_i \leftarrow_\$ \mathsf{Enc}_i(\mathsf{mpk}_i, x_i, c_{i-1})$.

    Finally, it outputs $c = \widetilde{\mathbb{C}}$ where $\widetilde{\mathbb{C}} \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{C}_{c_n}, y, m)$.

$\mathsf{Dec}(\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_n}, c)$: *Upon input n decryption keys $\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_n}$ and a ciphertext $c = \widetilde{\mathbb{C}}$, the deterministic decryption algorithm outputs $m = \mathsf{Eval}(\widetilde{\mathbb{C}}, (\mathsf{dk}_{v_1},\ldots,\mathsf{dk}_{v_n}))$.*

Correctness follows from the correctness of the underlying schemes. We establish the following result whose proof appears in Appendices E.1 and E.2

Theorem 4. *Let $n = \mathsf{poly}(\lambda)$, $\mathsf{PE}_1,\ldots,\mathsf{PE}_n$ and LOBF be as above.*

    1. If each $\mathsf{PE}_1,\ldots,\mathsf{PE}_n$ is CPA secure (Definition 8) and LOBF is secure (Definition 2), then the n-key PE scheme Π from Construction 1 is CPA-1-sided secure (Definition 11).

    2. If each $\mathsf{PE}_1,\ldots,\mathsf{PE}_n$ is CPA-2-sided secure (Definition 9) and LOBF is secure (Definition 2), then the n-key PE scheme Π from Construction 1 is CPA-2-sided secure (Definition 11).

5.2 Multi-input PE from PE, Lockable Obfuscation and SKE/PKE

Secret-key setting. First, we present our n-input PE construction that is CPA-1-sided secure in the secret-key setting without collusions, for $n = \mathsf{poly}(\lambda)$. First, we present our n-input PE construction that is CPA-1-sided secure in the secret-key setting without collusions, for $n = \mathsf{poly}(\lambda)$. It leverages a CPA-1-sided secure single-input PE, lockable obfuscation, and SKE. The same construction is CPA-2-sided secure in the secret-key setting without collusions for $n = O(\log(\lambda))$, if the initial single-input PE is CPA-2-sided secure.

Construction 2 (n-input PE in the secret-key setting). *Consider the following primitives:*

    1. A PE scheme $\mathsf{PE}_1 = (\mathsf{Setup}_1, \mathsf{KGen}_1, \mathsf{Enc}_1, \mathsf{Dec}_1)$ with message space $\mathcal{M}_1 = \{0,1\}^{m(\lambda)} \times \mathcal{M}'_1$, input space $\mathcal{X}_1 = \mathcal{X}_{1,1} \times \ldots \times \mathcal{X}_{1,n}$, and predicate space $\mathcal{P}_1 = \{\mathbb{P}(x_1,\ldots,x_n)\} = \{\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)\}$. Without loss of generality, we assume that PE has ciphertext space $\mathcal{M}_2$ and there exists a (single) wildcard input $(x_1^\star,\ldots,x_n^\star) \in \mathcal{X}_1$ such that $\forall(\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)) \in \mathcal{P}_1, \forall i \in [n], \mathbb{P}_i(x_i^\star) = 1$.

    2. A SKE scheme $\mathsf{SKE} = (\mathsf{KGen}_2, \mathsf{Enc}_2, \mathsf{Dec}_2)$ with message space $\mathcal{M}_2$. Without loss of generality, we assume that SKE has key space $\mathcal{M}'_1$

3. A lockable obfuscation scheme $\mathsf{LOBF} = (\mathsf{Obf}, \mathsf{Eval})$ with message space $\mathcal{M}_3$ for the family of circuits $\mathcal{C}_{n,s,d}(\lambda) = \{\mathbb{C}_{c,\mathsf{k}}\}$ as defined in Figure 6, where $n(\lambda)$, $s(\lambda)$, $d(\lambda)$ depends on the schemes PE and SKE used, and the circuit depth of the circuits $\mathcal{C}_{n,s,d}(\lambda)$.

We build a n-input PE scheme with message space $\mathcal{M} = \overbrace{\mathcal{M}_3 \times \ldots \times \mathcal{M}_3}^{n}$, input space $\mathcal{X} = \mathcal{X}_1$, and predicate space $\mathcal{P} = \mathcal{P}_1 = \{\mathbb{P}(x_1, \ldots, x_n)\} = \{\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)\}$ with wildcard (i.e., there exists a (single) wildcard $(x_1^\star, \ldots, x_n^\star) \in \mathcal{X}$ such that $\forall (\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)) \in \mathcal{P}, \forall i \in [n]$, $\mathbb{P}_i(x_i^\star) = 1$), as follows:

$\mathsf{Setup}(1^\lambda)$: Upon input the security parameter 1^λ, the randomized setup algorithm outputs $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n)$ and msk where $(\mathsf{mpk}, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}_1(1^\lambda)$, $\mathsf{ek}_i = (\mathsf{mpk}, \mathsf{k}_i, \mathsf{k}_{i+1})$, $\mathsf{k}_{n+1} = \mathsf{k}_1$, and $\mathsf{k}_i \leftarrow_\$ \mathsf{KGen}_2(1^\lambda)$ for $i \in [n]$.

$\mathsf{KGen}(\mathsf{msk}, \mathbb{P})$: Upon input the master secret key msk and a predicate $\mathbb{P} \in \mathcal{P}$, the randomized key generator outputs $\mathsf{dk}_\mathbb{P} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P})$.

$\mathsf{Enc}(\mathsf{ek}_i, x_i, m_i)$: Let $i \in [n]$. Upon input an encryption key $\mathsf{ek}_i = (\mathsf{mpk}, \mathsf{k}_i, \mathsf{k}_{i+1})$, an input $x_i \in \mathcal{X}_{1,i}$, and a message $m_i \in \mathcal{M}_3$, the randomized encryption algorithm samples $y_i \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and compute $c_i^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_i, \mathsf{k}_{i+1}))$ where $x_j = x_j^\star$ for any $j \in [n] \setminus \{i\}$. Finally, it outputs $c = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$ where $\widetilde{\mathbb{C}}_i \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{C}_{c_i^{(2)}, \mathsf{k}_{i+1}}, y_i, m_i)$ and $c_i^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_i, c_i^{(1)})$.

$\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, c_1, \ldots, c_n)$: Upon input a secret decryption key $\mathsf{dk}_\mathbb{P}$ for predicate $\mathbb{P} \in \mathcal{P}$, and n ciphertexts $(c_1, \ldots, c_n)$ such that $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$ for $i \in [n]$. The deterministic decryption returns $(m_1, \ldots, m_n)$ where $m_i = \mathsf{Eval}(\widetilde{\mathbb{C}}_i, (c_{i+1}^{(2)}, \ldots, c_n^{(2)}, c_1^{(2)}, \ldots, c_{i-1}^{(2)}, \mathsf{dk}_\mathbb{P}))$ for $i \in [n]$.

As usual, correctness follows from the correctness of the underlying primitives. Below, we establish the following result whose proof appear in Appendices E.3 and E.4.

Theorem 5. Let PE, SKE, and LOBF be as above.

1. For $n = \mathsf{poly}(\lambda)$, if PE is CPA-1-sided secure without collusions (Definition 9), SKE is CPA secure (Definition 4), and LOBF is secure (Definition 2), then the n-input PE scheme Π from Construction 2 is CPA-1-sided secure in the secret-key setting without collusions (Definition 13).

2. For $n = O(\log(\lambda))$, if PE is CPA-2-sided secure without collusions (Definition 9), SKE is CPA secure (Definition 4), and LOBF is secure (Definition 2), then the n-input PE scheme Π from Construction 2 is CPA-2-sided secure in the secret-key setting without collusions (Definition 13).

Corruption setting. We now move on to our construction of n-input PE that is CPA-1-sided secure in the $(n-1)$-corruptions setting without collusions. This construction handles constant-arity (i.e., $n \in O(1)$) since the decryption running time is $O(n^n)$. It is based on CPA secure single-input PE, lockable obfuscation, and PKE and it leverages the nested execution technique described in Section 1.2. Also, the same construction achieves CPA-2-sided security if the initial single-input PE is CPA-2-sided secure.

Construction 3 (n-input PE in the corruption setting). *Consider the following primitives:*
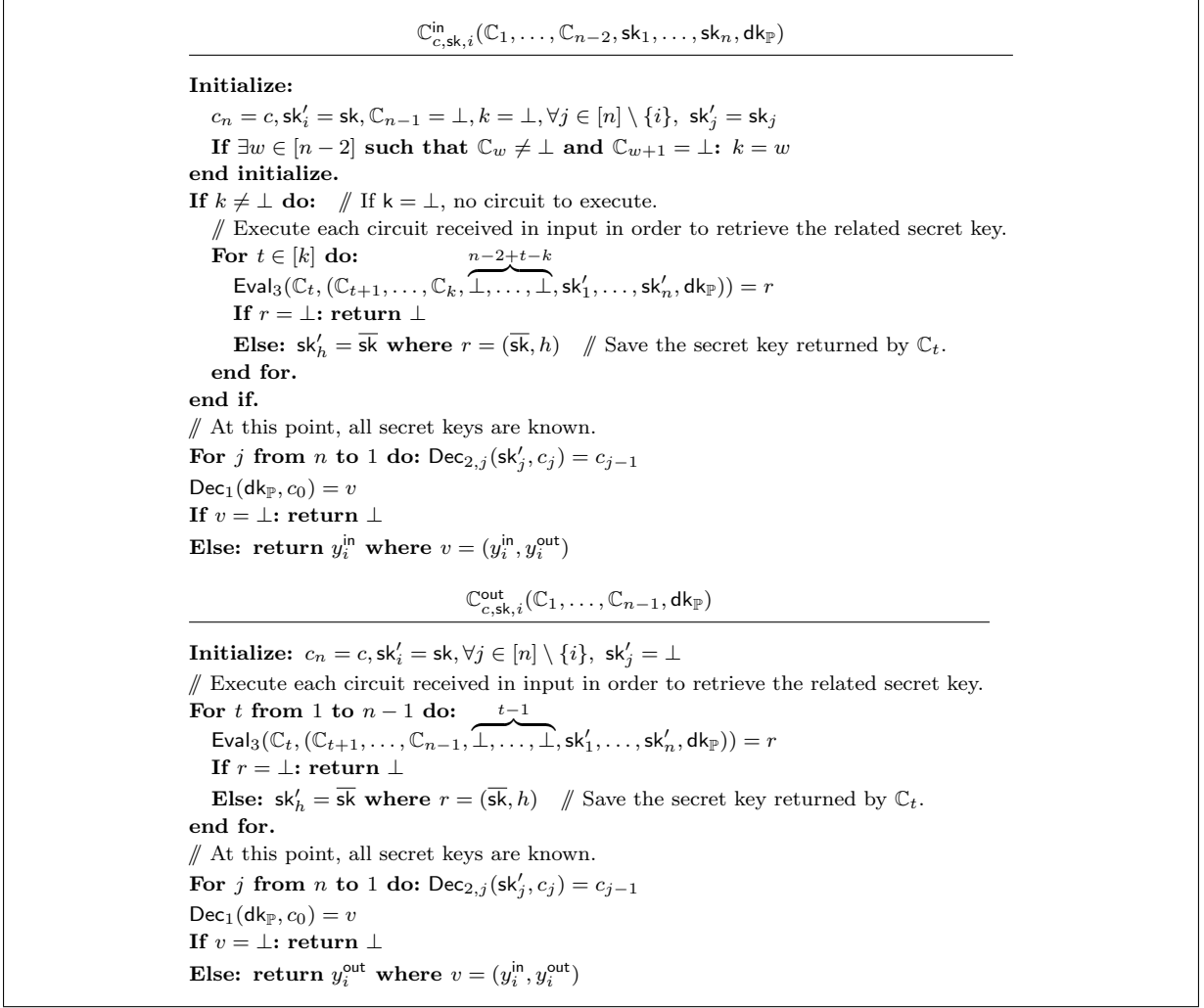
$$\mathbb{C}^{\mathsf{in}}_{c,\mathsf{sk},i}(\mathbb{C}_1,\ldots,\mathbb{C}_{n-2},\mathsf{sk}_1,\ldots,\mathsf{sk}_n,\mathsf{dk}_{\mathbb{P}})$$

Initialize:

$c_n = c, \mathsf{sk}'_i = \mathsf{sk}, \mathbb{C}_{n-1} = \bot, k = \bot, \forall j \in [n] \setminus \{i\}, \ \mathsf{sk}'_j = \mathsf{sk}_j$

If $\exists w \in [n-2]$ **such that** $\mathbb{C}_w \neq \bot$ **and** $\mathbb{C}_{w+1} = \bot$**:** $k = w$

end initialize.

If $k \neq \bot$ **do:** // If $\mathsf{k} = \bot$, no circuit to execute.

    // Execute each circuit received in input in order to retrieve the related secret key.

    For $t \in [k]$ **do:** $\qquad\qquad \overbrace{\qquad}^{n-2+t-k}$

        $\mathsf{Eval}_3(\mathbb{C}_t, (\mathbb{C}_{t+1}, \ldots, \mathbb{C}_k, \overbrace{\bot, \ldots, \bot}, \mathsf{sk}'_1, \ldots, \mathsf{sk}'_n, \mathsf{dk}_{\mathbb{P}})) = r$

        If $r = \bot$**: return** $\bot$

        Else: $\mathsf{sk}'_h = \overline{\mathsf{sk}}$ **where** $r = (\overline{\mathsf{sk}}, h)$ // Save the secret key returned by $\mathbb{C}_t$.

    end for.

end if.

// At this point, all secret keys are known.

For j **from** n **to** 1 **do:** $\mathsf{Dec}_{2,j}(\mathsf{sk}'_j, c_j) = c_{j-1}$

$\mathsf{Dec}_1(\mathsf{dk}_{\mathbb{P}}, c_0) = v$

If $v = \bot$**: return** $\bot$

Else: return y^{in}_i **where** $v = (y^{\mathsf{in}}_i, y^{\mathsf{out}}_i)$

$$\mathbb{C}^{\mathsf{out}}_{c,\mathsf{sk},i}(\mathbb{C}_1,\ldots,\mathbb{C}_{n-1},\mathsf{dk}_{\mathbb{P}})$$

Initialize: $c_n = c, \mathsf{sk}'_i = \mathsf{sk}, \forall j \in [n] \setminus \{i\}, \ \mathsf{sk}'_j = \bot$

// Execute each circuit received in input in order to retrieve the related secret key.

For t **from** 1 **to** $n-1$ **do:** $\qquad \overbrace{\qquad}^{t-1}$

    $\mathsf{Eval}_3(\mathbb{C}_t, (\mathbb{C}_{t+1}, \ldots, \mathbb{C}_{n-1}, \overbrace{\bot, \ldots, \bot}, \mathsf{sk}'_1, \ldots, \mathsf{sk}'_n, \mathsf{dk}_{\mathbb{P}})) = r$

    If $r = \bot$**: return** $\bot$

    Else: $\mathsf{sk}'_h = \overline{\mathsf{sk}}$ **where** $r = (\overline{\mathsf{sk}}, h)$ // Save the secret key returned by $\mathbb{C}_t$.

end for.

// At this point, all secret keys are known.

For j **from** n **to** 1 **do:** $\mathsf{Dec}_{2,j}(\mathsf{sk}'_j, c_j) = c_{j-1}$

$\mathsf{Dec}_1(\mathsf{dk}_{\mathbb{P}}, c_0) = v$

If $v = \bot$**: return** $\bot$

Else: return y^{out}_i **where** $v = (y^{\mathsf{in}}_i, y^{\mathsf{out}}_i)$

Figure 7: Definitions of the circuits $\mathbb{C}^{\mathsf{in}}_{c,\mathsf{sk},i}$ and $\mathbb{C}^{\mathsf{out}}_{c,\mathsf{sk},i}$ supported by the lockable obfuscation schemes LOBF_3 and LOBF_4 of Construction 3.

1. *A PE scheme* $\mathsf{PE} = (\mathsf{Setup}_1, \mathsf{KGen}_1, \mathsf{Enc}_1, \mathsf{Dec}_1)$ *with message space* $\mathcal{M}_1 = \{0,1\}^{m_3(\lambda)+m_4(\lambda)}$, *input space* $\mathcal{X}_1 = \mathcal{X}_{1,1} \times \ldots \times \mathcal{X}_{1,n}$, *and predicate space* $\mathcal{P}_1 = \{\mathbb{P}(x_1, \ldots, x_n)\} = \{\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)\}$. *Without loss of generality, we assume that* PE *has ciphertext space* $\mathcal{Y}_1$ *and there exists a (single) wildcard input* $(x^\star_1, \ldots, x^\star_n) \in \mathcal{X}_1$ *such that* $\forall(\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)) \in \mathcal{P}_1, \forall i \in [n], \mathbb{P}_i(x^\star_i) = 1$.

2. *For* $i \in [n]$, *a PKE scheme* $\mathsf{PKE}_{2,i} = (\mathsf{KGen}_{2,i}, \mathsf{Enc}_{2,i}, \mathsf{Dec}_{2,i})$ *with message space* $\mathcal{M}_{2,i}$. *Without loss of generality, we assume that* PKE_i *has ciphertext space* $\mathcal{Y}_{2,i}$ *and secret-key space* $\mathcal{K}_{2,i}$. *Moreover, we assume that* $\mathcal{M}_{2,1} = \mathcal{Y}_1$, *and* $\mathcal{M}_{2,i} = \mathcal{Y}_{2,i-1}$ *for every* $i \in [n] \setminus \{1\}$.

3. *A lockable obfuscation scheme* $\mathsf{LOBF}_3 = (\mathsf{Obf}_3, \mathsf{Eval}_3)$ *with message space* $\mathcal{M}_3 = (\mathcal{K}_{2,1} \cup \ldots \cup \mathcal{K}_{2,n}) \times \{0,1\}^{\lfloor \log_2(n) \rfloor + 1}$ *for the family of circuits* $\mathcal{C}^{\mathsf{in}}_{n_3,s_3,d_3}(\lambda) = \{\mathbb{C}^{\mathsf{in}}_{c,\mathsf{sk},i}\}$ *defined in Figure 7, where* $n_3(\lambda), s_3(\lambda), d_3(\lambda)$ *depends on the schemes* $\mathsf{PE}, \mathsf{PKE}_{2,1}, \ldots, \mathsf{PKE}_{2,n}$ *used, and the circuits* $\mathcal{C}^{\mathsf{in}}_{n_3,s_3,d_3}(\lambda)$.

4. *A lockable obfuscation scheme* $\mathsf{LOBF}_4 = (\mathsf{Obf}_4, \mathsf{Eval}_4)$ *with message space* $\mathcal{M}_4$ *for the family of circuits* $\mathcal{C}^{\mathsf{out}}_{n_4,s_4,d_4}(\lambda) = \{\mathbb{C}^{\mathsf{out}}_{c,\mathsf{sk},i}\}$ *defined in Figure 7, where* $n_4(\lambda), s_4(\lambda), d_4(\lambda)$ *depends on the schemes* $\mathsf{PE}, \mathsf{PKE}_{2,1}, \ldots, \mathsf{PKE}_{2,n}, \mathsf{LOBF}_3$ *used, and the circuits* $\mathcal{C}^{\mathsf{out}}_{n_4,s_4,d_4}(\lambda)$.

We build a n-input PE scheme with message space $\mathcal{M} = \overbrace{\mathcal{M}_4 \times \ldots \times \mathcal{M}_4}^{n}$, input space $\mathcal{X} = \mathcal{X}_1$, and predicate space $\mathcal{P} = \mathcal{P}_1 = \{\mathbb{P}(x_1, \ldots, x_n)\} = \{\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)\}$ with wildcard (i.e., there exists a (single) wildcard $(x_1^\star, \ldots, x_n^\star) \in \mathcal{X}$ such that $\forall (\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)) \in \mathcal{P}$, $\forall i \in [n]$, $\mathbb{P}_i(x_i^\star) = 1$), as follows:

$\mathsf{Setup}(1^\lambda)$: *Upon input the security parameter 1^λ the randomized setup algorithm outputs $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n)$ and msk where $(\mathsf{mpk}, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}_1(1^\lambda)$, $\mathsf{ek}_i = (\mathsf{mpk}, \mathsf{sk}_i, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$, and $(\mathsf{sk}_i, \mathsf{pk}_i) \leftarrow_\$ \mathsf{KGen}_{2,i}(1^\lambda)$ for $i \in [n]$.*

$\mathsf{KGen}(\mathsf{msk}, \mathbb{P})$: *Upon input the master secret key msk and a predicate $\mathbb{P} \in \mathcal{P}$, the randomized key generator algorithm outputs $\mathsf{dk}_\mathbb{P} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P})$.*

$\mathsf{Enc}(\mathsf{ek}_i, x_i, m_i)$: *Let $i \in [n]$. Upon input an encryption key $\mathsf{ek}_i = (\mathsf{mpk}, \mathsf{sk}_i, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$, an input $x_i \in \mathcal{X}_{1,i}$, and a message $m_i \in \mathcal{M}_4$, the randomized encryption algorithm samples $(y_i^{\mathsf{in}}, y_i^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda) + s_4(\lambda)}$ and proceeds as follows:*

   1. *Compute $c_i^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_i^{\mathsf{in}}, y_i^{\mathsf{out}}))$ where $x_j = x_j^\star$ for $j \in [n] \setminus \{i\}$.*

   2. *For $j \in [n]$, compute $c_i^{(j)} \leftarrow_\$ \mathsf{Enc}_{2,j}(\mathsf{pk}_j, c_i^{(j-1)})$.*

   Finally, it outputs $c_i = (\widetilde{\mathbb{C}}_i^{\mathsf{out}}, \widetilde{\mathbb{C}}_i^{\mathsf{in}})$, where $\widetilde{\mathbb{C}}_i^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{C}_{c_i^{(n)}, \mathsf{sk}_i, i}^{\mathsf{out}}, y_i^{\mathsf{out}}, m_i)$ and $\widetilde{\mathbb{C}}_i^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{C}_{c_i^{(n)}, \mathsf{sk}_i, i}^{\mathsf{in}}, y_i^{\mathsf{in}}, (\mathsf{sk}_i, i))$.

$\mathsf{Dec}(\mathsf{dk}_\mathbb{P}, c_1, \ldots, c_n)$: *Upon input a decryption key $\mathsf{dk}_\mathbb{P}$ for predicate $\mathbb{P} \in \mathcal{P}$, and n ciphertexts $(c_1, \ldots, c_n)$ such that $c_i = (\widetilde{\mathbb{C}}_i^{\mathsf{out}}, \widetilde{\mathbb{C}}_i^{\mathsf{in}})$ for $i \in [n]$. The deterministic decryption algorithm returns $(m_1, \ldots, m_n)$ where $m_i = \mathsf{Eval}_4(\widetilde{\mathbb{C}}_i^{\mathsf{out}}, (\widetilde{\mathbb{C}}_1^{\mathsf{in}}, \ldots, \widetilde{\mathbb{C}}_{i-1}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{i+1}^{\mathsf{in}}, \ldots, \widetilde{\mathbb{C}}_n^{\mathsf{in}}, \mathsf{dk}_\mathbb{P}))$ for $i \in [n]$.*

Correctness follows from the one of the underlying primitives (see also Figure 7 for the definitions of $\mathbb{C}_{c,\mathsf{sk},i}^{\mathsf{in}}$ and $\mathbb{C}_{c,\mathsf{sk},i}^{\mathsf{out}}$). Moreover, decryption is polynomial time when $n \in O(1)$. Below, we establish the following result whose proof appear in Appendices E.5 and E.6.

Theorem 6. *Let $n = O(1)$, PE, $\mathsf{PKE}_{2,1}, \ldots, \mathsf{PKE}_{2,n}$, LOBF_3, and LOBF_4 be as above.*

   1. *If PE is CPA secure without collusions (Definition 8), each $\mathsf{PKE}_{2,i}$ (for $i \in [n]$) is CPA secure (Definition 6), and both LOBF_3 and LOBF_4 are secure (Definition 2), then the n-input PE scheme Π from Construction 3 is CPA-1-sided secure in the $(n-1)$-corruptions setting without collusions (Definition 13).*

   2. *If PE is CPA-2-sided secure without collusions (Definition 9), each $\mathsf{PKE}_{2,i}$ (for $i \in [n]$) is CPA secure (Definition 6), and both LOBF_3 and LOBF_4 are secure (Definition 2), then the n-input PE scheme Π from Construction 3 is CPA-2-sided secure in the $(n-1)$-corruptions setting without collusions (Definition 13).*

Remark 3 (On wildcards). Wildcards affect the security guarantee and the expressiveness of the multi-input PE construction depending on the presence of corruptions. In the case of no corruptions (Construction 2), the (single) wildcard can be removed by simply requiring each i-th sender not to compute a ciphertext c_i under the corresponding i-th wildcard $x_i^\star$, i.e., $\mathsf{Enc}(\mathsf{ek}_i, x_i, m_i)$ outputs $\bot$ whenever $x_i = x_i^\star$. In other words, we can transform any secure multi-input PE for $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$ with wildcard $(x_1^\star, \ldots, x_n^\star)$ into a secure multi-input PE for the same predicate $\mathbb{P}(x_1, \ldots, x_n)$ without the wildcard. On the other hand,

this cannot be done when corruptions are in place (Construction 3). Indeed, if the adversary gets an encryption key ek_i, then it can use the latter to always produce a ciphertext c_i under $x_i^\star$. This means that the adversary can always use ek_i (of the corrupted user) and satisfy the i-th predicate $\mathbb{P}_i$ (this also affects the security proof of Construction 3. See Appendices E.5 and E.6).

Remark 4 (On unbounded collusions). For completeness, we highlight that if we start from an initial single-input PE scheme PE (of Theorems 5 and 6) that is CPA-1-sided secure against *unbounded collusions*, both our Construction 2 and Construction 3 are CPA-1-sided secure with respect to a *weaker* form of unbounded collusions (but still stronger than no collusions). For the sake of clarity, we focus on our secret-key Construction 2, but the same argument holds for our Construction 3 against corruptions.

In case of no collusions, at the beginning of the proof of Theorem 5 (see Appendix E.3), we show that the adversary's validity condition (of Definition 13) is equivalent to satisfying at least one of the following four conditions: for some $j_0, j_1 \in [n]$,

$\mathbf{Validity}_1$:
$$\mathbb{P}_1^*(x_1^0) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^0) = 0 \wedge \mathbb{P}_1^*(x_1^1) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^1) = 0 \tag{3}$$

$\mathbf{Validity}_{2,j_0,j_1} : \forall x'_{j_0} \in \mathcal{Q}_{j_0}, \forall x'_{j_1} \in \mathcal{Q}_{j_1},$
$$\mathbb{P}_{j_0}^*(x_{j_0}^0) = 0 \wedge \mathbb{P}_{j_0}^*(x'_{j_0}) = 0 \wedge \mathbb{P}_{j_1}^*(x_{j_1}^1) = 0 \wedge \mathbb{P}_{j_1}^*(x'_{j_1}) = 0 \tag{4}$$

$\mathbf{Validity}_{3,j_0} : \forall x'_{j_0} \in \mathcal{Q}_{j_0},$
$$\mathbb{P}_{j_0}^*(x_{j_0}^0) = 0 \wedge \mathbb{P}_{j_0}^*(x'_{j_0}) = 0 \wedge \mathbb{P}_1^*(x_1^1) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^1) = 0 \tag{5}$$

$\mathbf{Validity}_{4,j_1} : \forall x'_{j_1} \in \mathcal{Q}_{j_1},$
$$\mathbb{P}_1^*(x_1^0) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^0) = 0 \wedge \mathbb{P}_{j_1}^*(x_{j_1}^1) = 0 \wedge \mathbb{P}_{j_1}^*(x'_{j_1}) = 0 \tag{6}$$

where $\mathbb{P}^*(x_1, \ldots, x_n) = (\mathbb{P}_1^*(x_1) \wedge \ldots \wedge \mathbb{P}_n^*(x_n)) \in \mathcal{Q}_{\mathsf{KGen}}$ is the *single* key generation query submitted by the adversary A, $((x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ is the adversarial challenge inputs, and $\mathcal{Q}_i$ are the predicate inputs submitted to the encryption oracle $\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)$ for $i \in [n]$.

When working with CPA-1-sided security against (fully-fledged) unbounded collusions, a valid adversary can obtain two decryption keys for $\mathbb{P}$ and $\mathbb{P}'$ that satisfy Equation (4) (or Equations (5) and (6)) with respect to two different indexes $j_0, j_1 \in [n]$ and $j_0', j_1' \in [n]$, i.e., $(j_0, j_1) \neq (j_0', j_1')$. When this happens the proof fails since, as we discussed in Section 1.2, our reduction will make an invalid set of queries to the KGen oracle of the single-input PE. However, we observe that the exact same proof of Theorem 4 goes through when we allow A to asking for multiple decryption keys under the restriction that: $\exists j_0, j_1 \in [n], \forall \mathbb{P}(x_1, \ldots, x_n) = (\mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)) \in \mathcal{Q}_{\mathsf{KGen}}$, such that either one condition between Equations (3) to (6) is satisfied (i.e., the same indexes j_0, j_1 for all predicates $\mathbb{P} \in \mathcal{Q}_{\mathsf{KGen}}$).

References

[ABG19]   Michel Abdalla, Fabrice Benhamouda, and Romain Gay. From single-input to multi-client inner-product functional encryption. In Steven D. Galbraith and Shiho Moriai, editors, *ASIACRYPT 2019, Part III*, volume 11923 of *LNCS*, pages 552–582. Springer, Heidelberg, December 2019.

[ABKW19] Michel Abdalla, Fabrice Benhamouda, Markulf Kohlweiss, and Hendrik Waldner. Decentralizing inner-product functional encryption. In Dongdai Lin and Kazue Sako, editors, *PKC 2019, Part II*, volume 11443 of *LNCS*, pages 128–157. Springer, Heidelberg, April 2019.

[ACF⁺18]   Michel Abdalla, Dario Catalano, Dario Fiore, Romain Gay, and Bogdan Ursu. Multi-input functional encryption for inner products: Function-hiding realizations and constructions without pairings. In Hovav Shacham and Alexandra Boldyreva, editors, *CRYPTO 2018, Part I*, volume 10991 of *LNCS*, pages 597–627. Springer, Heidelberg, August 2018.

[AFNV19]   Giuseppe Ateniese, Danilo Francati, David Nuñez, and Daniele Venturi. Match me if you can: Matchmaking encryption and its applications. In Alexandra Boldyreva and Daniele Micciancio, editors, *CRYPTO 2019, Part II*, volume 11693 of *LNCS*, pages 701–731. Springer, Heidelberg, August 2019.

[AFNV21]   Giuseppe Ateniese, Danilo Francati, David Nuñez, and Daniele Venturi. Match me if you can: matchmaking encryption and its applications. *Journal of Cryptology*, 34(3):1–50, 2021.

[AFV11]   Shweta Agrawal, David Mandell Freeman, and Vinod Vaikuntanathan. Functional encryption for inner product predicates from learning with errors. In Dong Hoon Lee and Xiaoyun Wang, editors, *ASIACRYPT 2011*, volume 7073 of *LNCS*, pages 21–40. Springer, Heidelberg, December 2011.

[AGRW17]   Michel Abdalla, Romain Gay, Mariana Raykova, and Hoeteck Wee. Multi-input inner-product functional encryption from pairings. In Jean-Sébastien Coron and Jesper Buus Nielsen, editors, *EUROCRYPT 2017, Part I*, volume 10210 of *LNCS*, pages 601–626. Springer, Heidelberg, April / May 2017.

[AGT21]   Shweta Agrawal, Rishab Goyal, and Junichi Tomida. Multi-input quadratic functional encryption from pairings. In Tal Malkin and Chris Peikert, editors, *CRYPTO 2021, Part IV*, volume 12828 of *LNCS*, pages 208–238, Virtual Event, August 2021. Springer, Heidelberg.

[AGT22]   Shweta Agrawal, Rishab Goyal, and Junichi Tomida. Multi-input quadratic functional encryption: Stronger security, broader functionality. In *TCC 2022 (To appear)*. Springer, 2022.

[AJ15]   Prabhanjan Ananth and Abhishek Jain. Indistinguishability obfuscation from compact functional encryption. In Rosario Gennaro and Matthew J. B. Robshaw, editors, *CRYPTO 2015, Part I*, volume 9215 of *LNCS*, pages 308–326. Springer, Heidelberg, August 2015.

[Att14]   Nuttapong Attrapadung. Dual system encryption via doubly selective security: Framework, fully secure functional encryption for regular languages, and more. In Phong Q. Nguyen and Elisabeth Oswald, editors, *EUROCRYPT 2014*, volume 8441 of *LNCS*, pages 557–577. Springer, Heidelberg, May 2014.

[AYY22]   Shweta Agrawal, Anshu Yadav, and Shota Yamada. Multi-input attribute based encryption and predicate encryption. In *CRYPTO 2022 (To appear)*. Springer, 2022.

[BDGM22]   Zvika Brakerski, Nico Döttling, Sanjam Garg, and Giulio Malavolta. Factoring and pairings are not necessary for io: Circular-secure lwe suffices. ICALP, 2022.

[BGI⁺01]   Boaz Barak, Oded Goldreich, Russell Impagliazzo, Steven Rudich, Amit Sahai, Salil P. Vadhan, and Ke Yang. On the (im)possibility of obfuscating programs. In

Joe Kilian, editor, *CRYPTO 2001*, volume 2139 of *LNCS*, pages 1–18. Springer, Heidelberg, August 2001.

[BGI+14]   Amos Beimel, Ariel Gabizon, Yuval Ishai, Eyal Kushilevitz, Sigurd Meldgaard, and Anat Paskin-Cherniavsky. Non-interactive secure multiparty computation. In Juan A. Garay and Rosario Gennaro, editors, *CRYPTO 2014, Part II*, volume 8617 of *LNCS*, pages 387–404. Springer, Heidelberg, August 2014.

[BJK+18]   Zvika Brakerski, Aayush Jain, Ilan Komargodski, Alain Passelègue, and Daniel Wichs. Non-trivial witness encryption and null-iO from standard assumptions. In Dario Catalano and Roberto De Prisco, editors, *SCN 18*, volume 11035 of *LNCS*, pages 425–441. Springer, Heidelberg, September 2018.

[BLR+15]   Dan Boneh, Kevin Lewi, Mariana Raykova, Amit Sahai, Mark Zhandry, and Joe Zimmerman. Semantically secure order-revealing encryption: Multi-input functional encryption without obfuscation. In Elisabeth Oswald and Marc Fischlin, editors, *EUROCRYPT 2015, Part II*, volume 9057 of *LNCS*, pages 563–594. Springer, Heidelberg, April 2015.

[BV15]   Nir Bitansky and Vinod Vaikuntanathan. Indistinguishability obfuscation from functional encryption. In Venkatesan Guruswami, editor, *56th FOCS*, pages 171–190. IEEE Computer Society Press, October 2015.

[BW07]   Dan Boneh and Brent Waters. Conjunctive, subset, and range queries on encrypted data. In Salil P. Vadhan, editor, *TCC 2007*, volume 4392 of *LNCS*, pages 535–554. Springer, Heidelberg, February 2007.

[CDG+18]   Jérémy Chotard, Edouard Dufour Sans, Romain Gay, Duong Hieu Phan, and David Pointcheval. Decentralized multi-client functional encryption for inner product. In Thomas Peyrin and Steven Galbraith, editors, *ASIACRYPT 2018, Part II*, volume 11273 of *LNCS*, pages 703–732. Springer, Heidelberg, December 2018.

[CLWW22]   Jie Chen, Yu Li, Jinming Wen, and Jian Weng. Identity-based matchmaking encryption from standard assumptions. In *ASIACRYPT 2022 (To appear)*. Springer, 2022.

[CM15]   Michael Clear and Ciaran McGoldrick. Multi-identity and multi-key leveled FHE from learning with errors. In Rosario Gennaro and Matthew J. B. Robshaw, editors, *CRYPTO 2015, Part II*, volume 9216 of *LNCS*, pages 630–656. Springer, Heidelberg, August 2015.

[CSW21]   Michele Ciampi, Luisa Siniscalchi, and Hendrik Waldner. Multi-client functional encryption for separable functions. In Juan Garay, editor, *PKC 2021, Part I*, volume 12710 of *LNCS*, pages 724–753. Springer, Heidelberg, May 2021.

[DOT18]   Pratish Datta, Tatsuaki Okamoto, and Junichi Tomida. Full-hiding (unbounded) multi-input inner product functional encryption from the k-Linear assumption. In Michel Abdalla and Ricardo Dahab, editors, *PKC 2018, Part II*, volume 10770 of *LNCS*, pages 245–277. Springer, Heidelberg, March 2018.

[FGRV21]   Danilo Francati, Alessio Guidi, Luigi Russo, and Daniele Venturi. Identity-based matchmaking encryption without random oracles. In *INDOCRYPT 2021*, pages 415–435. Springer, 2021.

[GGG+14] Shafi Goldwasser, S. Dov Gordon, Vipul Goyal, Abhishek Jain, Jonathan Katz, Feng-Hao Liu, Amit Sahai, Elaine Shi, and Hong-Sheng Zhou. Multi-input functional encryption. In Phong Q. Nguyen and Elisabeth Oswald, editors, *EUROCRYPT 2014*, volume 8441 of *LNCS*, pages 578–602. Springer, Heidelberg, May 2014.

[GGSW13] Sanjam Garg, Craig Gentry, Amit Sahai, and Brent Waters. Witness encryption and its applications. In Dan Boneh, Tim Roughgarden, and Joan Feigenbaum, editors, *45th ACM STOC*, pages 467–476. ACM Press, June 2013.

[GKW17] Rishab Goyal, Venkata Koppula, and Brent Waters. Lockable obfuscation. In Chris Umans, editor, *58th FOCS*, pages 612–621. IEEE Computer Society Press, October 2017.

[GP21] Romain Gay and Rafael Pass. Indistinguishability obfuscation from circular security. In Samir Khuller and Virginia Vassilevska Williams, editors, *STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21-25, 2021*, pages 736–749. ACM, 2021.

[GVW15] Sergey Gorbunov, Vinod Vaikuntanathan, and Hoeteck Wee. Predicate encryption for circuits from LWE. In Rosario Gennaro and Matthew J. B. Robshaw, editors, *CRYPTO 2015, Part II*, volume 9216 of *LNCS*, pages 503–523. Springer, Heidelberg, August 2015.

[HIJ+16] Shai Halevi, Yuval Ishai, Abhishek Jain, Eyal Kushilevitz, and Tal Rabin. Secure multiparty computation with general interaction patterns. In Madhu Sudan, editor, *ITCS 2016*, pages 157–168. ACM, January 2016.

[HIJ+17] Shai Halevi, Yuval Ishai, Abhishek Jain, Ilan Komargodski, Amit Sahai, and Eylon Yogev. Non-interactive multiparty computation without correlated randomness. In Tsuyoshi Takagi and Thomas Peyrin, editors, *ASIACRYPT 2017, Part III*, volume 10626 of *LNCS*, pages 181–211. Springer, Heidelberg, December 2017.

[HLP11] Shai Halevi, Yehuda Lindell, and Benny Pinkas. Secure computation on the web: Computing without simultaneous interaction. In Phillip Rogaway, editor, *CRYPTO 2011*, volume 6841 of *LNCS*, pages 132–150. Springer, Heidelberg, August 2011.

[JLS21] Aayush Jain, Huijia Lin, and Amit Sahai. Indistinguishability obfuscation from well-founded assumptions. In Samir Khuller and Virginia Vassilevska Williams, editors, *STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21-25, 2021*, pages 60–73. ACM, 2021.

[JLS22] Aayush Jain, Huijia Lin, and Amit Sahai. Indistinguishability obfuscation from LPN over $\mathbb{F}_p$, DLIN, and PRGs in NC^0. In Orr Dunkelman and Stefan Dziembowski, editors, *EUROCRYPT 2022, Part I*, volume 13275 of *LNCS*, pages 670–699. Springer, Heidelberg, May / June 2022.

[KSW08] Jonathan Katz, Amit Sahai, and Brent Waters. Predicate encryption supporting disjunctions, polynomial equations, and inner products. In Nigel P. Smart, editor, *EUROCRYPT 2008*, volume 4965 of *LNCS*, pages 146–162. Springer, Heidelberg, April 2008.

[LOS+10]   Allison B. Lewko, Tatsuaki Okamoto, Amit Sahai, Katsuyuki Takashima, and Brent Waters. Fully secure functional encryption: Attribute-based encryption and (hierarchical) inner product encryption. In Henri Gilbert, editor, *EUROCRYPT 2010*, volume 6110 of *LNCS*, pages 62–91. Springer, Heidelberg, May / June 2010.

[LT19]   Benoît Libert and Radu Titiu. Multi-client functional encryption for linear functions in the standard model from LWE. In Steven D. Galbraith and Shiho Moriai, editors, *ASIACRYPT 2019, Part III*, volume 11923 of *LNCS*, pages 520–551. Springer, Heidelberg, December 2019.

[LTV12]   Adriana López-Alt, Eran Tromer, and Vinod Vaikuntanathan. On-the-fly multiparty computation on the cloud via multikey fully homomorphic encryption. In Howard J. Karloff and Toniann Pitassi, editors, *44th ACM STOC*, pages 1219–1234. ACM Press, May 2012.

[MW16]   Pratyay Mukherjee and Daniel Wichs. Two round multiparty computation via multi-key FHE. In Marc Fischlin and Jean-Sébastien Coron, editors, *EUROCRYPT 2016, Part II*, volume 9666 of *LNCS*, pages 735–763. Springer, Heidelberg, May 2016.

[OT10]   Tatsuaki Okamoto and Katsuyuki Takashima. Fully secure functional encryption with general relations from the decisional linear assumption. In Tal Rabin, editor, *CRYPTO 2010*, volume 6223 of *LNCS*, pages 191–208. Springer, Heidelberg, August 2010.

[OT12]   Tatsuaki Okamoto and Katsuyuki Takashima. Adaptively attribute-hiding (hierarchical) inner product encryption. In David Pointcheval and Thomas Johansson, editors, *EUROCRYPT 2012*, volume 7237 of *LNCS*, pages 591–608. Springer, Heidelberg, April 2012.

[Tom19]   Junichi Tomida. Tightly secure inner product functional encryption: Multi-input and function-hiding constructions. In Steven D. Galbraith and Shiho Moriai, editors, *ASIACRYPT 2019, Part III*, volume 11923 of *LNCS*, pages 459–488. Springer, Heidelberg, December 2019.

[Wat12]   Brent Waters. Functional encryption for regular languages. In Reihaneh Safavi-Naini and Ran Canetti, editors, *CRYPTO 2012*, volume 7417 of *LNCS*, pages 218–235. Springer, Heidelberg, August 2012.

[Wee14]   Hoeteck Wee. Dual system encryption via predicate encodings. In Yehuda Lindell, editor, *TCC 2014*, volume 8349 of *LNCS*, pages 616–637. Springer, Heidelberg, February 2014.

[WW21]   Hoeteck Wee and Daniel Wichs. Candidate obfuscation via oblivious LWE sampling. In Anne Canteaut and François-Xavier Standaert, editors, *EUROCRYPT 2021, Part III*, volume 12698 of *LNCS*, pages 127–156. Springer, Heidelberg, October 2021.

[WZ17]   Daniel Wichs and Giorgos Zirdelis. Obfuscating compute-and-compare programs under LWE. In Chris Umans, editor, *58th FOCS*, pages 600–611. IEEE Computer Society Press, October 2017.

A Matchmaking Encryption

A.1 Security of ME

In ME, a trusted authority generates a decryption key for the receiver, associated to an arbitrary policy of his choice. The receiver is able to decrypt the message if and only if a match occurs, i.e. the sender's attribute match the receiver policy, and vice-versa. Differently from [AFNV19, AFNV21], we consider honest senders (i.e., we do not consider authenticity security). Hence, the sender do not need to receive an encryption key from the authority, but can encrypt a message directly with the sender's attribute as an input. Security against malicious senders (i.e., authenticity) can be achieved by relying on similar techniques of [FGRV21, AFNV19, AFNV21], by combining non-interactive zero-knowledge proofs and digital signatures.

Formally, an ME with message space $\mathcal{M}$, sender's policy and attribute spaces $\mathcal{P}_1$ and $\mathcal{U}_1$, receiver's policy and attribute spaces $\mathcal{P}_2$ and $\mathcal{U}_2$, is composed of the following polynomial-time algorithms:

$\mathsf{Setup}(1^\lambda)$: Upon input the security parameter 1^λ the randomized setup algorithm outputs the master public key mpk and the master secret key msk.

$\mathsf{RKGen}(\mathsf{msk}, \rho)$: The randomized receiver-key generator takes as input the master secret key msk, and attributes $\rho \in \mathcal{U}_2$. The algorithm outputs a secret decryption key dk_ρ for attributes ρ.

$\mathsf{PolGen}(\mathsf{msk}, \mathbb{S})$: The randomized receiver policy generator takes as input the master secret key msk, and a policy $\mathbb{S} \in \mathcal{P}_2$. The algorithm outputs a secret decryption key $\mathsf{dk}_\mathbb{S}$ for the circuit $\mathbb{S}$.

$\mathsf{Enc}(\mathsf{mpk}, \sigma, \mathbb{R}, m)$: The randomized encryption algorithm takes as input the master public key mpk, attributes $\sigma \in \mathcal{U}_1$, a policy $\mathbb{R} \in \mathcal{P}_1$, and a message $m \in \mathcal{M}$. The algorithm produces a ciphertext c linked to both σ and $\mathbb{R}$.

$\mathsf{Dec}(\mathsf{dk}_\rho, \mathsf{dk}_\mathbb{S}, c)$: The deterministic decryption algorithm takes as input a secret decryption key dk_ρ for attributes $\rho \in \mathcal{U}_2$, a secret decryption key $\mathsf{dk}_\mathbb{S}$ for a circuit $\mathbb{S} \in \mathcal{P}_2$, and a ciphertext c. The algorithm outputs a message m.

Correctness states that the receiver can obtain the message with overwhelming probability if a match occurs. As for security, we consider the standard definition of ME, namely CPA-1-sided and CPA-2-sided security. Informally, CPA-1-sided security captures the secrecy of the sender's attributes, the sender's policy, and the message when a match does not occur. On the other hand, CPA-2-sided security extends this secrecy even when a match occurs.

Definition 14 (Correctness of ME). An ME with message space $\mathcal{M}$, sender's policy and attribute spaces $\mathcal{P}_1$ and $\mathcal{U}_1$, receiver's policy and attribute spaces $\mathcal{P}_2$ and $\mathcal{U}_2$, is correct if $\forall \lambda \in \mathbb{N}$, $\forall (\mathsf{mpk}, \mathsf{msk})$ output by $\mathsf{Setup}(1^\lambda)$, $\forall m \in \mathcal{M}$, $\forall \sigma \in \mathcal{U}_1, \forall \rho \in \mathcal{U}_2$, $\forall \mathbb{R} \in \mathcal{P}_1$, $\forall \mathbb{S} \in \mathcal{P}_2$ such that $\mathbb{S}(\sigma) = 1 \wedge \mathbb{R}(\rho) = 1$:

$$\mathbb{P}[\mathsf{Dec}(\mathsf{dk}_\rho, \mathsf{dk}_\mathbb{S}, \mathsf{Enc}(\mathsf{mpk}, \sigma, \mathbb{R}, m)) = m] \geq 1 - \mathsf{negl}(\lambda),$$

where $\mathsf{dk}_\rho \leftarrow_\$ \mathsf{RKGen}(\mathsf{msk}, \rho)$, and $\mathsf{dk}_\mathbb{S} \leftarrow_\$ \mathsf{PolGen}(\mathsf{msk}, \mathbb{S})$.

Definition 15 (CPA-1-sided and CPA-2-sided security of ME). Let $t \in [2]$. We say that an ME Π is CPA-t-sided secure if for all valid PPT adversaries $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$:

$$\left| \mathbb{P}\left[\mathbf{G}^{\mathsf{CPA}\text{-}t\text{-}\mathsf{ME}}_{\Pi, \mathsf{A}}(\lambda) = 1 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda),$$

$$\begin{array}{l}
\hline
\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}t\text{-}ME}}(\lambda) \\
\hline
(\mathsf{mpk}, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}(1^\lambda) \\
(m^0, m^1, \mathbb{R}^0, \mathbb{R}^1, \sigma^0, \sigma^1, \alpha) \leftarrow_\$ \mathsf{A}_0^{\mathsf{RKGen}(\mathsf{msk},\cdot),\mathsf{PolGen}(\mathsf{msk},\cdot)}(1^\lambda, \mathsf{mpk}) \\
b \leftarrow_\$ \{0,1\}, c \leftarrow_\$ \mathsf{Enc}(\mathsf{mpk}, \sigma^b, \mathbb{R}^b, m^b) \\
b' \leftarrow_\$ \mathsf{A}_1^{\mathsf{RKGen}(\mathsf{msk},\cdot),\mathsf{PolGen}(\mathsf{msk},\cdot)}(1^\lambda, c, \alpha) \\
\mathbf{If}\ (b' = b)\colon\ \mathbf{return}\ 1 \\
\mathbf{Else}\colon\ \mathbf{return}\ 0 \\
\hline
\end{array}$$

Figure 8: Games defining CPA-t-sided security of ME.

where game $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}t\text{-}ME}}(\lambda)$ is depicted in Figure 8. Adversary A is called valid if $\forall \rho \in \mathcal{Q}_{\mathsf{RKGen}}, \forall \mathbb{S} \in \mathcal{Q}_{\mathsf{PolGen}}$,

- **Case $t = 1$ (mismatch only):**

$$\begin{aligned}
(\mathbb{R}^0(\rho) = \mathbb{R}^1(\rho) = 0) &\vee (\mathbb{S}(\sigma^0) = \mathbb{S}(\sigma^1) = 0) \\
&\vee (\mathbb{R}^0(\rho) = \mathbb{S}(\sigma^1) = 0) \vee (\mathbb{R}^1(\rho) = \mathbb{S}(\sigma^0) = 0);
\end{aligned} \tag{7}$$

- **Case $t = 2$ (mismatch and match):** Either

$$\begin{aligned}
(\mathbb{R}^0(\rho) = \mathbb{R}^1(\rho) = 0) &\vee (\mathbb{S}(\sigma^0) = \mathbb{S}(\sigma^1) = 0) \\
&\vee (\mathbb{R}^0(\rho) = \mathbb{S}(\sigma^1) = 0) \vee (\mathbb{R}^1(\rho) = \mathbb{S}(\sigma^0) = 0)
\end{aligned}$$

$$\text{or } (\mathbb{R}^0(\rho) = \mathbb{R}^1(\rho)) \wedge (\mathbb{S}(\sigma^0) = \mathbb{S}(\sigma^1)) \wedge (m^0 = m^1). \tag{8}$$

We stress that CPA-1-sided and CPA-2-sided security reflects the "mismatch condition" and "match condition" of the original work of Ateniese et al. [AFNV19, Definition 5]. We chose to change their names to avoid confusion and make the notation consistent with respect to the one of PE.

A.2 ME from 2-Key PE

Construction 4. *Let* $\mathsf{kPE} = (\mathsf{Setup}_1, \mathsf{KGen}_1, \mathsf{Enc}_1, \mathsf{Dec}_1)$ *be a 2-key PE scheme with message space* $\mathcal{M}$, *input space* $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, *and predicate space* $\mathcal{P} = \{\mathbb{P}_{\rho,\mathbb{R}}(x_1, x_2)\}_{(\rho,\mathbb{R}) \in \mathcal{V}}$ *indexed by* $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ *such that*

$$\mathbb{P}_{\rho,\mathbb{R}}(\sigma, \mathbb{S}) = \mathbb{P}_\rho(\mathbb{S}) \wedge \mathbb{P}_\mathbb{R}(\sigma) = \mathbb{S}(\rho) \wedge \mathbb{R}(\sigma),$$

where $\sigma \in \mathcal{X}_1$, $\mathbb{S} \in \mathcal{X}_2$, $\rho \in \mathcal{V}_1$, *and* $\mathbb{R} \in \mathcal{V}_2$. *We build an ME scheme with message space* $\mathcal{M}$, *sender's policy and attribute spaces* $\mathcal{X}_2$ *and* $\mathcal{X}_1$, *and receiver's policy and attribute spaces* $\mathcal{V}_2$ *and* $\mathcal{V}_1$, *in the following way:*

$\mathsf{Setup}(1^\lambda)$**:** *Upon input the security parameter* 1^λ, *the randomized setup algorithm outputs* $\mathsf{mpk} = \mathsf{mpk}$ *and* $\mathsf{msk} = (\mathsf{msk}_1, \mathsf{msk}_2)$ *where* $(\mathsf{mpk}, \mathsf{msk}_1, \mathsf{msk}_2) \leftarrow_\$ \mathsf{Setup}_1(1^\lambda)$.

$\mathsf{RKGen}(\mathsf{msk}, \rho)$**:** *Upon input the master secret key* $\mathsf{msk} = (\mathsf{msk}_1, \mathsf{msk}_2)$ *and attributes* $\rho \in \mathcal{V}_1$, *the randomized receiver key generator outputs* $\mathsf{dk}_\rho \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}_1, \rho)$.

$\mathsf{PolGen}(\mathsf{msk}, \mathbb{S})$**:** *Upon input the master secret key* $\mathsf{msk} = (\mathsf{msk}_1, \mathsf{msk}_2)$ *and a policy* $\mathbb{S} \in \mathcal{V}_2$, *the randomized receiver policy generator outputs* $\mathsf{dk}_\mathbb{S} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}_2, \mathbb{S})$.

$\mathsf{Enc}(\mathsf{mpk}, \sigma, \mathbb{R}, m)$**:** *Upon input the master public key* mpk*, attributes* $\sigma \in \mathcal{X}_1$*, a policy* $\mathbb{R} \in \mathcal{X}_2$*, and a message* $m \in \mathcal{M}$*, the randomized encryption algorithm computes* $c \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (\sigma, \mathbb{R}), m)$*.*

$\mathsf{Dec}(\mathsf{dk}_\rho, \mathsf{dk}_{\mathbb{S}}, c)$**:** *Upon input a secret decryption key* dk_ρ *for attributes* $\rho \in \mathcal{V}_1$*, a secret decryption key* $\mathsf{dk}_{\mathbb{S}}$ *for a policy* $\mathbb{S} \in \mathcal{V}_2$*, and a ciphertext* c*, the deterministic decryption algorithm outputs* $m = \mathsf{Dec}_1(\mathsf{dk}_\rho, \mathsf{dk}_{\mathbb{S}}, c)$*.*

Correctness follows from the correctness of kPE. Below, we establish the following result.

Theorem 7. *Let* kPE *be as above.*

1. *If* kPE *is CPA-1-sided secure (Definition 11) then the ME scheme* Π *from Construction 4 is CPA-1-sided secure (Definition 15).*

2. *If* kPE *is CPA-2-sided secure (Definition 11) then the ME scheme* Π *from Construction 4 is CPA-2-sided secure (Definition 15).*

Proof. (CPA-1-sided security of Π*)* Suppose there exists a valid PPT adversary A with a non-negligible advantage in breaking the CPA-1-sided security of Π. We build an adversary A' that breaks the CPA-1-sided security of kPE. A' is defined as follows:

1. Receive mpk from the challenger and send it to A.

2. A' answers to the incoming oracle queries as follows:

   - On input $\rho \in \mathcal{V}_1$ for RKGen, forward the query ρ to $\mathsf{KGen}(\mathsf{msk}_1, \cdot)$ and return the answer dk_ρ.
   - On input $\mathbb{R} \in \mathcal{V}_2$ from PolGen, forward the query $\mathbb{R}$ to $\mathsf{KGen}(\mathsf{msk}_2, \cdot)$ and return the answer $\mathsf{dk}_{\mathbb{R}}$.

3. Receive the challenge $(m^0, m^1, \mathbb{R}^0, \mathbb{R}^1, \sigma^0, \sigma^1)$ from A. Send the challenge (m^0, m^1, x^0, x^1) where $x^i = (\sigma^i, \mathbb{S}^i)$ for $i \in \{0, 1\}$. Forward the challenge ciphertext c to A.

4. Answer to the incoming oracle queries as in Item 2.

5. Return the output of A.

Let d be the challenge bit sampled by the challenger. A' perfectly simulates the view of A. Moreover, A is a valid adversary, i.e., it satisfies the mismatch condition of Equation (7). This implies that $\forall \rho \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_1, \cdot)}, \mathbb{R} \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_2, \cdot)}, \mathbb{P}_{\rho, \mathbb{R}}(\sigma^0, \mathbb{S}^0) = \mathbb{S}^0(\rho) \wedge \mathbb{R}(\sigma^0) = 0$ and $\mathbb{P}_{\rho, \mathbb{R}}(\sigma^1, \mathbb{S}^1) = \mathbb{S}^1(\rho) \wedge \mathbb{R}^1(\sigma) = 0$. Hence, A' is a valid adversary for $\mathbf{G}_{\mathsf{kPE}, \mathsf{A'}}^{\mathsf{CPA-1-kPE}}(\lambda)$. This concludes the proof.

(CPA-2-sided security of Π*)* The reduction is identical. The only difference is the analysis of the validity of A'. Since A is a valid adversary with respect to the CPA-2-sided security experiment of kPE, i.e., it satisfies Equation (8). This implies that $\forall \rho \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_1, \cdot)}, \mathbb{R} \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_2, \cdot)}$, either $\mathbb{P}_{\rho, \mathbb{R}}(\sigma^0, \mathbb{S}^0) = \mathbb{P}_{\rho, \mathbb{R}}(\sigma^1, \mathbb{S}^1) = 0$ or $\mathbb{P}_{\rho, \mathbb{R}}(\sigma^0, \mathbb{S}^0) = \mathbb{P}_{\rho, \mathbb{R}}(\sigma^1, \mathbb{S}^1) \wedge m^0 = m^1$. Hence, A' is a valid adversary for $\mathbf{G}_{\mathsf{kPE}, \mathsf{A'}}^{\mathsf{CPA-2-kPE}}(\lambda)$. This concludes the proof.

B Non-Interactive Multi Party Computation (with Correlated Randomness)

B.1 Security of CPA-1-sided reusable k-robust NI-MPC for all-or-nothing functions

A NI-MPC protocol for a function $f : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \mathcal{Y}$ is a (non-interactive) protocol between n parties and an evaluator.[19] On initialization, a trusted party executes the setup algorithm $(\mathsf{crs}, \mathsf{ek}_1, \ldots, \mathsf{ek}_n) \leftarrow\!\!\$\ \mathsf{Setup}(1^\lambda, f)$. Then, it publishes the common reference string crs and sends the (possibly correlated) encryption keys to the corresponding parties, i.e., the i-th party receives the i-th encryption key ek_i. After the setup phase, each party, owning an input $v_i \in \mathcal{V}_i$, sends a single message $c_i \leftarrow\!\!\$\ \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, v_i)$ to the evaluator. The latter will be able to compute the output of the function f by executing $f(v_1, \ldots, v_n) = \mathsf{Eval}(\mathsf{crs}, c_1, \ldots, c_n)$. We focus on NI-MPC *without session identifiers*, i.e., the encryption algorithm do not takes in input the unique identifier for the current round. Hence, messages computed in different rounds can be interleaved by design (this will affect the security definition of NI-MPC).

Formally, a NI-MPC protocol Π for a function $f : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \mathcal{Y}$ consists of the following algorithms:

$\mathsf{Setup}(1^\lambda, f)$: Upon input the security parameter 1^λ and a function $f : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \mathcal{Y}$, the setup algorithm outputs the common reference string crs and n encryption keys $\mathsf{ek}_1, \ldots, \mathsf{ek}_n$.

$\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, v_i)$: Upon input a common reference string crs, an input $v_i \in \mathcal{V}_i$, and an encryption key ek_i, the randomized encryption algorithm outputs a ciphertext c_i.

$\mathsf{Eval}(\mathsf{crs}, c_1, \ldots, c_n)$: Upon input a common reference string crs and n ciphertexts $c_1, \ldots, c_n$, the deterministic evaluation algorithm outputs a value $y \in \mathcal{Y}$.

Correctness states that the evaluation of n ciphertext $(c_1, \ldots, c_n)$, computed over the inputs $(v_1, \ldots, v_n)$, outputs $f(v_1, \ldots, v_n)$

Definition 16 (Correctness of NI-MPC). A NI-MPC protocol for a function $f : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \mathcal{Y}$ is correct if $\forall \lambda \in \mathbb{N}$, $\forall (\mathsf{crs}, \mathsf{ek}_1, \ldots, \mathsf{ek}_n)$ output by $\mathsf{Setup}(1^\lambda, f)$, $\forall (v_1, \ldots, v_n) \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, we have:

$$\mathbb{P}[\mathsf{Eval}(\mathsf{crs}, c_1, \ldots, c_n) = f(v_1, \ldots, v_n)] = 1 - \mathsf{negl}(\lambda),$$

where $c_i \leftarrow\!\!\$\ \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, v_i)$ for $i \in [n]$.

As for security, a k-robust NI-MPC guarantees the secrecy of the inputs of honest parties even in the presence of an adversary that corrupts a set $\mathcal{Q}_{\mathsf{Corr}}$ of k parties (when an adversary corrupts the i-th party it obtains its encryption key ek_i and the latter gives to the adversary the ability of producing adversarially chosen messages using ek_i). Following the blueprint of Halevi et al. [HIJ+17] (see also [BGI+14]), this is formalized by an indistinguishability-based definition that states the infeasibility of distinguishing between $(\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_1, v_1^0), \ldots, \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_n, v_n^0))$ and $(\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_1, v_1^1), \ldots, \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_n, v_n^1))$,[20] so long as any interleaving of the honest inputs with any adversarially chosen input $v_i' \in \mathcal{V}_i$, belonging to a corrupted party $i \in \mathcal{Q}_{\mathsf{Corr}}$, produces the same function evaluation. In addition, security of NI-MPC can be formulated in two different settings, named non-reusable and reusable NI-MPC:

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[19]Depending on the scenario, the evaluator can be any of the parties running the NI-MPC protocol

[20]Simulation-based security of NI-MPC for general functions is impossible. Indeed, simulation-based NI-MPC implies virtual black-box (VBB) obfuscation [GGG+14, BGI+14, HIJ+17] and the latter is impossible for certain class of circuits/functions [BGI+01].

$$\begin{array}{l} \mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{ni\text{-}mpc}}(\lambda) \\ \hline (\mathsf{crs}, \mathsf{ek}_1, \ldots, \mathsf{ek}_n) \leftarrow\!\!\$ \; \mathsf{Setup}(1^\lambda, f_\mathbb{P}) \\ ((v_1^0, \ldots, v_n^0), (v_1^1, \ldots, v_n^1), \alpha) \leftarrow\!\!\$ \; \mathsf{A}_0^{\mathsf{Corr}(\cdot), \{\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, \cdot)\}_{i \in [n]}}(1^\lambda, \mathsf{crs}) \\ b \leftarrow\!\!\$ \; \{0,1\}, c_1 \leftarrow\!\!\$ \; \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_1, v_1^b), \ldots, c_n \leftarrow\!\!\$ \; \mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_n, v_n^b) \\ b' \leftarrow\!\!\$ \; \mathsf{A}_1^{\mathsf{Corr}(\cdot), \{\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, \cdot)\}_{i \in [n]}}(1^\lambda, c_1, \ldots, c_n, \alpha) \\ \mathbf{If}\;(b' = b)\mathbf{:}\; \mathbf{return}\; 1 \\ \mathbf{Else:}\; \mathbf{return}\; 0 \end{array}$$

Figure 9: Game defining (CPA-1-sided) reusable k-robust security of NI-MPC for all-or-nothing functions and without session identifiers. On input $i \in [n]$, the corruption oracle $\mathsf{Corr}(\cdot)$ returns the i-th encryption key ek_i.

- Non-reusable NI-MPC guarantees the secrecy of parties' inputs only if the setup is executed after each round (i.e., a single evaluation $f(v_1, \ldots, v_n)$ per setup is allowed).

- On the other hand, reusable NI-MPC provides a stronger security guarantees allowing parties to use the same setup in multiple rounds. As defined in [HIJ$^+$17], full-fledged reusability NI-MPC makes use of session identifiers in order to block interleaving of messages produced in different rounds. In particular, in each round of computation, the parties compute their messages $c_1, \ldots, c_n$ by attaching to them a unique session identifiers ℓ. Only messages $c_1, \ldots, c_n$ with the same identifier ℓ can be evaluated together yielding $f(v_1, \ldots, v_n) = \mathsf{Eval}(\mathsf{crs}, c_1, \ldots, c_n)$.

We focus on a weaker notion of reusability without session identifiers, specifically tailored for all-or-nothing functions, that allows to re-use the same setup until a certain condition is satisfied. An all-or-nothing function $f_\mathbb{P} : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to (\mathcal{M}_1 \times \ldots \times \mathcal{M}_n) \cup \{\bot\}$ returns parties' messages $(m_1, \ldots, m_n) \in \mathcal{M}_1 \times \ldots \times \mathcal{M}_n$ only if a predicate $\mathbb{P}(x_1, \ldots, x_n)$ is satisfied, i.e.,

$$f_\mathbb{P}(v_1, \ldots, v_n) = \begin{cases} (m_1, \ldots, m_n) & \text{if } \mathbb{P}(x_1, \ldots, x_n) = 1 \\ \bot & \text{otherwise} \end{cases} \tag{9}$$

where $v_i = (x_i, m_i) \in \mathcal{V}_i = \mathcal{X}_i \times \mathcal{M}_i$ for $i \in [n]$. We named our weaker notion of reusability *CPA-1-sided reusability* and, in a nutshell, it allows parties to reuse the same setup (without affecting the security of the protocol) so long as $f_\mathbb{P}$ evaluates $\bot$ for any combinations of the honest inputs and every input associated to the corrupted parties.[21] This condition resembles the CPA-1-sided security of multi-input PE (Definition 13).

Definition 17 (CPA-1-sided reusable k-robust security of NI-MPC for all-or-nothing functions)**.** Let $f_\mathbb{P} : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to (\mathcal{M}_1 \times \ldots \times \mathcal{M}_n) \cup \{\bot\}$ be an all-or-nothing function as defined in Equation (9). We say that a NI-MPC protocol Π for $f_\mathbb{P}$ is CPA-1-sided reusable k-robust secure if for any valid PPT adversary $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$ we have:

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{ni\text{-}mpc}}(\lambda) \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda)$$

where $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{ni\text{-}mpc}}(\lambda)$ is depicted in Fig. 9. Let $\mathcal{Q}_i = \mathcal{Q}_{\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, \cdot)}$ for $i \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$ and $\mathcal{Q}_i = \mathcal{X}_i$ for $i \in \mathcal{Q}_{\mathsf{Corr}}$. Adversary A is called valid if $|\mathcal{Q}_{\mathsf{Corr}}| \leq k$ and $\forall d \in \{0,1\}$, $\forall j \in [n]$, $\forall (v_1', \ldots, v_n') \in \mathcal{Q}_1 \cup \{v_1^d\} \times \ldots \times \mathcal{Q}_n \cup \{v_n^d\}$, we have that

$$f_\mathbb{P}(v_0', \ldots, v_{j-1}', v_j^d, v_{j+1}', \ldots, v_{n-1}') = \bot.$$

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[21] We consider every combination of the inputs due to the lack of session identifiers.

We stress that both the flavors of corruption and challenge selection considered in our Definition 17 are stronger than the one of Halevi et al. [HIJ+17]. In Definition 17, the adversary can both choose which parties want to corrupt and the challenge adaptively. On the other hand, [HIJ+17] only covers selective security on both aspects.

Remark 5 (On the relation between NI-MPC, iO, and null iO). As note by previous works [HIJ+17, BGI+14], NI-MPC has strong relations with iO. Taking into account full-fledged reusability, indistinguishability-based 0-robust NI-MPC for general functions that supports $n = \mathsf{poly}(\lambda)$ parties implies iO. The construction is reminiscent to that of iO from multi-input functional encryption [GGG+14]. Analogously, we can translate the above implications to the setting of CPA-1-sided reusability and null iO (and, in turn WE) [GKW17, WZ17, BJK+18], i.e., CPA-1-sided reusable 0-robust NI-MPC for general functions that supports $n = \mathsf{poly}(\lambda)$ parties implies null iO. This shows that nonetheless CPA-1-sided reusability is a weakening of standard reusability, it is non-trivial to achieve for general functions. Moreover, if we consider 1-robustness, we can get rid of both the (CPA-1-sided) reusability and $n = \mathsf{poly}(\lambda)$ parties requirements. In particular, as described in Section 1.4, we can build iO (resp. null iO) from indistinguishability-based (resp. CPA-1-sided) non-reusable 1-robust NI-MPC supporting $n = 2$ parties.[22]

B.2 NI-MPC for all-or-nothing functions from multi-input PE

In this section, we build a CPA-1-sided reusable k-robust NI-MPC protocol for $f_{\mathbb{P}} : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to (\mathcal{M}_1 \times \ldots \times \mathcal{M}_n) \cup \{\bot\}$ (defined as in Equation (9)) from any CPA-1-sided secure n-input PE in the k-corruptions setting without collusions.

Construction 5. *Let* $\mathsf{iPE}_1 = (\mathsf{Setup}_1, \mathsf{KGen}_1, \mathsf{Enc}_1, \mathsf{Dec}_1)$ *be a n-input PE scheme with message space* $\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_n$, *input space* $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$, *and predicate space* $\mathcal{P}_1 = \{\mathbb{P}(x_1, \ldots, x_n)\}$. *Let* $\mathcal{V}_i = \mathcal{X}_i \times \mathcal{M}_i$ *for* $i \in [n]$. *For every* $\mathbb{P} \in \mathcal{P}_1$, *we build a NI-MPC protocol for the function* $f_{\mathbb{P}} : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to (\mathcal{M}_1 \times \ldots \times \mathcal{M}_n) \cup \{\bot\}$ *(as defined in Equation (9)) in the following way:*

$\mathsf{Setup}(1^\lambda, f_{\mathbb{P}})$**:** *Upon input the security parameter* 1^λ *and a function* $f_{\mathbb{P}}$, *the randomized setup algorithm computes* $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_{\$} \mathsf{Setup}_1(1^\lambda)$ *and* $\mathsf{dk}_{\mathbb{P}} = \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P})$ *where* $\mathbb{P} \in \mathcal{P}_1$ *is the predicate defining the function* $f_{\mathbb{P}}$. *Finally, it returns* $\mathsf{crs} = \mathsf{dk}_{\mathbb{P}}$ *and* $\mathsf{ek}_1, \ldots, \mathsf{ek}_n$.

$\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, v_i)$**:** *Let* $i \in [n]$. *Upon input the common reference string* $\mathsf{crs} = \mathsf{dk}_{\mathbb{P}}$, *the encryption key* ek_i, *and the input* $v_i = (x_i, m_i) \in \mathcal{V}_i$, *the randomized encryption algorithm outputs* $c_i \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{ek}_i, x_i, m_i)$.

$\mathsf{Eval}(\mathsf{crs}, c_1, \ldots, c_n)$**:** *On input the common reference string* $\mathsf{crs} = \mathsf{dk}_{\mathbb{P}}$ *and n ciphertexts* $c_1, \ldots, c_n$, *the evaluation algorithm outputs* $\mathsf{Dec}_1(\mathsf{dk}_{\mathbb{P}}, c_1, \ldots, c_n)$.

Correctness follows from that of the underlying n-input PE iPE_1. In particular, correctness for the case $f_{\mathbb{P}}((x_1, m_1), \ldots, (x_n, m_n)) = \bot$ (i.e., $\mathbb{P}$ is not satisfied) can be obtained by extending the iPE_1's correctness to the case of $\mathbb{P}$ is not satisfied, i.e., $\mathsf{Dec}(\mathsf{dk}_{\mathbb{P}}, c_1, \ldots, c_n) = \bot$ whenever $\mathbb{P}(x_1, \ldots, x_n) = 0$.[23]

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[22]Non-reusable 1-robust security of NI-MPC means that the honest encryption key ek_i is used only once (i.e., to compute a single message) whereas ek_{1-i} is revealed to the adversary (i.e., the adversary can use it multiple times without breaking the security of the NI-MPC protocol).

[23]Correctness for the case $\mathbb{P}(x_1, \ldots, x_n) = 0$ can be seamlessly added to any multi-input PE scheme by applying an efficiently computable and invertible padding $\Phi(\cdot)$ (e.g., $\Phi(m) = m||1||0^\lambda$ where λ is the security parameter) before encrypting the message m_i, i.e., $\mathsf{Enc}(\mathsf{ek}_i, x_i, \Phi(m_i))$. On decryption, the n-input PE scheme will return $\bot$ whenever the padding is invalid.

Security of Construction 5 is formalized by Theorem 8. By combining Theorems 5 and 8 (and [GVW15]), we obtain a CPA-1-sided reusable 0-robust NI-MPC protocol for $n = \mathsf{poly}(\lambda)$ parties (based on the LWE assumption) for all-or-nothing functions $f_{\mathbb{P}}$ (Equation (9)) where $\mathbb{P}$ is a conjunctions of arbitrary predicates with wildcards. Similarly, by combining Theorems 6 and 8, we obtain a CPA-1-sided reusable $(n-1)$-robust NI-MPC protocol for $n = O(1)$ parties for the same class of functions. Both settings are non-trivial and they both imply null iO (and WE) in the case of NI-MPC for general functions (see Section 1.3 and Remark 5).

Theorem 8. *Let* iPE_1 *as above. If* iPE_1 *is CPA-1-sided secure in the k-corruptions setting without collusions (Definition 13), then* Π *of Construction 5 is CPA-1-sided reusable k-robust secure (Definition 17).*

Proof. Suppose there exists a valid PPT adversary A with a non-negligible advantage in breaking the partial reusability k-robust security of NI-MPC. we build an adversary A' that breaks the CPA-1-sided security in the k-corruptions setting without collusions of iPE_1. A' proceeds as follows:

1. Send $\mathbb{P}$ to the oracle $\mathsf{KGen}_1(\mathsf{msk}, \cdot)$ and receive $\mathsf{dk}_{\mathbb{P}}$.

2. Send $\mathsf{crs} = \mathsf{dk}_{\mathbb{P}}$ to A.

3. A' answers the incoming oracle queries as follows:

   - On input $v_i = (x, m) \in \mathcal{V}_i$ for $\mathsf{Enc}(\mathsf{crs}, \mathsf{ek}_i, \cdot)$ where $i \in [n]$, forward the query (x, m) to $\mathsf{Enc}_1(\mathsf{ek}_i, \cdot, \cdot)$ and return the answer c_i to A.

   - On input $i \in [n]$ for $\mathsf{Corr}(\cdot)$, forward the query i to oracle $\mathsf{Corr}_1(\cdot)$ and return the answer ek_i to A.

4. Receive the challenge $(v_1^0 = (x_1^0, m_1^0), \ldots, v_n^0 = (x_n^0, m_n^0))$ and $(v_1^1 = (x_1^1, m_1^1), \ldots, v_n^1 = (x_n^1, m_n^1))$.

5. Send $((m_1^0, \ldots, m_n^0), (x_1^0, \ldots, x_n^0))$ and $((m_1^1, \ldots, m_n^1), (x_1^1, \ldots, x_n^1))$ to the challenger.

6. Receive the ciphertexts $(c_1, \ldots, c_n)$ and forward them to A.

7. Answer to the incoming oracle queries as in Item 3.

8. Return the output of A.

The adversary A' perfectly simulates the view of A. Moreover, by combining $|\mathcal{Q}_{\mathsf{KGen}_1}| = 1$ (A submits a single query to the KGen_1 oracle) and A's validity, we can easily conclude that A' is a valid adversary for the experiment $\mathbf{G}_{\mathsf{iPE}, \mathsf{A}'}^{k\text{-}\mathsf{CPA}\text{-}1\text{-}\mathsf{iPE}}(\lambda)$ without collusions. This concludes the proof. $\qed$

C Relating Multi-key PE and Multi-input PE

Here, we show a construction of n-key PE from $(n+1)$-input PE supporting arbitrary predicates and tolerating 1 corruption. In more details, it suffices that the $(n + 1)$-input PE satisfies a weaker flavor of security under corruptions, named ℓ-*hybrid setting*.
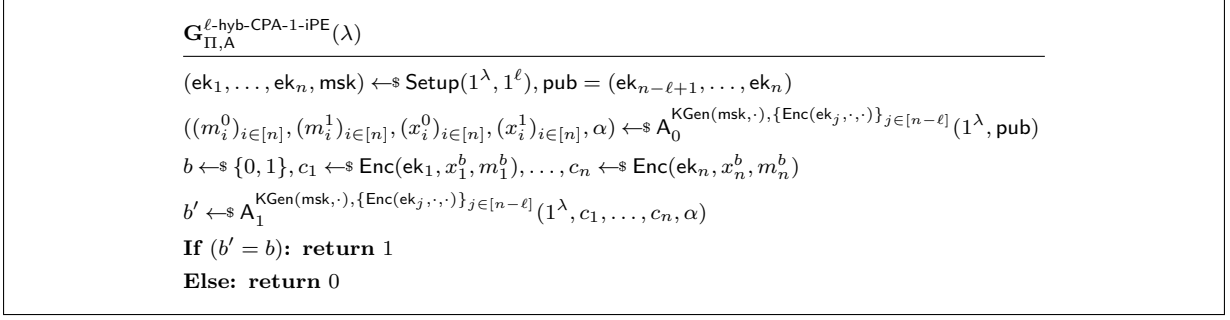
$$\underline{\mathbf{G}_{\Pi,\mathsf{A}}^{\ell\text{-hyb-CPA-1-iPE}}(\lambda)}$$

$(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow\!\!{}_\$ \mathsf{Setup}(1^\lambda, 1^\ell), \mathsf{pub} = (\mathsf{ek}_{n-\ell+1}, \ldots, \mathsf{ek}_n)$

$((m_i^0)_{i\in[n]}, (m_i^1)_{i\in[n]}, (x_i^0)_{i\in[n]}, (x_i^1)_{i\in[n]}, \alpha) \leftarrow\!\!{}_\$ \mathsf{A}_0^{\mathsf{KGen}(\mathsf{msk},\cdot),\{\mathsf{Enc}(\mathsf{ek}_j,\cdot,\cdot)\}_{j\in[n-\ell]}}(1^\lambda, \mathsf{pub})$

$b \leftarrow\!\!{}_\$ \{0,1\}, c_1 \leftarrow\!\!{}_\$ \mathsf{Enc}(\mathsf{ek}_1, x_1^b, m_1^b), \ldots, c_n \leftarrow\!\!{}_\$ \mathsf{Enc}(\mathsf{ek}_n, x_n^b, m_n^b)$

$b' \leftarrow\!\!{}_\$ \mathsf{A}_1^{\mathsf{KGen}(\mathsf{msk},\cdot),\{\mathsf{Enc}(\mathsf{ek}_j,\cdot,\cdot)\}_{j\in[n-\ell]}}(1^\lambda, c_1, \ldots, c_n, \alpha)$

If $(b' = b)$: **return** 1

Else: return 0

Figure 10: Game defining CPA-t-sided security of n-input PE in the ℓ-hybrid setting.

C.1 Multi-input PE in the ℓ-Hybrid Setting

A multi-input PE in the hybrid setting allows to generate (during setup) some encryptions keys that can be made public. The main difference between the hybrid setting and the corruption setting is that in the former the setup needs to know a priori which ones will be public (in other words, the setup depends on the keys that the adversary wants to leak/obtain). For this reason, it is easy to see that the hybrid setting is stronger than the secret-key one but significantly weaker than the corruption setting (in which the keys are leaked by the adversary in an adaptively fashion).

We assume that the Setup algorithm takes as input an additional parameter 1^ℓ denoting the number of keys that will be made public. Without loss of generality, we assume that the first $n - \ell$ keys $(\mathsf{ek}_1, \ldots, \mathsf{ek}_{n-\ell})$ are kept secret whereas the last ℓ keys $(\mathsf{ek}_{n-\ell+1}, \ldots, \mathsf{ek}_n)$ are published. Observe that, for $\ell = 0$, the hybrid setting corresponds to the secret-key setting (see Section 4.2).

Definition 18 (ℓ-Hybrid CPA-1-sided and CPA-2-side security of n-input PE)**.** Let $t \in [2]$. We say that a n-input PE Π is CPA-t-sided secure in the ℓ-hybrid setting if for all valid PPT adversaries $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$:

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\ell\text{-hyb-CPA-1-iPE}}(\lambda) = 1 \right] - \frac{1}{2} \right| \le \mathsf{negl}(\lambda),$$

where game $\mathbf{G}_{\Pi,\mathsf{A}}^{\ell\text{-hyb-CPA-1-iPE}}(\lambda)$ is depicted in Figure 10. Let $\mathcal{Q}_i = \{x | \exists (x,m) \in \mathcal{Q}_{\mathsf{Enc}(\mathsf{ek}_i,\cdot,\cdot)}\}$ for $i \in [n-\ell]$ and $\mathcal{Q}_i = \mathcal{X}_i$ for $i \in [n] \setminus [n-\ell]$. Moreover, let $\mathcal{Q}_i^d$ (for $d \in \{0,1\}$) be the ordered list composed of the predicate inputs $\mathcal{Q}_i$ and the challenge input x_i^d, i.e., $\mathcal{Q}_i^d = \{x_i^{(1,d)}, \ldots, x_i^{(k_i,d)}, x_i^{(k_i+1,d)} = x_i^d\}$ where $k_i = |\mathcal{Q}_i|$ and $x^{(j,d)} \in \mathcal{Q}_i$ for $j \in [k_i]$. Adversary A is called valid if $|\mathcal{Q}_{\mathsf{Corr}}| \le \ell$ and $\forall \mathbb{P} \in \mathcal{Q}_{\mathsf{KGen}}, \forall j \in [n], \forall i_1 \in [k_1 + 1], \ldots, \forall i_n \in [k_n + 1]$, we have

Case $t = 1$: $\mathbb{P}(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$

$\qquad \mathbb{P}(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 0.$

Case $t = 2$: Either

$\qquad \mathbb{P}(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$

$\qquad \mathbb{P}(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 0$

$\qquad$ or

$\qquad \mathbb{P}(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$

$\qquad \mathbb{P}(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) \wedge m_j^0 = m_j^1.$

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C.2 Multi-key PE from Multi-input PE

In the this section, we build a n-key PE from $(n+1)$-input PE that tolerates 1 public encryption key, i.e., 1-hybrid setting (Definition 18).

Construction 6. *Let* $\mathsf{iPE} = (\mathsf{Setup}_1, \mathsf{KGen}_1, \mathsf{Enc}_1, \mathsf{Dec}_1)$ *be a* $(n+1)$-*input PE scheme with message space* $\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_{n+1}$, *input space* $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_{n+1}$, *and predicate space* $\mathcal{P}_1 = \{\mathbb{P}(x_1, \ldots, x_{n+1})\}$ *such that*

$$\mathbb{P}(x_1, \ldots, x_{n+1}) = \mathbb{P}_{x_1, \ldots, x_n}(x_{n+1}),$$

where $x_i \in \mathcal{X}_i$ *for* $i \in [n+1]$. *We build a* n-*key PE scheme with message space* $\mathcal{M} = \mathcal{M}_{n+1}$, *input space* $\mathcal{X} = \mathcal{X}_{n+1}$, *and predicate space* $\mathcal{P} = \{\mathbb{P}_{v_1, \ldots, v_n}(x)\}_{(v_1, \ldots, v_n) \in \mathcal{V}}$ *indexed by* $\mathcal{V} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$, *in the following way:*

$\mathsf{Setup}(1^\lambda)$**:** *Upon input the security parameter* 1^λ *the randomized setup algorithm outputs* $\mathsf{mpk} = \mathsf{ek}_n$ *and* $\mathsf{msk}_1 = (\mathsf{ek}_1, \mathsf{dk}_\mathbb{P}), \ldots, \mathsf{msk}_n = (\mathsf{ek}_n, \mathsf{dk}_\mathbb{P})$ *where* $(\mathsf{mpk}', \mathsf{ek}_1, \ldots, \mathsf{ek}_{n+1}) \leftarrow_\$ \mathsf{Setup}_1(1^\lambda)$ *and* $\mathsf{dk}_\mathbb{P} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}', \mathbb{P})$ *for* $\mathbb{P} \in \mathcal{P}_1$.

$\mathsf{KGen}(\mathsf{msk}_i, v_i)$**:** *Let* $i \in [n]$. *Upon input the* i-*th master secret key* $\mathsf{msk}_i = (\mathsf{ek}_i, \mathsf{dk}_\mathbb{P})$, *and the* i-*th predicate index* $v_i \in \mathcal{X}_i$, *the randomized key generator outputs* $\mathsf{dk}_{v_i} = (c_{v_i}, \mathsf{dk}_\mathbb{P})$ *where* $c_{v_i} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{ek}_i, v_i, \bot)$.

$\mathsf{Enc}(\mathsf{mpk}, x, m)$**:** *Upon input the master public key* $\mathsf{mpk} = \mathsf{ek}_{n+1}$, *an input* $x \in \mathcal{X}_{n+1}$, *and a message* $m \in \mathcal{M}_{n+1}$, *the randomized encryption algorithm computes* $c \leftarrow_\$ \mathsf{Enc}_1(\mathsf{ek}_{n+1}, x, m)$.

$\mathsf{Dec}(\mathsf{dk}_{v_1}, \ldots, \mathsf{dk}_{v_n}, c)$**:** *Upon input* n *secret decryption keys* $\mathsf{dk}_{v_1} = (c_{v_1}, \mathsf{dk}_\mathbb{P}), \ldots, \mathsf{dk}_{v_n} = (c_{v_n}, \mathsf{dk}_\mathbb{P})$ *and a ciphertext* c, *the deterministic decryption algorithm outputs* m_{n+1} *where* $(m_1, \ldots, m_{n+1}) = \mathsf{Dec}_1(\mathsf{dk}_\mathbb{P}, c_{v_1}, \ldots, c_{v_n}, c)$.

Correctness follows from the correctness of iPE. As for security, we establish the following result.

Theorem 9. *Let* iPE *be as above. For* $t \in [2]$, *if* iPE *is CPA-t-sided secure in the 1-hybrid model without collusions (Definition 18) then the* n-*key PE scheme* Π *from Construction 6 is CPA-t-sided secure (Definition 11).*

Proof. (CPA-1-sided security of Π*)* Without loss of generality, we assume that the adversary A submits (at least) one query to each key generation oracle $\mathsf{KGen}(\mathsf{msk}_1, \cdot), \ldots, \mathsf{KGen}(\mathsf{msk}_n, \cdot)$ (proving the security of Π against this adversary implies the security of Π against any other adversary that does not query an oracle $\mathsf{KGen}(\mathsf{msk}_j, \cdot, \cdot)$ for a $j \in [n]$). Suppose there exists a valid PPT adversary A with a non-negligible advantage in breaking the CPA-1-sided security of Π. We build an adversary A' that breaks the 1-hybrid CPA-1-side security (without collusions) of iPE. A' is defined as follows:

1. Receive ek_{n+1} from the challenger and send it to A.

2. Send the query $\mathbb{P}$ (i.e., the predicate supported by iPE) to the KGen_1 oracle and receive $\mathsf{dk}_\mathbb{P}$.

3. Initialize $\mathcal{L}_i = \{\emptyset\}$ for $i \in [n]$. A' answers to the incoming oracle queries as follows:

   - On input $v_i \in \mathcal{X}_i$ for $\mathsf{KGen}(\mathsf{msk}_i, \cdot)$, forward the query $(v_i, \bot)$ to oracle $\mathsf{Enc}_1(\mathsf{ek}_i, \cdot, \cdot)$ and receive the answer c_{v_i}. Add v_i to $\mathcal{L}_i$ and return $\mathsf{dk}_{v_i} = (c_{v_i}, \mathsf{dk}_\mathbb{P})$.

4. Receive the challenge (m^0, m^1, x^0, x^1) from A. A' sends the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ where $m_1^i = \ldots = m_n^i = \bot$, $m_{n+1}^i = m^i$, $x_j^i = x_j^{1-i} = \bar{x}_j \leftarrow_\$ \mathcal{L}_j$, and $x_{n+1}^i = x^i$, for $j \in [n]$ and $i \in \{0, 1\}$.

5. Receive the challenge ciphertexts $c_1, \ldots, c_{n+1}$ and forward c_{n+1} to A.

6. Answer to the incoming oracle queries as in Item 3.

7. Return the output of A.

Let d be the challenge bit sampled by the challenger. A' perfectly simulates the view of A. Moreover, since A is a valid adversary, we have that $\forall v_1 \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_1, \cdot)}, \ldots, \forall v_n \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_n, \cdot)}$, we have $\mathbb{P}_{v_1, \ldots, v_n}(x^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^1) = 0$. In order to be valid, A' needs to satisfy the condition of Definition 18. Let $\mathcal{Q}_i^b$ as defined in Definition 18. First, note that, for $i \in [n]$, we have that $\mathcal{Q}_i^0 = \mathcal{Q}_i^1 = \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_i, \cdot)} = \mathcal{L}_i$ since $x_i^d = x_i^{1-d} = \bar{x}_i$ are sampled from $\mathcal{L}_i$ (i.e., $\mathcal{Q}_i$ does not contain any value that depends on the challenge bit d). Hence, the only case in which the adversary A' may evaluate the predicate $\mathbb{P}$ on an input that depends on the challenge bit d (i.e., the cases captured by Definition 18) is when A' uses the challenge ciphertext c_{n+1}. However, when c_{n+1} is used, the validity of A implies that $\forall (v_1, \ldots, v_n) \in \mathcal{Q}_1^b \times \ldots \times \mathcal{Q}_n^b$ (recall $\mathcal{Q}_i^b = \mathcal{Q}_i^{1-b}$ for $i \in [n]$),

$$\mathbb{P}(v_1, \ldots, v_n, x_{n+1}^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^0) = \mathbb{P}_{v_0, \ldots, v_n}(x^1) = \mathbb{P}(v_1, \ldots, v_n, x_{n+1}^1) = 0,$$

where $x_{n+1}^i = x^i$ for $i \in \{0, 1\}$. Hence, A' submits only a single query to oracle KGen_1 and is also a valid adversary for $\mathbf{G}_{\mathsf{iPE}, \mathsf{A}'}^{\ell\text{-hyb-CPA-1-iPE}}(\lambda)$. This concludes the proof.

(CPA-2-sided security of Π) The reduction is identical. The only difference is the analysis of the validity of A'. By definition A is a valid adversary with respect to the CPA-2-sided security of iPE, i.e., $\forall v_1 \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_1, \cdot)}, \ldots, \forall v_n \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_n, \cdot)}$, we have

$$\text{Either } \mathbb{P}_{v_1, \ldots, v_n}(x^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^1) = 0 \text{ or}$$
$$\mathbb{P}_{v_1, \ldots, v_n}(x^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^1) \wedge m^0 = m^1.$$

If A satisfies the first part of the above condition, then the analysis of A's validity is identical to that of CPA-1-sided security. On the other hand, if A satisfies the second part of the above condition, then the validity of A follows by using an similar argument to that of CPA-1-sided security and, in addition, observing that

$$\mathbb{P}(v_1, \ldots, v_n, x_{n+1}^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^0) = \mathbb{P}_{v_1, \ldots, v_n}(x^1) = \mathbb{P}(v_1, \ldots, v_n, x_{n+1}^1),$$

and $m_{n+1}^0 = m^0 = m^1 = m_{n+1}^1$. This concludes the proof.

D   Relating Multi-key ABE and Multi-input ABE

This work focuses on the definition of multi-key and multi-input PE whose security guarantees the secrecy of both the messages and the predicate inputs, i.e., CPA-t-sided security for $t \in [2]$. For completeness, we discuss also the relation between multi-key and multi-input schemes when moving to the weaker attribute-based encryption (ABE) setting, i.e., only the secrecy of the messages is guaranteed (inputs can be public).

First, we note that the discussion of Remark 2 and Appendix C naturally extends to the ABE setting, i.e., multi-input ABE, supporting arbitrary predicates and tolerating 1 corruption, implies multi-key ABE.

Second, we show that also multi-key ABE (for arbitrary predicates) implies secret-key multi-input ABE, i.e., no corruptions.[24] This follows by the following two observations:

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[24]Observe that multi-key does not imply multi-input when considering the PE setting, i.e., CPA-t-sided security.

1. We can first use an n-key ABE scheme, for arbitrary predicates, to build a secret-key n-input ABE such that only the i-th sender (for $i \in [n]$) can encrypt a message m_i (as considered in [AYY22]).

2. Then, as discussed in [AYY22], we can generalize the construction to support n messages by instantiating multiple times the n-input ABE and rotating the slot which contains the message.

Regarding Item 1, let $\Pi = (\mathsf{Setup}, \mathsf{KGen}, \mathsf{Enc}, \mathsf{Dec})$ be a n-key ABE, supporting the following class of predicates $\mathcal{P} = \{\mathbb{P}_{v_1,\ldots,v_n}(x)\}$:

$$\mathbb{P}_{v_1,\ldots,v_n}(x) = \mathbb{P}_{v_i}(v_1,\ldots,v_{i-1},x,v_{i+1},\ldots,v_n) = \mathbb{P}_{v_i}(x_1,\ldots,x_n), \tag{10}$$

where $x = x_i$, $v_j = x_j$ for $j \in [n] \setminus \{i\}$. We can build a secret-key n-input ABE $\Pi^* = (\mathsf{Setup}^*, \mathsf{KGen}^*, \mathsf{Enc}^*, \mathsf{Dec}^*)$ for $\mathcal{P}^* = \{\mathbb{P}_{v_i}(x_1,\ldots,x_n)\}$ as follows. Setup^* outputs $\mathsf{msk}^* = \mathsf{msk}_i$, $\mathsf{ek}_i = \mathsf{mpk}$, and $\mathsf{ek}_j = \mathsf{msk}_j$ for $j \in [n] \setminus \{i\}$ where $(\mathsf{mpk}, \mathsf{msk}_1, \ldots, \mathsf{msk}_n) \leftarrow\!\!{\scriptstyle\$}\, \mathsf{Setup}(1^\lambda)$. To generate the decryption key $\mathsf{dk}_{\mathbb{P}_{v_i}}$ for $\mathbb{P}_{v_i}$, KGen^* executes $\mathsf{dk}_{\mathbb{P}_{v_i}} = \mathsf{dk}_{v_i} \leftarrow\!\!{\scriptstyle\$}\, \mathsf{KGen}(\mathsf{msk}_i, v_i)$ where $\mathbb{P}_{v_i}$ as defined in Equation (10). The i-th sender can encrypt (x_i, m_i) by executing $\mathsf{Enc}^*(\mathsf{ek}_i, x_i, m_i) = \mathsf{Enc}(\mathsf{mpk}, x_i, m_i)$. On the other hand, for $j \in [n] \setminus \{i\}$, the j-th sender can encrypt its input x_j (recall only the i-th can encrypt a message) by computing $c_j = \mathsf{dk}_{x_j} \leftarrow\!\!{\scriptstyle\$}\, \mathsf{KGen}(\mathsf{msk}_j, x_j)$. Finally, on decryption, the receiver executes

$$\mathsf{Dec}^*(\mathsf{dk}_{\mathbb{P}_{v_i}}, c_1, \ldots, c_n) = \mathsf{Dec}(\mathsf{dk}_{x_1}, \ldots, \mathsf{dk}_{x_{i-1}}, \mathsf{dk}_{v_i}, \mathsf{dk}_{x_{i+1}}, \ldots, \mathsf{dk}_{x_n}, c_i)$$

where $c_j = \mathsf{dk}_{x_j}$ for $j \in [n] \setminus \{i\}$ and $\mathsf{dk}_{\mathbb{P}_{v_i}} = \mathsf{dk}_{v_i}$. Intuitively, the security of the above secret-key n-input ABE scheme follows from that of the n-key ABE.

By leveraging the above construction and Item 2, we obtain a standard secret-key n-input ABE supporting n messages.

E Missing Proofs

E.1 Proof of Theorem 4 (CPA-1-sided security of Π)

Consider the predicate space $\mathcal{P}$ of Construction 1, i.e.,

$$\begin{aligned}
\mathcal{P} &= \{\mathbb{P}_{v_1,\ldots,v_n}(x_1,\ldots,x_n)\}_{(v_1,\ldots,v_n)\in\mathcal{V}} \\
&= \{\mathbb{P}_{v_1}(x_1) \wedge \ldots \wedge \mathbb{P}_{v_n}(x_n)\}_{(\mathbb{P}_{v_1},\ldots,\mathbb{P}_{v_n})\in\mathcal{P}_1\times\ldots\times\mathcal{P}_n}.
\end{aligned} \tag{11}$$

Also, consider the validity condition of $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}1\text{-}kPE}}$ (Definition 11). We can write such a validity condition for the predicate space $\mathcal{P}$ as follows: $\forall v_1 \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_1,\cdot)}, \ldots, v_n \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_n,\cdot)}$,

$$\begin{aligned}
&\mathbb{P}_{v_1,\ldots,v_n}(x_1^0,\ldots,x_n^0) = \mathbb{P}_{v_1,\ldots,v_n}(x_1^1,\ldots,x_n^1) = \\
&\left(\mathbb{P}_{v_1}(x_1^0) \wedge \ldots \wedge \mathbb{P}_{v_n}(x_n^0)\right) = 0 \;\wedge\; \left(\mathbb{P}_{v_1}(x_1^0) \wedge \ldots \wedge \mathbb{P}_{v_n}(x_n^0)\right) = 0,
\end{aligned}$$

where $x_0 = (x_1^0, \ldots, x_n^0)$ and $x_1 = (x_1^1, \ldots, x_n^1)$ are the two input challenges output by the adversary. The above equation can be rewritten as follows: $\exists j_0, j_1 \in [n]$, $\forall v_{j_0} \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_{j_0},\cdot)}$, $\forall v_{j_1} \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_{j_1},\cdot)}$,

$$\mathbb{P}_{v_{j_0}}(x_{j_0}^0) = 0 \wedge \mathbb{P}_{v_{j_1}}(x_{j_1}^1) = 0.$$

Hence, in order to be valid with respect to $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}1\text{-}kPE}}$, A needs to satisfy the above equation. Let $\mathbf{Validity}_{j_0,j_1}$ the validity condition (as defined in Appendix E.1) with respect to some $j_0, j_1 \in [n]$. By taking into account the above point, the CPA-1-sided security of Construction 1 follows by proving the following lemma.

Lemma 1. *Let $j_0, j_1 \in [n]$. If both PE_{j_0} and PE_{j_1} are CPA secure (Definition 8) and LOBF is secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}1\text{-}kPE}}(\lambda) = 1 \middle| \mathbf{Validity}_{j_0,j_1} \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Consider the following hybrid experiments:

$\mathbf{H}_0^b(\lambda)$: This is exactly the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}1\text{-}kPE}}$ conditioned to $\mathbf{Validity}_{j_0,j_1}$ where the challenge bit is b.

$\mathbf{H}_1^b(\lambda)$: Same as $\mathbf{H}_0^b$, except that the challenger computes $c_{j_b} \leftarrow_{\$} \mathsf{Enc}_{j_b}(\mathsf{mpk}_{j_b}, x_{j_b}^b, w)$ where $w \leftarrow_{\$} \mathcal{M}_{j_b}$ (instead of $c_{j_b} \leftarrow_{\$} \mathsf{Enc}_{j_b}(\mathsf{mpk}_{j_b}, x_{j_b}^b, c_{j_b-1})$).

$\mathbf{H}_2^b(\lambda)$: Same as $\mathbf{H}_1^b$, except that the challenger simulates the challenge ciphertext $c = \widetilde{\mathbb{C}}$ using the simulator of the lockable obfuscation scheme LOBF, i.e., $\widetilde{\mathbb{C}} \leftarrow_{\$} \mathsf{S}(1^\lambda, 1^{|\mathbb{C}_c|}, 1^{|m_b|})$.

Claim 1. $\mathbf{H}_0^b(\lambda) \approx_c \mathbf{H}_1^b(\lambda)$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_0^b(\lambda)$ and $\mathbf{H}_1^b(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA security of PE_{j_b}. A is defined as follows:

1. Receive mpk_{j_b} from the challenger.

2. Send $\mathsf{mpk} = (\mathsf{mpk}_1, \ldots, \mathsf{mpk}_n)$ to D where $(\mathsf{mpk}_i, \mathsf{msk}_i) \leftarrow_{\$} \mathsf{Setup}_i(1^\lambda)$ for $i \in [n] \setminus \{j_b\}$.

3. A answers to the incoming oracle queries as follows:

    - On input $v_i \in \mathcal{V}_i$ for $\mathsf{KGen}(\mathsf{msk}_i, \cdot)$, A proceeds as follows: If $j_b = i$, it forwards the query $\mathbb{P}_{v_i} \in \mathcal{P}_{j_b}$ to KGen_{j_b} and returns the answer dk_{v_i}. Otherwise (if $j_b \neq i$), it returns $\mathsf{dk}_{v_i} \leftarrow_{\$} \mathsf{KGen}_i(\mathsf{msk}_i, \mathbb{P}_{v_i})$ for $\mathbb{P}_{v_i} \in \mathcal{P}_i$.

4. Receive the challenge $(m^0, m^1, (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

5. Sample $y \leftarrow_{\$} \{0,1\}^{s(\lambda)}$ and set $c_0 = y$.

6. For $i \in [j_b - 1]$, compute $c_i \leftarrow_{\$} \mathsf{Enc}_i(\mathsf{mpk}_i, x_i^b, c_{i-1})$.

7. Send the challenge $(m_*^0, m_*^1, x_{j_b}^b)$ where $m_*^0 = c_{j_b}$, $m_*^1 \leftarrow_{\$} \mathcal{M}_{j_b}$ and receive the challenge ciphertext c^*.

8. For $i \in [n] \setminus [j_b]$, compute $c_i \leftarrow_{\$} \mathsf{Enc}_i(\mathsf{mpk}_i, x_i^b, c_{i-1})$ where $c_{j_b} = c^*$.

9. Finally, send $c = \widetilde{\mathbb{C}} \leftarrow_{\$} \mathsf{Obf}(1^\lambda, \mathbb{C}_{c_n}, y, m^b)$ to D.

10. Answer to the incoming oracle queries as in Item 3.

11. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_0^b(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_1^b(\lambda)$. In addition, since D is conditioned to the event $\mathbf{Validity}_{j_0,j_1}$, we conclude that $\forall v_{j_0} \in \mathcal{V}_{j_0}, \mathbb{P}_{v_{j_0}}(x_{j_b}^0) = 0$. This implies that $\forall \mathbb{P} \in \mathcal{Q}_{\mathsf{KGen}_{j_b}}, \mathbb{P}(x_{j_b}^0) = 0$. Hence, A is a valid adversary with the same advantage of D. This concludes the proof.

Claim 2. $\mathbf{H}_1^b(\lambda) \approx_c \mathbf{H}_2^b(\lambda)$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_1^b(\lambda)$ and $\mathbf{H}_2^b(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of lockable obfuscation LOBF. A is defined as follows:

1. Send $\mathsf{mpk} = (\mathsf{mpk}_1, \ldots, \mathsf{mpk}_n)$ to D where $(\mathsf{mpk}_i, \mathsf{msk}_i) \leftarrow\!\!{}_\$ \, \mathsf{Setup}_i(1^\lambda)$ for $i \in [n]$.

2. A answers to the incoming oracle queries as follows:

   - On input $v_i \in \mathcal{V}_i$ for $\mathsf{RKGen}(\mathsf{msk}_i, \cdot)$, A returns $\mathsf{dk}_{v_i} \leftarrow\!\!{}_\$ \, \mathsf{KGen}_i(\mathsf{msk}_i, \mathbb{P}_{v_i})$ for $\mathbb{P}_{v_i} \in \mathcal{P}_i$.

3. Receive the challenge $(m^0, m^1, (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For $i \in [n] \setminus [j_b]$, compute $c_i \leftarrow\!\!{}_\$ \, \mathsf{Enc}_i(\mathsf{mpk}_i, x_i^b, c_{i-1})$ where $c_{j_b} \leftarrow\!\!{}_\$ \, \mathcal{M}_{j_b}$.

5. The adversary A send $(\mathbb{C}_{c_n}, m^b)$ to the challenger and receives back the obfuscated circuit $\widetilde{\mathbb{C}}$ from the challenger.

6. A returns $c = \widetilde{\mathbb{C}}$ to D.

7. Answer to the incoming oracle queries as in Item 2.

8. Return the output of D.

Let d be the challenge bit sampled by the challenger. When $d = 0$, A simulates $\mathbf{H}_1^b(\lambda)$; otherwise, if $d = 1$, A simulates $\mathbf{H}_2^b(\lambda)$. Thus, A has the same non-negligible advantage of D with respect to the experiment $\mathbf{G}_{\mathsf{LOBF},\mathsf{A},\mathsf{S}}^{\mathsf{lock\text{-}sim}}(\lambda)$. This concludes the proof.

Claim 3. $\mathbf{H}_2^b(\lambda) \equiv \mathbf{H}_2^{1-b}(\lambda)$.

Proof. The claim follows by observing that these experiments do not depend on the challenge bit b.

Lemma 1 follows by combining Claims 1 to 3.

By leveraging Lemma 1 we conclude that Π of Construction 1 is CPA-1-sided secure.

E.2 Proof of Theorem 4 (CPA-2-sided security of Π)

Consider the validity condition of $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{CPA\text{-}2\text{-}kPE}}$ (Definition 11). This can be rewritten with respect to the definition of $\mathcal{P}$ (Equation (11)) as follows: $\exists j_0, j_1 \in [n], \forall v_{j_0} \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_{j_0}, \cdot)}, \forall v_{j_1} \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_{j_1}, \cdot)}, \forall (v_1, \ldots, v_n) \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_1, \cdot)} \times \ldots \times \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_n, \cdot)}$,

$$\begin{aligned} \text{Either } & \mathbb{P}_{v_{j_0}}(x_{j_0}^0) = 0 \wedge \mathbb{P}_{v_{j_1}}(x_{j_1}^1) = 0 \\ \text{or } & \mathbb{P}_{v_1}(x_1^0) = \mathbb{P}_{v_1}(x_1^1) \wedge \ldots \wedge \mathbb{P}_{v_n}(x_n^0) = \mathbb{P}_{v_n}(x_n^1) \wedge m_0 = m_1 \end{aligned} \tag{12}$$

Consider the following conditions:

$$\begin{aligned} \mathbf{Validity}_{0,j_0,j_1} : & \forall v_{j_0} \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_{j_0}, \cdot)}, \forall v_{j_1} \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_{j_1}, \cdot)}, \\ & \mathbb{P}_{v_{j_0}}(x_{j_0}^0) = 0 \wedge \mathbb{P}_{v_{j_1}}(x_{j_1}^1) = 0 \\ \mathbf{Validity}_1 : & \forall (v_1, \ldots, v_n) \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_1, \cdot)} \times \ldots \times \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_n, \cdot)}, \\ & \mathbb{P}_{v_1}(x_1^0) = \mathbb{P}_{v_1}(x_1^1) \wedge \ldots \wedge \mathbb{P}_{v_n}(x_n^0) = \mathbb{P}_{v_n}(x_n^1) \wedge m^0 = m^1. \end{aligned}$$

By leveraging the above validity conditions we can rephrase Equation (12) as follows: $\exists j_0, j_1 \in [n]$ such that

$$\text{Either } \mathbf{Validity}_{0,j_0,j_1} \text{ or } \mathbf{Validity}_1.$$

Hence, in order to be valid with respect to $\mathbf{G}^{\mathsf{CPA\text{-}2\text{-}kPE}}_{\Pi,\mathsf{A}}$, A needs to satisfy the above equation. By taking into account the above point, the CPA-2-sided security of Construction 1 follows by proving the following lemmas.

Lemma 2. *Let $j_0, j_1 \in [n]$. If both PE_{j_0} and PE_{j_1} are CPA-2-sided secure (Definition 9) and LOBF is secure (Definition 2), then*

$$\left| \mathbb{P}\Big[\mathbf{G}^{\mathsf{CPA\text{-}2\text{-}kPE}}_{\Pi,\mathsf{A}}(\lambda) = 1 \Big| \mathbf{Validity}_{0,j_0,j_1} \Big] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. The lemma follows by using an identical argument to that of Lemma 1.

Lemma 3. *If each $\mathsf{PE}_1, \ldots, \mathsf{PE}_n$ are CPA-2-sided secure (Definition 9), then*

$$\left| \mathbb{P}\Big[\mathbf{G}^{\mathsf{CPA\text{-}2\text{-}kPE}}_{\Pi,\mathsf{A}}(\lambda) = 1 \Big| \mathbf{Validity}_1 \Big] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Consider the following hybrid experiments:

$\mathbf{H}^b_0(\lambda)$: This is exactly the experiment $\mathbf{G}^{\mathsf{CPA\text{-}2\text{-}kPE}}_{\Pi,\mathsf{A}}$ conditioned to $\mathbf{Validity}_1$ where the challenge bit is b.

$\mathbf{H}^b_i(\lambda)$ **for** $i \in [n]$: Same as $\mathbf{H}^b_{i-1}$, except that the challenger computes $c_i \leftarrow_\$ \mathsf{Enc}_i(\mathsf{mpk}_i, x^{1-b}_i, c_{i-1})$ (instead of $c_i \leftarrow_\$ \mathsf{Enc}_i(\mathsf{mpk}_i, x^b_i, c_{i-1})$.

Claim 4. *For $i \in [n]$, $\mathbf{H}^b_{i-1}(\lambda) \approx_c \mathbf{H}^b_i(\lambda)$.*

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}^b_{i-1}(\lambda)$ and $\mathbf{H}^b_i(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA-2-sided security of PE_i. A is defined as follows:

1. Receive mpk_i from the challenger.

2. Send $\mathsf{mpk} = (\mathsf{mpk}_1, \ldots, \mathsf{mpk}_n)$ to D where $(\mathsf{mpk}_j, \mathsf{msk}_j) \leftarrow_\$ \mathsf{Setup}_i(1^\lambda)$ for $j \in [n] \setminus \{i\}$.

3. A answers to the incoming oracle queries as follows:

   - On input $v_j \in \mathcal{V}_j$ for $\mathsf{RKGen}(\mathsf{msk}_j, \cdot)$, A proceeds as follows: If $j = i$, it forwards the query $\mathbb{P}_{v_j} \in \mathcal{P}_i$ to KGen_i and returns the answer dk_{v_j}. Otherwise (if $j \neq i$), it returns $\mathsf{dk}_{v_j} \leftarrow_\$ \mathsf{KGen}_j(\mathsf{msk}_j, \mathbb{P}_{v_j})$ for $\mathbb{P}_{v_j} \in \mathcal{P}_j$.

4. Receive the challenge $(m^0, m^1, (x^0_1, \ldots, x^0_n), (x^1_1, \ldots, x^1_n))$ from D.

5. Sample $y \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and set $c_0 = y$.

6. For $j \in [i-1]$, compute $c_j \leftarrow_\$ \mathsf{Enc}_j(\mathsf{mpk}_j, x^{1-b}_j, c_{j-1})$.

7. Send the challenge $(m^0_*, m^1_*, x^0 = x^b_i, x^1 = x^{1-b}_i)$ where $m^0_* = m^1_* = c_{i-1}$, and receive the challenge ciphertext c^*.

8. For $j \in [n] \setminus [i]$, compute $c_j \leftarrow_\$ \mathsf{Enc}_j(\mathsf{mpk}_j, x^b_j, c_{j-1})$ where $c_i = c^*$.

9. Finally, send $c = \widetilde{\mathbb{C}} \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{C}_{c_n}, y, m_b)$ to D.

10. Answer to the incoming oracle queries as in Item 3.

11. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{i-1}^b(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_i^b(\lambda)$. In addition, since D is conditioned to the event $\mathbf{Validity}_1$, we conclude that $\forall v_i \in \mathcal{Q}_{\mathsf{KGen}(\mathsf{msk}_i, \cdot)}, \mathbb{P}_{v_i}(x_i^0) = \mathbb{P}_{v_i}(x_i^1)$. Hence, A is a valid adversary with the same advantage of D. This concludes the proof.

Claim 5. $\mathbf{H}_n^b(\lambda) \equiv \mathbf{H}_n^{1-b}(\lambda)$.

Proof. Conditioned to $\mathbf{Validity}_1$, we know that $m_0 = m_1$. Hence, these experiments are identically distributed.

Lemma 3 follows by combining Claims 4 and 5.

By combining Lemmas 2 and 3 we conclude that Π of Construction 1 is CPA-2-sided secure.

E.3 Proof of Theorem 5 (CPA-1-sided security of Π)

Consider the predicate space $\mathcal{P} = \{\mathbb{P}(x_1, \ldots, x_n)\}$ of Construction 2 where $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$. Let $\mathbb{P}^* \in \mathcal{P}$ be the only predicate for which the adversary will ask the decryption key $\mathsf{dk}_{\mathbb{P}^*}$ during the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{0\text{-CPA-1-iPE}}$ (recall that we prove the security of Construction 2 in the scenario without collusions, i.e., $|\mathcal{Q}_{\mathsf{KGen}}| = 1$). Also, consider the validity condition of $\mathbf{G}_{\Pi,\mathsf{A}}^{0\text{-CPA-1-iPE}}$. We can write such a validity condition with respect to $\mathbb{P}^* \in \mathcal{Q}_{\mathsf{KGen}} = \{\mathbb{P}^*\}$ as follows: $\forall j \in [n], \forall i_1 \in [k_1 + 1], \ldots, \forall i_n \in [k_n + 1]$,

$$\mathbb{P}^*(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$$
$$\mathbb{P}^*(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) =$$
$$\mathbb{P}_1^*(x_1^{(i_1,0)}) \wedge \ldots \wedge \mathbb{P}_{j-1}^*(x_{j-1}^{(i_{j-1},0)}) \wedge \mathbb{P}_j^*(x_j^0) \wedge \mathbb{P}_{j+1}^*(x_{j+1}^{(i_{j+1},0)}) \wedge \ldots \wedge \mathbb{P}_{i_n}^*(x_n^{(i_n,0)}) =$$
$$\mathbb{P}_1^*(x_1^{(i_1,1)}) \wedge \ldots \wedge \mathbb{P}_{j-1}^*(x_{j-1}^{(i_{j-1},1)}) \wedge \mathbb{P}_j^*(x_j^1) \wedge \mathbb{P}_{j+1}^*(x_{j+1}^{(i_{j+1},1)}) \wedge \ldots \wedge \mathbb{P}_n^*(x_n^{(i_n,1)}) = 0,$$

where $\mathcal{Q}_i^b = \{x_i^{(1,b)}, \ldots, x_i^{(k_i,b)}, x_i^{(k_i+1,b)} = x_i^b\}$ is the ordered list composed of the k_i predicate inputs $\mathcal{Q}_i$ submitted to oracle $\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)$ and the challenge input x_i^b (as defined in Definition 13). The above equation can be rewritten as follows: $\exists j_0, j_1 \in [n], \forall (x_1', \ldots, x_n') \in \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n$,

$$\left(\left(\mathbb{P}_1^*(x_1^0) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^0) = 0 \right) \vee \left(\mathbb{P}_{j_0}^*(x_{j_0}^0) = 0 \wedge \mathbb{P}_{j_0}^*(x_{j_0}') = 0 \right) \right) \wedge$$
$$\left(\left(\mathbb{P}_1^*(x_1^1) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^1) = 0 \right) \vee \left(\mathbb{P}_{j_1}^*(x_{j_1}^1) = 0 \wedge \mathbb{P}_{j_1}^*(x_{j_1}') = 0 \right) \right). \tag{13}$$

Note that in the above equation we made explicit the challenge inputs and the inputs submitted to the encryption oracles. For this reason, it is enough to quantify over all $(x_1', \ldots, x_n') \in \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n$ where $\mathcal{Q}_i = \{x_i^{(1)}, \ldots, x_i^{(k_i)}\}$ are the inputs submitted to oracle $\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)$.

Hence, in order to be valid, A needs to satisfy the condition defined by Equation (13). These conditions are defined by the events below: for some $j_0, j_1 \in [n]$,

$\mathbf{Validity}_1$:
$$\mathbb{P}_1^*(x_1^0) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^0) = 0 \wedge \mathbb{P}_1^*(x_1^1) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^1) = 0.$$
$\mathbf{Validity}_{2,j_0,j_1} : \forall x_{j_0}' \in \mathcal{Q}_{j_0}, \forall x_{j_1}' \in \mathcal{Q}_{j_1},$
$$\mathbb{P}_{j_0}^*(x_{j_0}^0) = 0 \wedge \mathbb{P}_{j_0}^*(x_{j_0}') = 0 \wedge \mathbb{P}_{j_1}^*(x_{j_1}^1) = 0 \wedge \mathbb{P}_{j_1}^*(x_{j_1}') = 0.$$
$\mathbf{Validity}_{3,j_0} : \forall x_{j_0}' \in \mathcal{Q}_{j_0},$
$$\mathbb{P}_{j_0}^*(x_{j_0}^0) = 0 \wedge \mathbb{P}_{j_0}^*(x_{j_0}') = 0 \wedge \mathbb{P}_1^*(x_1^1) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^1) = 0.$$
$\mathbf{Validity}_{4,j_1} : \forall x_{j_1}' \in \mathcal{Q}_{j_1},$
$$\mathbb{P}_1^*(x_1^0) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^0) = 0 \wedge \mathbb{P}_{j_1}^*(x_{j_1}^1) = 0 \wedge \mathbb{P}_{j_1}^*(x_{j_1}') = 0.$$

For the sake of clarity, in the rest of this proof, we use the notation $\mathbb{V}_i \stackrel{\text{def}}{=} \mathbb{C}_{c_i^{(2)}, \mathsf{k}_{i+1}}$ where $c_i^{(2)}$ and k_{i+1} will be clear from the context. Also, $[a:b]_n^+ = \{a, a+1, \ldots, n, 1, 2, \ldots, b\}$. If $1 \le a \le b \le n$, we have $[a:b]_n^+ = \{a, a+1, \ldots, b\}$. Similarly, $[a:b]_n^- = \{a, a-1, \ldots, 1, n, n-1, \ldots, b\}$. If $1 \le b \le a \le n$, we have $[a:b]_n^- = \{a, a-1, \ldots, b\}$.

Lemma 4. *If* PE *is CPA-1-sided secure without collusions (Definition 9) and* LOBF *is secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\text{0-CPA-1-iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \, \middle| \, \mathbf{Validity}_1 \right] - \frac{1}{2} \right| \le \mathsf{negl}(\lambda).$$

Proof. Consider the following hybrid experiments:

$\mathbf{H}_0^{b,0}(\lambda)$**:** This is exactly the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{\text{0-CPA-1-iPE}}(\lambda)$ conditioned to the event $\mathbf{Validity}_1$ where the challenge bit is b.

$\mathbf{H}_0^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_0^{b,i-1}$, except that the challenger changes how it computes the challenger ciphertext c_i. The value $c_i^{(1)}$ challenge ciphertext $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$ is computed as $c_i^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ (instead of $c_i^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_i, \mathsf{k}_{i+1}))$) where $0^{s(\lambda)+k(\lambda)} \in \mathcal{M}_1$ (for some function k), $x_i = x_i^0$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{i\}$. Observe that $c_i^{(1)}$ is computed by fixing $x_i = x_i^0$ (instead of $x_i = x_i^b$), i.e., the input $(x_1, \ldots, x_n)$ used to compute the i-th challenge ciphertext is fixed and does not depend on the challenge bit b.

$\mathbf{H}_1^{b,0}(\lambda)$**:** Identical to $\mathbf{H}_0^{b,n}(\lambda)$.

$\mathbf{H}_1^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_1^{b,i-1}$, except that the challenger changes how it computes the challenger ciphertext c_i. Formally, the value $\widetilde{\mathbb{C}}_i$ of challenge ciphertext $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$ is simulated by the challenger using the simulator of the lockable obfuscation scheme LOBF, i.e., $\widetilde{\mathbb{C}}_i \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_i|}, 1^{|m_i^b|})$.

Claim 6. $\mathbf{H}_0^{b,i-1}(\lambda) \approx_c \mathbf{H}_0^{b,i}(\lambda)$ *for* $i \in [n]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_0^{b,1-i}(\lambda)$ and $\mathbf{H}_0^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA-1-sided security without collusions of PE. A is defined as follows:

1. Receive mpk from the challenger.

2. Computes $\mathsf{k}_j \leftarrow_\$ \mathsf{KGen}_2(1^\lambda)$ for $j \in [n]$. Let $\mathsf{ek}_i = (\mathsf{mpk}, \mathsf{k}_i, \mathsf{k}_{i+1})$ for $i \in [n]$ where $\mathsf{k}_{n+1} = \mathsf{k}_1$.

3. A answers to the incoming oracle queries as follows:

    - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_\mathbb{P}$.

    - On input $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)}) \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

4. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

5. For any $j \in [n]$, A proceeds as follows:

    Case $j < i$: Sample $y_j \leftarrow_\$ \{0, 1\}^{s(\lambda)}$. Execute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

    Case $j = i$: Send the challenge $(m_*^0 = (y_i, \mathsf{k}_{i+1}), m_*^1 = 0^{s(\lambda)+k(\lambda)}, x_*^0 = (x_{*1}^0, \ldots, x_{*n}^0),$ $x_*^1 = (x_{*1}^1, \ldots, x_{*n}^1))$ where $y_i \leftarrow_\$ \{0, 1\}^{s(\lambda)}$, $0^{s(\lambda)+k(\lambda)} \in \mathcal{M}_1$, $x_{*i}^0 = x_i^b$, $x_{*i}^1 = x_i^0$, and $x_{*j}^0 = x_{*j}^1 = x_j^\star$ for $j \in [n] \setminus \{i\}$. Receive the challenge ciphertext c^* from the challenger. Set $c_i^{(1)} = c^*$.

    Case $j > i$: Sample $y_j \leftarrow_\$ \{0, 1\}^{s(\lambda)}$ and compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

6. Compute $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{ek}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ for any $j \in [n]$.

7. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

8. Answer to the incoming oracle queries as in .

9. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_0^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_0^{b,i}(\lambda)$. Moreover, conditioned to the event $\mathbf{Validity}_1$, we know that D asks for a single decryption key $\mathsf{dk}_{\mathbb{P}^*}$ for $\mathbb{P}^*$ and $\mathbb{P}_i^*(x_i^0) = 0 \wedge \mathbb{P}_i^*(x_i^1) = 0$. Because of this, A submits a single query $\mathbb{P}^*$ to oracle $\mathsf{KGen}(\mathsf{msk}, \cdot)$ and it is also a valid adversary for the experiment $\mathbf{G}_{\mathsf{PE},\mathsf{A}}^{\mathsf{CPA}\text{-}1\text{-}\mathsf{PE}}(\lambda)$ with the same advantage of D. This concludes the proof.

Claim 7. $\mathbf{H}_1^{b,i-1}(\lambda) \approx_c \mathbf{H}_1^{b,i}(\lambda)$ for $i \in [n]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_1^{b,1-i}(\lambda)$ and $\mathbf{H}_1^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF. A is defined as follows:

1. Computes $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{k}_j, \mathsf{k}_{j-1})$ for $j \in [n]$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

    - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}(\mathsf{msk}, \mathbb{P}^*)$.

    - On input $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)}) \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For any $j \in [n]$, compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j,$ $c_j^{(1)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

5. For any $j \in [n] \setminus \{i\}$, A proceeds as follows:

   Case $j < i$: Compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^b|})$.

   Case $j = i$: Send the challenge $(\mathbb{V}_i, m_i^b)$ to the challenger and receive $\widetilde{\mathbb{C}}$. Set $\widetilde{\mathbb{C}}_i = \widetilde{\mathbb{C}}$.

   Case $j > i$: Compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$.

6. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

7. Answer to the incoming oracle queries as in Item 2.

8. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_1^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_1^{b,i}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof. $\qquad\square$

Claim 8. $\mathbf{H}_1^{b,n}(\lambda) \equiv \mathbf{H}_1^{1-b,n}(\lambda)$.

Proof. The distribution of these two experiments do not depend on the bit b. $\qquad\square$

By combining Claims 6 to 8 and conditioned to the event $\mathbf{Validity}_1$, we conclude that

$$\mathbf{H}_0^{b,0} \approx_c \ldots \approx_c \mathbf{H}_0^{b,n} \equiv \mathbf{H}_1^{b,0} \approx_c \ldots \approx_c \mathbf{H}_1^{b,n} \equiv \mathbf{H}_1^{1-b,n}.$$

This concludes the proof. $\qquad\square$

Lemma 5. *Let $j_0, j_1 \in [n]$. If* PE *is CPA-1-sided secure without collusions (Definition 9),* SKE *is CPA secure (Definition 4), and* LOBF *is secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{0\text{-}CPA\text{-}1\text{-}iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \,\middle|\, \mathbf{Validity}_{2,j_0,j_1} \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Without loss of generality, let $q = |\mathcal{Q}_1| = \ldots = |\mathcal{Q}_n| \in \mathsf{poly}(\lambda)$. Consider the following hybrid experiments:

$\mathbf{H}_0^b(\lambda)$**:** This is exactly the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{0\text{-}CPA\text{-}1\text{-}iPE}}(\lambda)$ conditioned to the event $\mathbf{Validity}_{2,j_0,j_1}$ where the challenge bit is b.

$\mathbf{H}_1^b(\lambda)$**:** Same as $\mathbf{H}_0^b$, except that the challenger changes how it computes the challenger ciphertext c_{j_b}. Formally, the value $c_{j_b}^{(1)}$ of the challenge ciphertext $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}, c_{j_b}^{(2)})$ is computed as $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ (instead of $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_{j_b}, \mathsf{k}_{j_b+1}))$) where $0^{s(\lambda)+k(\lambda)} \in \mathcal{M}_1$ (for some function k), $x_{j_b} = x_{j_b}^b$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{j_b\}$. Note that $c_{j_b}^{(1)}$ still depends on the challenge bit b since it is computed over the input $(x_1, \ldots x_n)$ where $x_{j_b} = x_{j_b}^b$. We will remove this dependency in $\mathbf{H}_{5+n-1}^{b,0,0,0}$.[25]

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[25] This allow us to reuse the proof in Lemmas 6 and 7.

$\mathbf{H}_2^{b,0}$: Identical to $\mathbf{H}_1^b(\lambda)$.

$\mathbf{H}_2^{b,i}(\lambda)$ **for** $i \in [q]$**:** Same as $\mathbf{H}_2^{b,i-1}(\lambda)$ except that the challenger changes how it answers to the first i queries for oracle $\mathsf{Enc}(\mathsf{ek}_{j_b}, \cdot, \cdot)$. Formally, on input the i'-th query (x, m) such that $i' \leq i$, the challenger computes $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_{j_b} = x$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{j_b\}$. Finally, the challenger returns $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}, c_{j_b}^{(2)})$ where $c_{j_b}^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_{j_b}, c_{j_b}^{(1)})$, $y_{j_b} \leftarrow_\$ \{0,1\}^{s(\lambda)}$, and $\widetilde{\mathbb{C}}_{j_b} \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_{j_b}, y_{j_b}, m)$. Otherwise, on input the i'-th query (x, m) such that $i' > i$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_2^{b,0}$.

$\mathbf{H}_3^b(\lambda)$**:** Same as $\mathbf{H}_2^{b,q}$, except that the challenger changes how it computes the challenger ciphertext c_{j_b}. Formally, the value $\widetilde{\mathbb{C}}_{j_b}$ of challenge ciphertext $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}, c_{j_b}^{(2)})$ is simulated by the challenger using the simulator of the lockable obfuscation scheme LOBF, i.e., $\widetilde{\mathbb{C}}_{j_b} \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_{j_b}|}, 1^{|m_{j_b}^b|})$.

$\mathbf{H}_4^{b,0}$: Identical to $\mathbf{H}_3^b(\lambda)$.

$\mathbf{H}_4^{b,i}(\lambda)$ **for** $i \in [q]$**:** Same as $\mathbf{H}_4^{b,i-1}(\lambda)$ except that the challenger changes how it answers to the first i queries for oracle $\mathsf{Enc}(\mathsf{ek}_{j_b}, \cdot, \cdot)$. Formally, on input the i'-th query (x, m) such that $i' \leq i$, the challenger returns $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}, c_{j_b}^{(2)})$ where $\widetilde{\mathbb{C}}_{j_b}$ is computed using the simulator of the lockable obfuscator scheme LOBF, i.e., $\widetilde{\mathbb{C}}_{j_b} \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_{j_b}|}, 1^{|m|})$. Otherwise, on input the i'-th query (x, m) such that $i' > i$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_4^{b,0}$.

$\mathbf{H}_4^{b,q,q,1}$**:** Identical to $\mathbf{H}_4^{b,q}(\lambda)$.

$\mathbf{H}_{5+i}^{b,0,0,0}$ **for** $i \in \{0\} \cup [n-1]$**:** Same as $\mathbf{H}_{5+i-1}^{b,q,q,1}$ except that the challenger changes how it computes the challenger ciphertext c_v where $v = (j_b + i \mod n) + 1$. Formally, the value $c_v^{(1)}$ is computed as $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $0^{s(\lambda)+k(\lambda)} \in \mathcal{M}_1$ (for some function k), $x_v = x_v^0$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{v\}$. Observe that $c_v^{(1)}$ is computed by fixing $x_v = x_v^0$ (instead of $x_v = x_v^b$), i.e., the predicate input $(x_1, \ldots, x_n)$ used to compute the v-th challenge ciphertext is fixed and does not depend on the challenge bit b.

$\mathbf{H}_{5+i}^{b,t_1,0,0}$ **for** $t_1 \in [q]$**,** $i \in \{0\} \cup [n-2]$**:** Same as $\mathbf{H}_{5+i}^{b,t_1-1,0,0}(\lambda)$ except that the challenger changes how it answers to the first t_1 queries for oracle $\mathsf{Enc}(\mathsf{ek}_v, \cdot, \cdot)$ where $v = (j_b + i \mod n) + 1$. On input the t_1'-th query (x, m) such that $t_1' \leq t_1$, the challenger computes $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{v\}$. Finally, the challenger returns $c_v = (\widetilde{\mathbb{C}}_v, c_v^{(2)})$ where $c_v^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$, $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$, and $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$. Otherwise, on input the t_1'-th query (x, m) such that $t_1' > t_1$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_{5+i}^{b,0,0,0}$.

$\mathbf{H}_{5+i}^{b,q,t_2,0}$ **for** $t_2 \in [q]$**,**$i \in \{0\} \cup [n-2]$**:** Same as $\mathbf{H}_{5+i}^{b,q,t_2-1,0}(\lambda)$ except that the challenger changes how it answers to the first t_2 queries for oracle $\mathsf{Enc}(\mathsf{ek}_v, \cdot, \cdot)$ where $v = (j_b + i \mod n) + 1$. Formally, on input the t_2'-th query (x, m) such that $t_2' \leq t_2$, the challenger returns $c_v = (\widetilde{\mathbb{C}}_v, c_v^{(2)})$ where $\widetilde{\mathbb{C}}_v$ is computed using the simulator of the lockable obfuscator scheme LOBF, i.e., $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_v|}, 1^{|m|})$. Otherwise, on input the t_2'-th query (x, m) such that $t_2' > t_2$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_{5+i}^{b,q,0,0}$.

$\mathbf{H}_{5+i}^{b,q,1}$ **for** $i \in \{0\} \cup [n-2]$**:** Same as $\mathbf{H}_{5+i}^{b,q,0}(\lambda)$ except that the challenger computes the challenger ciphertext c_v differently for $v = (j_b + i \mod n) + 1$. Formally, the value $\widetilde{\mathbb{C}}_v$ of challenge ciphertext $c_v = (\widetilde{\mathbb{C}}_v, c_v^{(2)})$ is simulated by the challenger using the simulator of the lockable obfuscation scheme LOBF, i.e., $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_v|}, 1^{|m_v^b|})$.

Claim 9. $\mathbf{H}_0^b(\lambda) \approx_c \mathbf{H}_1^b(\lambda)$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_0^b(\lambda)$ and $\mathbf{H}_1^b(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA-1-sided security without collusions of PE. A is defined as follows:

1. Receive mpk from the challenger.

2. Computes $\mathsf{k}_j \leftarrow_\$ \mathsf{KGen}_2(1^\lambda)$ for $j \in [n]$. Let $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{k}_j, \mathsf{k}_{j+1})$ for $j \in [n]$ where $\mathsf{k}_{n+1} = \mathsf{k}_1$.

3. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_{\mathbb{P}^*}$.
   - On input $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)}) \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

4. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D. Send the challenge $(m_*^0 = (y_{j_b}, \mathsf{k}_{j_b+1}), m_*^1 = 0^{s(\lambda)+k(\lambda)}, x_*^0 = (x_{*1}^0, \ldots, x_{*n}^0), x_*^1 = (x_{*1}^1, \ldots, x_{*n}^1))$ where $y_{j_b} \leftarrow_\$ \{0, 1\}^{s(\lambda)}$, $0^{s(\lambda)+k(\lambda)} \in \mathcal{M}_1$, $x_{*j_b}^0 = x_{*j_b}^1 = x_{j_b}^b$ and $x_{*j}^0 = x_{*j}^1 = x_j^\star$ for $j \in [n] \setminus \{j_b\}$.

5. Receive the challenge ciphertext c^* from the challenger. Set $c_{j_b}^{(1)} = c^*$.

6. For any $j \in [n] \setminus \{j_b\}$, compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0, 1\}^{s(\lambda)}$, $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

7. Compute $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{ek}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ for any $j \in [n]$.

8. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

9. Answer to the incoming oracle queries as in Item 3.

10. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_0^b(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_1^b(\lambda)$. Moreover, D submits a single query $\mathbb{P}^*$ to oracle $\mathsf{KGen}(\mathsf{msk}, \cdot)$ and, conditioned to the event $\mathbf{Validity}_{2,j_0,j_1}$, we know that $\mathbb{P}_{j_b}^*(x_{j_b}^b) = 0$. Because of this, A submits only a query to oracle $\mathsf{KGen}_1(\mathsf{msk}, \cdot)$ (i.e., security without collusions) and, it is also a valid adversary for the experiment $\mathbf{G}_{\mathsf{PE},\mathsf{A}}^{\mathsf{CPA-1-PE}}(\lambda)$ with the same advantage of D. This concludes the proof.

Claim 10. $\mathbf{H}_2^{b,i-1}(\lambda) \approx_c \mathbf{H}_2^{b,i}(\lambda)$ *for* $i \in [q]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_2^{b,i-1}(\lambda)$ and $\mathbf{H}_2^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA-1-sided security without collusions of PE. A is defined as follows:

1. Receive mpk from the challenger.

2. Computes $\mathsf{k}_j \leftarrow_\$ \mathsf{KGen}_2(1^\lambda)$ for $j \in [n]$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

3. A answers to the incoming oracle queries as follows:

    - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_{\mathbb{P}^*}$.

    - On input i'-th query $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, A proceeds as follows:

        Case $j \neq j_b$: Sample $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$. Compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $x_j = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

        Case $j = j_b$ and $i' < i$: Sample $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$. Compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_{j_b} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$.

        Case $j = j_b$ and $i' = i$: Sample $y_{j_b} \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and send $(m_*^0 = (y_{j_b}, \mathsf{k}_{j_b+1}), m_*^1 = 0^{s(\lambda)+k(\lambda)}, x_*^0 = (x_{*1}^0, \ldots, x_{*n}^0), x_*^1 = (x_{*1}^1, \ldots, x_{*n}^1))$ to the challenger where $x_{*j_b}^0 = x_{*j_b}^1 = x$ and $x_{*j'}^0 = x_{*j'}^1 = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$. Receive the challenge ciphertext c^* and $c_{j_b}^{(1)} = c^*$.

        Case $j = j_b$ and $i' > i$: Sample $y_{j_b} \leftarrow_\$ \{0,1\}^{s(\lambda)}$. Compute $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_{j_b}, \mathsf{k}_{j_b+1}))$ where $x_{j_b} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$.

        Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$.

4. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1)$ from D.

5. For every $j \in [n] \setminus \{j_b\}$, sample $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $x_j = x_j^b$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

6. Sample $y_{j_b} \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and compute the ciphertext $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_{j_b} = x_{j_b}^b$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$.

7. Compute the ciphertext $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ for any $j \in [n]$.

8. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

9. Answer to the incoming oracle queries as in Item 3.

10. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_2^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_2^{b,i}(\lambda)$. Moreover, we know that D submits a single query $\mathbb{P}^*$ to oracle $\mathsf{KGen}(\mathsf{msk}, \cdot)$ and, conditioned to the event $\mathbf{Validity}_{2,j_0,j_1}$, we know that $\forall x'_{j_b} \in \mathcal{Q}_{j_b}, \mathbb{P}^*_{j_b}(x'_{j_b}) = 0$. Because of this, A submits a single query to oracle $\mathsf{KGen}_1(\mathsf{msk}, \cdot)$ and it is also a valid adversary for the experiment $\mathbf{G}_{\mathsf{PE},\mathsf{A}}^{\mathsf{CPA}\text{-}1\text{-}\mathsf{PE}}(\lambda)$ with the same advantage of D. This concludes the proof.

Claim 11. $\mathbf{H}_2^{b,q}(\lambda) \approx_c \mathbf{H}_3^b(\lambda)$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_2^{b,q}(\lambda)$ and $\mathbf{H}_3^b(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF. A is defined as follows:

1. Computes $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_{\$} \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{k}_j, \mathsf{k}_{j+1})$ for $j \in [n]$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_{\$} \mathsf{KGen}(\mathsf{msk}, \mathbb{P}^*)$.
   - On input $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

     Case $j = j_b$: Sample $y_{j_b} \leftarrow_{\$} \{0,1\}^{s(\lambda)}$. Compute $c_{j_b}^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_{j_b} = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j_b\}$.

     Case $j \neq j_b$: Run $c_j^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_{\$} \{0,1\}^{s(\lambda)}$, $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

     Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow_{\$} \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. Compute $c_{j_b}^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ and $c_{j_b}^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_j, c_{j_b}^{(1)})$ where $x_{j_b} = x_{j_b}^b$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{j_b\}$.

5. For any $j \in [n] \setminus \{j_b\}$, sample $y_j \leftarrow_{\$} \{0,1\}^{s(\lambda)}$ and compute $c_j^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$, $c_j^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $\widetilde{\mathbb{C}}_j \leftarrow_{\$} \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ where $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

6. Send the challenge $(\mathbb{V}_{j_b}, m_{j_b}^b)$ to the challenger and receive $\widetilde{\mathbb{C}}$. Set $\widetilde{\mathbb{C}}_{j_b} = \widetilde{\mathbb{C}}$.

7. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

8. Answer to the incoming oracle queries as in Item 2.

9. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_2^{b,q}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_3^b(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 12. $\mathbf{H}_4^{b,i-1}(\lambda) \approx_c \mathbf{H}_4^{b,i}(\lambda)$ *for $i \in [q]$.*

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_4^{b,i-1}(\lambda)$ and $\mathbf{H}_4^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF. A is defined as follows:

1. Computes $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_{\$} \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{ek}_j, \mathsf{ek}_{j-1})$ for $j \in [n]$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_{\$} \mathsf{KGen}(\mathsf{msk}, \mathbb{P}^*)$.

- On input the i'-th query $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

  Case $j = j_b$ and $i' < i$: Run $\widetilde{\mathbb{C}}_{j_b} \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_{j_b}|}, 1^{|m|})$, $c_{j_b}^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_{j_b}, c_{j_b}^{(1)})$, and $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_{j_b} = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j_b\}$.

  Case $j = j_b$ and $i' = i$: Compute $c_{j_b}^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_{j_b}, c_{j_b}^{(1)})$ and $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_{j_b} = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j_b\}$. Send the challenge $(\mathbb{V}_{j_b}, m)$ to the challenger and receive the answer $\widetilde{\mathbb{C}}^*$. Set $\widetilde{\mathbb{C}}_{j_b} = \widetilde{\mathbb{C}}^*$.

  Case $j = j_b$ and $i' > i$: Compute $\widetilde{\mathbb{C}}_{j_b} \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_{j_b}, m)$, $c_{j_b}^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_{j_b}, c_{j_b}^{(1)})$, and $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $y_{j_b} \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_{j_b} = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j_b\}$.

  Case $j \neq j_b$: Compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

  Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. Run $\widetilde{\mathbb{C}}_{j_b} \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_{j_b}|}, 1^{|m_{j_b}^b|})$, $c_{j_b}^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$, and $c_{j_b}^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_{j_b}^{(1)})$ where $x_{j_b} = x_{j_b}^b$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{j_b\}$.

5. For any $j \in [n] \setminus \{j_b\}$, sample $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ where $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

6. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

7. Answer to the incoming oracle queries as in Item 2.

8. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_4^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_4^{b,i}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 13. $\mathbf{H}_{5+i-1}^{b,q,q,1}(\lambda) \approx_c \mathbf{H}_{5+i}^{b,0,0,0}(\lambda)$ for $i \in \{0\} \cup [n-1]$.

Proof. Let $v = (j_b + i \mod n) + 1$. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{5+i-1}^{b,q,q,1}(\lambda)$ and $\mathbf{H}_{5+i}^{b,0,0,0}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA security of SKE. A is defined as follows:

1. Computes $(\mathsf{mpk}, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}_1(1^\lambda)$ and $\mathsf{ek}_j = (\mathsf{ek}, \mathsf{k}_j, \mathsf{k}_{j-1})$ for $j \in [n] \setminus \{v\}$. If $v \neq 1$, let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P}^*)$.

   - On input $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

Case $j \in [j_b : v-1]_n^+$**:** Compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m|})$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and
$c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

Case $j = v$**:** Run $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_v, \mathsf{k}_{v+1}))$ where $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$. Send the query $c_v^{(1)}$ to the oracle Enc_2 and receive the answer $c_v^{(2)}$. Compute $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$.

Case $i < n-2$ **(hence,** $v \notin \{j_b - 1, j_b\}$**) and** $j \in [v+1 : j_b - 1]_n^+$**:** Run $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$, the ciphertext $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. **Case** $i < n-1$ **(hence,** $v \neq j_b$**):** For every $j \in [n]$, the adversary A proceeds as follows:

   Case $j \in [j_b, v-1]_n^+$**:** Run $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^b|})$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

   Case $j = v$**:** Run $c_v^{(1,0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_{*1}^0, \ldots, x_{*n}^0), (y_v, \mathsf{k}_{v+1}))$ and $c_v^{(1,1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_{*1}^1, \ldots, x_{*n}^1), 0^{s(\lambda)+k(\lambda)})$ where $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_{*v}^0 = x_v^b$, $x_{*v}^1 = x_v^0$, and $x_{*j'}^0 = x_{*j'}^1 = x_{j'}^\star$ for $j' \in [n] \setminus \{v\}$. Send the challenge $(m^0 = c_v^{(1,0)}, m^1 = c_v^{(1,1)})$ to the challenger and receive the answer c^*. Set $c_v^{(2)}$ and compute $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, m_v^b)$.

   Case $i < n-2$ **(hence,** $v \notin \{j_b - 1, j_b\}$**) and** $j \in [v+1 : j_b - 1]_n^+$**:** Run $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

5. **Otherwise, case** $i = n-1$ **(hence,** $v = j_b$**):** For every $j \in [n]$, the adversary A proceeds as follows:

   Case $j \in [j_b + 1 : j_b - 1]_n^+$**:** Execute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^b|})$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

   Case $j = j_b$**:** Run $c_{j_b}^{(1,0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_{*1}^0, \ldots, x_{*n}^0), 0^{s(\lambda)+k(\lambda)})$ and $c_{j_b}^{(1,1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_{*1}^1, \ldots, x_{*n}^1), 0^{s(\lambda)+k(\lambda)})$ where $x_{*j_b}^0 = x_{j_b}^b$, $x_{*j_b}^1 = x_{j_b}^0$, and $x_{*j'}^0 = x_{*j'}^1 = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$. Send the challenge $(m^0 = c_{j_b}^{(1,0)}, m^1 = c_{j_b}^{(1,1)})$ to the challenger and receive the answer c^*. Set $c_{j_b}^{(2)}$. Finally, compute $\widetilde{\mathbb{C}}_{j_b} \leftarrow_\$ \widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_{j_b}|}, 1^{|m_{j_b}^b|})$.

6. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

7. Answer to the incoming oracle queries as in .

8. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{5+i-1}^{b,q,q,1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{5+i}^{b,0,0,0}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 14. $\mathbf{H}_{5+i}^{b,t_1-1,0,0}(\lambda) \approx_c \mathbf{H}_{5+i}^{b,t_1,0,0}(\lambda)$ *for* $t_1 \in [q]$ *and* $i \in \{0\} \cup [n-2]$.

Proof. Let $v = (j_b + i \mod n) + 1$ Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{5+i}^{b,t_1-1,0,0}(\lambda)$ and $\mathbf{H}_{5+i}^{b,t_1,0,0}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA security of SKE. A is defined as follows:

1. Computes $(\mathsf{mpk}, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}_1(1^\lambda)$ and $\mathsf{ek}_j = (\mathsf{ek}, \mathsf{k}_j, \mathsf{k}_{j-1})$ for $j \in [n] \setminus \{v\}$. If $v \neq 1$, let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P}^*)$.

   - On input the t_1'-th query $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

     Case $j \in [j_b : v-1]_n^+$: Execute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m|})$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(, \mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

     Case $j = v$ and $t_1' < t_1$: Sample $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$. Run $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$. Send the query $c_v^{(1)}$ to the oracle Enc_2 and receive the answer $c_v^{(2)}$. Compute $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$.

     Case $j = v$ and $t_1' = t_1$: Compute $c_v^{(1,0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_v, \mathsf{k}_{v+1}))$ and $c_v^{(1,1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_v = x$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{v\}$. Send the challenge $(m^0 = c_v^{(1,0)}, m^1 = c_v^{(1,1)})$ to the challenger and receive the answer c^*. Set $c_v^{(2)}$. Finally, compute $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$.

     Case $j = v$ and $t_1' > t_1$: Sample $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$. Run $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (\mathsf{k}_v, \mathsf{k}_{v+1}))$ where $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$. Send the query $c_v^{(1)}$ to the oracle Enc_2 and receive the answer $c_v^{(2)}$. Compute $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$.

     Case $i < n-2$ (hence, $v \neq j_b - 1$) and $j \in [v+1 : j_b - 1]_n^+$: Run $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

   Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For every $j \in [n]$, the adversary A proceeds as follows:

   Case $j \in [j_b : v-1]_n^+$: Run $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^b|})$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

   Case $j = v$: Sample $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)+k(\lambda)}$ and compute $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_v = x_v^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{v\}$. Send the query $c_v^{(1)}$ to the oracle Enc_2 and receive the answer $c_v^{(2)}$. Compute $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$.

Case $i < n-2$ (hence, $v \neq j_b - 1$) and $j \in [v+1 : j_b - 1]_n^+$: Run $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

5. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

6. Answer to the incoming oracle queries as in Item 2.

7. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{5+i}^{b,t_1-1,0,0}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{5+i}^{b,t_1,0,0}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 15. $\mathbf{H}_{5+i}^{b,q,t_2-1,0}(\lambda) \approx_c \mathbf{H}_{5+i}^{b,q,t_2,0}(\lambda)$ for $t_2 \in [q]$ and $i \in \{0\} \cup [n-2]$.

Proof. Let $v = (j_b + i \mod n) + 1$. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{5+i}^{b,q,t_2-1,0}(\lambda)$ and $\mathbf{H}_{5+i}^{b,q,t_2,0}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscator scheme LOBF. A is defined as follows:

1. Computes $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{k}_j, \mathsf{k}_{j+1})$ for $j \in [n]$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

    - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P}^*)$.
    - On input the t_2'-th query $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

    Case $j \in [j_b : v-1]_n^+$: Execute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m|})$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

    Case $j = v$ and $t_2' < t_2$: Run $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_v|}, 1^{|m|})$, $c_v^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$, $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$.

    Case $j = v$ and $t_2' = t_2$: Compute $c_v^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$ and $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{v\}$. Send the challenge $(\mathbb{V}_v, m)$ to the challenger and receive the answer $\widetilde{\mathbb{C}}^*$. Set $\widetilde{\mathbb{C}}_v = \widetilde{\mathbb{C}}^*$.

    Case $j = v$ and $t_2' > t_2$: Sample $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$. Compute $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$. Send the query $c_v^{(1)}$ to the oracle Enc_2 and receive the answer $c_v^{(2)}$. Compute $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{C}_{c_v^{(2)}, \mathsf{k}_{v+1}}, y_v, m)$.

    Case $i < n-2$ (hence, $v \neq j_b - 1$) and $j \in [v+1 : j_b - 1]_n^+$: Run $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

    Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For every $j \in [n]$, the adversary A proceeds as follows:

Case $j \in [j_b : v-1]_n^+$**:** Run $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^b|})$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

Case $j = v$**:** Sample $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)+k(\lambda)}$ and compute $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_v = x_v^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{v\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m_v^b)$ and $c_v^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$.

Case $i < n-2$ **(hence,** $v \neq j_b - 1$**) and** $j \in [v+1 : j_b - 1]_n^+$**:** Run $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

5. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

6. Answer to the incoming oracle queries as in .

7. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{5+i}^{b,q,t_2-1,0}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{5+i}^{b,q,t_2,0}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 16. $\mathbf{H}_{5+i}^{b,q,q,0}(\lambda) \approx_c \mathbf{H}_{5+i}^{b,q,q,1}(\lambda)$ *for* $i \in \{0\} \cup [n-2]$.

Proof. Let $v = (j_b + i \mod n) + 1$. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{5+i}^{b,q,q,0}(\lambda)$ and $\mathbf{H}_{5+i}^{b,q,q,1}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscator scheme LOBF. A is defined as follows:

1. Computes $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{k}_j, \mathsf{k}_{j+1})$ for $j \in [n]$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P}^*)$.

   - On input $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

     Case $j \in [j_b : v]_n^+$**:** Run $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m|})$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

     Case $i < n-2$ **(hence,** $v \neq j_b - 1$**) and** $j \in [v+1 : j_b - 1]_n^+$**:** Run $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

     Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For every $j \in [n]$, the adversary A proceeds as follows:

   Case $j \in [j_b : v-1]_n^+$**:** Run $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^b|})$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

Case $j = v$: Run $c_v^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$ and $c_v^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x_v^0$ and $x_{j'} = x_{j'}^{\star}$ for $j' \in [n] \setminus \{v\}$. Send the challenge $(\mathbb{V}_v, m_v^b)$ to the challenger and receive the answer $\widetilde{\mathbb{C}}^*$. Set $\widetilde{\mathbb{C}}_v = \widetilde{\mathbb{C}}^*$.

Case $i < n - 2$ (hence, $v \neq j_b - 1$) and $j \in [v + 1 : j_b - 1]_n^+$: Run $c_j^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_{\$} \{0,1\}^{s(\lambda)}$, $x_j = x_j^b$, and $x_{j'} = x_{j'}^{\star}$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_{\$} \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ and $c_j^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

5. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

6. Answer to the incoming oracle queries as in Item 2.

7. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{5+i}^{b,q,q,0}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{5+i}^{b,q,q,1}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 17. $\mathbf{H}_{5+n}^{1-b,q,q,q}(\lambda) \equiv \mathbf{H}_{5+n}^{b,q,q,1}(\lambda)$.

Proof. The distributions of these two experiments do not depend on the bit b.

By combining Claims 9 to 17 and conditioned to the event $\mathbf{Validity}_{2,j_0,j_1}$, we conclude that

$$\mathbf{H}_0^b \approx_c \mathbf{H}_1^b \equiv \mathbf{H}_2^{b,0} \approx_c \ldots \approx_c \mathbf{H}_2^{b,q} \approx_c \mathbf{H}_3^b \equiv \mathbf{H}_4^{b,0} \approx_c \ldots \approx_c \mathbf{H}_4^{b,q} \equiv$$
$$\mathbf{H}_4^{b,q,q,1} \approx_c \mathbf{H}_5^{b,0,0,0} \approx_c \ldots \approx_c \mathbf{H}_5^{b,q,0,0} \approx_c \ldots \approx_c \mathbf{H}_5^{b,q,q,0} \approx_c$$
$$\mathbf{H}_5^{b,q,q,1} \approx_c \ldots \approx_c \mathbf{H}_{5+n-1}^{b,0,0,0} \equiv \mathbf{H}_{5+n-1}^{1-b,0,0,0}.$$

This concludes the proof.

Lemma 6. *Let $j_0 \in [n]$. If PE is CPA-1-sided secure without collusions (Definition 9), SKE is CPA secure (Definition 4), and LOBF is secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\mathsf{0\text{-}CPA\text{-}1\text{-}iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \,\middle|\, \mathbf{Validity}_{3,j_0} \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Without loss of generality, let $q = |\mathcal{Q}_1| = \ldots = |\mathcal{Q}_{nim}| \in \mathsf{poly}(\lambda)$. Consider the hybrid experiments of Lemma 4 and Lemma 5. Formally,

- Let $\mathbf{H}_0^{1,i}(\lambda)$ and $\mathbf{H}_1^{1,i}(\lambda)$ for $i \in \{0\} \cup [n]$ be the hybrids of Lemma 4 (for the challenge bit $b = 1$) except that are conditioned to the event $\mathbf{Validity}_{3,j_0}$ (instead of $\mathbf{Validity}_1$)

- Let $\mathbf{H}_0^0(\lambda)$, $\mathbf{H}_1^0(\lambda)$, $\mathbf{H}_2^{0,i}(\lambda)$, $\mathbf{H}_3^0(\lambda)$, $\mathbf{H}_4^{0,i}(\lambda)$, $\mathbf{H}_4^{0,q,q,1}(\lambda)$, $\mathbf{H}_{5+j}^{0,i,0,0}(\lambda)$, $\mathbf{H}_{5+j}^{0,q,i,0}(\lambda)$, $\mathbf{H}_{5+j}^{0,q,q,k}(\lambda)$, and $\mathbf{H}_{5+n-1}^{0,0,0,0}(\lambda)$, for $(i, j, k) \in (\{0\} \cup [q]) \times (\{0\} \cup [n-2]) \times \{0,1\}$, be the hybrids of Lemma 5 (for the challenge bit $b = 0$) except that are conditioned to the event $\mathbf{Validity}_{3,j_0}$ (instead of $\mathbf{Validity}_{2,j_0,j_1}$).

In addition, consider the following additional hybrids experiments:

$\mathbf{H}_{5+n}^{0,q,q}$: Identical to $\mathbf{H}_{5+n-1}^{0,0,0,0}$.

$\mathbf{H}_{5+n+i}^{0,0,0}$ **for $i \in [n]$:** Identical to $\mathbf{H}_{5+n+i-1}^{0,q,q}$.

$\mathbf{H}_{5+n+i}^{0,0,t_2}$ **for** $t_2 \in [q], i \in [n]$**:** Same as $\mathbf{H}_{5+n+i}^{0,0,t_2-1}$ except that the challenger changes how it answers to the first t_2 queries for oracle $\mathsf{Enc}(\mathsf{ek}_v, \cdot, \cdot)$ where $v = (j_0 - i - 1 \mod n) + 1$. Formally, on input the t_2'-th query (x, m) such that $t_2' \le t_2$, the challenger returns $c_v = (\widetilde{\mathbb{C}}_v, c_v^{(2)})$ where $\widetilde{\mathbb{C}}_v \leftarrow_{\$} \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$ where $y_v \leftarrow_{\$} \{0,1\}^{s(\lambda)}$. Otherwise, on input the t_2'-th query (x, m) such that $t_2' > t_2$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_{5+n+i}^{0,0,0}$.

$\mathbf{H}_{5+n+i}^{0,t_1,q}$ **for** $t_1 \in [q], i \in [n]$**:** Same as $\mathbf{H}_{5+n+i}^{0,t_1-1,q}$ except that the challenger changes how it answers to the first t_1 queries for oracle $\mathsf{Enc}(\mathsf{ek}_v, \cdot, \cdot)$ where $v = (j_0 - i - 1 \mod n) + 1$. Formally, on input the t_1'-th query (x, m) such that $t_1' \le t_1$, the challenger computes $c_v^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_v, \mathsf{k}_{v+1}))$ where $y_v \leftarrow_{\$} \{0,1\}^{s(\lambda)}$, $x_v = x$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{v\}$. Finally, the challenger returns $c_v = (\widetilde{\mathbb{C}}_v, c_v^{(2)})$ where $c_v^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$, $\widetilde{\mathbb{C}}_v \leftarrow_{\$} \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$. Otherwise, on input the t_1'-th query (x, m) such that $t_1' > t_1$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_{5+n+i}^{0,0,q}$.

Claim 18. $\mathbf{H}_0^0(\lambda) \approx_c \mathbf{H}_{5+n-1}^{0,0,0,0}(\lambda)$.

Proof. The proof of Claim 18 is identical to that of Lemma 5 where the challenge bit is $b = 0$.

Claim 19. $\mathbf{H}_{5+n+i}^{0,0,t_2-1}(\lambda) \approx_c \mathbf{H}_{5+n+i}^{0,0,t_2}(\lambda)$ *for* $t_2 \in [q]$ *and* $i \in [n]$.

Proof. Let $v = (j_0 - i - 1 \mod n) + 1$. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{5+n+i}^{0,0,t_2-1}(\lambda)$ and $\mathbf{H}_{5+n+i}^{0,0,t_2}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscator scheme LOBF. A is defined as follows:

1. Computes $(\mathsf{ek}_1, \ldots, \mathsf{ek}_1, \mathsf{msk}) \leftarrow_{\$} \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{k}_j, \mathsf{k}_{j+1})$ for $j \in [n]$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_{\$} \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P}^*)$.
   - On input the t_2'-th query $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

     Case $i > 1$ and $j \in [j_0 - 1 : v + 1]_n^-$: Compute $\widetilde{\mathbb{C}}_j \leftarrow_{\$} \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$, $c_j^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_{\$} \{0,1\}^{s(\lambda)}$, $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

     Case $j = v$ and $t_2' < t_2$: Compute $\widetilde{\mathbb{C}}_j \leftarrow_{\$} \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$, $c_v^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$, and $c_v^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $y_v \leftarrow_{\$} \{0,1\}^{s(\lambda)}$, $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$.

     Case $j = v$ and $t_2' = t_2$: Compute $c_v^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$, and $c_v^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$. Send the challenge $(\mathbb{C}_{c_v^{(2)}, \mathsf{k}_{v+1}}, m)$ to the challenger and receive $\widetilde{\mathbb{C}}^*$. Set $\widetilde{\mathbb{C}}_v = \widetilde{\mathbb{C}}^*$.

     Case $j = v$ and $t_2' > t_2$: Run $\widetilde{\mathbb{C}}_v \leftarrow_{\$} \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_v|}, 1^{|m|})$, $c_v^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_v, c_v^{(1)})$, $c_v^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$.

     Case $i \ne n$ and $j \in [v - 1 : j_0]_n^-$: Compute $\widetilde{\mathbb{C}}_j \leftarrow_{\$} \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m|})$, $c_j^{(2)} \leftarrow_{\$} \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For every $j \in [n]$, the adversary A computes $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^0|})$ and $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

5. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

6. Answer to the incoming oracle queries as in Item 2.

7. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{5+n+i}^{0,0,t_2}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{5+n+i}^{0,0,t_2-1}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 20. $\mathbf{H}_{5+n+i}^{0,t_1-1,q}(\lambda) \approx_c \mathbf{H}_{5+n+i}^{0,t_1,q}(\lambda)$ *for* $t_1 \in [q]$ *and* $i \in [n-1]$.

Proof. Let $v = (j_0 - i - 1 \mod n) + 1$. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{5+n+i}^{0,t_1,q}(\lambda)$ and $\mathbf{H}_{5+n+i}^{0,t_1-1,q}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA security of SKE. A is defined as follows:

1. Computes $(\mathsf{mpk}, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}_1(1^\lambda)$ and $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{ek}_j, \mathsf{ek}_{j-1})$ for $j \in [n] \setminus \{v\}$. If $v \neq 1$, let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P}^*)$.
   - On input the t_1'-th query $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

     Case $i > 1$ and $j \in [j_0 - 1 : v + 1]_n^-$: Run $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_j, m)$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

     Case $j = v$ and $t_1' < t_1$: Run $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_v, \mathsf{k}_{v+1}))$ where $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$. Send the query $c_v^{(1)}$ to the oracle Enc_2 and receive the answer $c_v^{(2)}$. Compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$.

     Case $j = v$ and $t_1' = t_1$: Run $c_v^{(1,0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ and $c_v^{(1,1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_v, \mathsf{k}_{v+1}))$ where $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_v = x$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{v\}$. Send the challenge $(m^0 = c_v^{(1,0)}, m^1 = c_v^{(1,1)})$ to the challenger and receive the answer c^*. Set $c_v^{(2)}$ and compute $\widetilde{\mathbb{C}}_v \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$.

     Case $j = v$ and $t_1' > t_1$: Run $c_v^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{v\}$. Send the query $c_v^{(1)}$ to the oracle Enc_2 and receive the answer $c_v^{(2)}$. Compute $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_v, y_v, m)$ where $y_v \leftarrow_\$ \{0,1\}^{s(\lambda)}$.

     Case $j \in [v - 1 : j_0]_n^-$: Run $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_v|}, 1^{|m|})$, $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$, and $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{j\}$.

Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For every $j \in [n]$, the adversary A proceeds as follows:

   Case $j = v$: Computes $c_v^{(1)} \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_v = x_v^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{v\}$. Send the query $c_v^{(1)}$ to the oracle Enc_2 and receive the answer $c_v^{(2)}$. Finally, compute $\widetilde{\mathbb{C}}_j \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^0|})$.

   Case $j \neq v$: Computes $c_j^{(1)} \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, run $\widetilde{\mathbb{C}}_j \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^0|})$ and $c_j^{(2)} \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$.

5. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

6. Answer to the incoming oracle queries as in Item 2.

7. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{5+n+i}^{0,t_1-1,q}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{5+n+i}^{0,t_1,q}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 21. $\mathbf{H}_{5+2n}^{0,t_1-1,q}(\lambda) \approx_c \mathbf{H}_{5+2n}^{0,t_1,q}(\lambda)$ *for $t_1 \in [q]$.*

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{5+2n}^{0,t_1,q}(\lambda)$ and $\mathbf{H}_{5+2n}^{0,t_1-1,q}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA-1-sided security without collusions of PE. A is defined as follows:

1. Receive mpk from the challenger.

2. Computes $\mathsf{k}_j \leftarrow\!\!{\scriptstyle\$}\ \mathsf{KGen}_2(1^\lambda)$ for $j \in [n]$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

3. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_{\mathbb{P}^*}$.

   - On input t_1'-th query $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, A proceeds as follows:

     Case $j \neq j_0$: Sample $y_j \leftarrow\!\!{\scriptstyle\$}\ \{0,1\}^{s(\lambda)}$ and compute $c_j^{(1)} \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $x_j = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

     Case $j = j_0$ and $t_1' < t_1$: Compute $c_{j_0}^{(1)} \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_{j_0}, \mathsf{k}_{j_0+1}))$ where $x_{j_0} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_0\}$.

     Case $j = j_0$ and $t_1' = t_1$: Sample $y_{j_0} \leftarrow\!\!{\scriptstyle\$}\ \{0,1\}^{s(\lambda)}$ and send the challenge $(m_*^0 = 0^{s(\lambda)+k(\lambda)}, m_*^1 = (y_{j_0}, \mathsf{k}_{j_0+1}), x_*^0 = (x_{*1}^0, \ldots, x_{*n}^0), x_*^1 = (x_{*1}^1, \ldots, x_{*n}^1))$ to the challenger where $x_{*j_0}^0 = x_{*j_0}^1 = x$ and $x_{*j'}^0 = x_{*j'}^1 = x_{j'}^\star$ for $j' \in [n] \setminus \{j_0\}$. Receive the challenge ciphertext c^* and $c_{j_0}^{(1)} = c^*$.

     Case $j = j_0$ and $t_1' > t_1$: Sample $y_{j_0} \leftarrow\!\!{\scriptstyle\$}\ \{0,1\}^{s(\lambda)}$ and compute $c_{j_0}^{(1)} \leftarrow\!\!{\scriptstyle\$}\ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_{j_0} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_0\}$.

Finally, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m)$.

4. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1)$ from D.

5. For every $j \in [n]$, the adversary A computes $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

6. For every $j \in [n]$, the adversary A computes $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{ek}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^0|})$.

7. Set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

8. Answer to the incoming oracle queries as in Item 3.

9. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{5+2n}^{0,t_1-1,q}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{5+2n}^{0,t_1,q}(\lambda)$. Moreover, D submits a single query $\mathcal{P}^*$ to oracle $\mathsf{KGen}(\mathsf{msk}, \cdot)$ and, conditioned to the event $\mathbf{Validity}_{3,j_0}$, we know that $\forall x_{j_0}' \in \mathcal{Q}_{j_0}, \mathbb{P}_{j_0}^*(x_{j_0}') = 0$. Because of this, A submits a single query to oracle $\mathsf{KGen}_1(\mathsf{msk}, \cdot)$ and it is also a valid adversary for the experiment $\mathbf{G}_{\mathsf{PE},\mathsf{A}}^{\mathsf{CPA}\text{-}1\text{-}\mathsf{PE}}(\lambda)$ with the same advantage of D. This concludes the proof.

Claim 22. $\mathbf{H}_0^{1,0}(\lambda) \approx_c \mathbf{H}_1^{1,q}(\lambda)$.

Proof. The proof of Claim 22 is identical to that of Lemma 4 where the challenge bit is $b = 1$.

Claim 23. $\mathbf{H}_{5+2n}^{0,q,q}(\lambda) \equiv \mathbf{H}_1^{1,q}(\lambda)$.

Proof. Claim 23 follows by observing that experiments $\mathbf{H}_{5+2n}^{0,q,q}(\lambda)$ and $\mathbf{H}_1^{1,q}(\lambda)$ are identical (and does not depend on the bit b).

By combining Claims 18 to 23 and conditioned to the event $\mathbf{Validity}_{3,j_0}$, we conclude that

$$\mathbf{H}_0^0 \approx_c \mathbf{H}_{5+n-1}^{0,0,0,0} \equiv \mathbf{H}_{5+n}^{0,q,q} \equiv \mathbf{H}_{5+n+1}^{0,0,0} \approx_c \ldots \approx_c \mathbf{H}_{5+n+1}^{0,0,q} \approx_c \ldots \approx_c$$
$$\mathbf{H}_{5+n+1}^{0,q,q} \equiv \mathbf{H}_{5+n+2}^{0,0,0} \approx_c \ldots \approx_c \mathbf{H}_{5+2n}^{0,0,q} \approx_c \ldots \approx_c \mathbf{H}_{5+2n}^{0,q,q} \equiv \mathbf{H}_1^{1,q} \approx_c \mathbf{H}_0^{1,0}$$

This concludes the proof.

Lemma 7. *Let $j_1 \in [n]$. If PE is CPA-1-sided-secure without collusions (Definition 9), SKE is CPA-secure (Definition 4), and LOBF is secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{0\text{-}\mathsf{CPA}\text{-}1\text{-}\mathsf{iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \,\Big|\, \mathbf{Validity}_{4,j_1} \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Lemma 7 follows by using a symmetrical argument to that of Lemma 6.

By combining Lemmas 4 to 7 we conclude that Π is CPA-1-sided secure without collusions.

E.4 Proof of Theorem 5 (CPA-2-sided security of Π for $n = O(\log(\lambda))$)

As usual, consider the predicate space $\mathcal{P} = \{\mathbb{P}(x_1, \ldots, x_n)\}$ of Construction 2 where $\mathbb{P}(x_1, \ldots, x_n)$ $= \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$. Let $\mathbb{P}^* \in \mathcal{P}$ be the only predicate for which the adversary will ask for the decryption key $\mathsf{dk}_{\mathbb{P}^*}$ during the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{\text{0-CPA-2-iPE}}$ (recall that we prove the security of Construction 2 in the scenario without collusions, i.e., $|\mathcal{Q}_{\mathsf{KGen}}| = 1$). Also, consider the validity condition of $\mathbf{G}_{\Pi,\mathsf{A}}^{\text{0-CPA-2-iPE}}$ and consider the following observations:

1. Suppose that $\forall j \in [n], \forall i_1 \in [k_1 + 1], \ldots, \forall i_n \in [k_n + 1]$, we have

$$\mathbb{P}^*(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$$
$$\mathbb{P}^*(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 0,$$

where $\mathcal{Q}_i^b = \{x_i^{(1,b)}, \ldots, x_i^{(k_i,b)}, x_i^{(k_i+1,b)} = x_i^b\}$ for $i \in [n]$, $b \in \{0,1\}$ as defined in Definition 13. This means that the adversary cannot decrypt any part of the challenge ciphertext.

2. Otherwise, if $\exists j \in [n], \exists i_1 \in [k_1 + 1], \ldots, \exists i_n \in [k_n + 1]$ such that

$$\mathbb{P}^*(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$$
$$\mathbb{P}^*(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 1, \tag{14}$$

we are guaranteed that the adversary can retrieve the message m_j^b contained into the j-th challenge ciphertext c_j. By taking into account the definition of $\mathbb{P}^*(x_1, \ldots, x_n) = \mathbb{P}_1^*(x_1) \wedge \ldots \wedge \mathbb{P}_n^*(x_n)$, Equation (14) implies that, for any $j' \in [n] \setminus [j]$, the adversary can satisfy the i-th predicate $\mathbb{P}_i^*$ for $i \in [n] \setminus [j']$ (e.g., by taking the ciphertexts corresponding to the indexes $i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n$ and the j-th challenge ciphertext c_j). Hence, the secrecy of the challenge message $m_{j'}^b$ solely depends on the evaluation of $\mathbb{P}_{j'}^*$ over the challenge input $x_{j'}^b$.

By taking into account the following observations, we can rewrite the validity condition of $\mathbf{G}_{\Pi,\mathsf{A}}^{\text{0-CPA-2-iPE}}$ (Definition 13) in the following way:

$$\text{Either } \mathbf{Validity_1} \text{ or } \mathbf{Validity_2}$$

where $\mathbf{Validity}_1$ and $\mathbf{Validity}_2$ formalize the observations of Items 1 and 2 respectively, i.e.,

$$\mathbf{Validity}_1 : \forall j \in [n], \forall i_1 \in [k_1 + 1], \ldots, \forall i_n \in [k_n + 1],$$
$$\mathbb{P}^*(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$$
$$\mathbb{P}^*(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 0$$
$$\mathbf{Validity}_2 : \forall j \in [n], \text{Either } \mathbb{P}_j^*(x_j^0) = \mathbb{P}_j^*(x_j^1) = 0 \text{ or } \mathbb{P}_j^*(x_j^0) = \mathbb{P}_j^*(x_j^1) \wedge m_j^0 = m_j^1$$

where $\mathcal{Q}_i^b = \{x_i^{(1,b)}, \ldots, x_i^{(k_i,b)}, x_i^{(k_i+1,b)} = x_i^b\}$ for $i \in [n]$, $b \in \{0,1\}$ as defined in Definition 13. Hence, the CPA-2-sided security of Construction 2 follows by proving the following lemmas.

Lemma 8. *If* PE *is CPA-1-sided secure without collusions (Definition 9),* SKE *is CPA secure (Definition 4), and* LOBF *is secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{\text{0-CPA-2-iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \middle| \mathbf{Validity}_1 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Note that **Validity**$_1$ is equivalent to the validity condition of CPA-1-sided security. Hence, the lemma follows by leveraging an identical argument to that of the CPA-1-sided case (Appendix E.3).

Lemma 9. *If* PE *is CPA-2-sided secure without collusions (Definition 9) and* LOBF *is secure (Definition 2), then*

$$\left| \mathbb{P}\Big[\mathbf{G}_{\Pi,\mathsf{A}}^{\text{0-CPA-2-iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \Big| \mathbf{Validity}_2 \Big] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. In this lemma, we restrict the adversary to submit the (single) query to KGen only before the challenge phase, i.e., the oracle KGen is not available after the challenge phase. Under this restriction, we prove Lemma 9 for any $n = \mathsf{poly}(\lambda)$. Then, we use complexity leveraging to show that the lemma holds when $n = O(\log(\lambda))$ and the oracle KGen is available after the challenge phase. Without loss of generality, we assume the adversary always submit a query to KGen. Finally, for the sake of clarity, in the rest of this proof we use the notation $\mathbb{V}_i \stackrel{\text{def}}{=} \mathbb{C}_{c_i^{(2)}, \mathsf{k}_{i+1}}$ where $c_i^{(2)}$ and k_{i+1} will be clear from the context.

Consider the following hybrid experiments:

$\mathbf{H}_0^{b,0}(\lambda)$**:** This is exactly the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{\text{0-CPA-2-iPE}}(\lambda)$ conditioned to the event **Validity**$_2$ where the challenge bit is b. Recall that the oracle KGen is not available after the challenge phase.

$\mathbf{H}_0^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_0^{b,i-1}$, except that the challenger changes how it computes the challenger ciphertext c_i. Let $\mathbb{P}^* \in \mathcal{Q}_{\mathsf{KGen}}$ and $((x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ be the predicate submitted to the oracle KGen before the challenge phase and the challenge inputs chosen by the adversary. If $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 0$, the value $c_i^{(1)}$ challenge ciphertext $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$ is computed as $c_i^{(1)} \leftarrow\!\!\$ \; \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda) + k(\lambda)})$ where $0^{s(\lambda) + k(\lambda)} \in \mathcal{M}_1$ (for some function k) $x_i = x_i^0$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{i\}$. Otherwise, if $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 1$, the value $c_i^{(1)}$ challenge ciphertext $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$ is computed as $c_i^{(1)} \leftarrow\!\!\$ \; \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_i, \mathsf{k}_{i+1}))$ where $y_i \leftarrow\!\!\$ \; \{0,1\}^{s(\lambda)}$, $x_i = x_i^0$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{i\}$. Observe that $c_i^{(1)}$ is computed by fixing $x_i = x_i^0$ (instead of $x_i = x_i^b$), i.e., the input $(x_1, \ldots, x_n)$ used to compute the i-th challenge ciphertext is fixed and does not depend on the challenge bit b.

$\mathbf{H}_1^{b,0}(\lambda)$**:** Identical to $\mathbf{H}_1^{b,n}(\lambda)$.

$\mathbf{H}_1^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_1^{b,i-1}$, except that the challenger changes how it computes the challenger ciphertext c_i. Let $\mathbb{P}^* \in \mathcal{Q}_{\mathsf{KGen}}$ and $((x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ be the predicate submitted to the oracle KGen before the challenge phase and the challenge inputs chosen by the adversary. If $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 0$, the value $\widetilde{\mathbb{C}}_i$ of challenge ciphertext $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$ is simulated by the challenger using the simulator of the lockable obfuscation scheme LOBF, i.e., $\widetilde{\mathbb{C}}_i \leftarrow\!\!\$ \; \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_i|}, 1^{|m_i^b|})$. Otherwise, if $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 1$, the value $\widetilde{\mathbb{C}}_i$ is computed as in $\mathbf{H}_1^{b,0}(\lambda)$.

Claim 24. $\mathbf{H}_0^{b,i-1}(\lambda) \approx_c \mathbf{H}_0^{b,i}(\lambda)$ *for* $i \in [n]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_0^{b,1-i}(\lambda)$ and $\mathbf{H}_0^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA-2-sided security without collusions of PE. A is defined as follows:

1. Receive mpk from the challenger.

2. Computes $\mathsf{k}_j \leftarrow_\$ \mathsf{KGen}_2(1^\lambda)$ for $j \in [n]$. Let $\mathsf{ek}_i = (\mathsf{mpk}, \mathsf{k}_i, \mathsf{k}_{i+1})$ for $i \in [n]$ where $\mathsf{k}_{n+1} = \mathsf{k}_1$.

3. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_\mathbb{P}$.
   - On input $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)}) \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

4. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

5. Let $\mathbb{P}^*(x_1, \ldots, x_n) = \mathbb{P}_1^*(x_1) \wedge \ldots \wedge \mathbb{P}_n^*(x_n)$ be the predicate submitted by A to the oracle KGen. For any $j \in [n]$, A proceeds as follows:

   Case $j < i$ and $\mathbb{P}^*(x_j^0) = \mathbb{P}^*(x_j^1) = 0$: Compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

   Case $j < i$ and $\mathbb{P}^*(x_j^0) = \mathbb{P}^*(x_j^1) = 1$: Sample $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and execute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

   Case $j = i$ and $\mathbb{P}^*(x_j^0) = \mathbb{P}^*(x_j^1) = 0$: Send the challenge $(m_*^0 = (y_i, \mathsf{k}_{i+1}), m_*^1 = 0^{s(\lambda)+k(\lambda)}, x_*^0 = (x_{*1}^0, \ldots, x_{*n}^0), x_*^1 = (x_{*1}^1, \ldots, x_{*n}^1))$ where $y_i \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $0^{s(\lambda)+k(\lambda)} \in \mathcal{M}_1$, $x_{*i}^0 = x_i^b$, $x_{*i}^1 = x_i^0$, and $x_{*j}^0 = x_{*j}^1 = x_j^\star$ for $j \in [n] \setminus \{i\}$. Receive the challenge ciphertext c^* from the challenger. Set $c_i^{(1)} = c^*$.

   Case $j = i$ and $\mathbb{P}^*(x_j^0) = \mathbb{P}^*(x_j^1) = 1$: Send the challenge $(m_*^0 = (y_i, \mathsf{k}_{i+1}), m_*^1 = (y_i, \mathsf{k}_{i+1}), x_*^0 = (x_{*1}^0, \ldots, x_{*n}^0), x_*^1 = (x_{*1}^1, \ldots, x_{*n}^1))$ where $y_i \leftarrow_\$ \{0,1\}^{s(\lambda)}$, $x_{*i}^0 = x_i^b$, $x_{*i}^1 = x_i^0$, and $x_{*j}^0 = x_{*j}^1 = x_j^\star$ for $j \in [n] \setminus \{i\}$. Receive the challenge ciphertext c^* from the challenger. Set $c_i^{(1)} = c^*$.

   Case $j > i$: Sample $y_j \leftarrow_\$ \{0,1\}^{s(\lambda)}$ and compute $c_j^{(1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j, \mathsf{k}_{j+1}))$ where $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

6. Compute $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow_\$ \mathsf{Enc}_2(\mathsf{ek}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow_\$ \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ for any $j \in [n]$.

7. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

8. Answer to the incoming oracle queries for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ as in Item 3.

9. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_0^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_0^{b,i}(\lambda)$. Moreover, conditioned to the event $\mathbf{Validity}_2$, we know that D asks for a single decryption key $\mathsf{dk}_{\mathbb{P}^*}$ for $\mathbb{P}^*$ and either $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 0$ or $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) \wedge m_i^0 = m_i^1$. This implies that A is a valid adversary for the experiment $\mathbf{G}_{\mathsf{PE},\mathsf{A}}^{\mathsf{CPA-2-PE}}(\lambda)$ with the same advantage of D. This concludes the proof.

Claim 25. $\mathbf{H}_1^{b,i-1}(\lambda) \approx_c \mathbf{H}_1^{b,i}(\lambda)$ *for* $i \in [n]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_1^{b,1-i}(\lambda)$ and $\mathbf{H}_1^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF. A is defined as follows:

1. Compute $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow\!\!{\$}\, \mathsf{Setup}(1^\lambda)$ for $j \in [n]$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{k}_j, \mathsf{k}_{j+1})$. Let $\mathsf{k}_{n+1} = \mathsf{k}_1$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_\mathbb{P} \leftarrow\!\!{\$}\, \mathsf{KGen}(\mathsf{msk}, \mathbb{P})$.

   - On input $(x, m) \in \mathcal{X}_1 \times \mathcal{M}_3$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, return $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)}) \leftarrow\!\!{\$}\, \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. Let $\mathbb{P}^*(x_1, \ldots, x_n) = \mathbb{P}_1^*(x_1) \wedge \ldots \wedge \mathbb{P}_n^*(x)$ be the predicate submitted by A to the oracle KGen. For any $j \in [n]$, A proceeds as follows:

   Case $j < i$ and $\mathbb{P}^*(x_j^0) = \mathbb{P}^*(x_j^1) = 0$: Compute $c_j^{(1)} \leftarrow\!\!{\$}\, \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow\!\!{\$}\, \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow\!\!{\$}\, \mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_j|}, 1^{|m_j^b|})$.

   Case $j = i$ and $\mathbb{P}^*(x_j^0) = \mathbb{P}^*(x_j^1) = 0$: Compute $c_i^{(1)} \leftarrow\!\!{\$}\, \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ and $c_j^{(2)} \leftarrow\!\!{\$}\, \mathsf{Enc}_2(\mathsf{k}_i, c_i^{(1)})$ where $x_i = x_i^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{i\}$. Send the challenge $(\mathbb{V}_i, m_i^b)$ to the challenger and receive $\widetilde{\mathbb{C}}_i$. Set $c_i = (\widetilde{\mathbb{C}}_i, c_i^{(2)})$.

   Case $j > i$ and $\mathbb{P}^*(x_j^0) = \mathbb{P}^*(x_j^1) = 0$: Compute $c_j^{(1)} \leftarrow\!\!{\$}\, \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s(\lambda)+k(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$. Finally, set $c_j = (\widetilde{\mathbb{C}}_j, c_j^{(2)})$ where $c_j^{(2)} \leftarrow\!\!{\$}\, \mathsf{Enc}_2(\mathsf{k}_j, c_j^{(1)})$ and $\widetilde{\mathbb{C}}_j \leftarrow\!\!{\$}\, \mathsf{Obf}(1^\lambda, \mathbb{V}_j, y_j, m_j^b)$ where $y_j \leftarrow\!\!{\$}\, \{0,1\}^{s(\lambda)}$.

   Case $\mathbb{P}^*(x_j^0) = \mathbb{P}^*(x_j^1) = 1$: Compute $c_j \leftarrow\!\!{\$}\, \mathsf{Enc}(\mathsf{ek}_j, x_j^0, m_j^b)$.

5. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

6. Answer to the incoming oracle queries for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ as in Item 2.

7. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_1^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_1^{b,i}(\lambda)$. Hence, A retains the same advantage of D. This concludes the proof.

Claim 26. $\mathbf{H}_1^{b,n}(\lambda) \equiv \mathbf{H}_1^{1-b,n}(\lambda)$.

Proof. The claim follows by leveraging the validity condition $\mathbf{Validity}_2$. Indeed, for every $i \in [n]$, if $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 0$ we have that the j-th ciphertext c_j does not depend on the bit b. On the other hand, if $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 1$, we have that the j-th ciphertext c_j depends on either m_j^0 or m_j^1. However, by the validity condition $\mathbf{Validity}_2$ we have that $m_j^0 = m_j^1$. Hence, $\mathbf{H}_1^{b,n}(\lambda)$ and $\mathbf{H}_1^{1-b,n}(\lambda)$ are identically distributed. This concludes the proof.

By combining Claims 24 to 25 and conditioned to the event $\mathbf{Validity}_2$, we conclude that $\mathbf{H}_0^{0,0} \approx_c \ldots \approx_c \mathbf{H}_0^{0,n} \equiv \mathbf{H}_1^{0,0} \approx_c \ldots \approx_c \mathbf{H}_1^{0,n} \equiv \mathbf{H}_1^{1,n}$. Note that this holds if $n = \mathsf{poly}(\lambda)$ and the adversary is restricted to submitting the (single) key generation query before the challenge

phase, i.e., KGen oracle not available after challenge phase. By using complexity leveraging, we conclude that the same result holds also when the KGen oracle is available after the challenge phase when $n = O(\log(\lambda))$. This concludes the proof.

By leveraging Lemmas 8 and 9, we conclude that Π of Construction 1 is CPA-2-sided secure for $n = O(\log(\lambda))$.

E.5 Proof of Theorem 6 (CPA-1-sided security of Π)

Consider the predicate space $\mathcal{P} = \{\mathbb{P}(x_1, \ldots, x_n)\}$ of Construction 3 where $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$. Let $\mathbb{P}^* \in \mathcal{P}$ be the only predicate for which the adversary will ask the decryption key $\mathsf{dk}_{\mathbb{P}^*}$ during the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-CPA-1-iPE}}$ (recall that we prove the security of Construction 3 in the ℓ-corruptions setting without collusions (i.e., $|\mathcal{Q}_{\mathsf{KGen}}| = 1$). Consider the validity condition of $\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-CPA-1-iPE}}$ and let $\mathcal{Q}_i = \{x | \exists (x, m) \in \mathcal{Q}_{\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)}\}$ for $i \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$, and $\mathcal{Q}_i = \mathcal{X}_{1,i}$ for $i \in \mathcal{Q}_{\mathsf{Corr}}$ (recall that $|\mathcal{Q}_{\mathsf{Corr}}| \le n - 1$) as defined in Definition 13. We can write such a validity condition with respect to $\mathbb{P}^* \in \mathcal{Q}_{\mathsf{KGen}} = \{\mathbb{P}^*\}$ as follows: $\forall j \in [n]$, $\forall i_1 \in [k_1 + 1], \ldots, \forall i_n \in [k_n + 1]$,

$$
\begin{aligned}
&\mathbb{P}^*(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) = \\
&\mathbb{P}^*(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = \\
&\mathbb{P}_1^*(x_1^{(i_1,0)}) \wedge \ldots \wedge \mathbb{P}_{j-1}^*(x_{j-1}^{(i_{j-1},0)}) \wedge \mathbb{P}_j^*(x_j^0) \wedge \mathbb{P}_{j+1}^*(x_{j+1}^{(i_{j+1},0)}) \wedge \ldots \wedge \mathbb{P}_{i_n}^*(x_n^{(i_n,0)}) = \\
&\mathbb{P}_1^*(x_1^{(i_1,1)}) \wedge \ldots \wedge \mathbb{P}_{j-1}^*(x_{j-1}^{(i_{j-1},1)}) \wedge \mathbb{P}_j^*(x_j^1) \wedge \mathbb{P}_{j+1}^*(x_{j+1}^{(i_{j+1},1)}) \wedge \ldots \wedge \mathbb{P}_n^*(x_n^{(i_n,1)}) = 0,
\end{aligned}
$$

where $\mathcal{Q}_i^b = \{x_i^{(1,b)}, \ldots, x_i^{(k_i,b)}, x_i^{(k_i+1,b)} = x_i^b\}$ is the ordered list composed of the k_i predicate inputs $\mathcal{Q}_i$ and the challenge input x_i^b (as defined in Definition 13). Note that Construction 3 has input space $\mathcal{X}_1 = \mathcal{X}_{1,1} \times \ldots \times \mathcal{X}_{1,n}$ (that is identical to the one of the underlying PE). Hence, we can conclude that for each $\mathcal{X}_{1,i}$ for $i \in [n]$ there exists $x_i^\star \in \mathcal{X}_{1,i}$ such that $\mathbb{P}_i^*(x_i^\star) = 1$. As a consequence, an adversary is valid only if there exists $j_0, j_1 \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$ such that $\mathbb{P}_{j_0}^*(x_{j_0}^0) = \mathbb{P}_{j_1}^*(x_{j_1}^1) = 0$. Otherwise, an adversary is able to decrypt at least one out the two challenges by leveraging the corrupted encryption keys $\{\mathsf{ek}_i\}_{i \in \mathcal{Q}_{\mathsf{Corr}}}$ and computing $|\mathcal{Q}_{\mathsf{Corr}}|$ ciphertexts, each under the i-th predicate wildcard $x_i^\star \in \mathcal{X}_{1,i}$ for $i \in \mathcal{Q}_{\mathsf{Corr}}$.

According to the above observation, the A's validity can be rewritten as follows: $\exists j_0, j_1 \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}, \forall (x_1', \ldots, x_n') \in \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n$,

$$
\begin{aligned}
&\left(\left(\mathbb{P}_1^*(x_1^0) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^0) = 0 \right) \vee \left(\mathbb{P}_{j_0}^*(x_{j_0}^0) = 0 \wedge \mathbb{P}_{j_0}^*(x_{j_0}') = 0 \right) \right) \wedge \\
&\left(\left(\mathbb{P}_1^*(x_)^1) = 0 \wedge \ldots \wedge \mathbb{P}_n^*(x_n^1) = 0 \right) \vee \left(\mathbb{P}_{j_1}^*(x_{j_1}^1) = 0 \wedge \mathbb{P}_{j_1}^*(x_{j_1}') = 0 \right) \right). \quad (15)
\end{aligned}
$$

Note that in the above equation we made explicit the challenge inputs and the inputs of $\mathcal{Q}_i$. For this reason, it is enough to quantify over all $(x_1', \ldots, x_n') \in \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n$ where $\mathcal{Q}_i$ is equal to the inputs $\{x_i^{(1)}, \ldots, x_i^{(k_i)}\}$ submitted to oracle $\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)$, if $i \notin \mathcal{Q}_{\mathsf{Corr}}$. Otherwise (if $i \in \mathcal{Q}_{\mathsf{Corr}}$), $\mathcal{Q}_i$ is equal to the i-th input space $\mathcal{X}_{1,i}$. Hence, in order to be valid, A needs to satisfy the condition defined by Equation (15). This is equivalent to considering the events

below: For some $j_0, j_1 \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$,[26]

Validity$_1$:

$$\mathbb{P}_1^*(x_1^0) = 0 \land \ldots \land \mathbb{P}_n^*(x_n^0) = 0 \land \mathbb{P}_1^*(x_1^1) = 0 \land \ldots \land \mathbb{P}_n^*(x_n^1) = 0.$$

Validity$_{2,j_0,j_1}$: $\forall x'_{j_0} \in \mathcal{Q}_{j_0}, \forall x'_{j_1} \in \mathcal{Q}_{j_1}$,

$$\mathbb{P}_{j_0}^*(x_{j_0}^0) = 0 \land \mathbb{P}_{j_0}^*(x'_{j_0}) = 0 \land \mathbb{P}_{j_1}^*(x_{j_1}^1) = 0 \land \mathbb{P}_{j_1}^*(x'_{j_1}) = 0.$$

Validity$_{3,j_0}$: $\forall x'_{j_0} \in \mathcal{Q}_{j_0}$,

$$\mathbb{P}_{j_0}^*(x_{j_0}^0) = 0 \land \mathbb{P}_{j_0}^*(x'_{j_0}) = 0 \land \mathbb{P}_1^*(x_1^1) = 0 \land \ldots \land \mathbb{P}_n^*(x_n^1) = 0.$$

Validity$_{4,j_1}$: $\forall x'_{j_1} \in \mathcal{Q}_{j_1}$,

$$\mathbb{P}_1^*(x_1^0) = 0 \land \ldots \land \mathbb{P}_n^*(x_n^0) = 0 \land \mathbb{P}_{j_1}^*(x_{j_1}^1) = 0 \land \mathbb{P}_{j_1}^*(x'_{j_1}) = 0.$$

For the sake of clarity, in the rest of this proof, we use the notation $\mathbb{V}_i^{\mathsf{in}} \stackrel{\mathsf{def}}{=} \mathbb{C}_{c_i^{(n)},\mathsf{sk}_i,i}^{\mathsf{in}}$ (resp. $\mathbb{V}_i^{\mathsf{out}} \stackrel{\mathsf{def}}{=} \mathbb{C}_{c_i^{(n)},\mathsf{sk}_i,i}^{\mathsf{out}}$) where $c_i^{(n)}$, sk_i, and i will be clear from the context. Also, $[a : b]_n^+ = \{a, a+1, \ldots, n, 1, 2, \ldots, b\}$. If $1 \le a \le b \le n$, we have $[a : b]_n^+ = \{a, a+1, \ldots, b\}$.

Lemma 10. *If* PE *is CPA secure without collusions (Definition 8),* LOBF_3 *and* LOBF_4 *are secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-}\mathsf{CPA}\text{-}1\text{-}\mathsf{iPE}}(\lambda) = 1 \land |\mathcal{Q}_{\mathsf{KGen}}| = 1 \middle| \mathbf{Validity}_1 \right] - \frac{1}{2} \right| \le \mathsf{negl}(\lambda).$$

Proof. Consider the following hybrid experiments:

$\mathbf{H}_0^{b,0}(\lambda)$**:** This is exactly the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-}\mathsf{CPA}\text{-}1\text{-}\mathsf{iPE}}(\lambda)$ conditioned to the validity event **Validity$_1$** where the challenge bit is b.

$\mathbf{H}_0^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_0^{b,i-1}$, except that the challenger changes how it computes the challenger ciphertext c_i. Formally, it computes value $c_i^{(0)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ (instead of $c_i^{(0)} \leftarrow_{\$} \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_i^{\mathsf{in}}, y_i^{\mathsf{out}}))$) where $c_i^{(0)}$ is the value used to compute the challenge ciphertext $x_i = x_i^0$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{i\}$. Observe that c_i is computed by fixing $x_i = x_i^0$ (instead of $x_i = x_i^b$), i.e., the predicate input $(x_1, \ldots, x_n)$ used to compute the i-th challenge ciphertext is fixed and does not depend on the challenge bit b.

$\mathbf{H}_1^{b,0}(\lambda)$**:** Identical to $\mathbf{H}_0^{b,n}(\lambda)$.

$\mathbf{H}_1^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_1^{b,i-1}$, except that the challenger changes how it computes the challenger ciphertext c_i. Formally, the value $\widetilde{\mathbb{C}}_i^{\mathsf{in}}$ of challenge ciphertext $c_i = (\widetilde{\mathbb{C}}_i^{\mathsf{in}}, \widetilde{\mathbb{C}}_i^{\mathsf{out}})$ is simulated by the challenger using the simulator Sim_3 of the lockable obfuscation scheme LOBF_3, i.e., $\mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_i^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_i,i)|})$.

$\mathbf{H}_2^{b,0}(\lambda)$**:** Identical to $\mathbf{H}_1^{b,n}(\lambda)$.

$\mathbf{H}_2^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_2^{b,i-1}$, except that the challenger changes how it computes the challenger ciphertext c_i. Formally, the value $\widetilde{\mathbb{C}}_i^{\mathsf{out}}$ of challenge ciphertext $c_i = (\widetilde{\mathbb{C}}_i^{\mathsf{in}}, \widetilde{\mathbb{C}}_i^{\mathsf{out}})$ is simulated by the challenger using the simulator Sim_4 of the lockable obfuscation scheme LOBF_4, i.e., $\mathsf{Sim}(1^\lambda, 1^{|\mathbb{V}_i^{\mathsf{out}}|}, 1^{|m_i^b|})$.

---

[26]Since we are in the $(n-1)$-corruptions setting (i.e, $|\mathcal{Q}_{\mathsf{Corr}}| \le n-1$) such as $j_0, j_1 \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$ always exist.

Claim 27. $\mathbf{H}_0^{b,i-1}(\lambda) \approx_c \mathbf{H}_0^{b,i}(\lambda)$ *for* $i \in [n]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_0^{b,1-i}(\lambda)$ and $\mathbf{H}_0^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA security without collusions of PE. A is defined as follows:

1. Receive mpk from the challenger.

2. Compute $(\mathsf{sk}_j, \mathsf{pk}_j) \leftarrow_\$ \mathsf{KGen}_{2,j}(1^\lambda)$ and set $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{sk}_j, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$ for $j \in [n]$.

3. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_{\mathbb{P}^*}$.
   - On input $j \in [n]$ for Corr, return ek_j.
   - On input $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, return $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

4. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

5. For any $j \in [n]$, A proceeds as follows:

   Case $j < i$: Sample $(y_j^{\mathsf{in}}, y_j^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$. Compute $c_j^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

   Case $j = i$: Send the challenge $(m^0 = (y_i^{\mathsf{in}}, y_i^{\mathsf{out}}), m^1 = 0^{s_3(\lambda)+s_4(\lambda)}, x = (x_1, \ldots, x_n))$ where $(y_j^{\mathsf{in}}, y_j^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$, $x_i = x_i^b$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{i\}$. Receive the challenge ciphertext c^* from the challenger. Set $c_i^{(0)} = c^*$.

   Case $j > i$: Sample $(y_j^{\mathsf{in}}, y_j^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$. Compute $c_j^{(-1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j^{\mathsf{in}}, y_j^{\mathsf{out}}))$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

6. For every $j \in [n]$, compute $c_j^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_j^{(v-1)})$ for $v \in [n]$.

7. Compute $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_j^{\mathsf{in}}, y_j^{\mathsf{in}}, (\mathsf{sk}_j, j))$ and $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_j^{\mathsf{out}}, y_j^{\mathsf{out}}, m_j^b)$ for any $j \in [n]$.

8. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

9. Answer to the incoming oracle queries as in Item 3.

10. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_0^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_1^{b,i}(\lambda)$. Moreover, conditioned to the event $\mathbf{Validity}_1$, we know that D asks for a single decryption key $\mathsf{dk}_{\mathbb{P}^*}$ for $\mathbb{P}^*$ and $\mathbb{P}_i^*(x_i^0) = 0 \wedge \mathbb{P}_i^*(x_i^1) = 0$. Because of this, A submits a single query $\mathbb{P}^*$ to oracle KGen_1 and it is also a valid adversary for the experiment $\mathbf{G}_{\mathsf{PE},\mathsf{A}}^{\mathsf{CPA}\text{-}\mathsf{PE}}(\lambda)$ with the same advantage of D. This concludes the proof.

Claim 28. $\mathbf{H}_1^{b,i-1}(\lambda) \approx_c \mathbf{H}_1^{b,i}(\lambda)$ *for* $i \in [n]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_1^{b,1-i}(\lambda)$ and $\mathbf{H}_1^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF_3. A is defined as follows:

1. Computes $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{sk}_j, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$ for $j \in [n]$.

2. A answers to the incoming oracle queries as follows:

    - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}(\mathsf{msk}, \mathbb{P}^*)$.
    - On input $j \in [n]$ for Corr, return ek_j.
    - On input $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, return $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For every $j \in [n]$, run $c_j^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

5. For every $j \in [n]$, compute $c_j^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_j^{(v-1)})$ for $v \in [n]$.

6. For any $j \in [n]$, A proceeds as follows:

    Case $j < i$: Compute $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow_\$ \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_j, j)|})$.

    Case $j = i$: Send the challenge $(\mathbb{V}_i^{\mathsf{in}}, (\mathsf{sk}_i, i))$ to the challenger and receive $\widetilde{\mathbb{C}}_i^{\mathsf{in}}$.

    Case $j > i$: Compute $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_j^{\mathsf{in}}, y_j^{\mathsf{in}}, (\mathsf{sk}_j, j))$ where $y_j^{\mathsf{in}} \leftarrow_\$ \{0, 1\}^{s_3(\lambda)}$.

7. For every $j \in [n]$, compute $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_j^{\mathsf{out}}, y_j^{\mathsf{out}}, m_j^b)$ where $y_j^{\mathsf{out}} \leftarrow_\$ \{0, 1\}^{s_4(\lambda)}$.

8. Set $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

9. Answer to the incoming oracle queries as in Item 2.

10. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_1^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_1^{b,i}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof. $\qed$

Claim 29. $\mathbf{H}_2^{b,i-1}(\lambda) \approx_c \mathbf{H}_2^{b,i}(\lambda)$ *for $i \in [n]$.*

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_2^{b,1-i}(\lambda)$ and $\mathbf{H}_2^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF_4. A is defined as follows:

1. Computes $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{sk}_j, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$ for $j \in [n]$.

2. A answers to the incoming oracle queries as follows:

    - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}(\mathsf{msk}, \mathbb{P}^*)$.
    - On input $j \in [n]$ for Corr, return ek_j.
    - On input $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, return $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For every $j \in [n]$, compute $c_j^{(0)} \leftarrow\!\!{}_\$ \, \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_j = x_j^0$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

5. For every $j \in [n]$, run $c_j^{(v)} \leftarrow\!\!{}_\$ \, \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_j^{(v-1)})$ for $v \in [n]$.

6. For every $j \in [n]$, compute $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow\!\!{}_\$ \, \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_j, j)|})$.

7. For every $j \in [n]$, A proceeds as follows:

   Case $j < i$: Compute $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow\!\!{}_\$ \, \mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{out}}|}, 1^{|m_j^b|})$.

   Case $j = i$: Send the challenge $(\mathbb{V}_i^{\mathsf{out}}, m_i^b)$ to the challenger and receive $\widetilde{\mathbb{C}}_i^{\mathsf{out}}$.

   Case $j > i$: Compute $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow\!\!{}_\$ \, \mathsf{Obf}_4(1^\lambda, \mathbb{V}_j^{\mathsf{out}}, y_j^{\mathsf{out}}, m_j^b)$ where $y_j^{\mathsf{out}} \leftarrow\!\!{}_\$ \, \{0,1\}^{s_4(\lambda)}$.

8. Set $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ for $j \in [n]$ and send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

9. Answer to the incoming oracle queries as in Item 2.

10. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_2^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_2^{b,i}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 30. $\mathbf{H}_2^{b,n}(\lambda) \equiv \mathbf{H}_2^{1-b,n}(\lambda)$.

Proof. The distribution of these two experiments do not depend on the bit b.

By combining Claims 27 to 30 and conditioned to the event $\mathbf{Validity}_1$, we conclude that

$$\mathbf{H}_0^{b,0} \approx_c \ldots \approx_c \mathbf{H}_0^{b,n} \equiv \mathbf{H}_1^{b,0} \ldots \approx_c \mathbf{H}_1^{b,n} \equiv \mathbf{H}_2^{b,0} \approx_c \ldots \approx_c \mathbf{H}_2^{b,n} \equiv \mathbf{H}_2^{1-b,n}.$$

This concludes the proof.

Lemma 11. *Let $j_0, j_1 \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$. If PE is CPA secure without collusions (Definition 8), PKE_{2,j_0} and PKE_{2,j_1} are CPA secure (Definition 6), LOBF_3 and LOBF_4 are secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-CPA-1-iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \middle| \mathbf{Validity}_{2,j_0,j_1} \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Without loss of generality, let $q = |\mathcal{Q}_{j_0}| = |\mathcal{Q}_{j_1}| \in \mathsf{poly}(\lambda)$ (recall $j_0, j_1 \notin \mathcal{Q}_{\mathsf{Corr}}$). Consider the following hybrid experiments:

$\mathbf{H}_0^b(\lambda)$**:** This is exactly the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-CPA-1-iPE}}(\lambda)$ conditioned to the validity event $\mathbf{Validity}_{2,j_0,j_1}$ where the challenge bit is b.

$\mathbf{H}_1^b(\lambda)$**:** Same as $\mathbf{H}_0^b$, except that the challenger changes how it computes the challenge j_b-th ciphertext c_{j_b}. Specifically, it computes $c_{j_b}^{(0)} \leftarrow\!\!{}_\$ \, \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ (instead of $c_{j_b}^{(0)} \leftarrow\!\!{}_\$ \, \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_i^{\mathsf{in}}, y_i^{\mathsf{out}}))$) where the value $c_{j_b}^{(0)}$ is used to compute the challenge ciphertext, $x_i = x_i^b$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{j_b\}$.

$\mathbf{H}_2^{b,0}$**:** Identical to $\mathbf{H}_1^b(\lambda)$.

$\mathbf{H}_2^{b,i}(\lambda)$ **for** $i \in [q]$**:** Same as $\mathbf{H}_2^{b,i-1}(\lambda)$ except that the challenger changes how it answers to the first i queries for oracle $\mathsf{Enc}(\mathsf{ek}_{j_b}, \cdot, \cdot)$. Formally, on input the i'-th query (x, m) such that $i' \le i$, the challenger computes $c_{j_b}^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_{j_b} = x$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{j_b\}$. Finally, the challenger returns $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ where $c_{j_b}^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_{j_b}^{(b-1)})$ for $v \in [n]$, $(y_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$, $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{in}}, (\mathsf{sk}_{j_b}, j_b))$, and $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_{j_b}^{\mathsf{out}}, y_{j_b}^{\mathsf{out}}, m_{j_b}^b)$. Otherwise, on input the i'-th query (x, m) such that $i' > i$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_2^{b,0}$.

$\mathbf{H}_3^b(\lambda)$**:** Same as $\mathbf{H}_2^{b,q}$, except that the challenger changes how it computes the challenge j_b-th ciphertext c_{j_b}. Formally, the value $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}$ of challenge j_b-th ciphertext $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ is simulated by the challenger using the simulator Sim_3 of the lockable obfuscation scheme LOBF_3, i.e., $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}$ is computed by executing $\mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_{j_b}^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_{j_b}, j_b)|})$.

$\mathbf{H}_4^b(\lambda)$**:** Same as $\mathbf{H}_3^b$, except that the challenger changes how it computes the challenge j_b-th ciphertext c_{j_b}. Formally, the value $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}}$ of challenge j_b-th ciphertext $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ is simulated by the challenger using the simulator Sim_4 of the lockable obfuscation scheme LOBF_4, i.e., $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}}$ is computed by executing $\mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_{j_b}^{\mathsf{out}}|}, 1^{|m_{j_b}^b|})$.

$\mathbf{H}_5^{b,0}$**:** Identical to $\mathbf{H}_4^b(\lambda)$.

$\mathbf{H}_5^{b,i}(\lambda)$ **for** $i \in [q]$**:** Same as $\mathbf{H}_5^{b,i-1}(\lambda)$ except that the challenger changes how it answers to the first i queries for oracle $\mathsf{Enc}(\mathsf{ek}_{j_b}, \cdot, \cdot)$. Formally, on input the i'-th query (x, m) such that $i' \le i$, the challenger returns $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}$ is computed using the simulator Sim_3 of the lockable obfuscator scheme LOBF_3, i.e., $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}} \leftarrow_\$ \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_{j_b}^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_{j_b}, j_b)|})$. Otherwise, on input the i'-th query (x, m) such that $i' > i$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_5^{b,0}$.

$\mathbf{H}_6^{b,0}$**:** Identical to $\mathbf{H}_5^{b,q}(\lambda)$.

$\mathbf{H}_6^{b,i}(\lambda)$ **for** $i \in [q]$**:** Same as $\mathbf{H}_6^{b,i-1}(\lambda)$ except that the challenger changes how it answers to the first i queries for oracle $\mathsf{Enc}(\mathsf{ek}_{j_b}, \cdot, \cdot)$. Formally, on input the i'-th query (x, m) such that $i' \le i$, the challenger returns $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}}$ is computed using the simulator Sim_4 of the lockable obfuscator scheme LOBF_4, i.e., $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}} \leftarrow_\$ \mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_{j_b}^{\mathsf{out}}|}, 1^{|m|})$. Otherwise, on input the i'-th query (x, m) such that $i' > i$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_6^{b,0}$.

$\mathbf{H}_6^{b,1,1}$**:** Identical to $\mathbf{H}_6^{b,q}(\lambda)$.

$\mathbf{H}_{7+i}^{b,0,0}$ **for** $i \in \{0\} \cup [n-2]$**:** Same as $\mathbf{H}_{7+i-1}^{b,1,1}$ except that the challenger changes how it computes the challenge ciphertext c_r where $r = (j_b + i \mod n) + 1$. Formally, the value $c_r^{(j_b)}$ is computed as $c_r^{(j_b)} \leftarrow_\$ \mathsf{Enc}_{2,j_b}(\mathsf{pk}_{j_b}, w)$ where $w \leftarrow_\$ \mathcal{M}_{2,j_b}$.

$\mathbf{H}_{7+i}^{b,1,0}$ **for** $i \in \{0\} \cup [n-2]$**:** Same as $\mathbf{H}_{7+i}^{b,1,0}(\lambda)$ except that the challenger changes how it computes the challenge ciphertext c_r where $r = (j_b + i \mod n) + 1$. Formally, the value $\widetilde{\mathbb{C}}_v^{\mathsf{in}}$ of challenge ciphertext $c_r = (\widetilde{\mathbb{C}}_r^{\mathsf{in}}, \widetilde{\mathbb{C}}_r^{\mathsf{out}})$ is simulated by the challenger using the simulator of the lockable obfuscation scheme LOBF_3, i.e., $\widetilde{\mathbb{C}}_v^{\mathsf{in}}$ is computed by executing $\mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_r^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_r, r)|})$.

$\mathbf{H}_{7+i}^{b,1,1}$ **for** $i \in \{0\} \cup [n-2]$**:** Same as $\mathbf{H}_{7+i}^{b,1,0}(\lambda)$ except that the challenger changes how it computes the challenge ciphertext c_r where $r = (j_b + i \mod n) + 1$. Formally, the value $\widetilde{\mathbb{C}}_v^{\mathsf{out}}$ of challenge ciphertext $c_r = (\widetilde{\mathbb{C}}_r^{\mathsf{in}}, \widetilde{\mathbb{C}}_r^{\mathsf{out}})$ is simulated by the challenger using the simulator of the lockable obfuscation scheme LOBF_4, i.e., $\widetilde{\mathbb{C}}_v^{\mathsf{out}}$ is computed by executing $\mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_r^{\mathsf{out}}|}, 1^{|m_r^b|})$.

Claim 31. $\mathbf{H}_0^b(\lambda) \approx_c \mathbf{H}_1^b(\lambda)$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_0^b(\lambda)$ and $\mathbf{H}_1^b(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA security without collusions of PE. A is defined as follows:

1. Receive mpk from the challenger.

2. Compute $(\mathsf{sk}_j, \mathsf{pk}_j) \leftarrow_\$ \mathsf{KGen}_{2,j}(1^\lambda)$ and set $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{sk}_j, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$ for $j \in [n]$.

3. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_{\mathbb{P}^*}$.

   - On input $j \in [n]$ for Corr, return ek_j.

   - On input $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, return $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

4. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D. Send the challenge $(m^0 = (y_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{out}}), m^1 = 0^{s_3(\lambda) + s_4(\lambda)}, x = (x_1, \ldots, x_n))$ where $(y_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{out}}) \leftarrow_\$ \{0, 1\}^{s_3(\lambda) + s_4(\lambda)}$, $x_{j_b} = x_{j_b}^b$ and $x_j = x_j^\star$ for $j \in [n] \setminus \{j_b\}$.

5. Receive the challenge ciphertext c^* from the challenger. Set $c_{j_b}^{(0)} = c^*$.

6. For every $j \in [n] \setminus \{j_b\}$, compute $c_j^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_j^{\mathsf{in}}, y_j^{\mathsf{out}}))$ where $(y_j^{\mathsf{in}}, y_j^{\mathsf{out}}) \leftarrow_\$ \{0, 1\}^{s_3(\lambda) + s_4(\lambda)}$, $x_j = x_j^b$, and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

7. For every $j \in [n]$, compute $c_j^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_j^{(v-1)})$ for $v \in [n]$.

8. Compute $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_j^{\mathsf{in}}, y_j^{\mathsf{in}}, (\mathsf{sk}_j, j))$ and $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_j^{\mathsf{out}}, y_j^{\mathsf{out}}, m_j^b)$ for any $j \in [n]$.

9. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

10. Answer to the incoming oracle queries as in Item 3.

11. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_0^b(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_1^b(\lambda)$. Moreover, D submits a single query $\mathbb{P}^*$ to oracle KGen and, conditioned to the event $\mathbf{Validity}_{2,j_0,j_1}$, we know that $\mathbb{P}_{j_b}^*(x_{j_b}^b) = 0$. Because of this, A submits a single query to oracle KGen_1 and, it is also a valid adversary for the experiment $\mathbf{G}_{\mathsf{PE},\mathsf{A}}^{\mathsf{CPA\text{-}PE}}(\lambda)$ with the same advantage of D. This concludes the proof.

Claim 32. $\mathbf{H}_2^{b,i-1}(\lambda) \approx_c \mathbf{H}_2^{b,i}(\lambda)$ *for* $i \in [q]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_2^{b,i-1}(\lambda)$ and $\mathbf{H}_2^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA security without collusions of PE. A is defined as follows:

1. Receive mpk from the challenger.

2. Compute $(\mathsf{sk}_j, \mathsf{pk}_j) \leftarrow_\$ \mathsf{KGen}_{2,j}(1^\lambda)$ and set $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{sk}_j, \mathsf{pk}_1, \dots, \mathsf{pk}_n)$ for $j \in [n]$.

3. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_{\mathbb{P}^*}$.

   - On input $j \in [n]$ for Corr, return ek_j.

   - On input i'-th query $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, A proceeds as follows:

     Case $j \neq j_b$: Sample $(y_j^{\mathsf{in}} y_j^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$. Run $c_j^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \dots, x_n), (y_j^{\mathsf{in}}, y_j^{\mathsf{out}}))$ where $x_j = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j\}$.

     Case $j = j_b$ and $i' < i$: Sample $(y_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$. Compute $c_{j_b}^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \dots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_{j_b} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$.

     Case $j = j_b$ and $i' = i$: Sample $(y_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$. Send the challenge $(m^0 = (y_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{out}}), m^1 = 0^{s_3(\lambda)+s_4(\lambda)}, x = (x_1, \dots, x_n))$ to the challenger where $x_{j_b} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$. Receive the challenge ciphertext c^* and set $c_{j_b}^{(0)} = c^*$.

     Case $j = j_b$ and $i' > i$: Sample $(y_j^{\mathsf{in}}, y_j^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$. Compute $c_{j_b}^{(-1)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \dots, x_n), (y_j^{\mathsf{in}}, y_j^{\mathsf{out}}))$ where $x_{j_b} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$.

     Finally, return $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ where $c_j^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_j^{(v-1)})$ for $v \in [n]$, $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_j^{\mathsf{in}}, y_j^{\mathsf{in}}, (\mathsf{sk}_j, j))$ and $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_j^{\mathsf{out}}, y_j^{\mathsf{out}}, m)$.

4. Receive the challenge $((m_1^0, \dots, m_n^0), (m_1^1, \dots, m_n^1), (x_1^0, \dots, x_n^0), (x_1^1, \dots, x_n^1))$ from D.

5. Compute $c_{j_b}^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \dots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_{j_b} = x_{j_b}^b$, $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$, and $(y_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$.

6. Compute $c_{j_b}^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_{j_b}^{(v-1)})$ for $v \in [n]$.

7. Compute $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{in}}, (\mathsf{sk}_{j_b}, j_b))$ and $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_{j_b}^{\mathsf{out}}, y_{j_b}^{\mathsf{out}}, m_{j_b}^b)$.

8. For every $j \in [n] \setminus \{j_b\}$, compute $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x_j^b, m_j^b)$.

9. Send the challenge ciphertexts $(c_1, \dots, c_n)$ to D.

10. Answer to the incoming oracle queries as in Item 3.

11. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_2^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_2^{b,i}(\lambda)$. Moreover, we know that D submits a single query $\mathbb{P}^*$ to oracle KGen and, conditioned

to the event $\mathbf{Validity}_{2,j_0,j_1}$, we know that $j_b \notin \mathcal{Q}_{\mathsf{Corr}}$ and $\forall x'_{j_b} \in \mathcal{Q}_{j_b} \subset \mathcal{X}_{1,j_b}, \mathbb{P}^*_{j_b}(x'_{j_b}) = 0$. Because of this, A submits a single query to oracle KGen_1 and it is also a valid adversary for the experiment $\mathbf{G}^{\mathsf{CPA\text{-}PE}}_{\mathsf{PE},\mathsf{A}}(\lambda)$ with the same advantage of D. This concludes the proof.

Claim 33. $\mathbf{H}^{b,q}_2(\lambda) \approx_c \mathbf{H}^b_3(\lambda)$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}^{b,q}_2(\lambda)$ and $\mathbf{H}^b_3(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF_3. A is defined as follows:

1. Compute $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}(1^\lambda)$ where $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{sk}_j, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$ for $j \in [n]$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}(\mathsf{msk}, \mathbb{P}^*)$.

   - On input $j \in [n]$ for Corr, return ek_j.

   -

   - On input $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

     Case $j = j_b$: Sample $(y^{\mathsf{in}}_{j_b}, y^{\mathsf{out}}_{j_b}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$. Run $c^{(0)}_{j_b} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_{j_b} = x$, $x_{j'} = x^\star_{j'}$ for any $j' \in [n] \setminus \{j_b\}$.

     Case $j \neq j_b$: Compute $c^{(0)}_j \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y^{\mathsf{in}}_j, y^{\mathsf{out}}_j))$ where $(y^{\mathsf{in}}_j, y^{\mathsf{out}}_j) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$, $x_j = x$, $x_{j'} = x^\star_{j'}$ for any $j' \in [n] \setminus \{j\}$.

     Finally, return $c_j = (\widetilde{\mathbb{C}}^{\mathsf{in}}_j, \widetilde{\mathbb{C}}^{\mathsf{out}}_j)$ where $c^{(v)}_j \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c^{(v-1)}_j)$ for $v \in [n]$, $\widetilde{\mathbb{C}}^{\mathsf{in}}_j \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}^{\mathsf{in}}_j, y^{\mathsf{in}}_j, (\mathsf{sk}_j, j))$ and $\widetilde{\mathbb{C}}^{\mathsf{out}}_j \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}^{\mathsf{out}}_j, y^{\mathsf{out}}_j, m)$.

3. Receive the challenge $((m^0_1, \ldots, m^0_n), (m^1_1, \ldots, m^1_n), (x^0_1, \ldots, x^0_n), (x^1_1, \ldots, x^1_n))$ from D.

4. Compute $c^{(0)}_{j_b} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_{j_b} = x^b_{j_b}$ and $x_{j'} = x^\star_{j'}$ for $j' \in [n] \setminus \{j_b\}$.

5. Compute $c^{(v)}_{j_b} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c^{(v-1)}_{j_b})$ for $v \in [n]$.

6. Send the challenge $(\mathbb{V}^{\mathsf{in}}_{j_b}, (\mathsf{sk}_{j_b}, j_b))$ to the challenger and receive $\widetilde{\mathbb{C}}$. Compute $c_{j_b} = (\widetilde{\mathbb{C}}^{\mathsf{in}}_{j_b}, \widetilde{\mathbb{C}}^{\mathsf{out}}_{j_b})$ where $\widetilde{\mathbb{C}}^{\mathsf{in}}_{j_b} = \widetilde{\mathbb{C}}$, $\widetilde{\mathbb{C}}^{\mathsf{out}}_{j_b} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}^{\mathsf{out}}_{j_b}, y^{\mathsf{out}}_{j_b}, m^b_{j_b})$ and $y^{\mathsf{out}}_{j_b} \leftarrow_\$ \{0,1\}^{s_4(\lambda)}$.

7. For every $j \in [n] \setminus \{j_b\}$, compute $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x^b_j, m^b_j)$.

8. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

9. Answer to the incoming oracle queries as in Item 2.

10. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}^{b,q}_2(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}^b_3(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 34. $\mathbf{H}^{b,i-1}_4(\lambda) \approx_c \mathbf{H}^{b,i}_4(\lambda)$.

Proof. Claim 34 follows by leveraging a similar argument to that of Claim 33.

Claim 35. $\mathbf{H}_5^{b,i-1}(\lambda) \approx_c \mathbf{H}_5^{b,i}(\lambda)$ *for* $i \in [q]$.

Proof. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_5^{b,i-1}(\lambda)$ and $\mathbf{H}_5^{b,i}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF_3. A is defined as follows:

1. Compute $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}(1^\lambda)$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, forward the query $\mathbb{P}^*$ to KGen_1 and return the answer $\mathsf{dk}_{\mathbb{P}^*}$.

   - On input $j \in [n]$ for Corr, return ek_j.

   - On input i'-th query $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$ where $j \in [n]$, A proceeds as follows:

     Case $j \neq j_b$: Return $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}}) \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

     Case $j = j_b$ and $i' < i$: Sample $y_{j_b}^{\mathsf{out}} \leftarrow_\$ \{0,1\}^{s_4(\lambda)}$. Run $c_{j_b}^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_{j_b} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$. Return $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ where $c_{j_b}^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_{j_b}^{(v-1)})$ for $v \in [n]$, $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}} \leftarrow_\$ \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_{j_b}^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_{j_b}, j_b)|})$, and $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_{j_b}^{\mathsf{out}}, y_{j_b}^{\mathsf{out}}, m)$.

     Case $j = j_b$ and $i' = i$: Sample $y_{j_b}^{\mathsf{out}} \leftarrow_\$ \{0,1\}^{s_4(\lambda)}$. Run $c_{j_b}^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_{j_b} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$. Send the challenge $(\mathbb{V}_{j_b}^{\mathsf{in}}, (\mathsf{sk}_{j_b}, j_b))$ to the challenger where $c_{j_b}^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_{j_b}^{(v-1)})$ for $v \in [n]$. Receive the challenge ciphertext $\widetilde{\mathbb{C}}$ and set $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}} = \widetilde{\mathbb{C}}$. Return $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_{j_b}^{\mathsf{out}}, y_{j_b}^{\mathsf{out}}, m)$.

     Case $j = j_b$ and $i' > i$: Sample $(y_j^{\mathsf{in}}, y_j^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$. Compute $c_{j_b}^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_{j_b} = x$ and $x_{j'} = x_{j'}^\star$ for $j' \in [n] \setminus \{j_b\}$. Return $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ where $c_{j_b}^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_{j_b}^{(v-1)})$ for $v \in [n]$, $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_{j_b}^{\mathsf{in}}, y_{j_b}^{\mathsf{in}}, (\mathsf{sk}_{j_b}, j_b))$, and $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_{j_b}^{\mathsf{out}}, y_{j_b}^{\mathsf{out}}, m)$.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. Compute $c_{j_b} = (\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{in}} \leftarrow_\$ \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_{j_b}^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_{j_b}, j_b)|})$ and $\widetilde{\mathbb{C}}_{j_b}^{\mathsf{out}} \leftarrow_\$ \mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_{j_b}^{\mathsf{out}}|}, 1^{|m_{j_b}^b|})$.

5. For every $j \in [n] \setminus \{j_b\}$, compute $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x_j^b, m_j^b)$.

6. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

7. Answer to the incoming oracle queries as in Item 2.

8. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_5^{b,i-1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_5^{b,i}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 36. $\mathbf{H}_6^{b,i-1}(\lambda) \approx_c \mathbf{H}_6^{b,i}(\lambda)$ *for* $i \in [q]$.

Proof. Claim 36 follows by leveraging a similar argument to that of Claim 35.

Claim 37. $\mathbf{H}_{7+i-1}^{b,1,1}(\lambda) \approx_c \mathbf{H}_{7+i}^{b,0,0}(\lambda)$ *for* $i \in \{0\} \cup [n-2]$.

Proof. Let $r = (j_b+i \mod n)+1$. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{7+i-1}^{b,1,1}(\lambda)$ and $\mathbf{H}_{7+i}^{b,0,0}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the CPA security of PKE_{2,j_b}. A is defined as follows:

1. Compute $(\mathsf{mpk}, \mathsf{msk}) \leftarrow_\$ \mathsf{Setup}_1(1^\lambda)$ and $(\mathsf{sk}_j, \mathsf{pk}_j) \leftarrow_\$ \mathsf{KGen}_{2,j}(1^\lambda)$ for $j \in [n] \setminus \{j_b\}$.

2. Receive pk_{j_b} from the challenger.

3. Set $\mathsf{ek}_j = (\mathsf{mpk}, \mathsf{sk}_j, \mathsf{pk}_1, \ldots, \mathsf{pk}_n)$ for $j \in [n] \setminus \{j_b\}$.

4. A answers to the incoming oracle queries as follows:

    - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P}^*)$.
    - On input $j \in [n]$ for Corr, return ek_j.
    - On input $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

        Case $j = j_b$: Run $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow_\$ \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_j, j)|})$ and $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow_\$ \mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{out}}|}, 1^{|m|})$.

        Case $j \neq j_b$: Compute $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

        Finally, return c_j.

5. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

6. For every $j \in [n]$, the adversary A proceeds as follows:

    Case $j \in [j_b : r-1]_n^+$: Compute $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow_\$ \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_j, j)|})$ and $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow_\$ \mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{out}}|}, 1^{|m_j^b|})$.

    Case $j = r$: Sample $(y_r^{\mathsf{in}}, y_r^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$ and compute $c_r^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_r^{\mathsf{in}}, y_r^{\mathsf{out}}))$ where $x_r = x_r^b$, $x_{j'} = x_{j'}^\star$ for any $j' \in [n] \setminus \{r\}$. Compute $c_r^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_r^{(v-1)})$ for $v \in [j_b - 1]$. Send the challenge $(m^0 = c_r^{(v)}, m^1 = w)$ to the challenger where $w \leftarrow_\$ \mathcal{M}_{2,j_b}$. Receive the answer c^* and set $c_r^{(j_b)} = c^*$. Compute $c_r^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_r^{(v-1)})$ for $v \in [n] \setminus [j_b]$. Set $c_r = (\widetilde{\mathbb{C}}_r^{\mathsf{in}}, \widetilde{\mathbb{C}}_r^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_r^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_r^{\mathsf{in}}, y_r^{\mathsf{in}}, (\mathsf{sk}_r, r))$ and $\widetilde{\mathbb{C}}_r^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_r^{\mathsf{out}}, y_r^{\mathsf{out}}, m_r^b)$.

    Case $i < n-2$ and $j \in [r+1 : j_b - 1]_n^+$: Compute $c_j \leftarrow_\$ \mathsf{Enc}(\mathsf{ek}_j, x_j^b, m_j^b)$.

7. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

8. Answer to the incoming oracle queries as in Item 4.

9. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. This is because, by the **Validity**$_{2,j_0,j_1}$ we have that $j_b \notin \mathcal{Q}_{\mathsf{Corr}}$, i.e., A can simulate the view of D without knowing sk_{j_b} (sampled by the challenger). Moreover, if $d = 0$, A simulates $\mathbf{H}_{7+i-1}^{b,1,1}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{7+i}^{b,0,0}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 38. $\mathbf{H}_{7+i}^{b,0,0}(\lambda) \approx_c \mathbf{H}_{7+i}^{b,1,0}(\lambda)$ *for* $i \in \{0\} \cup [n-2]$.

Proof. Let $r = (j_b + i \mod n) + 1$. Suppose there exists a PPT distinguisher D that distinguishes between $\mathbf{H}_{7+i}^{b,0,0}(\lambda)$ and $\mathbf{H}_{7+i}^{b,1,0}(\lambda)$ with non-negligible probability. We build an adversary A that breaks the security of the lockable obfuscation scheme LOBF_3. A is defined as follows:

1. Compute $(\mathsf{ek}_1, \ldots, \mathsf{ek}_n, \mathsf{msk}) \leftarrow\!\!{}_\$ \mathsf{Setup}(1^\lambda)$.

2. A answers to the incoming oracle queries as follows:

   - On input $\mathbb{P}^* \in \mathcal{P}$ for KGen, return $\mathsf{dk}_{\mathbb{P}^*} \leftarrow\!\!{}_\$ \mathsf{KGen}_1(\mathsf{msk}, \mathbb{P}^*)$.
   - On input $j \in [n]$ for Corr, return ek_j.
   - On input $(x, m) \in \mathcal{X}_{1,j} \times \mathcal{M}_4$ for $\mathsf{Enc}(\mathsf{ek}_j, \cdot, \cdot)$, A proceeds as follows:

     Case $j = j_b$: Run $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow\!\!{}_\$ \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_j, j)|})$ and $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow\!\!{}_\$$
     $\mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{out}}|}, 1^{|m|})$.

     Case $j \neq j_b$: Compute $c_j \leftarrow\!\!{}_\$ \mathsf{Enc}(\mathsf{ek}_j, x, m)$.

     Finally, return c_j.

3. Receive the challenge $((m_1^0, \ldots, m_n^0), (m_1^1, \ldots, m_n^1), (x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ from D.

4. For every $j \in [n]$, the adversary A proceeds as follows:

   Case $j \in [j_b : r-1]_n^+$: Compute $c_j = (\widetilde{\mathbb{C}}_j^{\mathsf{in}}, \widetilde{\mathbb{C}}_j^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_j^{\mathsf{in}} \leftarrow\!\!{}_\$ \mathsf{Sim}_3(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{in}}|}, 1^{|(\mathsf{sk}_j, j)|})$ and
   $\widetilde{\mathbb{C}}_j^{\mathsf{out}} \leftarrow\!\!{}_\$ \mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_j^{\mathsf{out}}|}, 1^{|m_j^b|})$.

   Case $j = r$: Compute $c_r^{(v)} \leftarrow\!\!{}_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_r^{(v-1)})$ for $v \in [n] \setminus [j_b - 1]$ where $c_r^{(j_b - 1)} = w \leftarrow\!\!{}_\$ \mathcal{M}_{2,j_b}$. Send the challenge $(\mathbb{V}_r^{\mathsf{in}}, (\mathsf{sk}_r, r))$ to the challenger and receive the answer $\widetilde{\mathbb{C}}^*$. Set $c_r = (\widetilde{\mathbb{C}}_r^{\mathsf{in}}, \widetilde{\mathbb{C}}_r^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_r^{\mathsf{in}} = \widetilde{\mathbb{C}}^*$, $y_r^{\mathsf{out}} \leftarrow\!\!{}_\$ \{0,1\}^{s_4(\lambda)}$, and $\widetilde{\mathbb{C}}_r^{\mathsf{out}} \leftarrow\!\!{}_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_r^{\mathsf{out}}, y_r^{\mathsf{out}}, m_r^b)$.

   Case $i < n - 2$ and $j \in [r+1 : j_b - 1]_n^+$: Compute $c_j \leftarrow\!\!{}_\$ \mathsf{Enc}(\mathsf{ek}_j, x_j^b, m_j^b)$.

5. Send the challenge ciphertexts $(c_1, \ldots, c_n)$ to D.

6. Answer to the incoming oracle queries as in Item 2.

7. Return the output of D.

Let d be the challenge bit sampled by the challenger. The adversary A perfectly simulates the view of D. In particular, if $d = 0$, A simulates $\mathbf{H}_{7+i}^{b,0,0}(\lambda)$. On the other hand, if $d = 1$, A simulates $\mathbf{H}_{7+i}^{b,1,0}(\lambda)$. Hence, A has the same advantage of D. This concludes the proof.

Claim 39. $\mathbf{H}_{7+i}^{b,1,0}(\lambda) \approx_c \mathbf{H}_{7+i}^{b,1,1}(\lambda)$ *for $i \in \{0\} \cup [n-2]$.*

Proof. Claim 39 follows by leveraging a similar argument to that of Claim 38.

Claim 40. $\mathbf{H}_{7+n-2}^{1-b,1,1}(\lambda) \approx_c \mathbf{H}_{7+n-2}^{b,1,1}(\lambda)$.

Proof. The distribution of these two experiments do not depend on the bit b.

By combining Claims 31 to 40 and conditioned to the event $\mathbf{Validity}_{2,j_0,j_1}$, we conclude that

$$\mathbf{H}_0^b \approx_c \mathbf{H}_1^b \equiv \mathbf{H}_2^{b,0} \approx_c \ldots \approx_c \mathbf{H}_2^{b,q} \approx_c \mathbf{H}_3^b \approx_c \mathbf{H}_4^b \equiv \mathbf{H}_5^{b,0} \approx_c \ldots \approx_c \mathbf{H}_5^{b,q} \equiv$$
$$\mathbf{H}_6^{b,0} \approx_c \ldots \approx_c \mathbf{H}_6^{b,q} \equiv \mathbf{H}_6^{b,1,1} \approx_c \mathbf{H}_7^{b,0,0} \approx_c \ldots \approx_c \mathbf{H}_{7+n-2}^{b,1,1} \equiv \mathbf{H}_{7+n-2}^{1-b,1,1}.$$

This concludes the proof.

Lemma 12. *Let $j_0 \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$. If PE is CPA secure without collusions (Definition 8), PKE_{2,j_0} is CPA secure (Definition 6), LOBF_3 and LOBF_4 are secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-}\mathsf{CPA}\text{-}1\text{-}\mathsf{iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \Big| \mathbf{Validity}_{3,j_0} \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Without loss of generality, let $q = |\mathcal{Q}_{j_0}| \in \mathsf{poly}(\lambda)$ (recall $j_0 \notin \mathcal{Q}_{\mathsf{Corr}}$). Consider the hybrid experiments of Lemma 4 and Lemma 11. Formally,

- Let $\mathbf{H}_0^{1,i}(\lambda), \mathbf{H}_1^{1,i}(\lambda), \mathbf{H}_2^{1,i}(\lambda)$ for $i \in [n]$ be the hybrid of Lemma 10 (for the challenge bit $b = 1$) except that are conditioned to the event $\mathbf{Validity}_{3,j_0}$ (instead of $\mathbf{Validity}_1$)

- Let $\mathbf{H}_0^0(\lambda), \mathbf{H}_1^0(\lambda), \mathbf{H}_2^{0,t}(\lambda), \mathbf{H}_3^0(\lambda), \mathbf{H}_4^0(\lambda), \mathbf{H}_5^{0,t}(\lambda), \mathbf{H}_6^{0,t}(\lambda), \mathbf{H}_6^{0,1,1}(\lambda), \mathbf{H}_{7+j}^{0,k_1,k_2}(\lambda)$, for $i \in [n]$, $t \in [q], j \in \{0\} \cup [n-2], (k_1, k_2) \times \{0,1\}^2$, be the hybrids of Lemma 11 (for the challenge bit $b = 0$) except that are conditioned to the event $\mathbf{Validity}_{3,j_0}$ (instead of $\mathbf{Validity}_{2,j_0,j_1}$).

In addition, consider the following additional hybrids experiments:

$\mathbf{H}_{7+n-1}^{0,0}$**:** Identical to $\mathbf{H}_{7+n-2}^{0,1,1}$.

$\mathbf{H}_{7+n-1}^{0,i}$ **for $i \in [q]$:** Same as $\mathbf{H}_{7+n-1}^{0,i-1}$ except that the challenger changes how it answers to the first i queries for oracle $\mathsf{Enc}(\mathsf{ek}_{j_0}, \cdot, \cdot)$. Formally, on input the i'-th query (x, m) such that $i' \leq i$, the challenger returns $c_{j_0} = (\widetilde{\mathbb{C}}_{j_0}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_0}^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_v^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_{j_0}^{\mathsf{out}}, y_{j_0}^{\mathsf{out}}, m)$ where $y_{j_0}^{\mathsf{out}} \leftarrow_\$ \{0,1\}^{s_4(\lambda)}$. Otherwise, on input the i'-th query (x, m) such that $i' > i$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_{7+n-1}^{0,0}$.

$\mathbf{H}_{7+n}^{0,0}$**:** Identical to $\mathbf{H}_{7+n-1}^{0,q}$.

$\mathbf{H}_{7+n}^{0,i}$ **for $i \in [q]$:** Same as $\mathbf{H}_{7+n}^{0,i-1}$ except that the challenger changes how it answers to the first i queries for oracle $\mathsf{Enc}(\mathsf{ek}_{j_0}, \cdot, \cdot)$. Formally, on input the i'-th query (x, m) such that $i' \leq i$, the challenger returns $c_{j_0} = (\widetilde{\mathbb{C}}_{j_0}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_0}^{\mathsf{out}})$ where $\widetilde{\mathbb{C}}_{j_0}^{\mathsf{in}} \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_{j_0}^{\mathsf{in}}, y_{j_0}^{\mathsf{in}}, (\mathsf{sk}_{j_0}, j_0))$ where $y_{j_0}^{\mathsf{in}} \leftarrow_\$ \{0,1\}^{s_3(\lambda)}$. Otherwise, on input the i'-th query (x, m) such that $i' > i$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_{7+n}^{0,0}$.

$\mathbf{H}_{7+n+1}^{0,0}$**:** Identical to $\mathbf{H}_{7+n}^{0,q}$.

$\mathbf{H}_{7+n+1}^{0,i}$ **for $i \in [q]$:** Same as $\mathbf{H}_{7+n+1}^{0,i-1}$ except that the challenger changes how it answers to the first i queries for oracle $\mathsf{Enc}(\mathsf{ek}_{j_0}, \cdot, \cdot)$. On input the i'-th query (x, m) such that $i' \leq i$, the challenger samples $(y_{j_0}^{\mathsf{in}}, y_{j_0}^{\mathsf{out}}) \leftarrow_\$ \{0,1\}^{s_3(\lambda)+s_4(\lambda)}$ and computes $c_{j_0}^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_{j_0}^{\mathsf{in}}, y_{j_0}^{\mathsf{out}}))$ where $x_{j_0} = x$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{j_0\}$. Finally, the challenger returns $c_{j_0} = (\widetilde{\mathbb{C}}_{j_0}^{\mathsf{in}}, \widetilde{\mathbb{C}}_{j_0}^{\mathsf{out}})$ where $c_{j_0}^{(v)} \leftarrow_\$ \mathsf{Enc}_{2,v}(\mathsf{pk}_v, c_{j_0}^{(v-1)})$ for $v \in [n]$, $\widetilde{\mathbb{C}}_{j_0}^{\mathsf{in}} \leftarrow_\$ \leftarrow_\$ \mathsf{Obf}_3(1^\lambda, \mathbb{V}_{j_0}^{\mathsf{in}}, y_{j_0}^{\mathsf{in}}, (\mathsf{sk}_{j_0}, j_0))$, $\widetilde{\mathbb{C}}_v^{\mathsf{out}} \leftarrow_\$ \mathsf{Obf}_4(1^\lambda, \mathbb{V}_{j_0}^{\mathsf{out}}, y_{j_0}^{\mathsf{out}}, m)$. Otherwise, on input the i'-th query (x, m) such that $i' > i$, the challenger answers as usual, i.e., as defined in $\mathbf{H}_{7+n+1}^{0,0}$.

Claim 41. $\mathbf{H}_0^0(\lambda) \approx_c \mathbf{H}_{7+n-2}^{0,1,1}(\lambda)$.

Proof. The proof of Claim 41 is identical to that of Lemma 5 where the challenge bit is $b = 0$.

Claim 42. $\mathbf{H}_{7+n-1}^{0,i-1}(\lambda) \approx_c \mathbf{H}_{7+n-1}^{0,i}(\lambda)$ *for $i \in [q]$.*

Proof. Claim 42 follows by leveraging a similar argument to that of Claim 36.

Claim 43. $\mathbf{H}_{7+n}^{0,i-1}(\lambda) \approx_c \mathbf{H}_{7+n}^{0,i}(\lambda)$ *for* $i \in [q]$.

Proof. Claim 43 follows by leveraging a similar argument to that of Claim 35.

Claim 44. $\mathbf{H}_{7+n+1}^{0,i-1}(\lambda) \approx_c \mathbf{H}_{7+n+1}^{0,i-1}(\lambda)$ *for* $i \in [q]$.

Proof. Claim 44 follows by leveraging a similar argument to that of Claim 32.

Claim 45. $\mathbf{H}_0^{1,0}(\lambda) \approx_c \mathbf{H}_2^{1,q}(\lambda)$.

Proof. The proof of Claim 45 is identical to that of Lemma 4 where the challenge bit is $b = 1$.

Claim 46. $\mathbf{H}_{7+n+1}^{0,q}(\lambda) \equiv \mathbf{H}_2^{1,q}(\lambda)$.

Proof. Claim 46 follows by observing that experiments $\mathbf{H}_{7+n+1}^{0,q}(\lambda)$ and $\mathbf{H}_2^{1,q}(\lambda)$ are identical (and does not depend on the bit b).

By combining Claims 41 to 46 and conditioned to the event $\mathbf{Validity}_{3,j_0,}$, we conclude that

$$\mathbf{H}_0^0 \approx_c \mathbf{H}_{7+n-2}^{0,1,1} \equiv \mathbf{H}_{7+n-1}^{b,0} \approx_c \ldots \approx_c \mathbf{H}_{7+n-1}^{b,q} \equiv \mathbf{H}_{7+n}^{b,0} \approx_c \ldots \approx_c \mathbf{H}_{7+n}^{b,q}$$
$$\equiv \mathbf{H}_{7+n+1}^{b,0} \approx_c \ldots \approx_c \mathbf{H}_{7+n+1}^{b,q} \equiv \mathbf{H}_2^{1,q} \approx_c \mathbf{H}_0^{1,0}.$$

This concludes the proof.

Lemma 13. *Let* $j_1 \in [n] \setminus \mathcal{Q}_{\mathsf{Corr}}$. *If* PE *is CPA secure without collusions (Definition 8),* PKE_{2,j_1} *is CPA secure (Definition 6),* LOBF_3 *and* LOBF_4 *are secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-CPA-1-iPE}}(\lambda) = 1 \wedge |\mathcal{Q}_{\mathsf{KGen}}| = 1 \Big| \mathbf{Validity}_{4,j_1} \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Lemma 13 follows by using a symmetrical argument to that of Lemma 12.

By combining Lemmas 10 to 13 we conclude that Π is CPA secure in the $(n-1)$-corruptions setting without collusions.

E.6 Proof of Theorem 6 (CPA-2-sided security of Π)

As usual, consider the predicate space $\mathcal{P} = \{\mathbb{P}(x_1, \ldots, x_n)\}$ of Construction 3 where $\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}_1(x_1) \wedge \ldots \wedge \mathbb{P}_n(x_n)$. Let $\mathbb{P}^* \in \mathcal{P}$ be the only predicate for which the adversary will ask for the decryption key $\mathsf{dk}_{\mathbb{P}^*}$ during the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-CPA-2-iPE}}$ (recall that we prove the security of Construction 3 in the scenario without collusions, i.e., $|\mathcal{Q}_{\mathsf{KGen}}| = 1$). We can leverage a similar argument to that used to prove Theorem 5 for the CPA-2-sided case (see Appendix E.4) in order to rewrite the validity condition of $\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-CPA-2-iPE}}$ (Definition 13) as follows:

$$\text{Either } \mathbf{Validity}_1 \text{ or } \mathbf{Validity}_2$$

where

$$\mathbf{Validity}_1 : \ \forall j \in [n], \forall i_1 \in [k_1 + 1], \ldots, \forall i_n \in [k_n + 1],$$
$$\mathbb{P}^*(x_1^{(i_1,0)}, \ldots, x_{j-1}^{(i_{j-1},0)}, x_j^0, x_{j+1}^{(i_{j+1},0)}, \ldots, x_n^{(i_n,0)}) =$$
$$\mathbb{P}^*(x_1^{(i_1,1)}, \ldots, x_{j-1}^{(i_{j-1},1)}, x_j^1, x_{j+1}^{(i_{j+1},1)}, \ldots, x_n^{(i_n,1)}) = 0$$
$$\mathbf{Validity}_2 : \ \forall j \in [n], \text{Either } \mathbb{P}_j^*(x_j^0) = \mathbb{P}_j^*(x_j^1) = 0 \text{ or } \mathbb{P}_j^*(x_j^0) = \mathbb{P}_j^*(x_j^1) \wedge m_j^0 = m_j^1$$

for $\mathcal{Q}_i^b = \{x_i^{(1,b)}, \ldots, x_i^{(k_i,b)}, x_i^{(k_i+1,b)} = x_i^b\}$ for $i \in [n]$, $b \in \{0, 1\}$ as defined in Definition 13. Recall that, if $i \notin \mathcal{Q}_{\mathsf{Corr}}$, then $\mathcal{Q}_i^b$ is the ordered list composed of the inputs submitted to the oracle $\mathsf{Enc}(\mathsf{ek}_i, \cdot, \cdot)$ and the challenge input x_i^b. Otherwise, if $i \in \mathcal{Q}_{\mathsf{Corr}}$, then $\mathcal{Q}_i^b$ is equal to the i-th input space $\mathcal{X}_{1,i}$ that, in turn, contains also the challenge input x_i^b. Hence, the CPA-2-sided security of Construction 3 follows by proving the following lemmas.

Lemma 14. *If* PE *is CPA secure without collusions (Definition 8),* SKE *is CPA secure (Definition 4), and* LOBF *is secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-}\mathsf{CPA}\text{-}2\text{-}\mathsf{iPE}}(\lambda) = 1 \land |\mathcal{Q}_{\mathsf{KGen}}| = 1 \Big| \mathbf{Validity}_1 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Note that $\mathbf{Validity}_1$ is equivalent to the validity condition of CPA-1-sided security. Hence, the lemma follows by leveraging an identical argument to that of the CPA-1-sided case (Appendix E.5). ∎

Lemma 15. *If* PE *is CPA-2-sided secure without collusions (Definition 9) and* LOBF *is secure (Definition 2), then*

$$\left| \mathbb{P}\left[\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-}\mathsf{CPA}\text{-}2\text{-}\mathsf{iPE}}(\lambda) = 1 \land |\mathcal{Q}_{\mathsf{KGen}}| = 1 \Big| \mathbf{Validity}_2 \right] - \frac{1}{2} \right| \leq \mathsf{negl}(\lambda).$$

Proof. Let $\mathbb{P}^* \in \mathcal{Q}_{\mathsf{KGen}}$ and $((x_1^0, \ldots, x_n^0), (x_1^1, \ldots, x_n^1))$ be the predicate submitted to the oracle KGen and the challenge inputs chosen by the adversary, respectively. Despite $\mathbb{P}^*$ is chosen adaptively, we assume that the values $\{z_i\}_{i \in [n]}$ such that $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = z_i$ are known before the challenge phase. Indeed, $\{z_i\}_{i \in [n]}$ can be guessed with non-negligible probability since $n = O(1)$.

Consider the following hybrid experiments:

$\mathbf{H}_0^{b,0}(\lambda)$**:** This is exactly the experiment $\mathbf{G}_{\Pi,\mathsf{A}}^{(n-1)\text{-}\mathsf{CPA}\text{-}2\text{-}\mathsf{iPE}}(\lambda)$ conditioned to the event $\mathbf{Validity}_2$ where the challenge bit is b.

$\mathbf{H}_0^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_0^{b,i-1}$, except that the challenger changes how it computes the challenge ciphertext c_i with respect to z_i. If $z_i = 0$ (i.e., $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 0$), the value $c_i^{(0)}$ is computed as $c_i^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), 0^{s_3(\lambda)+s_4(\lambda)})$ where $x_i = x_i^0$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{i\}$. Otherwise, if $z_i = 1$ (i.e., $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 1$), the value $c_i^{(0)}$ is computed as $c_i^{(0)} \leftarrow_\$ \mathsf{Enc}_1(\mathsf{mpk}, (x_1, \ldots, x_n), (y_i^{\mathsf{in}}, y_1^{\mathsf{out}}))$ where $(y_i^{\mathsf{in}}, y_i^{\mathsf{out}}) \leftarrow_\$ \{0, 1\}^{s_3(\lambda)+s_4(\lambda)}$, $x_i = x_i^0$, and $x_j = x_j^\star$ for $j \in [n] \setminus \{i\}$. Observe that $c_i^{(0)}$ is computed by fixing $x_i = x_i^0$ (instead of $x_i = x_i^b$), i.e., the input $(x_1, \ldots, x_n)$ used to compute the i-th challenge ciphertext is fixed and does not depend on the challenge bit b.

$\mathbf{H}_1^{b,0}(\lambda)$**:** Identical to $\mathbf{H}_1^{b,n}(\lambda)$.

$\mathbf{H}_1^{b,i}(\lambda)$ **for** $i \in [n]$**:** Same as $\mathbf{H}_1^{b,i-1}$, except that the challenger changes how it computes the challenger ciphertext c_i with respect to z_i. If $z_i = 0$ (i.e., $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 0$), the value $\widetilde{\mathbb{C}}_i^{\mathsf{out}}$ of challenge ciphertext $c_i = (\widetilde{\mathbb{C}}_i^{\mathsf{in}}, \widetilde{\mathbb{C}}_i^{\mathsf{out}})$ is simulated by the challenger using the simulator of the lockable obfuscation scheme LOBF_4, i.e., $\widetilde{\mathbb{C}}_i^{\mathsf{out}} \leftarrow_\$ \mathsf{Sim}_4(1^\lambda, 1^{|\mathbb{V}_i^{\mathsf{out}}|}, 1^{|m_i^b|})$ where $\mathbb{V}_i^{\mathsf{out}} = \mathbb{C}_{c_i^{(n)}, \mathsf{sk}_i, i}^{\mathsf{out}}$. Otherwise, if $z_i = 1$ (i.e., $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = 1$), the value $\widetilde{\mathbb{C}}_i^{\mathsf{out}}$ is computed as in $\mathbf{H}_1^{b,0}(\lambda)$.

We can prove that the indistinguishability of the above hybrids by leveraging similar techniques to that of Appendices E.4 and E.5.

Claim 47. $\mathbf{H}_0^{b,i-1}(\lambda) \approx_c \mathbf{H}_0^{b,i}(\lambda)$ *for* $i \in [n]$.

Proof. Note that the values $\{z_i\}_{i\in[n]}$ (i.e., $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = z_i$), can be correctly guessed with non-negligible probability since $n = O(1)$. Conditioned to the above, the claim follows from the CPA-2-sided security of PE.

Claim 48. $\mathbf{H}_1^{b,i-1}(\lambda) \approx_c \mathbf{H}_1^{b,i}(\lambda)$ *for* $i \in [n]$.

Proof. As usual, the values $\{z_i\}_{i\in[n]}$ (i.e., $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = z_i$), can be correctly guessed with non-negligible probability since $n = O(1)$. Conditioned to the above, the claim follows from the security of the lockable obfuscation scheme LOBF_4.

Claim 49. $\mathbf{H}_1^{b,n}(\lambda) \equiv \mathbf{H}_1^{1-b,n}(\lambda)$.

Proof. The claim follows by leveraging the validity condition $\mathbf{Validity}_2$ and observing that the values $\{z_i\}_{i\in[n]}$ (i.e., $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = z_i$), can be correctly guessed with non-negligible probability since $n = O(1)$. Conditioned to the above, for every $i \in [n]$, if $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = z_i = 0$ we have that the j-th challenge ciphertext c_j does not depend on the bit b. On the other hand, if $\mathbb{P}_i^*(x_i^0) = \mathbb{P}_i^*(x_i^1) = z_i = 1$, we have that the j-th challenge ciphertext c_j depends on either m_j^0 or m_j^1. However, by the validity condition $\mathbf{Validity}_2$ we have that $m_j^0 = m_j^1$. Hence, $\mathbf{H}_1^{b,n}(\lambda)$ and $\mathbf{H}_1^{1-b,n}(\lambda)$ are identically distributed. This concludes the proof.

By combining Claims 47 to 49 and conditioned to the event $\mathbf{Validity}_2$, we conclude that $\mathbf{H}_0^{0,0} \approx_c \ldots \approx_c \mathbf{H}_0^{0,n} \equiv \mathbf{H}_1^{0,0} \approx_c \ldots \approx_c \mathbf{H}_1^{0,n} \equiv \mathbf{H}_1^{1,n}$. This concludes the proof.

By leveraging Lemmas 14 and 15, we conclude that Π of Construction 3 is CPA-2-sided secure.