A quantum algorithm for semidirect discrete logarithm problem on elliptic curves

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Abstract

Shor's algorithm efficiently solves the discrete logarithm problem (DLP) by taking advantage of the commutativity structure of the group underlying the problem. To counter Shor's algorithm, Horan *et al.* propose a DLP analogue in the semidirect product semigroup $G \rtimes \mathrm{End}(G)$, given a (semi)group G, to construct a quantum-resistant Diffie-Hellman key exchange based on it. For general (semi)groups, the semidirect product discrete logarithm problem (SDLP) can be reduced to the hidden shift problem where Kuperberg's subexponential quantum algorithm is applicable. In this paper, we consider a specific case where G is an elliptic curve over a finite field and we show that SDLP on elliptic curves can be solved efficiently using an adaptation of Shor's algorithm for the standard elliptic curve discrete logarithm problem (ECDLP). The key implication of the result is that one should not use elliptic curves as the platforms for the semidirect product key exchange.

Keywords: Quantum algorithm, Semidirect discrete logarithm problem, Elliptic curve.

1 Introduction

The presumed difficulty of computing discrete logarithm problem (DLP) in certain groups is essential for the security of Diffie-Hellman key exchange which is the basis for a number of communication protocols deployed today. One of the modern choice is a group of rational points of an elliptic curve defined over a finite field, which is the centre of elliptic curve cryptography, one of the most popular public key cryptography today. However, since the invention of Shor's algorithm [Sho94], the problem of computing discrete logarithm can be solved efficiently in the domain of quantum computing.

A massive efforts has been done in order to construct a DLP analogue that enables Diffie-Hellman key exchange in which Shor's algorithm is not applicable. Since Shor's algorithm takes advantage of the group structure underlying the problem, a DLP analoge in the framework of commutative group actions has been proposed. It is called the *vectorization problem*. The framework originally appears in [Cou06] and it becomes a centre of isogeny-based cryptography, CSIDH [CLM+18] for example.

Another natural approach which is worth consideration to escape from the quantum attack is a DLP analogue in non-commutative groups. It is natural in a sense that Shor's algorithm crucially depends on the commutativity of the underlying groups. A promising proposal for this direction is to use semidirect product groups. The proposal firstly appears in its full generality in [HKKS13].

Specifically, let G be a group and $\mathrm{Aut}(G)$ be the group of automorphisms of G. Then we have the holomorph of G as the semidirect product $G \rtimes \mathrm{Aut}(G)$ where the multiplication is defined by $(g, \phi)(h, \psi) = (g\phi(h), \phi\psi)$. Moreover, we have the formula for exponentiation

$$(g, \phi)^n = \left(\prod_{i=0}^{n-1} \phi^{n-(i+1)}(g), \phi^n \right).$$

This leads us to a discrete logarithm analogue in the semidirect product group defined as follows.

Problem 1 (Semidirect discrete logarithm problem)**.** *Given $g \in G, \phi \in \mathrm{Aut}(G)$, and $A = \prod_{i=0}^{n-1} \phi^{n-(i+1)}(g)$ for some integer n. Find n.*

SDLP is interesting as it allows us to perform a Diffie-Hellman key exchange procedure. Suppose two parties, Alice and Bob, agree on a public group G, an element $g \in G$, and an automorphism $\phi \in \mathrm{Aut}(G)$. Then they can arrive at the same $G-$element:

1. Alice picks a random positive integer x and computes $(g, \phi)^x = (A, \phi^x)$. Then, Alice sends $A = \prod_{i=0}^{x-1} \phi^{x-(i+1)}(g)$ to Bob.
2. Bob also picks a random positive integer y, computes $(g, \phi)^y = (B, \phi^y)$ and sends $B = \prod_{i=0}^{y-1} \phi^{y-(i+1)}(g)$ to Alice.

3 Alice computes its shared key $K_A = \varphi^x(B)A$.

4 Bob computes its shared key $K_B = \varphi^y(A)B$.

Note that $K_A = K_B$ based on the following computations

$$\begin{aligned}
\varphi^x(B)A &= \prod_{i=0}^{y-1} \varphi^{x+y-i-1}(P) \prod_{i=0}^{x-1} \varphi^{x-i-1}(P) \\
&= \prod_{i=0}^{x-1} \varphi^{x+y-i-1}(P) \prod_{i=0}^{y-1} \varphi^{y-i-1}(P) \\
&= \varphi^y(A)B.
\end{aligned}$$

The problem above can even be generalized to semigroups G by taking the semigroup of endomorphisms $\mathrm{End}(G)$ under the composition operation instead of $\mathrm{Aut}(G)$.

Battarbee *et al* [BKPS22] present a subexponential quantum attack for SDLP by giving a reduction to the vectorization problem and hence use the fact that the vectorization problem reduces to the Abelian hidden shift problem in which Kuperbeg's subexponential time algorithm [Kup05] is available.

There are several proposed platforms for SDLP including matrices over group rings $M_3(\mathbb{Z}_7[A_5])$ [HKKS13], free nilpotent p-group [KS16], tropical algebra [GS14, GS19], matrices over finite filed $\mathbb{Z}_p$ [RS22], and matrices over bit strings [RS21]. Some of the proposed platforms are vulnerable by some attacks using the structure of the platforms. See [BKS22] for more detailed survey on the semidirect product key exchange.

In this paper, we consider SDLP on elliptic curves. Particularly, we consider the semidirect product $E \rtimes \mathrm{End}(E)$ where E is an elliptic curve. Moreover, we show that SDLP on elliptic curves can be solved efficiently using an adaptation of the standard Shor's quantum algorithm for discrete logarithm problem.

2 Preliminaries

Below, we give a brief introduction to some necessary mathematical background. More details on elliptic curves and endomorphism can be found in [Sil09]. The textbook of Michael Nielsen and Isaac Chuang [NC10] or the lecture note [DW19] by Ronald De Wolf are good references regarding quantum algorithms.

2.1 Elliptic curves

Let $\mathbb{F}_p$ be a finite field of characteristic p and $\overline{\mathbb{F}}_p$ be the algebraically closed field of $\mathbb{F}_p$. In the following we assume $p > 3$ and therefore an elliptic curve E over $\mathbb{F}_p$ can be defined by its short Weierstrass form

$$E(\overline{\mathbb{F}}_p) = \{(x, y) \in \overline{\mathbb{F}}_p^2 \mid y^2 = x^3 + ax + b\} \cup \{O_E\}$$

where $a, b \in \mathbb{F}_p$ such that $4a^3 + 27b^2 \neq 0$ and O_E is the point $(X : Y : Z) = (0 : 1 : 0)$ on the associated projective curve. The set of points of an elliptic curve forms an abelian group with the chord and tangent rule: *three points of E on a line sum to zero, which is the point at infinity O_E.*

For any pair E_1 and E_2 of elliptic curves over $\mathbb{F}_p$, the group $\mathrm{Hom}(E_1, E_2)$ consists of all morphisms of curves $E_1 \to E_2$ that are also group homomorphisms of $E_1(\overline{\mathbb{F}}_p) \to E_2(\overline{\mathbb{F}}_p)$, such a curve morphism is also called *isogeny*. Given $\varphi \in \mathrm{Hom}(E_1, E_2)$, the degree of φ is the degree of φ as a curve morphism and we have the dual of φ denoted by $\widehat{\varphi} \in \mathrm{Hom}(E_2, E_1)$ such that $\varphi \circ \widehat{\varphi} = \widehat{\varphi} \circ \varphi = \deg \varphi$.

The endomorphism ring of an elliptic curve E is defined as $\mathrm{End}(E) = \mathrm{Hom}(E, E)$ with multiplication defined by composition $\alpha\beta = \alpha \circ \beta$. For any

$\alpha \in \mathrm{End}(E)$, we have $\mathrm{tr}(\alpha) = \alpha + \widehat{\alpha} = 1 + \deg(\alpha) - \deg(1-\alpha)$. This can be shown directly using the fact that $\widehat{(\alpha + \beta)} = \widehat{\alpha} + \widehat{\beta}$. Thus,

$$\deg(1-\alpha) = (1-\alpha)(\widehat{1} - \widehat{\alpha}) = 1 - (\alpha + \widehat{\alpha}) + \deg(\alpha).$$

Moreover, both α and its dual $\widehat{\alpha}$ are the roots of the quadratic polynomial

$$\lambda^2 - \mathrm{tr}(\alpha)\lambda + \deg(\alpha). \tag{1}$$

Indeed, as $\alpha^2 - \mathrm{tr}(\alpha)\alpha + \deg(\alpha) = \alpha^2 - (\alpha + \widehat{\alpha})\alpha + \alpha\widehat{\alpha} = 0$.

Kohel shows in [Koh96, Theorem 81] that there exists a polynomial time algorithm that can compute the trace $\mathrm{tr}(\varphi)$ of a given endomorphism φ using a modified Schoof's algorithm. Furthermore, Wills [Wil21] presents an explicit algorithm built upon Kohel's result.

The group $\mathrm{Aut}(E)$ of authomorphisms of E consists of all invertible endomorphisms α of E, i.e., there exists $\beta \in \mathrm{End}(E)$ such that $\alpha\beta = \beta\alpha = 1$. The group of automorphisms of elliptic curves over a field K is well known and it has order dividing 24. It is classified based on the j-invariant of $E : y^2 = x^3 + ax + b$ which is defined as $j(E) = 1728\frac{4a^3}{4a^3+27b^2}$ and the character of the underlying field. Particularly we have the following classifications.

$\lvert\mathrm{Aut}(E)\rvert$	$j(E)$	$\mathrm{char}(K)$
2	$j(E) \neq 0, 1728$	-
4	$j(E) = 1728$	$\mathrm{char}(K) \neq 2, 3$
6	$j(E) = 0$	$\mathrm{char}(K) \neq 2, 3$
12	$j(E) = 0, 1728$	$\mathrm{char}(K) = 3$
24	$j(E) = 0, 1728$	$\mathrm{char}(K) = 2$

2.2 Shor's algorithm for discrete logarithm problem

Shor's algorithm for DLP uses the fact that the problem can be translated to the problem of finding period of a function. Particularly, let G be a group and $g \in G$. Given $h \in \langle g \rangle$, we would like to find the smallest integer a such that $g^a = h$. Hence, we have an equivalent problem as follows. Let $N = \mathrm{ord}(x)$. We define the function $f(x, y) = g^x h^y$ on $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. Note that $f(x, y) = f(x', y')$ if and only if $(x - x', y - y') \in \langle (a, -1) \rangle$. Thus, finding the period of f on the group $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ solves the corresponding DLP.

The latter problem is also called the hidden subgroup problem on the group $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. The key tool of the quantum algorithm is *Fourier sampling method*. The details of the quantum algorithm is out of the scope of this work, we refer the readers to the textbooks, e.g., [NC10] or a nice lecture note [DW19]. Proos and Zalka [PZ03] present a specific case of Shor's algorithm for elliptic curve discrete logarithm problem.

3 A quantum algorithm for SDLP on elliptic curves

In this section, we show that there exists an efficient quantum algorithm that solves semidirect discrete logarithm problem on elliptic curves over finite fields.

Let E be an elliptic curve over a finite field. We consider the semidirect discrete logarithm on the semigroup $E \rtimes \mathrm{End}(E)$ where the multiplication is defined by $(P,\varphi)(Q,\psi) = (P+\varphi(Q), \varphi\psi)$. Therefore, we have the formula for

$$(P,\varphi)^n = \left(\sum_{i=0}^{n-1} \varphi^{n-i-1}(P), \varphi^n\right).$$

The semidirect discrete logarithm problem on this semigroup can be restated as follows.

Problem 2. *Given an elliptic curve E over a finite field, $P \in E$, $\varphi \in \mathrm{End}(E)$, and the element $A = \sum_{i=0}^{n-1} \varphi^{n-i-1}(P)$ for some integer n. The task is to find n.*

3.1 Easy instances

First we observe the easyinstances of problem 2. If φ is the identity endomorphism, then problem 2 is the standard elliptic curve discrete logarithm problem (ECDLP). Moreover, if φ is an endomorphism given by a scalar multiplication $[m]$, then

$$A = \sum_{i=0}^{n-1} \left[m^{n-(i+1)}\right] P = \left[\frac{m^n - 1}{m-1}\right] P.$$

Hence, knowing P and m will reveal n by Shor's algorithm. Specifically, we can use the standard Shor's algorithm to $[m-1]A + P = [m^n]\,P$ to get m^n. Thus, another application of Shor's algorithm on m^n with the knowledge of m will reveal n.

Another straight forward applications of Shor's algorithm can be performed when φ is an automorphism of E. Recall that given an elliptic curve, the endomorphism ring $\mathrm{End}(E)$ has order at most 24. Let φ be an automorphism of order $m \leq 24$ and let $Q = \sum_{i=0}^{m-1} \varphi^i(P)$. Then $A = \sum_{i=0}^{n-1} \varphi^{n-i-1}(P)$ is one of $kQ, kQ+P, kQ+P+\varphi(P), \ldots, kQ + \sum_{i=0}^{m-2} \varphi^i(P)$ where $k = \lfloor n/m \rfloor$. Hence, one can reveal n by using Shor's algorithm on all possible forms, i.e., applying Shor's algorithm to $A - \sum_{i=0}^{j-1} \varphi^i(P)$ for $1 \leq j \leq m-1$ will reveal k and hence reveal n.

We restate the easy cases discussed above in the following theorem.

Theorem 1. *Let E be an elliptic curve over a finite field $\mathbb{F}_p$. Given $P \in E(\mathbb{F}_p)$, $\varphi \in \mathrm{End}(E)$ which is either a scalar multiple or an automorphism, and the element $A = \sum_{i=0}^{n-1} \varphi^{n-i-1}(P)$ for some integer n. Then there exists an efficient quantum algorithm that finds n.*

3.2 General endomorphisms

Now present an efficient quantum algorithm for that solves problem 2 for any endomorphism of E. Recall that any endomorphism $\varphi \in \mathrm{End}(E)$ satisfies the quadratic equation (1). Thus, we have $\varphi^2 = \mathrm{tr}(\varphi)\varphi - \deg\varphi$ and this gives the recursive formula for $\varphi^n = \mathrm{tr}(\varphi)\varphi^{n-1} - \deg(\varphi)\varphi^{n-2}$.

By expanding the reqursive formula for φ^n to the linear form in φ, the summation $A = \sum_{i=0}^{n-1} \varphi^{n-i-1}(P)$ can be simplified as a linear combination of P and $\varphi(P)$, i.e.,

$$A = a_n(t,d)P + b_n(t,d)\varphi(P),$$

where a_n and b_n are polynomial in $t = \mathrm{tr}(\varphi)$ and $d = \deg(\varphi)$. If we can explicitly write down the formula for a_n and b_n, then we can solve the problem 2 using the generalization of Shor's algorithm for the abelian hidden subgroup problem in $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ where $N = \mathrm{lcm}\,(\mathrm{ord}(P), \mathrm{ord}(\varphi(P))$ using the following hiding function

$$F(x,y) = a_x(t,d)P + b_x(t,d)\varphi(P) + yA,$$

where the hidden subgroup is $\langle (n,-1)\rangle$.

In the expansion of φ^n in term of linear form, we have that both the coefficient of φ and the scalar satisfy the recursive function $f(n) = \mathrm{tr}(\varphi)f(n-1) - \deg(\varphi)f(n-2)$. Let us first compute the explicit formula of the scalar in the expansion. We have that the recursive function for the scalar satisfies the initial value $f(1) = 0$ and $f(2) = -\deg(\varphi)$. Hence, we can write it in the matrix form as $F_n = M \cdot F_{n-1}$ where

$$F_n = \begin{pmatrix} f(n) \\ f(n-1) \end{pmatrix}, \text{ and } M = \begin{pmatrix} \mathrm{tr}(\varphi) & -\deg(\varphi) \\ 1 & 0 \end{pmatrix}.$$

Therefore, we have $F_n = M^{n-1} \cdot F_2$, where $F_2 = \begin{pmatrix} -\deg\varphi \\ 0 \end{pmatrix}$.

By the above discussion, calculating M^{n-1} will give the explicit formula for the scalar in the expansion of φ^n. This can be done using the eigenvalue decomposition of M. Let V be the matrix of eigenvectors of M and D be the diagonal matrix of eigenvalues of M. Then $M = V \cdot D \cdot V^{-1}$ and thus $M^n = V \cdot D^n \cdot V^{-1}$. Therefore, by summing up all the explicit formula of the scalar in each expansion of φ^i for $0 \le i \le n-1$, we have the explicit formula for $a_n(t,d)$.

Similar computations, using the recursive function $f(n) = \mathrm{tr}(\varphi)f(n-1) - \deg(\varphi)f(n-2)$ where $f(1) = 1$ and $f(2) = \mathrm{tr}(\varphi)$, will give $F_n = M^n \cdot F_1$ where $F_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the explicit formula for the coefficient of φ in the expansion of φ^n. Hence, we can also obtain the explicit formula for $a_n(t,d)$ in the obvious way.

4 Conclusion

The semidirect discrete logarithm problem on elliptic curves can be seen as a natural generalization of the standard elliptic curve discrete logarithm problem. However, prior to this work, elliptic curves have never been considered as a platform for the semidirect product key exchange which was introduced in [HKKS13]. One of the reasons might be that this requires an efficient way to generate and evaluate an endomorphism of a given elliptic curve E on arbitrary points of E. As shown by Wesolowski [Wes22], generating a random endomorphism is as hard as computing the full endomorphism ring which is a central problem in isogeny-based cryptography. Moreover, given an endomorphism of E, it is not clear how to efficiently evaluate the endomorphism on arbitrary points of E. This depends on how the endomorphism is represented.

The two obstacles can be fixed by using some efficient tools developed in isogeny-based cryptography. Particularly, Petit and Lauter in [PL17] present an algorithm that given a prime $p > 2$ computes a supersingular elliptic curve E such that $\mathrm{End}(E)$ is isomorphic to a maximal order $\mathcal{O}$ in the quaternion algebra $B_{p,\infty}$ as classified by Pizer [Piz80]. This way, we can efficiently generate and evaluate a random endomorphism of E which is represented as an element of the quaternion algebra.

Unfortunately, as we analyzed the complexity of the semidirect discrete logarithm problem in $E \rtimes \mathrm{End}(E)$ for arbitrary elliptic curves E, we show that the structure of endomorphisms of elliptic curves allows us to efficiently solve the SPDLP on $E \rtimes \mathrm{End}(E)$ using an adaptation of Shor's algorithm. Therefore, the semidirect product key exchange using elliptic curves over finite fields does not belong to the realm of post-quantum cryptography.

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