ON THE SCHOLZ CONJECTURE ON ADDITION CHAINS

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Abstract. Applying the pothole method on the factors of numbers of the form $2^n - 1$, we prove the stronger inequality

$$\iota(2^n - 1) \leq n + 1 - \sum_{j=1}^{\lfloor \frac{\log n}{\log 2} \rfloor} \xi(n,j) + 3\lfloor \frac{\log n}{\log 2} \rfloor$$

for all $n \in \mathbb{N}$ with $n \geq 64$ for $0 \leq \xi(n,j) < 1$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$. This inequality is stronger than

$$\iota(r) < \frac{\log r}{\log 2}(1 + \frac{1}{\log \log r} + \frac{2 \log 2}{(\log r)^{1-\log 2}})$$

in the case $r = 2^n - 1$ but slightly weaker than the conjectured inequality

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

1. Introduction

An addition chain producing $n \geq 3$, roughly speaking, is a sequence of numbers of the form $1, 2, s_3, s_4, \ldots, s_{k-1}, s_k = n$ where each term is the sum of two earlier terms- not necessarily distinct - in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The length of the chain is determined by the number of entries in the sequence excluding the mandatory first term 1, since it is the only term which cannot be expressed as the sum of two previous terms in the sequence. There are numerous addition chains that result in a fixed number n; In other words, it is always possible to construct as many addition chains producing a fixed number positive integer n as n grows in magnitude. The shortest among these possible chains producing n is regarded as the optimal or the shortest addition chain producing n. There is currently no efficient method for getting the shortest addition yielding a given number, thus reducing an addition chain might be a difficult task, thereby making addition chain theory a fascinating subject to study. By letting $\iota(n)$ denotes the length of the shortest addition chain producing n, then Arnold Scholz conjectured and alfred Braurer proved the following inequalities

Theorem 1.1 (Braurer). *The inequality*

$$m + 1 \leq \iota(n) \leq 2m$$

for $2^m + 1 \leq n \leq 2^{m+1}$ *holds for* $m \geq 1$.

Date: July 8, 2023.

2010 *Mathematics Subject Classification.* Primary 11P83, 11B50 ; Secondary 11B75, 11B30.

Key words and phrases. sub-addition chain; determiners; regulators; length; generators; partition; complete.

Conjecture 1.1 (Scholz)**.** The inequality

$$\iota(2^n-1)\leq n-1+\iota(n)$$

holds for all $n\geq 2$.

It has been shown computationally by Neill Clift, that the conjecture holds for all $n\leq 5784688$ and in fact it is an equality for all exponents $n\leq 64$ [2]. Alfred Brauer proved the Scholz conjecture for the star addition chain, a special type of addition chain where each term in the sequence obtained by summing uses the immediately subsequent number in the chain. By denoting with $\iota^*(n)$ as the length of the shortest star addition chain producing n, it is shown that (See [1])

Theorem 1.2. *The inequality*

$$\iota^*(2^n-1)\leq n-1+\iota^*(n)$$

holds for all $n\geq 2$.

In relation to Conjecture 1.1, Arnold Scholz postulated that Conjecture 1.1 can be improved in general. In particular, Alfred Braurer [1] proved the inequality

$$\iota(n)<\frac{\log n}{\log 2}(1+\frac{1}{\log\log n}+\frac{2\log 2}{(\log n)^{1-\log 2}})$$

for $2^m\leq n<2^{m+1}$ for all sufficiently large n.
Quite a particular special cases of the conjecture has also be studied by many authors in the past. For instance, it is shown in [4] that the scholz conjecture holds for all numbers of the form 2^n-1 with $n=2^q$ and $n=2^s(2^q+1)$ for $s,q\geq 0$. If we let $\nu(n)$ denotes the number of $1's$ in the binary expansion of n for $m=2^n-1$, then it is shown in [3] that the Scholz conjecture holds in the case $\nu(n)=5$.

In this paper, we combine the factor method and the newly introduced "fill in the pothole" method to study the shortest or the optimal addition chains producing numbers of the form 2^n-1 and the Scholz conjecture. Given any number of the form 2^n-1, we obtain the general decomposition

$$2^n-1=(2^{\frac{n-(1-(-1)^n)\frac{1}{2}}{2}}-1)(2^{\frac{n-(1-(-1)^n\frac{1}{2})}{2}}+1)+\frac{(1-(-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}})$$

which eventually yield the following decomposition $2^n-1=(2^{\frac{n}{2}}-1)(2^{\frac{n}{2}}+1)$ in the case $n\equiv 0 \pmod 2$ and

$$2^n-1=(2^{\frac{n-1}{2}}-1)(2^{\frac{n-1}{2}}+1)+2^{n-1}$$

in the case $n\equiv 1 \pmod 2$. We iterate this decomposition up to a certain desired frequency and apply the factor method on all the factors obtained from this decomposition. We then apply the pothole method to obtain a bound for the shortest addition chain producing the only factor of form 2^v-1. The length of the shortest addition chains of numbers of the form 2^v+1 is easy to construct, by first constructing the shortest addition chain producing 2^v, adding the first term of the chain to the last term and adjoining to the chain. That is, the chain

$$1,2,2^2,\cdots,2^v,2^v+1$$

is the shortest addition chain producing 2^v+1 of length $\iota(2^v+1)=\iota(2^v)+1=v+1$. We combine the method of **filling the potholes** and the factor method to prove the improved inequality

Theorem 1.3.

$$\iota(2^n-1)\leq n+1-\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\xi(n,j)+3\lfloor \frac{\log n}{\log 2}\rfloor$$

for all $n\in \mathbb{N}$ with $n\geq 64$ for $0\leq \xi(n,j)<1$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

1.1. **Summary sketch and idea of proof.** In this section we describe in a somewhat intuitive fashion the mode of operation of the method of **filling the potholes**, which is employed to obtain our upper bound. We lay them down chronologically as follows.

- We first construct a complete sub-addition chain producing 2^n-1. For technical reasons which will become clear later, we stop the chain prematurely at 2^{n-1}.
- We extend this addition chain by a length of logarithm order.
- This extension has missing terms to qualify as addition chain producing 2^n-1. We fill in the missing terms thereby obtaining what one might refer to as spoof addition chain producing 2^n-1.
- Creating this spoof addition chain comes at a cost. The remaining step will be to cover the cost and render an account to obtain the upper bound.

2. Sub-addition chains

In this section we introduce the notion of sub-addition chains.

Definition 2.1. Let $n\geq 3$, then by the addition chain of length $k-1$ producing n, we mean the sequence of positive integers

$$1,2,\ldots,s_{k-1},s_k$$

where each term s_j $(j\geq 3)$ in the sequence is the sum of two earlier terms in the sequence, with the corresponding sequence of partition

$$2=1+1,\ldots,s_{k-1}=a_{k-1}+r_{k-1},s_k=a_k+r_k=n$$

with $a_{i+1}=a_i+r_i$ and $a_{i+1}=s_i$ for $2\leq i\leq k$. We call the partition a_i+r_i the i^{th} **generator** of the chain for $2\leq i\leq k$. We call a_i the **determiners** and r_i the **regulator** of the i^{th} generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2\leq i\leq k$. The **determiners** are the terms produced by summing of previous terms, whereas the **regulators** are chosen from previous terms in the sequence.

Definition 2.2. Let the sequence $1,2,\ldots,s_{k-1},s_k=n$ be an addition chain producing n with the corresponding sequence of partition

$$2=1+1,\ldots,s_{k-1}=a_{k-1}+r_{k-1},s_k=a_k+r_k=n.$$

Then, we call the sub-sequence (s_{j_m}) for $1\leq j\leq k$ and $1\leq m\leq t\leq k$ a **sub-addition** chain of the addition chain producing n. We say it is **complete** sub-addition chain of the addition chain producing n if it contains exactly the first t terms of the addition chain. Otherwise we say it is an **incomplete** sub-addition chain.

2.1. **Addition chains of numbers of special forms and Main result.** In this section, we prove an explicit upper bound for the length of the shortest addition chain producing numbers of the form 2^n-1. We begin with the following important but fundamental result.

Lemma 2.3. *Let $\iota(n)$ denotes the length of the shortest addition chain producing n. Then we have the inequality*

$$\lfloor \frac{\log n}{\log 2}\rfloor \leq \iota(n).$$

Proof. The proof of this Lemma can be found in [1]. □

Lemma 2.4. *Let $\iota(n)$ denotes the length of the shortest addition chain producing n. If $a,b\in\mathbb{N}$ then*

$$\iota(ab)\leq \iota(a)+\iota(b).$$

Proof. The proof of this Lemma can be found in [1]. □

Theorem 2.5. *The inequality*

$$\iota(2^n-1)\leq n+1-\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\xi(n,j)+3\lfloor\frac{\log n}{\log 2}\rfloor$$

holds for all $n\in\mathbb{N}$ with $n\geq 64$ for $0\leq \xi(n,j)<1$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

Proof. First, we consider the number 2^n-1 and obtain the decomposition

$$2^n-1=(2^{\frac{n-(1-(-1)^n)\frac{1}{2}}{2}}-1)(2^{\frac{n-(1-(-1)^n\frac{1}{2})}{2}}+1)+\frac{(1-(-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}}).$$

It is easy to see that we can recover the general factorization of 2^n-1 from this identity according to the parity of the exponent n. In particular, if $n\equiv 0 \pmod 2$, then we have

$$2^n-1=(2^{\frac{n}{2}}-1)(2^{\frac{n}{2}}+1)$$

and

$$2^n-1=(2^{\frac{n-1}{2}}-1)(2^{\frac{n-1}{2}}+1)+2^{n-1}$$

if $n\equiv 1 \pmod 2$. By combining both cases, we obtain the inequality

$$\iota(2^n-1)\leq \iota((2^{\frac{n-(1-(-1)^n)\frac{1}{2}}{2}}-1)(2^{\frac{n-(1-(-1)^n\frac{1}{2})}{2}}+1))+2$$

obtained by constructing an addition chain producing $2^{n-1}-1$, adding $2^{n-1}-1$ to $2^{n-1}-1$, adding 1 and adjoining the result in the case $n\equiv 1 \pmod 2$. Applying Lemma 2.4, we obtain further the inequality

$$\iota(2^n-1)\leq \iota(2^{\frac{n-(1-(-1)^n)\frac{1}{2}}{2}}-1)+\iota(2^{\frac{n-(1-(-1)^n)\frac{1}{2}}{2}}+1)+2 \tag{2.1}$$

Again let us set $\frac{n-(1-(-1)^n)\frac{1}{2}}{2}=k$ in (2.1), then we obtain the general decomposition

$$2^k-1=(2^{\frac{k-(1-(-1)^k)\frac{1}{2}}{2}}-1)(2^{\frac{k-(1-(-1)^k\frac{1}{2})}{2}}+1)+\frac{(1-(-1)^k)}{2}(2^{k-(1-(-1)^k)\frac{1}{2}}).$$

It is easy to see that we can recover the general factorization of 2^k-1 from this identity according to the parity of the exponent k. In particular, if $k\equiv 0 \pmod 2$, then we have

$$2^k-1=(2^{\frac{k}{2}}-1)(2^{\frac{k}{2}}+1)$$

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$$2^k-1=(2^{\frac{k-1}{2}}-1)(2^{\frac{k-1}{2}}+1)+2^{k-1}$$

if $k\equiv 1 \pmod 2$. By combining both cases, we obtain the inequality

$$\iota(2^k-1)\leq \iota((2^{\frac{k-(1-(-1)^k)\frac{1}{2}}{2}}-1)(2^{\frac{k-(1-(-1)^k\frac{1}{2})}{2}}+1))+2$$

obtained by constructing an addition chain producing $2^{k-1}-1$, adding $2^{k-1}-1$ to $2^{k-1}-1$, adding 1 and adjoining the result in the case $k\equiv 1 \pmod 2$. Applying Lemma 2.4, we obtain further the inequality

$$\iota(2^k-1)\leq \iota(2^{\frac{k-(1-(-1)^k)\frac{1}{2}}{2}}-1)+\iota(2^{\frac{k-(1-(-1)^k)\frac{1}{2}}{2}}+1)+2$$

$$=\iota(2^{\frac{n}{4}-(1-(-1)^n)\frac{1}{8}-(1-(-1)^k)\frac{1}{4}}-1)+\iota(2^{\frac{n}{4}-(1-(-1)^n)\frac{1}{8}-(1-(-1)^k)\frac{1}{4}}+1)+2 \tag{2.2}$$

so that by inserting (2.2) into (2.1), we obtain the inequality

$$\iota(2^n-1)\leq \iota(2^{\frac{n}{4}-(1-(-1)^n)\frac{1}{8}-(1-(-1)^k)\frac{1}{4}}-1)+\iota(2^{\frac{n}{4}-(1-(-1)^n)\frac{1}{8}-(1-(-1)^k)\frac{1}{4}}+1)+2$$

$$+\,\iota(2^{\frac{n-(1-(-1)^n)\frac{1}{2}}{2}}+1)+2. \tag{2.3}$$

Next we iterate the factorization up to frequency s to obtain

$$\iota(2^n-1)\leq \iota(2^{\frac{n-(1-(-1)^n)\frac{1}{2}}{2}}+1)+2+\iota(2^{\frac{n}{4}-(1-(-1)^n)\frac{1}{8}-(1-(-1)^k)\frac{1}{4}}+1)+2$$

$$++\cdots+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}+1)+2 \tag{2.4}$$

where $0\leq \xi(n,s)<1$ for an integer $2\leq s:=s(n)$ fixed to be chosen later. For instance,

$$\xi(n,1)=(1-(-1)^n)\frac{1}{4}<1$$

and

$$\xi(n,2)=(1-(-1)^n)\frac{1}{8}+(1-(-1)^k)\frac{1}{4}<1$$

with

$$k:=\frac{n-(1-(-1)^n)\frac{1}{2}}{2}$$

and so on. Indeed the function $\xi(n,s)$ for values of $s\geq 3$ can be read from exponents of the terms arising from the iteration process. It follows from (2.4) the inequality

$$\iota(2^n-1)\leq \sum_{v=1}^{s}\frac{n}{2^v}+3s-\theta(n,s)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1)$$

$$=n(1-\frac{1}{2^s})+3s-\theta(n,s)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1) \tag{2.5}$$

for some $0\leq \theta(n,s):=\sum\limits_{j=1}^{s}\xi(n,j)$ and $2\leq s:=s(n)$ fixed, an integer to be chosen later. It is worth noting that

$$\theta(n,s):=\sum_{j=1}^{s}\xi(n,j)=0$$

if $n=2^r$ for some $r\in\mathbb{N}$, since $\xi(n,j)=0$ for each $1\leq j\leq s$ for all n which are powers of 2. It is also important to note that the $2s$ term is obtained by noting that there are at most s terms with odd exponents under the iteration process and each term with odd exponent contributes 2, and the other s term comes from summing 1 with frequency s finding the total length of the short addition chains producing numbers of the form 2^v+1. Now, we set $k=\frac{n}{2^s}-\xi(n,s)$ and construct the addition chain producing 2^k as $1,2,2^2,\ldots,2^{k-1},2^k$ with corresponding sequence of partition

$$2=1+1,2+2=2^2,2^2+2^2=2^3\ldots,2^{k-1}=2^{k-2}+2^{k-2},2^k=2^{k-1}+2^{k-1}$$

with $a_i=2^{i-2}=r_i$ for $2\leq i\leq k+1$, where a_i and r_i denotes the determiner and the regulator of the i^{th} generator of the chain. Let us consider only the complete sub-addition chain

$$2=1+1,2+2=2^2,2^2+2^2=2^3,\ldots,2^{k-1}=2^{k-2}+2^{k-2}.$$

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor},2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor},\ldots,2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor}+\cdots+2^1.$$

Since $\xi(n,s)=0$ if $n=2^r$ and $0\leq\xi(n,s)<1$ if $n\neq 2^r$, we note that the adjoined sequence contributes at most

$$\lfloor\frac{\log k}{\log 2}\rfloor=\lfloor\frac{\log(\frac{n}{2^s}-\xi(n,s))}{\log 2}\rfloor=\lfloor\frac{\log n-s\log 2}{\log 2}\rfloor=\lfloor\frac{\log n}{\log 2}\rfloor-s$$

terms to the original complete sub-addition chain, where the upper bound follows by virtue of Lemma 2.3. Since the inequality holds

$$2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor}+\cdots+2^1<\sum_{i=1}^{k-1}2^i$$
$$=2^k-2$$

we insert terms into the sum

$$2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor}+\cdots+2^1 \tag{2.6}$$

so that we have

$$\sum_{i=1}^{k-1}2^i=2^k-2.$$

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert $2^{k-2}+2^{k-3}+\cdots+2^{\lfloor\frac{k-1}{2}\rfloor+1}$ into (2.6) and this comes at the cost of adjoining

$$k-2-\lfloor\frac{k-1}{2}\rfloor$$

terms to the term in (2.6). The last term of the adjoined sequence is given by

$$2^{k-1}+(2^{k-2}+2^{k-3}+\cdots+2^{\lfloor\frac{k-1}{2}\rfloor+1})+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor}+\cdots+2^1. \tag{2.7}$$

Again we have to insert $2^{\lfloor\frac{k-1}{2}\rfloor-1}+\cdots+2^{\lfloor\frac{k-1}{2^2}\rfloor+1}$ into (2.7) and this comes at the cost of adjoining

$$\lfloor\frac{k-1}{2}\rfloor-\lfloor\frac{k-1}{2^2}\rfloor-1$$

terms to the term in (2.7). The last term of the adjoined sequence is given by

$$2^{k-1}+(2^{k-2}+2^{k-3}+\cdots+2^{\lfloor\frac{k-1}{2}\rfloor+1})+2^{\lfloor\frac{k-1}{2}\rfloor}+(2^{\lfloor\frac{k-1}{2}\rfloor-1}+\cdots+2^{\lfloor\frac{k-1}{2^2}\rfloor+1})+2^{\lfloor\frac{k-1}{2^2}\rfloor}+\cdots+2^1. \tag{2.8}$$

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (2.8) and this comes at the cost of adjoining

$$\lfloor\frac{k-1}{2^j}\rfloor-\lfloor\frac{k-1}{2^{j+1}}\rfloor-1$$

terms to the term in (2.8) for $j\leq\lfloor\frac{\log n}{\log 2}\rfloor-s$, since we are filling in at most $\lfloor\frac{\log k}{\log 2}\rfloor$ blocks with $k=\frac{n}{2^s}-\xi(n,s)$. It follows that the contribution of these new terms is at most

$$k-1-\left\lfloor\frac{k-1}{2^{\lfloor\frac{\log k}{\log 2}\rfloor}}\right\rfloor-\lfloor\frac{\log k}{\log 2}\rfloor \tag{2.9}$$

obtained by adding the numbers in the chain

$$k-1-\lfloor\frac{k-1}{2}\rfloor-1$$

$$\lfloor\frac{k-1}{2}\rfloor-\lfloor\frac{k-1}{2^2}\rfloor-1$$

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$$\lfloor\frac{k-1}{2^{\lfloor\frac{\log k}{\log 2}\rfloor}}\rfloor-\lfloor\frac{k-1}{2^{\lfloor\frac{\log k}{\log 2}\rfloor+1}}\rfloor-1.$$

By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain producing 2^k-1 with $k=\frac{n}{2^s}-\xi(n,s)$ is

$$\begin{aligned}\delta(2^k-1)&\leq k+k-1-\left\lfloor\frac{k-1}{2^{\lfloor\frac{\log k}{\log 2}\rfloor+1}}\right\rfloor-\lfloor\frac{\log k}{\log 2}\rfloor+\lfloor\frac{\log n}{\log 2}\rfloor-s\\&\leq\frac{n}{2^{s-1}}-1-\left\lfloor\frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor\frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor-\lfloor\frac{\log n}{\log 2}\rfloor+s+\lfloor\frac{\log n}{\log 2}\rfloor-s\\&=\frac{n}{2^{s-1}}-1-\left\lfloor\frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor\frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor.\end{aligned} \tag{2.10}$$

By plugging the inequality (2.10) into the inequalities in (2.5) and noting that $\iota(\cdot)\leq\delta(\cdot)$, we obtain the inequality

$$\begin{aligned}\iota(2^n-1)&\leq\sum_{v=1}^{s}\frac{n}{2^v}+3s-\theta(n,s)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1)\\&=n(1-\frac{1}{2^s})+\frac{n}{2^{s-1}}-1+3s-\theta(n,s)-\left\lfloor\frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor\frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor\\&=n-1+\frac{n}{2^s}+3s-\theta(n,s)-\left\lfloor\frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor\frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor.\end{aligned} \tag{2.11}$$

By taking $2 \leq s := s(n)$ such that $s = \lfloor \frac{\log n}{\log 2} \rfloor$ then

$$\left\lfloor \frac{\frac{n}{2^s} - \xi(n,s) - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor + 1 - s}} \right\rfloor = 0$$

and we obtained further the inequality

$$\iota(2^n - 1) \leq n - 1 - \theta(n, \lfloor \frac{\log n}{\log 2} \rfloor) + 2 + 3\lfloor \frac{\log n}{\log 2} \rfloor$$

for $\theta(n, \lfloor \frac{\log n}{\log 2} \rfloor) := \sum_{j=1}^{\lfloor \frac{\log n}{\log 2} \rfloor} \xi(n,j)$ with $n > 64$ and the claimed inequality follows as a consequence. □

[1].

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