

# Fast batched asynchronous distributed key generation

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**Abstract.** We present new protocols for threshold Schnorr signatures that work in an *asynchronous* communication setting, providing *robustness* and *optimal resilience*. These protocols provide unprecedented performance in terms of communication and computational complexity. In terms of communication complexity, for each signature, a single party must transmit a few dozen group elements and scalars across the network (independent of the size of the signing committee). In terms of computational complexity, the amortized cost for one party to generate a signature is actually less than that of just running the standard Schnorr signing or verification algorithm (at least for moderately sized signing committees, say, up to 100).

For example, we estimate that with a signing committee of 49 parties, at most 16 of which are corrupt, we can generate *50,000 Schnorr signatures per second* (assuming each party can dedicate one standard CPU core and 500Mbps of network bandwidth to signing). Importantly, this estimate includes both the cost of an offline precomputation phase (which just churns out message independent “presignatures”) and an online signature generation phase. Also, the online signing phase can generate a signature with very little network latency (just one to three rounds, depending on how throughput and latency are balanced).

To achieve this result, we provide two new innovations. One is a new secret sharing protocol (again, asynchronous, robust, optimally resilient) that allows the dealer to securely distribute shares of a large batch of ephemeral secret keys, and to publish the corresponding ephemeral public keys. To achieve better performance, our protocol minimizes public-key operations, and in particular, is based on a novel technique that does *not* use the traditional technique based on “polynomial commitments”. The second innovation is a new algorithm to efficiently combine ephemeral public keys contributed by different parties (some possibly corrupt) into a smaller number of secure ephemeral public keys. This new algorithm is based on a novel construction of a so-called “super-invertible matrix” along with a corresponding highly-efficient algorithm for multiplying this matrix by a vector of group elements.

As protocols for verifiably sharing a secret key with an associated public key and the technology of super-invertible matrices both play a major role in threshold cryptography and multi-party computation, our two new innovations should have applicability well beyond that of threshold Schnorr signatures.

## 1 Introduction

The main motivation for our work is to design efficient protocols for threshold Schnorr signatures that work in an *asynchronous* communication setting, providing *robustness* and *optimal resilience*. As will be explained in detail below, our results and techniques result in threshold Schnorr protocols with extremely high throughput and low latency. These protocols follow the usual offline/online paradigm, where higher-latency, message-independent precomputations are performed in the offline phase, and lower-latency, message-dependent computations are performed in the online phase. The resulting protocol has linear communication complexity per signature in both phases — each party in the signing committee essentially transmits and receives a total number of 18 scalars and 9 group elements per signature in the offline phase, and 6 scalars per signature in the online phase.<sup>1</sup>

<sup>1</sup> We stress that we are not assuming any type of “broadcast channel”. When we say  $P$  transmits a certain amount of data, we are counting the sum over all parties  $Q$  of the amount of data that  $P$  sends to  $Q$  over a point-to-point channel.

Moreover, for moderately sized signing committees (in the range 10–100), they enjoy extremely good computational complexity. In particular, over a group of order  $q \approx 2^\lambda$ , and with a signing committee of size  $n$ , the running time per signature of each party in the signing committee in the offline phase is dominated by the cost of performing  $O(n + \lambda/n)$  group additions (we use additive notation and terminology throughout), and in the online phase is dominated by the cost of  $O(n)$  arithmetic operations mod  $q$ . Note that these estimates assume some batching is done in the offline phase and a small amount of batching is done in the online phase. Somewhat surprisingly, this result says that for such moderately sized  $n$ , the running time per party decreases as  $n$  increases, and is actually less than the time used to just compute a Schnorr signature. Note that our results and techniques have much broader applicability. For example, they can also be applied to threshold ECDSA signatures and certainly other problems in threshold cryptography.

The big-O constants here are quite small. For example, with  $n = 49$  and  $\lambda = 256$ , the number of group additions per party per signature in the offline phase is just 23. Note that this group addition count does not presume an exorbitant amount of batching in the offline phase, nor does it assume a particularly sophisticated multi-scalar/group multiplication algorithm. If the group is an elliptic curve such as `secp256k1` [Cer10], a single group addition can be performed by a reasonably good library in well under  $0.5\mu\text{s}$  on commonly available machines (see details in [Appendix A.1](#)). This translates to a total of  $11.5\mu\text{s}$  per signature, and we will conservatively round this up to  $20\mu\text{s}$  to account for other overheads. This translates into a throughput of 50,000 signatures per second. Now, as mentioned above each party transmits a total of 24 scalars and 9 group elements per signature, so roughly 10Kb (10,000 bits). So in order to sustain a throughput of 50,000 signatures per second, a network bandwidth of 500Mbps suffices, which is not unreasonable.

## 1.1 An MPC engine geared towards Schnorr

Let us first recall the Schnorr signature scheme. Let  $E$  be a group of prime order  $q$  generated by  $\mathcal{G} \in E$ . As mentioned already, we use additive notation for the group operation of  $E$ , and denote the identity element of  $E$  by  $\mathcal{O}$ . The secret key is a random  $x \in \mathbb{Z}_q$  and the public key is  $\mathcal{X} \leftarrow x\mathcal{G} \in E$ . To sign a message  $m$ , the signer chooses  $r \in \mathbb{Z}_q$  at random, computes

$$\mathcal{R} \leftarrow r\mathcal{G} \in E, \quad h \leftarrow \text{Hash}(\mathcal{X}, \mathcal{R}, m) \in \mathbb{Z}_q, \quad s \leftarrow r + xh \in E,$$

and outputs the signature  $(\mathcal{R}, s) \in E \times \mathbb{Z}_q$ .

The approach we take to designing a threshold Schnorr signing protocol in the asynchronous communication setting is to build it on top of a highly optimized **MPC (multi-party computation) engine**. That is, rather than designing and analyzing a monolithic protocol, we design an MPC engine that supports operations well suited to threshold Schnorr signatures (and other threshold cryptography tasks as well), and that can be efficiently implemented while providing robustness and optimal resilience in an asynchronous communication setting.

At a high level, we need an MPC engine that, as an ideal functionality  $\mathfrak{F}_{\text{MPC}}$ , supports the following operations:

- $([r], \mathcal{R}) \leftarrow \text{RandomKeyGen}()$ :  
 $\mathfrak{F}_{\text{MPC}}$  chooses  $r \in \mathbb{Z}_q$  at random, computes  $\mathcal{R} \leftarrow r\mathcal{G} \in E$ , gives  $\mathcal{R}$  immediately to the adversary, and gives  $\mathcal{R}$  to each party as a delayed output (i.e., the adversary indicates when the output is given to any given party). The ideal functionality  $\mathfrak{F}_{\text{MPC}}$  also stores  $r$  for future use.
- $[z] \leftarrow \text{LinearOp}(a, [x], b, [y])$ :  
 For public inputs  $a, b \in \mathbb{Z}_q$ , and previously stored values  $x, y \in \mathbb{Z}_q$ ,  $\mathfrak{F}_{\text{MPC}}$  computes  $z \leftarrow ax + by \in \mathbb{Z}_q$  and stores  $z$  for future use.  
 This is typically implemented as a local computation.
- $z \leftarrow \text{Open}([z])$ :  
 For a previously stored value  $z \in \mathbb{Z}_q$ ,  $\mathfrak{F}_{\text{MPC}}$  gives  $z$  to each party as a delayed output.

We assume we have  $n$  parties  $P_1, \dots, P_n$ , at most  $t < n/3$  of which may be corrupt. We also assume static corruptions — that is at the beginning of the attack and before the start of the protocol the adversary corrupts some subset of  $t^* \leq t$  parties. We envision a signing protocol that is driven by a blockchain or any other BFT protocol that orders the signing requests and the execution of the  $\mathfrak{F}_{\text{MPC}}$ -operations, so that each party receives the same signing requests and initiates the same  $\mathfrak{F}_{\text{MPC}}$ -operations in the same order. It is not important here which method is used for the parties to agree on request and activation ordering.

Using the above MPC engine, we can easily implement threshold Schnorr signatures as follows. The protocol to generate the key is:

$$([x], \mathcal{X}) \leftarrow \text{RandomKeyGen}()$$

The protocol to sign a message  $m$  is:

$$\begin{aligned} &([r], \mathcal{R}) \leftarrow \text{RandomKeyGen}() \\ &h \leftarrow \text{Hash}(\mathcal{X}, \mathcal{R}, m) \in \mathbb{Z}_q \text{ //local computation} \\ &[s] \leftarrow \text{LinearOp}(1, [r], h, [x]) \text{ //local computation} \\ &s \leftarrow \text{Open}([s]) \\ &\text{output } (\mathcal{R}, s) \end{aligned}$$

While the above is very simple, it is typically not very efficient. The problem is that in a typical implementation of  $\mathfrak{F}_{\text{MPC}}$ , the `RandomKeyGen` operation is fairly expensive. One way of improving this situation is to observe that the value  $r$  is independent of  $m$ , and so we might move the computation of  $([r], \mathcal{R}) \leftarrow \text{RandomKeyGen}()$  to an “offline” precomputation phase. In this case, we refer to the pair  $([r], \mathcal{R})$  as a “presignature”. Moreover, we might be able to exploit “batching” techniques to more efficiently produce such presignatures in batches, and then consume them in an “online” phase as signing requests are made.

The problem with this approach is that by computing these presignatures  $([r], \mathcal{R})$  in advance and revealing the group element  $\mathcal{R}$  to the adversary before the corresponding signing request is made, the signature scheme becomes insecure. This is a well-known problem and a number of good mitigation strategies have been devised. See [Sho23] for details on how this can be done efficiently and securely (see also Section 7.2).

So we can safely ignore these issues for now and focus on the remaining challenge: how to generate large batches of presignatures efficiently. In computing batches of presignatures, we are not so much concerned about the latency of producing a batch, as this occurs in the “offline”

phase. We are, however, concerned about the throughput, that is, the rate at which we can produce presignatures, as this will be the limiting factor on the signing throughput, that is, the rate at which we can process signing requests.

So the problem we focus on is this: *high-throughput generation of presignatures*. This problem may be simplified by the observation that for a presignature  $([r], \mathcal{R})$ , we do not require that the value  $r$  is perfectly random. In fact, as was first observed in a specific context in [GJKR07], and then again in an another specific context in [BHK<sup>+</sup>23], and then more recently in a much more general context in [Sho23], the threshold signing protocol will still be secure even if the adversary is allowed to *bias* the presignatures in a particular *benign* way, and moreover, it is much easier to generate such benignly biased presignatures than it is to generate unbiased presignatures.

One form of biased presignature that is relevant can be captured by following operation, which we can add to our MPC engine:

–  $([r'], \mathcal{R}') \leftarrow \text{BiasedKeyGen}()$ :  
 $\mathfrak{F}_{\text{MPC}}$  chooses  $r \in \mathbb{Z}_q$  at random, computes  $\mathcal{R} \leftarrow r\mathcal{G} \in E$ , gives  $\mathcal{R}$  immediately to the adversary.  
 The adversary later responds with a “bias”  $(a, b) \in \mathbb{Z}_q^* \times \mathbb{Z}_q$ .  
 $\mathfrak{F}_{\text{MPC}}$  then computes  $r' \leftarrow ar + b \in \mathbb{Z}_q$  and  $\mathcal{R}' \leftarrow r'\mathcal{G} \in E$ , and gives  $\mathcal{R}'$  to each party as a delayed output.  $\mathfrak{F}_{\text{MPC}}$  also stores  $r'$  for future use.

One can securely realize this functionality based on a simpler operation and a consensus protocol. The simpler operation is this:

–  $([r_i], \mathcal{R}_i) \leftarrow \text{InputKey}_i(r_i)$ :  
 Party  $P_i$  inputs  $r_i \in \mathbb{Z}_q$  to  $\mathfrak{F}_{\text{MPC}}$ , who computes  $\mathcal{R}_i \leftarrow r_i\mathcal{G} \in E$ , gives  $\mathcal{R}_i$  immediately to the adversary, and gives  $\mathcal{R}_i$  to each party as a delayed output.  $\mathfrak{F}_{\text{MPC}}$  also stores  $r_i$  for future use.

To implement `BiasedKeyGen` with this operation, each party  $P_i$  inputs a random secret key  $r_i$  to the ideal functionality via `InputKeyi` so that every party, including the adversary, learns the public key  $\mathcal{R}_i$ . Note that while honest parties input random secret keys, the corrupt parties may choose arbitrary secret keys in a way that depends on the public keys of the honest parties. The parties then use a consensus protocol to agree on a set  $\mathcal{I}$  of  $t + 1$  indices, where for each  $i \in \mathcal{I}$ , the operation `InputKeyi` has successfully completed.<sup>2</sup> The resulting biased key-pair is  $([r'], \mathcal{R}')$ , where  $r' = \sum_{i \in \mathcal{I}} r_i$  and  $\mathcal{R}' = \sum_{i \in \mathcal{I}} \mathcal{R}_i$ . This is computed locally, using `LinearOp` to compute  $[r']$  from  $\{[r_i]\}_{i \in \mathcal{I}}$ , and computing  $\mathcal{R}'$  directly using the known values  $\{\mathcal{R}_i\}_{i \in \mathcal{I}}$  as output by  $\{\text{InputKey}\}_{i \in \mathcal{I}}$ . That this securely realizes `BiasedKeyGen` is fairly straightforward to see (this was observed implicitly in [Gro21] and explicitly in Section A.3.6 of [GS22]).

Yet better performance can be obtained by utilizing the well-known “batch randomness extraction” technique — an idea that goes back at least to [HN06], but first applied to Schnorr signatures in [BHK<sup>+</sup>23], and then analyzed more fully in the context of Schnorr signatures in [Sho23]. Here, we choose a certain  $M \times N$  matrix  $W$  over  $\mathbb{Z}_q$  (whose entries are public constants — see below), where  $M = n - 2t$  and  $N = n - t$ . As above, each party  $P_i$  inputs a random secret key  $r_i$  to the ideal

<sup>2</sup> More specifically, the parties run an ACS (Agreement on a Common Set) protocol. See [DWZ23] for a state-of-the-art ACS protocol and for useful historical perspective on the ACS problem. With such a protocol, ACS will not be the bottleneck in any of our protocols, in terms of either communication, computational, or round complexity.

functionality via  $\text{InputKey}_i$ , so that every party, including the adversary, learns the public key  $\mathcal{R}_i$ . As above, while honest parties input random secret keys, the corrupt parties may choose arbitrary secret keys in a way that depends on the public keys of the honest parties. The parties then use a consensus protocol to agree on a set  $\mathcal{I}$  of  $N$  indices, where for each  $i \in \mathcal{I}$ , the operation  $\text{InputKey}_i$  has successfully completed. Let us write

$$\mathcal{I} = \{i_1, \dots, i_N\}.$$

The parties then locally compute  $M$  biased key-pairs

$$([r'_1], \mathcal{R}'_1), \dots, ([r'_M], \mathcal{R}'_M)$$

where

$$\begin{pmatrix} r'_1 \\ \vdots \\ r'_M \end{pmatrix} = W \cdot \begin{pmatrix} r_{i_1} \\ \vdots \\ r_{i_N} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathcal{R}'_1 \\ \vdots \\ \mathcal{R}'_M \end{pmatrix} = W \cdot \begin{pmatrix} \mathcal{R}_{i_1} \\ \vdots \\ \mathcal{R}_{i_N} \end{pmatrix}.$$

This is computed locally, using  $\text{LinearOp}$  to compute  $[r'_1], \dots, [r'_M]$  from  $[r_1], \dots, [r_N]$ , and computing  $\mathcal{R}'_1, \dots, \mathcal{R}'_M$  directly using the known values  $\{\mathcal{R}_i\}_{i \in \mathcal{I}}$ .

The property that the matrix  $W$  must satisfy is called **super-invertibility**, which simply means that every subset of  $M$  columns of  $A$  is linearly independent.

The security property that the above protocol satisfies can be elegantly captured by adding the following operation to our MPC engine, where we define  $M := n - 2t$  and  $N^* := n - t^*$ , where  $t^* \leq t$  is the number of actual corrupted parties.

–  $(([r'_1], \mathcal{R}'_1), \dots, ([r'_M], \mathcal{R}'_M)) \leftarrow \text{BatchedBiasedKeyGen}()$ :  
 $\mathfrak{F}_{\text{MPC}}$  chooses  $r_1, \dots, r_{N^*} \in \mathbb{Z}_q$  at random, computes

$$\mathcal{R}_1 \leftarrow r_1 \mathcal{G}, \dots, \mathcal{R}_{N^*} \leftarrow r_{N^*} \mathcal{G}$$

and gives these group elements immediately to the adversary.  
The adversary later responds with a “bias”  $(A, \mathbf{b})$ , where  $A \in \mathbb{Z}_q^{M \times N^*}$  is a full rank matrix and  $\mathbf{b} \in \mathbb{Z}_q^{M \times 1}$  is an arbitrary vector.  
 $\mathfrak{F}_{\text{MPC}}$  then computes

$$\begin{pmatrix} r'_1 \\ \vdots \\ r'_M \end{pmatrix} = A \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_{N^*} \end{pmatrix} + \mathbf{b}$$

and

$$\mathcal{R}'_1 \leftarrow r'_1 \mathcal{G}, \dots, \mathcal{R}'_M \leftarrow r'_M \mathcal{G}$$

and gives  $(\mathcal{R}'_1, \dots, \mathcal{R}'_M)$  to each party as a delayed output.  $\mathfrak{F}_{\text{MPC}}$  also stores the values  $r'_1, \dots, r'_M$  for future use.

Note that one can define this operation more generally, in that the parameter  $M$  could be set to a different value, possibly determined adaptively by the adversary. This can be useful to model variations on the above protocol for batch randomness extraction.

## 1.2 Two problems

So we now have reduced the problem of building threshold Schnorr to the following two problems:

**Problem 1:** Securely and efficiently implementing the  $\mathfrak{F}_{\text{MPC}}$ -operations `InputKey`, `LinearOp`, and `Open`.

**Problem 2:** Designing a super-invertible matrix equipped with an efficient algorithm for multiplication on the right by vector of group elements.

Indeed, with solutions to both of these problems, we can directly implement `BiasedKeyGen` and `BatchedBiasKeyGen`, as outlined above. Now, the papers [Sho23] and [BHK<sup>+</sup>23] do not analyze the usage of a biased keys as signing keys, but only as presignatures. However, the analysis in [Sho23] can be extended to allow for biased signing keys. That said, it is arguably not that important, since we presumably generate signing keys very occasionally, and so we could afford to use a more expensive implementation of an unbiased key. Moreover, for other applications, such as ECDSA, the use of an unbiased signing key is essential — see Section 3.6 of [GS22]. Note that to support ECDSA, we would also have to extend our MPC engine to include a multiplication operations. That is a issue we do not consider in the paper. Nevertheless, our techniques here can be extended to cover this as well, although with certain limitations. In any case, in this paper we do not consider any implementation of `RandomKeyGen`, as it is not needed to implement threshold Schnorr, but in any case, it is easy enough to implement in a way that is (a) not too horribly inefficient, and (b) is compatible with the other operations in our MPC engine.

## 2 Our contributions

We present new solutions to Problems 1 and 2 above.

### 2.1 Solution to Problem 1

Our new solution to Problem 1 is a new type of protocol, which we call a **GoAVSS protocol**. Here, GoAVSS stands for **group-oriented asynchronous verifiable secret sharing**. We assume parties  $P_1, \dots, P_n$ , one of which is designated as the dealer, and at most  $t < n/3$  of which may be (statically) corrupted. We also assume we have fixed evaluation points  $e_1, \dots, e_n$ , which are distinct, nonzero elements of  $\mathbb{Z}_q$ . A GoAVSS protocol should securely realize the following ideal functionality.

- The dealer inputs polynomials  $f_1, \dots, f_L \in \mathbb{Z}_q[x]$  to the GoAVSS functionality, each of which must have degree at most  $t$  (this condition is enforced by the functionality).
- For each  $\ell \in [L]$ , the polynomial  $f_\ell$  defines a secret key  $s_\ell := f_\ell(0) \in \mathbb{Z}_q$  and a public key  $\mathcal{S}_\ell := s_\ell \mathcal{G} \in E$ , and the GoAVSS functionality immediately reveals the public keys  $\mathcal{S}_1, \dots, \mathcal{S}_L$  to the adversary.
- The GoAVSS functionality gives to each party  $P_j$  as a delayed output the public keys  $\mathcal{S}_1, \dots, \mathcal{S}_L$ , as well as its shares of the secret keys  $s_1, \dots, s_L$ , that is, the values  $f_1(e_j), \dots, f_L(e_j)$ .

In addition, a GoAVSS protocol should satisfy a **completeness property**, which means that if the dealer is honest or any honest party outputs a value, then all honest parties eventually output a value. Here, “eventually” means if and when all honest parties initiate the protocol and all messages sent between honest parties are delivered.

Any GoAVSS protocol gives a solution to Problem 1. Indeed, such a protocol can be used directly to implement the `InputKey` operation of our MPC engine. In fact, one run of the GoAVSS protocol actually yields  $L$  instances of `InputKey`. For our application to threshold signatures, such batching is perfectly fine, as parties just input random secret keys. Moreover, such batching can be used to get much more efficient protocols. Since a GoAVSS protocol distributes traditional Shamir shares of the secrets, the `LinearOp` is also easily and efficiently implemented.

As for the `Open` operation, the reader may notice that our GoAVSS functionality does not make any explicit mention of a “polynomial commitment” of any form that can be used to verify the secret shares revealed during the `Open` operation. This is intentional: our new GoAVSS protocol is very efficient precisely because it does not use a polynomial commitment at all. This is also very different from just about every other DKG protocol in the literature (going back to Feldman [Fel87] and Pedersen [Ped91]).<sup>3</sup> Nevertheless, since we are assuming  $n > 3t$  (which is necessary in the asynchronous setting), we do not need to use polynomial commitments to verify the secret shares revealed during the `Open` operation — we can instead just use error correcting codes. This is a standard technique in the field of information-theoretic asynchronous MPC. Below in Section 7.1, we review this technique, and discuss how this impacts the practical performance of the online phase of a threshold signing protocol. As we will see, to get the best throughput in the online phase, we have to increase the latency of the online phase just a bit (but this is not because we use error correcting codes). Note that the completeness property of our GoAVSS property is essential to make it possible to forgo polynomial commitments and rely on error correction.

As already mentioned, our GoAVSS protocol achieves very good performance by avoiding polynomial commitments. Instead, at a very high level, it works as follows:

- First, the dealer runs a “plain” AVSS protocol, which just works with scalars, rather than group elements, to distribute shares of the polynomials  $f_1, \dots, f_\ell \in \mathbb{Z}_q[x]$ . The dealer then simply broadcasts the group elements  $\mathcal{S}_1, \dots, \mathcal{S}_L$  to the receivers.

To get very high performance for this step, we may use the recently developed AVSS protocol from [SS23], which uses very lightweight cryptography (i.e., hash functions) and generally has very good communication and computational complexity.

- At this point, we may assume that the “plain” AVSS protocol ensures that receivers are holding shares of polynomials  $f_1, \dots, f_\ell$  of the right degree. However, there is no guarantee that  $f_\ell(0)\mathcal{G} = \mathcal{S}_\ell$  for all  $\ell \in [L]$ . So we run another subprotocol that performs a simple statistical test. Moreover, this test is designed so that we can distribute the work among the receivers, so that each party performs a *very* small number of group additions — just  $O(\lambda/n)$  additions per individual sharing, if  $q \approx 2^\lambda$ . The work actually decreases as  $n$  increases! This is, of course, too good to be true. Indeed, there is a trade-off, in that each party must perform  $O(n)$  *very* simple scalar operations per individual sharing (each such operation is not much more expensive than the addition of two  $\lambda$ -bit numbers, and we estimate that such a scalar operation takes just a few percentage points of the time to perform a single group addition). For moderately sized  $n$ , this turns out to be an excellent trade-off.

We analyze in detail both the computational and communication complexity of our GoAVSS protocol, with certain subprotocols implemented as in [SS23]. These subprotocols have an “optimistic path”, where no party misbehaves in a publicly provable way, and we only consider the

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<sup>3</sup> One notable exception to this is the DKG protocol in the very recent work [ABCP23], which presents a DKG protocol in the synchronous model. We compare our work to theirs later in the paper.



cost on this optimistic path. Arguably, over a long run of the system where parties that provably misbehave are effectively removed from the system, this is the only cost that matters. Now, each run of our GoAVSS protocol produces  $L$  “raw sharings” created by a single dealer. In the intended usage as a subprotocol in our implementation of the `BatchedBiasedKeyGen`, every party will run the GoAVSS protocol  $n$  times: once playing role of both dealer and receiver, and  $n - 1$  times just as a receiver, and this will yield a total of  $L \cdot (n - 2t) \geq L \cdot n/3$  “processed sharings”. So we naturally measure the amortized cost per “processed sharing”, which in the application to threshold signatures, represents the amortized cost per presignature of the offline phase contributed by the GoAVSS protocol (but does not include the computational cost of applying the super-invertible matrix). Our stated amortized costs assume that  $n$  is not too big relative to  $L$  and  $\lambda$ . In particular, to achieve the stated amortized costs, we require  $L = \Omega(n^2)$  and  $\lambda = \Omega(n)$  — actually, for larger  $n$ , we can use a variant of our main protocol for which we only require  $L = \Omega(n \log n)$ , rather than  $L = \Omega(n^2)$ .

**Communication complexity** We define *communication complexity* as the sum, over all honest parties  $P$  and all parties  $Q$ , of the total number of bits that  $P$  sends to  $Q$  over a point-to-point channel.

The amortized communication complexity per processed sharing of our GoAVSS protocol is  $O(n\lambda)$ , where  $q \approx 2^\lambda$  and we assume group elements are encoded as  $O(\lambda)$ -bit strings. Also, the communication complexity is well balanced: each party sends (and receives)  $O(\lambda)$  bits per of data per processed sharing. In fact, each party essentially transmits (and receives) a total of 18 scalars and 9 group elements per processed sharing.

**Computational complexity** We define *computational complexity* to be the maximum running time of any one individual party.

The amortized computational complexity per processed sharing of our GoAVSS protocol is dominated by the cost of performing  $O(n/\lambda + 1)$  additions in the group  $E$ , and  $O(n)$  arithmetic operations in  $\mathbb{Z}_q$ .

To state the computational complexity more precisely, we introduce some terminology.

**Full scalar/group multiplication:** This is a scalar/group multiplication, where the scalar is a secret, full-sized element of  $\mathbb{Z}_q$ , and the group element is the generator  $\mathcal{G}$ . This means that (a) we can use precomputation on  $\mathcal{G}$  to make the algorithm faster, but (b) we have to be careful to use a constant-time algorithm. For typical parameter settings and implementations, we can estimate the cost of a full scalar/group multiplication to be about 64 group additions (see [Appendix A](#)).

**Tiny scalar/group multiplication:** Let  $\rho$  be a parameter (usually clear from context). This is a scalar/group multiplication where the scalar is a public, random  $\rho$ -bit number, and the group element is variable and public.

Moreover, the resulting products are only used as terms in a long summation, so special optimized algorithms may be used. Using Pippenger’s simple “bucket method” (described in [Appendix B](#)), for typical parameter settings, the amortized cost of each tiny/scalar group multiplication is just 1 or 2 group additions.

**Tiny scalar/scalar multiplication:** Again, let  $\rho$  be a parameter (usually clear from context). This is a scalar/scalar multiplication where one scalar input is a public, random  $\rho$ -bit number, and the other scalar input is a secret, full-sized element of  $\mathbb{Z}_q$ .



Moreover, the resulting products are only used as terms in a long summation, so special optimized algorithms may be used. In particular, the cost of such an operation is essentially that of multiplying a  $\rho$ -bit integer by a  $\lambda$ -bit integer (without a modular reduction). For typical parameter settings, the amortized cost of each tiny/scalar group multiplication is just a very small fraction (less than  $1/40$ ) of the cost of a group addition.

Let  $\sigma$  be a statistical security parameter (typically  $\sigma = 80$ ). Set the parameter  $\rho = \lceil 3\sigma/n \rceil + 2$ , used in defining tiny scalar/group and scalar/scalar multiplications, as above. The amortized computational complexity per processed sharing of our GoAVSS protocol can be stated as follows.

- $3/n$  full scalar/group multiplications,
- 3 tiny scalar/group multiplications, and
- $4n$  tiny scalar/scalar multiplications.

In addition to the communication and computational complexity of our GoAVSS protocol, also of interest is the **round complexity**. This is a (reasonably small) constant. Because of this, any performance degradation caused by network latency (which could cause the network and CPU bandwidth to be under-utilized) can be minimized by increasing the batch size  $L$ .

## 2.2 Solution to Problem 2

Our solution to problem two can be stated very simply. For any  $M \leq N \leq q$ , we give an explicit construction of an  $M \times N$  super-invertible matrix over  $\mathbb{Z}_q$ , along with an algorithm for multiplying this matrix on the right by an  $N \times 1$  column vector over the group  $E$  that uses precisely  $M \cdot (N - 1 - (M + 1)/2) + 1$  additions in the group  $E$ . In the application to batch randomness extraction, where  $M = n - 2t$  and  $N = n - t$ , this contributes a amortized cost of about  $n/2 - 3/2$  group additions per processed sharing. Moreover, as we discuss, in some special cases, this amortized cost can be reduced a bit further, to just  $t$  group additions per processed sharing.

## 2.3 Combining the two solutions

Combining our solutions to Problems 1 and 2, we immediately get a protocol for computing presignatures for Schnorr threshold signing with the following properties:

- its amortized communication complexity per presignature is  $O(n\lambda)$ , i.e.,  $O(\lambda)$  bits per party and
- a party’s amortized computational complexity per presignature is dominated by the cost of performing  $O(n + \lambda/n)$  additions in the group  $E$

The online complexity of the resulting protocol depends on a number of design choices with regard to how the Open protocol is implemented. If we insist on one round of communication in the online phase, we obtain

- an amortized communication complexity per signature of  $O(n^2\lambda)$ , with each party essentially transmitting  $n$  scalars per signature, and
- an amortized computational complexity per signature dominated by the cost of performing  $O(n^2)$  arithmetic operations in  $\mathbb{Z}_q$ .

If we allow two rounds of communication in the online phase, and also allow for some batching of signing requests (which is reasonable in a heavily loaded system), we obtain

- an amortized communication complexity per signature of  $O(n\lambda)$ , with each party essentially transmitting 6 scalars per signature, and
- an amortized computational complexity per signature dominated by the cost of performing  $O(n)$  arithmetic operations in  $\mathbb{Z}_q$ .

This is discussed in more detail in [Section 7.1](#).

All of the above computational estimates assume only naive, quadratic-time polynomial arithmetic.

## 2.4 The rest of the paper

In [Section 3](#), we present the precise security definitions of AVSS and GoAVSS that we will use throughout the paper. In [Section 4](#), we review the subprotocols we make use of to build our new GoAVSS protocol. In [Section 5](#), we present our new GoAVSS protocol, provide a careful security and complexity analysis, and look a couple of variations as well. We also present an analyze of simple variant in [Section 5.3](#) that may be preferred in when  $n$  is very large. In [Section 6](#), we present our new constructions of super-invertible matrices with corresponding algorithms for multiplying such a matrix on the right by a vector of group elements. In [Section 7](#), we discuss some details about the online signing phase of the threshold Schnorr signature derived from our protocols. In [Section 8](#), we make an in-depth comparison (with numerical examples) with our closest competitor, SPRINT [[BHK<sup>+</sup>23](#)]; in [Section 8.4](#), we look beyond SPRINT, and combine various ideas (from [[GS22](#)] as well as from this paper) to get the best possible “batch Feldman” GoAVSS protocol based on polynomial commitments, and compare its performance to that of our new GoAVSS protocol. Finally, in [Section 9](#), we compare our techniques to those in [[ABCP23](#)]. This paper presents (among other things) DKG and threshold Schnorr protocols that work in the synchronous communication model. While the goals and results of this paper are in many ways incomparable with our results, there is some overlap in techniques.

## 3 Preliminaries

### 3.1 Asynchronous verifiable secret sharing

We recall the notion of *asynchronous verifiable secret sharing (AVSS)*. We have  $n$  parties  $P_1, \dots, P_n$ , of which at most  $t < n/3$  may be corrupt. We assume *static* corruptions. Let  $\mathcal{H}$  denote the indices of the honest parties, and let  $\mathcal{C}$  denote the indices of the corrupt parties.

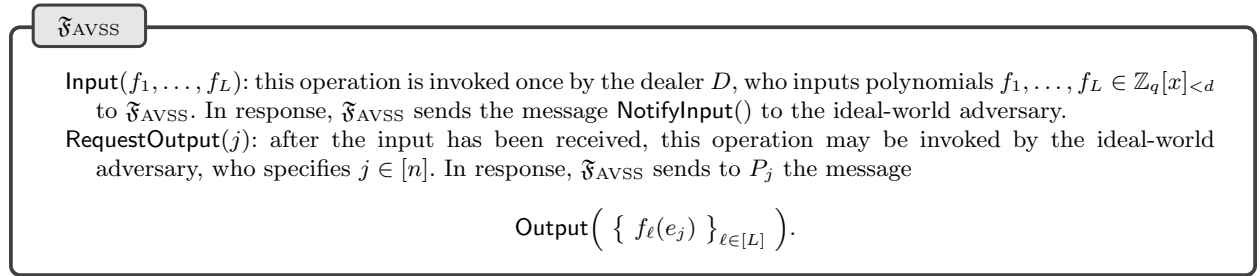
We assume the parties are connected by secure point-to-point channels, which provide both privacy and authentication. As we are working exclusively in the asynchronous communication model, there is no bound on the time required to deliver messages between honest parties.

Let  $\mathbb{Z}_q$  be finite field with  $q$  elements. Let  $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{Z}_q^n$  be a sequence of distinct, nonzero elements of  $\mathbb{Z}_q$ . Let  $d$  be a positive integer, and let  $\mathbb{Z}_q[x]_{<d}$  denote the  $\mathbb{Z}_q$ -subspace of  $\mathbb{Z}_q[x]$  consisting of all polynomials of degree less than  $d$ . Let  $L$  be a positive integer.

An  $(n, d, L)$ -**AVSS protocol** over  $\mathbb{Z}_q$  (with respect to  $\mathbf{e}$ ) allows a dealer  $D \in \{P_1, \dots, P_n\}$  to share  $a$  polynomials  $f_1, \dots, f_L \in \mathbb{Z}_q[x]_{<d}$  in such a way that each party  $P_j$  learns only  $f_1(e_j), \dots, f_L(e_j)$ . More precisely, it should satisfy a *security* property and a *completeness* property. The security property is captured by the ideal functionality  $\mathfrak{F}_{\text{AVSS}}$  in [Fig. 1](#). The completeness property is as follows: if the dealer is honest or any honest party outputs a value, then all honest

parties eventually output a value. Here, “eventually” means if and when all honest parties initiate the protocol and all messages sent between honest parties are delivered.

One can show that like security, completeness is a composable property in the UC framework. That is, one can naturally define the notion of completeness for a hybrid protocol which makes use of ideal functionalities as subprotocols, and it is a simple exercise to show that if we replace these idealized subprotocols by complete, concrete protocols that securely realize these functionalities, the resulting protocol is a concrete protocol that is secure and complete. Completeness for such a hybrid protocol is the same as for a concrete protocol, except that in addition to the condition that all messages sent between honest parties have been delivered, we also add the condition that all `RequestOutput` messages for the ideal functionalities have been delivered — in fact, concrete protocols are really hybrid protocols with an appropriate secure messaging ideal functionality, and so this is actually the same condition.



**Fig. 1.** The AVSS Ideal Functionality (parameterized by  $n, d, L, \mathbb{Z}_q, \mathbf{e}$ , and  $D$ )

### 3.2 Group-oriented AVSS

We now introduce the notion of a **group-oriented AVSS (GoAVSS)**. In this setting,  $q$  is a prime, and we are working over a group  $E$  of order  $q$  generated by  $\mathcal{G} \in E$ . We use additive notation for the group operation of  $E$ , and denote the identity element of  $E$  by  $\mathcal{O}$ .

Essentially, an  $(n, d, L)$ -**GoAVSS protocol** over  $E$  (with respect to  $\mathbf{e}$ ) is the same as an  $(n, d, L)$ -GoAVSS protocol over  $\mathbb{Z}_q$ , and each party (including the adversary) also obtains  $\mathcal{S}_\ell := f_\ell(0)\mathcal{G} \in E$  for each  $\ell \in [L]$ .

The security property of a GoAVSS protocol are captured by the ideal functionality  $\mathfrak{F}_{\text{GoAVSS}}$  in Fig. 2. We also require a completeness property, which is identical to that for an AVSS protocol.

## 4 Subprotocols

In this section, we review the subprotocols that our new GoAVSS protocol will need.

### 4.1 AVSS

Our GoAVSS protocol over  $E$  will be built using an ordinary AVSS protocol over  $\mathbb{Z}_q$  (as defined above in Section 3.1). In principle, any such AVSS protocol could be used. However, the protocol in [SS23] is well suited to the task, as it uses only “lightweight” cryptography (namely, hash functions)

$\mathfrak{F}_{\text{GoAVSS}}$

**Input**( $f_1, \dots, f_L$ ): this operation is invoked once by the dealer  $D$ , who inputs polynomials  $f_1, \dots, f_L \in \mathbb{Z}_q[x]_{<d}$  to  $\mathfrak{F}_{\text{GoAVSS}}$ . In response,  $\mathfrak{F}_{\text{GoAVSS}}$  sends the message  $\text{NotifyInput}(\{\mathcal{S}_\ell\}_{\ell \in [L]})$  to the ideal-world adversary, where  $\mathcal{S}_\ell := f_\ell(0)\mathcal{G}$ , for  $\ell \in [L]$ .

**RequestOutput**( $j$ ): after the input has been received, this operation may be invoked by the ideal-world adversary, who specifies  $j \in [n]$ . In response,  $\mathfrak{F}_{\text{GoAVSS}}$  sends to  $P_j$  the message

$$\text{Output}\left(\left\{\left(\mathcal{S}_\ell, f_\ell(e_j)\right)\right\}_{\ell \in [L]}\right).$$

**Fig. 2.** The GoAVSS Ideal Functionality (parameterized by  $n, d, L, E$  (which defines  $\mathbb{Z}_q$ ),  $e$ , and  $D$ )

and is quite efficient, both in terms of communication and computational complexity, especially on the so-called “optimistic path”, where no party provably misbehaves. More concretely, on the “optimistic path”, if  $q \approx 2^\lambda$ , the communication complexity (total number of bits transmitted in aggregate by all honest parties)

$$6nL\lambda + O(\lambda \cdot n^2 \log n + n^3).$$

Also, the computational complexity (running time of any one individual honest party) is as follows, assuming  $L = \Omega(n \log n)$ :

- for a party acting in its role as a receiver, the cost of decoding, encoding, hashing, and decrypting  $O(L\lambda)$  bits of data, performing  $O(L(1 + n/\lambda))$  arithmetic operations in  $\mathbb{Z}_q$ , and generating  $O(\lambda L(1 + n/\lambda))$  pseudorandom bits.
- for a party acting in its role as a dealer, the cost of encrypting, encoding, and hashing  $O(nL\lambda)$  bits of data, performing  $O(dL(1 + n/\lambda))$  arithmetic operations in  $\mathbb{Z}_q$ , and generating  $O(d\lambda L(1 + n/\lambda))$  pseudorandom bits.

Note that the dealer plays a role both as a dealer and as a receiver. Here, encoding and decoding refers to encoding and decoding data using an  $(n, n - 2t)$ -erasure code.

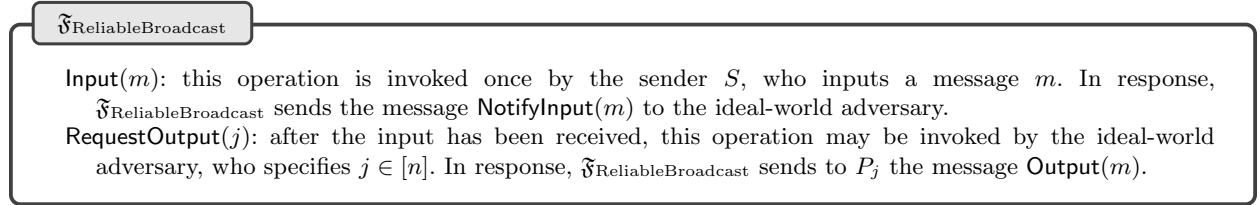
Note that this protocol makes use of a collision resistant hash function with a  $\kappa$ -bit output and a statistical security parameter  $\sigma$ , and we assume  $\max\{\kappa, \sigma\} \leq \lambda$ . This protocol uses a statistical test that has error bound of  $2^n/q$ . For large  $n$ , this test must be repeated several times, and this is what gives rise to the additive term  $n^3$  appearing in the communication complexity bound and the additive term  $n/\lambda$  appearing in the computational complexity bounds.

Note that when the protocol falls off the “optimistic path”, the communication and computational complexity may increase by a factor of  $O(n)$ . However, at least one misbehaving party will be identified and can be effectively removed from participating any further in the protocol.

In the above computational complexity estimates, we have not included the cost for the dealer of actually computing the shares  $f_\ell(e_j)$  for  $\ell \in [L]$ ,  $j \in [n]$ , which is a part of any AVSS protocol. For very large  $n$ , an asymptotically fast multi-point evaluation algorithm may be useful. However, for small to moderate sized  $n$ , it is more practical to use a simple Horner’s rule evaluation, which adds to the dealer’s cost  $\approx dnL$  multiplications and additions mod  $q$ . Moreover, if the evaluation points are  $1, \dots, n$  (which is typical), we can run many steps of Horner without any reductions mod  $q$ , making these steps relatively inexpensive.

## 4.2 Reliable broadcast

A **reliable broadcast** protocol allows a sender  $S$  to broadcast a single message  $m$  to  $P_1, \dots, P_n$  in such a way that all parties are guaranteed to receive the same message. More precisely, it should satisfy a *security* property and a *completeness* property. The security property is captured by the ideal functionality  $\mathfrak{F}_{\text{ReliableBroadcast}}$  in Fig. 3. The completeness property says that: if one honest party outputs a message, then every honest party eventually outputs a message; moreover, if  $S$  is honest, then every honest party eventually outputs a message.



**Fig. 3.** The Reliable Broadcast Ideal Functionality (parameterized by  $S$ )

A well-known Reliable Broadcast protocol is due to Bracha [Bra87]. However, its communication complexity (total number of bits transmitted in aggregate by all honest parties) is  $O(n^2 \cdot |m|)$ . Better communication complexity can be obtained using an protocol based on erasure codes. This approach was initially considered in [CT05], who give a protocol with communication complexity

$$3n|m| + O(\kappa \cdot n^2 \log n),$$

where  $\kappa$  is the output length of a collision-resistant hash. Also, the computational complexity (running time of any one individual honest party) is as follows, assuming  $|m| = \Omega(\kappa \cdot n \log n)$ :

- for a party acting in its role as a receiver, the cost of decoding, encoding, and hashing  $O(|m|)$  bits of data;
- for a party acting in its role as a dealer, the cost of encoding and hashing  $O(n|m|)$  bits of data operations in  $\mathbb{Z}_q$ .

Note that the dealer plays a role both as a dealer and as a receiver. Here, encoding and decoding refers to encoding and decoding data using an  $(n, n - 2t)$ -erasure code.

While there have been various subsequent improvements on reliable broadcast, [CT05] is efficient enough for our purposes.

## 4.3 Random Beacon

This is a protocol that reveals a value  $\omega$  chosen at random from an **output space**  $\Omega$ , in such a way that satisfies a *security* property and a *completeness* property. The security property is captured by the ideal functionality  $\mathfrak{F}_{\text{Beacon}}$  in Fig. 4. The main idea is that the adversary learns nothing about  $\omega$  until after at least one honest party initiates the protocol. The completeness property says that if at least  $t + 1$  honest parties initiate the protocol, every honest party eventually outputs a value.

A random beacon may be efficiently implemented using any AVSS protocol and any consensus protocol, using the standard technique of agreeing on a set of  $t + 1$  secret sharings, and then opening

$\mathfrak{F}_{\text{Beacon}}$

**Input():** This operation may be invoked once by each party  $P_j$ . If this is the first time this is invoked by any honest party,  $\mathfrak{F}_{\text{Beacon}}$  chooses  $\omega \in \Omega$  at random and sends  $\text{NotifyInput}(j, \omega)$  to the ideal-world adversary; otherwise,  $\mathfrak{F}_{\text{Beacon}}$  sends  $\text{NotifyInput}(j)$  to the ideal-world adversary.

**RequestOutput( $j$ ):** after the value  $\omega$  has been generated, this operation may be invoked by the ideal-world adversary, who specifies  $j \in [n]$ . In response,  $\mathfrak{F}_{\text{AVSS}}$  sends to  $P_j$  the message  $\text{Output}(\omega)$ .

**Fig. 4.** The Random Beacon Functionality  $\mathfrak{F}_{\text{Beacon}}$  (parameterized by output space  $\Omega$ )

all of them and adding them up. This will yield a beacon output that is an element of a finite field. If the output of the random beacon needs to be longer, it can be passed through a PRG or a hash function.

## 5 Our new GoAVSS protocol

Our new GoAVSS protocol is presented in Fig. 5.

- The protocol is parameterized by a subset  $R \subseteq \mathbb{Z}_q$ .
- The protocol makes use of
  - an instance of an AVSS subprotocol, which is invoked as an ideal functionality  $\mathfrak{F}_{\text{AVSS}}$ ;
  - two instances of a reliable broadcast subprotocol, both of which are invoked as the ideal functionality  $\mathfrak{F}_{\text{ReliableBroadcast}}$  (but there are really two separate instances here of this ideal functionality);
  - an instance of a random beacon, which is invoked as an ideal functionality  $\mathfrak{F}_{\text{beacon}}$ ; this beacon outputs

$$\{\gamma_\ell^{(k)}\}_{\ell \in [L], k \in [n]},$$

where each  $\gamma_\ell^{(k)} \in R \subseteq \mathbb{Z}_q$ ; an implementation may choose to instead use a beacon that outputs a short seed to a PRG, and then use the PRG to derive these values.

### 5.1 Security analysis of protocol $\Pi_{\text{GoAVSS1}}$

In the security analysis of this protocol, we make use of the following version of the Chernoff bound. Let  $S_{N,p}$  be the number of successes among  $N$  independent Bernoulli trials, each with success probability  $p$ . Let  $M$  be a nonnegative integer. We define

$$\text{Tail}(N, M, p) := \Pr[S_{N,p} \geq M]. \tag{1}$$

Then assuming  $0 < p < M/N < 1$ , we have

$$\text{Tail}(N, M, p) \leq 2^{-N \cdot H(M/N, p)}, \tag{2}$$

where

$$H(a, p) := a \log_2 \left( \frac{a}{p} \right) + (1 - a) \log_2 \left( \frac{1 - a}{1 - p} \right).$$

The dealer  $D \in \{P_1, \dots, P_n\}$  has input  $f_1, \dots, f_L \in \mathbb{Z}_q[x]_{<d}$ .

1. Dealer  $D$ :

- (a) Choose  $g^{(k)} \in \mathbb{Z}_q[x]_{<d}$  at random for  $k \in [n]$
- (b) Invoke the operation  $\text{Input}(\{g^{(k)}\}_{k \in [n]}, \{f_\ell\}_{\ell \in [L]})$  on  $\mathfrak{F}_{\text{AVSS}}$ .
- (c) Compute  $\mathcal{T}^{(k)} \leftarrow g^{(k)}(0)\mathcal{G} \in E$  for  $k \in [n]$  and  $\mathcal{S}_\ell \leftarrow f_\ell(0)\mathcal{G} \in E$  for  $\ell \in [L]$ .
- (d) Invoke the operation

$$\text{Input}(\{\mathcal{T}^{(k)}\}_{k \in [n]}, \{\mathcal{S}_\ell\}_{\ell \in [L]})$$

on  $\mathfrak{F}_{\text{ReliableBroadcast}}$ .

2. Each  $P_j$ :

- (a) Wait for  $\mathfrak{F}_{\text{AVSS}}$  to deliver the message

$$\text{Output}(\{w_j^{(k)}\}_{k \in [n]}, \{v_{\ell,j}\}_{\ell \in [L]})$$

and for  $\mathfrak{F}_{\text{ReliableBroadcast}}$  to deliver the message

$$\text{Output}(\{\mathcal{T}^{(k)}\}_{k \in [n]}, \{\mathcal{S}_\ell\}_{\ell \in [L]}).$$

- (b) Invoke the operation  $\text{Input}()$  on  $\mathfrak{F}_{\text{Beacon}}$ .

3. Each  $P_j$ : Wait for  $\mathfrak{F}_{\text{Beacon}}$  to deliver the message

$$\text{Output}(\{\gamma_\ell^{(k)}\}_{\ell \in [L], k \in [n]}),$$

with each  $\gamma_\ell^{(k)} \in R \subseteq \mathbb{Z}_q$ .

4. Dealer  $D$ :

- (a) For each  $k \in [n]$ , compute

$$h^{(k)} \leftarrow g^{(k)} + \sum_{\ell \in [L]} \gamma_\ell^{(k)} \cdot f_\ell \in \mathbb{Z}_q[x]_{<d}.$$

- (b) Invoke the operation  $\text{Input}(\{h^{(k)}\}_{k \in [n]})$  on  $\mathfrak{F}_{\text{ReliableBroadcast}}$ .

5. Each  $P_j$ :

- (a) Wait for  $\mathfrak{F}_{\text{ReliableBroadcast}}$  to deliver the message  $\text{Output}(\{h^{(k)}\}_{k \in [n]})$ .
- (b) For each  $k \in [n]$ , check that  $h^{(k)} \in \mathbb{Z}_q[x]_{<d}$  and that

$$h^{(k)}(e_j) = w_j^{(k)} + \sum_{\ell \in [L]} \gamma_\ell^{(k)} \cdot v_{\ell,j}.$$

- (c) Check that

$$h^{(j)}(0)\mathcal{G} = \mathcal{T}^{(j)} + \sum_{\ell \in [L]} \gamma_\ell^{(j)} \mathcal{S}_\ell.$$

- (d) If these checks do not pass then *abort*. Otherwise:

- i. send a “vote” message to all parties;
- ii. wait for “vote” messages from  $n - t$  distinct parties;
- iii. output  $\{(\mathcal{S}_\ell, v_{\ell,j})\}_{\ell \in [L]}$ .

**Fig. 5.** A GoAVSS protocol



**Theorem 5.1 (Security of  $\Pi_{\text{GoAVSS1}}$ ).** *Suppose that  $t < d \leq n - 2t$  and  $|R| \geq 2^\rho$ . Then assuming that*

$$\text{Tail}(n - t, n - 2t, 1/|R|) \leq 2^{-n \cdot (\rho - 2)/3} \quad (3)$$

*is negligible, protocol  $\Pi_{\text{GoAVSS1}}$  securely and completely realizes  $\mathfrak{F}_{\text{GoAVSS}}$  in the  $(\mathfrak{F}_{\text{AVSS}}, \mathfrak{F}_{\text{Beacon}}, \mathfrak{F}_{\text{ReliableBroadcast}})$ -hybrid model.*

*Proof.* If the dealer is honest, we just have to show how to simulate the information that the protocol leaks to the adversary, given the information leaked by  $\mathfrak{F}_{\text{GoAVSS}}$ , which is  $\{v_{\ell,j}\}_{\ell \in [L], j \in \mathcal{C}}$  and  $\{\mathcal{S}_\ell\}_{\ell \in [L]}$ , and given the fact that the simulator is also allowed to generate the output  $\{\gamma_\ell^{(k)}\}_{\ell \in [L], k \in [n]}$  of the random beacon in advance. This is straightforward. For each  $k \in [n]$ , the simulator can simply generate the polynomial  $h^{(k)}$  at random, and then compute the adversary's shares of  $g^{(k)}$  as

$$w_j^{(k)} \leftarrow h^{(k)}(e_j) - \sum_{\ell \in [L]} \gamma_\ell^{(k)} v_{\ell,j}$$

for  $j \in \mathcal{C}$ , and the group element  $\mathcal{T}^{(k)}$  as

$$\mathcal{T}^{(k)} \leftarrow h^{(k)}(0)\mathcal{G} - \sum_{\ell \in [L]} \gamma_\ell^{(k)} \mathcal{S}_\ell.$$

Now assume the dealer is corrupt. Consider the point in time where some honest party collects “vote” messages from  $n - t$  distinct parties in Step 5(d) for the very first time. Let  $\mathfrak{E}$  be the event that  $\mathcal{S}_\ell \neq f_\ell(0)\mathcal{G}$  for some  $\ell \in [L]$ . So long as  $\mathfrak{E}$  does not occur, correctness and completeness will hold. We argue that  $\Pr[\mathfrak{E}]$  is at most (3), from which the theorem will follow.

By the logic of the protocol, at this point in time, at least  $n - 2t$  out of a set of  $n - t$  honest parties must have found that all checks passed in Step 5. Call these parties “accepting” parties. Since  $n - 2t \geq d$ , and all polynomials here have degree less than  $d$ , this means all of the polynomials  $h^{(k)}$  for  $k \in [n]$  were correctly computed, that is,

$$h^{(k)} = g^{(k)} + \sum_{\ell \in [L]} \gamma_\ell^{(k)} f_\ell$$

for all  $k \in [n]$ . Therefore, for each “accepting” party  $P_j$ , we have

$$\mathcal{T}^{(j)} + \sum_{\ell \in [L]} \gamma_\ell^{(j)} \mathcal{S}_\ell = h^{(j)}(0)\mathcal{G} = g^{(j)}(0)\mathcal{G} + \sum_{\ell \in [L]} \gamma_\ell^{(j)} f_\ell(0)\mathcal{G}.$$

Here, the first equality holds by the logic of test in Step 5(c), and the second holds because of the fact that the polynomials  $h^{(k)}$  for  $k \in [n]$  were correctly computed.

For each  $k \in [n]$ , if the values

$$\{f_\ell\}_{\ell \in [L]}, \{\mathcal{S}_\ell\}_{\ell \in [L]}, \mathcal{T}^{(k)}, \text{ and, } g^{(k)}$$

are fixed, and  $\mathcal{S}_\ell \neq f_\ell(0)\mathcal{G}$  for some  $\ell \in [L]$ , and if we choose the values  $\gamma_\ell^{(k)} \in R$  at random for  $\ell \in [L]$ , then the probability that

$$\mathcal{T}^{(k)} + \sum_{\ell \in [L]} \gamma_\ell^{(k)} \mathcal{S}_\ell = g^{(k)}(0)\mathcal{G} + \sum_{\ell \in [L]} \gamma_\ell^{(k)} f_\ell(0)\mathcal{G}$$

is clearly at most  $2^{-\rho}$ .

Thus, if  $\mathfrak{E}$  is to occur, at least  $n - 2t$  parties out of a set of  $n - t$  honest parties must each find that a certain event occurs, where these events are independent of one another and each occurs with probability  $\leq 2^{-\rho}$ . Therefore,  $\Pr[\mathfrak{E}] \leq \text{Tail}(n - t, n - 2t, 2^{-\rho})$ , and the bound (3) follows from the Chernoff bound (2).  $\square$

*Remark 5.1.* The error bound (3) is the probability that a corrupt dealer breaks either the correctness or completeness property of the protocol. So to achieve an error bound of  $2^{-\sigma}$ , where  $\sigma$  is a statistical security parameter, we can set  $\rho := \lceil 3\sigma/n \rceil + 2$ . In a typical implementation,  $R$  will be chosen to be the set of all  $\rho$ -bit numbers. For a typical setting such as  $\sigma = 80$ , and with a network of size, say,  $n = 49 = 3 \cdot 16 + 1$ , we can set  $\rho = 7$ . By using such small numbers, we get a protocol that is very attractive from a computational complexity perspective. We analyze the computational complexity of this protocol in detail below in Section 5.2.  $\square$

*Remark 5.2.* Note that the analysis here is in the static corruption model, and the error bound (3) reflects that. If we carried out the analysis in an adaptive corruption model, it would be

$$\text{Tail}(n, n - 2t, 1/|R|) \leq 2^{-n \cdot (\rho - 3)/3}.$$

Indeed, in a adaptive corruption model, before corrupting any parties, the adversary could then hope that  $n - 2t$  of these parties cast a positive vote, and then corrupt  $t$  of the remaining parties and make them cast positive votes as well. While our recommended implementation of the AVSS subprotocol in [SS23] is only analyzed in the static corruption model, it is suggested there that it may be secure against adaptive corruptions in the random oracle model. If this adaptive error bound is used, we would have to add 1 to the value of  $\rho$  to achieve the same level of security, so this would generally have a mild impact on the performance of the protocol.  $\square$

*Remark 5.3.* The random beacon may output a short seed, which is passed to a PRG to derive all of the challenge values  $\gamma_\ell^{(k)}$ . When we do this, we need to add to the error probability in Theorem 5.1 a term that measures advantage of a certain adversary (whose running time is essentially the same as that of the original adversary) in distinguishing the PRG output from random.  $\square$

*Remark 5.4.* We could replace the voting step in Step 5(d) by a simple Bracha-like agreement protocol (see, for example, Section 3.3 of [SS23]). This would make the completeness property of the protocol unconditional, at the cost of an extra round of communication.  $\square$

## 5.2 Complexity analysis of protocol $\Pi_{\text{GoAVSS1}}$

We analyze both computational and communication complexity. For each complexity metric, we consider the cost contributed by the body of the protocol, as well the cost contributed by the recommended implementation of the subprotocols. For subprotocols that have an “optimistic path”, where no party provably misbehaves, we consider only the cost on this optimistic path. Arguably, over a long run of the system where parties that provably misbehave are effectively removed from the system, this is the only cost that matters. To simplify matters, we assume the  $d = t + 1$  and  $n = 3t + 1$ .

We will also consider the “amortized” cost, in two different senses.

- One is the amortized cost per “raw sharing” in one run of  $\Pi_{\text{GoAVSS1}}$ , where one party is the designated dealer, and the other parties are receivers. As there are  $L$  sharings generated in one run of  $\Pi_{\text{GoAVSS1}}$ , and it is assumed that  $L$  is very large, we can effectively ignore the cost of any protocol steps and subprotocols whose cost does not grow with  $L$  (such as the random beacon). We will generally assume that  $L = \Omega(n^2)$ . This assumption is needed only to account for the overhead associated with the polynomials  $h^{(k)}$  for  $k \in [n]$ . Otherwise, we could get by just under the weaker assumption that  $L = \Omega(n \log n)$  to account for the overheads in the subprotocols. For large  $n$ , we can use the protocol in [Section 5.3](#) to get by with a weaker assumption on  $L$ .
- The other is the amortized cost per “processed sharing”. This is the sum of the amortized cost per “raw sharing” over  $n$  runs of the protocol, with different designated dealer in each run, divided by the number of sharings output by the batch randomness extraction procedure (which is at least  $t + 1$ ). This is ultimately the number of interest in applications.

Throughout this section,  $\lambda := \lceil \log_2(q) \rceil$  and  $\sigma$  is a statistical security parameter. We assume  $\sigma \leq \lambda$ . In typical settings, we will have  $\lambda = 256$  and  $\sigma = 80$ .

**Communication complexity** As usual, we measure communication complexity as the sum, over all honest parties  $P$  and all parties  $Q$ , of the total number of bits that  $P$  sends to  $Q$  over a point-to-point channel. With AVSS and reliable broadcast implemented as described in [Section 4](#), it is easily seen that the amortized communication complexity per processed sharing is  $O(n\lambda)$ . This assumes elements of  $E$  can be encoded in  $O(\lambda)$  bits. Also, the communication complexity is well balanced: each party transmits (and receives)  $O(\lambda)$  bits of data per processed sharing. In fact, it is easy to see from the estimates given in [Section 4](#) that (for large enough  $L$ ), each party essentially transmits (and receives) a total of 18 scalars and 9 group elements per processed sharing.

**Computational complexity** To state computational costs, we use the terms “full scalar/group multiplication”, “tiny scalar/group multiplication”, and “tiny scalar/scalar multiplication” introduced in [Section 2.1](#).

*Amortized cost per raw sharing.*

- For a party acting in its role as a receiver, the amortized cost per raw sharing in the body of our protocol is:
  - 1 tiny scalar/group multiplication and addition.
  - $n$  tiny scalar/scalar multiplications.
- For a party acting in its role as a dealer, the amortized cost per raw sharing in the body of our protocol is:
  - 1 full scalar/group multiplication.
  - $n(t + 1)$  tiny scalar/scalar multiplications.

Note that the dealer plays a role both as a dealer and as a receiver.

*Amortized cost per processed sharing.* The amortized cost per processed sharing is equal to  $n/(t + 1)$  times the amortized receiver cost per raw sharing plus  $1/(t + 1)$  times the amortized dealer cost per raw sharing. This gives us an amortized cost per processed sharing of:

- $1/(t + 1)$  full scalar/group multiplications.

- 3 tiny scalar/group multiplications.
- $4n$  tiny scalar/scalar multiplications.

*Example 5.1.* Consider  $n = 49 = 3 \cdot 16 + 1$  and we set the statistical security parameter  $\sigma := 80$ . So we set  $\rho := 7$ . Let us also assume that  $\lambda = 256$ . We compute the amortized cost per processed sharing. This is:

- 1/17 full scalar/group multiplications.  
As discussed in [Appendix A](#), we will assume that we can compute a full scalar/group multiplication using 64 group additions. Dividing by 17, we get an amortized cost per processed sharing of less than 4 group additions.
- 3 tiny scalar/group multiplications.  
Using the bucket method in [Appendix B](#), using just a single window of size 7, the amortized cost of one tiny scalar/group multiplication is just one group addition. Multiplying by 3, we get an amortized cost per processed sharing of 3 group additions.
- 196 tiny scalar/scalar multiplications.

So we estimate the total amortized cost per processed sharing to be So we get a total of 7 group additions plus 196 tiny scalar/scalar multiplications.

Now let us add in the amortized cost of applying our new constructions super-invertible matrices (see [Section 6](#)). Using the construction  $U'_{17,32,q}$  in [Section 6.2](#), this amortized cost is  $(17 \cdot (32 - 9) + 1)/17 \approx 23$  group additions. However, because  $t$  is small enough, we can use the better construction  $S'_{17,16,q}$  in [Section 6.3](#), which has an amortized of just 16 group additions.

This is 16 group additions and 16 additions in  $\mathbb{Z}_q$  per processed sharing. So this gives us:

- $23 = 7 + 16$  group additions;
- 196 tiny scalar/scalar multiplications;
- 16 additions in  $\mathbb{Z}_q$ .

Note that we perform no more than almost 10 times as many tiny scalar/scalar multiplications as we do group additions. However, each of these is tiny scalar/scalar multiplications relatively cheap (essentially, the cost of multiplying a  $\rho$ -bit integer by a  $\lambda$ -bit integer, with no modular reduction), so that in any reasonably optimized implementation, the cost of these should be significantly less than the cost of the group additions.

Indeed, we conservatively estimate that each tiny scalar/scalar multiplication is about 1/40th the cost of a group addition over an elliptic curve such as `secp2565k1`. In such a group addition, we have to perform several multiplications mod  $p$ , where  $p$  is also a 256-bit prime. We estimate the number of such multiplications mod  $p$  to be about 10 (although this depends on many factors). On a 64-bit machine, assuming 256 bit numbers are represented as four 64-bit limbs (i.e., four base- $2^{64}$  digits), one multiplication mod  $p$  costs four 1-limb-by-4-limb multiplications. While reduction mod  $p$  is not free, we ignore that cost here. In contrast, a tiny scalar/scalar multiplication should take about the same cost as one 1-limb-by-4-limb multiplication. This is where we get the factor of  $40 = 10 \cdot 4$ .<sup>4</sup> Thus, we estimate the cost of the tiny scalar/scalar multiplications to be the equivalent of about 5 group additions.

Note that if we use the adaptive error bound as discussed in [Remark 5.2](#), we have to use  $\rho = 8$ . However, this does not change the amortized cost of our algorithm at all (but we may have to use a slightly somewhat larger batch size to compensate).  $\square$

<sup>4</sup> On CPUs that support SIMD instructions, a single tiny scalar/scalar multiplication can take just a few cycles, by using “redundant representation”, so that products can be accumulated within a SIMD register without carries.

*Other costs.* The above cost analysis left out some costs that we shall now discuss. As we shall see, these other costs should not have any significant impact on the overall performance.

*Cost of subprotocols.* In terms of amortized computational complexity per sharing (raw or processed), the only subprotocols that matter are AVSS and reliable broadcast (specifically, the first reliable broadcast invoked by the dealer in Step 1(d)). We assume all subprotocols are implemented as in [SS23]. We focus here on the AVSS subprotocol, as this is the most expensive. This protocol uses a statistical test that has error bound of  $2^n/q$ . For  $n$  of the size we are mostly interested in here, this error probability will be sufficiently small, and the test only needs to be repeated once. If we count the number of arithmetic operations in  $\mathbb{Z}_q$  performed by this AVSS protocol, the amortized number of such operations per processed sharing is easily seen to be 4. Thus, this will not have any measurable impact on the overall GoAVSS protocol.

For larger values of  $n$ , this test has to be repeated several times. However, for  $\lambda = 256$  and  $\sigma = 80$ , and for  $n$  up to 1000, we have to repeat the test at most 5 times, giving an amortized cost per processed sharing of 20 arithmetic operations in  $\mathbb{Z}_q$ , which still has no measurable impact on the overall GoAVSS protocol.

The above does not take into account the cost for the dealer of actually computing the shares  $f_\ell(e_j)$  for  $\ell \in [L]$ ,  $j \in [n]$ , which is a part of any AVSS protocol. As discussed in Section 4.1, for small to moderate size  $n$ , we can use Horner’s rule, which will add the cost of performing  $\approx n$  multiplications and additions mod  $q$  to the amortized cost per processed sharing. Moreover, if the evaluation points are  $1, \dots, n$  (which is typical), we can run many steps of Horner without any reductions mod  $q$ , making these steps relatively inexpensive (similar in cost to a tiny scalar/scalar multiplication).

*Cost of erasure codes.* We also consider the cost of performing the encoding and decoding operations for the erasure codes used in the reliable broadcast protocol and similar subprotocols used in the AVSS protocol in [SS23]. Note that all of these encoding and decoding operations work only on public data, so there are no restrictions on the type of codes and algorithms that may be used. The computational cost of these protocols is very implementation dependent. But even if we use naive, quadratic-time algorithms, the amortized computational cost per processed sharing will be essentially  $O(nw)$  word operations per processed sharing, where  $w$  is the number of machine words needed to represent  $q$ . Here, the implied big-O constant is quite reasonable. Note that such erasure codes generally will not perform arithmetic operations in  $\mathbb{Z}_q$  — rather, they work over a domain that is conveniently suited to the machine instruction set. For example, chip manufacturers have recently added special-purpose instructions to more efficiently support the type of operations needed to implement erasure codes [DGK18a]. For larger  $n$ , asymptotically fast algorithms may be used. In any case, it seems likely that in any good implementation of erasure codes, this cost will not greatly impact the overall cost of GoAVSS.

*Cost of PRGs and hashing.* We also must consider the cost of generating pseudorandom bits as a the output for the random beacon, both in the GoAVSS protocol and in the implementation of the AVSS subprotocol. In typical settings, where  $n = O(\lambda)$ , the amortized number of bits per processed sharing that need to generated is  $O(\lambda)$ . It seems likely that with any good implementation of a pseudorandom bit generator, especially on a CPU with good cryptographic hardware support [FLdO18,DGK18b], this cost will not greatly impact the overall cost of GoAVSS. Similarly, in the underlying AVSS and reliable broadcast subprotocols, much of the data sent and received needs to be hashed. The amortized number of bits per processed sharing that need to be hashed is  $O(\lambda)$ . It seems likely that with any good implementation of a hash function,

especially on a CPU with good cryptographic hardware support [FLdO18], this cost will not greatly impact the overall cost of GoAVSS.

### 5.3 A variation for large $n$

For large  $n$ , the following variation of  $\Pi_{\text{GoAVSS1}}$  may be preferred.

Suppose that instead of having  $k$  range over  $[n]$  in  $\Pi_{\text{GoAVSS1}}$ , it instead ranges over  $[K]$  for some parameter  $K$  in the range  $1, \dots, n$ . Moreover, in Step 5(c) of the protocol, Party  $P_j$  checks that

$$h^{(k)}(0)\mathcal{G} = \mathcal{T}^{(k)} + \sum_{\ell \in [L]} \gamma_{\ell}^{(k)} \mathcal{S}_{\ell}, \quad (4)$$

where  $k := (j \bmod K) + 1 \in [K]$ . Let us call this variation  $\Pi_{\text{GoAVSS2}}$ .

**Theorem 5.2 (Security of  $\Pi_{\text{GoAVSS2}}$ ).** *Suppose that  $t < d \leq n - 2t$  and  $|R| \geq 2^{\rho}$ . Then assuming that*

$$\text{Tail} \left( K, \left\lceil (n - 2t) / \lceil n/K \rceil \right\rceil, 1/|R| \right) \quad (5)$$

*is negligible, protocol  $\Pi_{\text{GoAVSS2}}$  securely and completely realizes  $\mathfrak{F}_{\text{GoAVSS}}$  in the  $(\mathfrak{F}_{\text{AVSS}}, \mathfrak{F}_{\text{Beacon}}, \mathfrak{F}_{\text{ReliableBroadcast}})$ -hybrid model.*

*Proof.* The main thing we have to do is to bound the probability that a corrupt dealer “wins” by breaking either the correctness or completeness property of the protocol. As it simplifies the proof, we will assume the adversary adaptively corrupts parties. For an index  $k \in [K]$ , let us say that party  $P_j$  is “associated with”  $k$  if  $k = (j \bmod K) + 1$ , and let us say that  $k$  is “good” if the equality (4) holds. To win, there must be at least  $n - 2t$  parties associated with good indices, and the adversary can corrupt  $t$  other parties. Since there are at most  $\lceil n/K \rceil$  parties associated with any one index, this means there must be at least  $\lceil (n - 2t) / \lceil n/K \rceil \rceil$  good indices.  $\square$

*Remark 5.5.* The error bound (5) is the probability that a corrupt dealer breaks either the correctness or completeness property of the protocol.  $\square$

*Remark 5.6.* The point of this protocol is that when  $n = 3t + 1$  amortized computational cost of this protocol per processed sharing becomes

- $1/(t + 1)$  full scalar/group multiplications,
- 3 tiny scalar/group multiplications,
- $4K$  tiny scalar/scalar multiplications,

instead of

- $1/(t + 1)$  full scalar/group multiplications,
- 3 tiny scalar/group multiplications,
- $4n$  tiny scalar/scalar multiplications.

However, the value of  $\rho$  defining the size of the “tiny” scalars may be smaller in  $\Pi_{\text{GoAVSS1}}$  than in  $\Pi_{\text{GoAVSS2}}$ .  $\square$

*Example 5.2.* Let us return to the parameters in [Example 5.1](#), with  $n = 49 = 3 \cdot 16 + 1$ ,  $\lambda = 256$ , and  $\sigma = 80$ . We will estimate the amortized cost per processed sharing contributed by the tiny scalar/group multiplications and the tiny scalar/scalar multiplications. In that example, we estimated this cost to be (the equivalent of) 8 group additions (3 for the tiny scalar/group multiplications and the equivalent of at most 5 for the tiny scalar/scalar multiplications). In this example, we shall estimate the amortized cost per tiny scalar/group multiplication for a given value of  $\rho$  to be  $\lceil \rho/8 \rceil$ , using the bucket method in [Appendix B](#) with a window size of 8.

We can set  $K = 1$  and  $\rho = 80$ , and the amortized cost per cost per tiny scalar/group multiplication is 10 group additions, which contributes a total of 30 group additions to the amortized cost per processed sharing. The cost contributed by the tiny scalar/scalar multiplications may be safely ignored.

We can set  $K = 16$  and  $\rho = 16$  — although the statement of [Theorem 5.2](#) does not directly imply an error bound of  $2^{-80}$ , the proof is easily adapted (one sees that at least 6 out of 16 indices must be good in order for the adversary to win). So the amortized cost per cost per tiny scalar/group multiplication is 2 group additions, which contribute a total of 6 group additions to the amortized cost per processed sharing. As we did in [Example 5.1](#), we may safely estimate the cost of the 64 tiny scalar/scalar multiplications per processed sharing to be the equivalent of less than 2 group additions. So this setting yields roughly the same computational performance as the settings in [Example 5.1](#).  $\square$

As the above example illustrates, for large values of  $n$ , and  $\lambda \approx 256$ , it probably makes sense to use  $\Pi_{\text{GoAVSS2}}$  with values of  $K$  in the range 10–20 rather than  $\Pi_{\text{GoAVSS1}}$ . One advantage of doing so is that in order to achieve the stated amortized complexity bounds, we only need to assume  $L = \Omega(n \cdot \max\{\log n, K\})$ , rather  $L = \Omega(n^2)$  as we did with  $\Pi_{\text{GoAVSS1}}$ . Perhaps in many situations it even makes sense to just use  $K = 1$ , which is the simplest version of any of our protocols and still achieves quite good performance.

#### 5.4 A variation with packing

As we will discuss below in [Section 8.3](#), there are situations where we may want to do “packed” secret sharing. The idea is that instead of having a polynomial  $f \in \mathbb{Z}_q[x]$  encode a single secret as the value of  $f$  at the point 0, we instead have it encode several secrets, as the value of  $f$  at several points, say  $e'_1, \dots, e'_p$ . In this setting, we can easily generalize the notion of a “packed” GoAVSS protocol. It is an easy exercise to generalize our GoAVSS protocol to this case as well. Instead of broadcasting  $\{g^{(k)}(0)\mathcal{G}\}_k$  and  $\{f_\ell(0)\}_\ell$  in Step 1(d), the dealer would instead broadcast  $\{(g^{(k)}(e'_1)\mathcal{G}, \dots, g^{(k)}(e'_p)\mathcal{G})\}_k$  and  $\{(f_\ell(e'_1), \dots, f_\ell(e'_p))\}_\ell$ . The test in Step 5(c) would then be performed  $p$  times, once for each evaluation point  $e'_1, \dots, e'_p$  (but we use the same values  $\gamma_\ell^{(k)}$  for each test). The error probability (3) would have to be replaced with

$$\text{Tail}(n - t, n - 2t, p/|R|).$$

If we now amortize over the number  $p$  of packed secrets, the amortized number of group operations does not change at all (but the fact that the error bound increases means that we may have to increase slightly the size of  $\rho$  to get the same error bound, so it will actually increase if we want to maintain the same overall error bound). However, the amortized cost of all other operations will decrease by a factor of  $p$ .



## 6 Super-invertible matrices from Pascal

Let  $A \in \mathbb{Z}_q^{M \times N}$  be a matrix with  $M$  rows and  $N \geq M$  columns. The matrix  $A$  is called **super-invertible** if every collection of  $M$  of its columns is linearly independent. The definition of super-invertible matrices and their application to multi-party computation comes from [HN06]. Such matrices also have been studied in the context of error-correcting codes, as they are precisely the generator matrices of MDS (maximum distance separable) codes — see, for example, [RL89].

### 6.1 The symmetric Pascal matrix

The well-known symmetric Pascal matrix  $S_N \in \mathbb{Z}^{N \times N}$  is an  $N \times N$  matrix whose  $i, j$  entry is defined to be

$$S_{i,j} := C_{i+j,i} = C_{i+j,j} = \frac{(i+j)!}{i!j!},$$

where the indices  $i, j$  start at 0 and  $C_{n,k}$  is the binomial coefficient

$$C_{n,k} = \binom{n}{k}.$$

For example,

$$S_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

Define the matrix  $S_{M,N,q} \in \mathbb{Z}_q^{M \times N}$  to be the matrix consisting of the first  $M$  rows of  $S_N$  with entries mapped from  $\mathbb{Z}$  to  $\mathbb{Z}_q$ .

**Theorem 6.1 (The symmetric Pascal matrix is super-invertible).** *Assuming  $q \geq N$ , the matrix  $S_{M,N,q}$  is super-invertible.*

*Proof.* First, for  $i = 0, 1, \dots, M-1$  define the polynomial

$$p_i(x) := \frac{(x+1)(x+2) \cdots (x+i)}{1 \cdot 2 \cdots i} \in \mathbb{Z}_q[x].$$

Note that since  $M \leq N \leq q$ , the denominator in the definition of  $p_i(x)$ , which is the image of  $i!$  in  $\mathbb{Z}_q$ , is nonzero and hence invertible. So we see that  $p_i(x)$  is a polynomial of degree  $i$ .

Second, for  $j = 0, 1, \dots, N-1$ , observe that the  $i, j$  entry of  $S_{M,N,q}$  is equal to  $p_i(j)$ . This follows from the definition, as the  $i, j$  entry of  $S_{M,N,q}$  is

$$\frac{(i+j)!}{i!j!} = \frac{(j+1)(j+2) \cdots (j+i)}{1 \cdot 2 \cdots i} = p_i(j).$$

So now consider any subset  $\mathcal{J} \subseteq \{0, \dots, N-1\}$  of cardinality  $M$  and form the matrix  $R \in \mathbb{Z}_q^{M \times M}$  consisting of the columns of  $S_{M,N,q}$  indexed by  $j \in \mathcal{J}$ . By the above observations, we can write

$$R = P \cdot V, \tag{6}$$

where  $P \in \mathbb{Z}_q^{M \times M}$  is the matrix whose  $i$ th row, for  $i = 0, 1, \dots, M - 1$ , is the coefficient vector of the polynomial  $p_i(x)$ , and

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ j_1 & j_2 & \dots & j_M \\ \vdots & \vdots & & \vdots \\ j_1^{M-1} & j_2^{M-1} & \dots & j_M^{M-1} \end{pmatrix} \in \mathbb{Z}_q^{M \times M}$$

is a Vandermonde matrix. Since  $N \leq q$ , the entries  $j_1, \dots, j_M \in \mathcal{J}$  are distinct when mapped to  $\mathbb{Z}_q$  and hence  $V$  is nonsingular. Since  $P$  is a lower diagonal matrix with nonzero entries along the diagonal,  $P$  is also nonsingular. Since  $R$  is a product of nonsingular matrices, it follows that  $R$  is nonsingular, which proves the theorem.  $\square$

Now suppose we are given as input a column vector of group elements  $\mathbf{x} = (\mathcal{X}_0, \dots, \mathcal{X}_{N-1})^\top \in E^{N \times 1}$  and want to compute a column vector of group elements  $\mathbf{y} = (\mathcal{Y}_0, \dots, \mathcal{Y}_{M-1})^\top \in E^{M \times 1}$ , where  $\mathbf{y} = S_{M,N,q}\mathbf{x}$ . We next show how this can be done using  $M \cdot (N - 1)$  group additions.

Define group elements  $\mathcal{X}_k^{(i)}$  for  $k = 0, \dots, N - 1$  and  $i = -1, 0, \dots, M - 1$ , as follows:

$$\mathcal{X}_k^{(-1)} := \mathcal{X}_k \quad \text{for } k = 0, \dots, N - 1 \quad (7)$$

and for  $i = 0, \dots, M - 1$

$$\mathcal{X}_{N-1}^{(i)} := \mathcal{X}_{N-1}^{(i-1)} \quad \text{and} \quad \mathcal{X}_k^{(i)} := \mathcal{X}_k^{(i-1)} + \mathcal{X}_{k+1}^{(i)} \quad \text{for } k = N - 2, \dots, 1, 0. \quad (8)$$

Then for  $i = 0, \dots, M - 1$  and  $k = 0, \dots, N - 1$ , we have:

$$\mathcal{X}_k^{(i)} = \sum_{j=k}^{N-1} C_{i+j-k,i} \mathcal{X}_j. \quad (9)$$

This follows from a simple induction proof and Pascal's rule  $C_{a,b} = C_{a-1,b-1} + C_{a-1,b}$ . It follows that  $\mathcal{Y}_i = \mathcal{X}_0^{(i)}$  for  $i = 0, \dots, M - 1$ , which we can clearly compute using a total of  $M \cdot (N - 1)$  group additions.

## 6.2 The upper-triangular Pascal matrix

The upper-triangular Pascal matrix  $U_N \in \mathbb{Z}^{N \times N}$  is an  $N \times N$  upper-triangular matrix whose  $i, j$  entry is defined to be

$$U_{i,j} := C_{j,i},$$

where the indices  $i, j$  start at 0 and we define  $C_{j,i} := 0$  for  $j < 0$ . For example,

$$U_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define the matrix  $U_{M,N,q} \in \mathbb{Z}_q^{M \times N}$  to be the matrix consisting of the first  $M$  rows of  $U_N$  with entries mapped from  $\mathbb{Z}$  to  $\mathbb{Z}_q$ .

**Theorem 6.2 (The upper Pascal matrix is super-invertible).** *Assuming  $q \geq N$ , the matrix  $U_{M,N,q}$  is super-invertible.*

*Proof.* The proof is the same as that of Theorem 6.1, except we define

$$p_i(x) := \frac{(x-0)(x-1)\cdots(x-(i-1))}{1 \cdot 2 \cdots i} \in \mathbb{Z}_q[x].$$

Now suppose we are given as input a column vector of group elements  $\mathbf{x} = (\mathcal{X}_0, \dots, \mathcal{X}_{N-1})^\top \in E^{N \times 1}$  and want to compute a column vector of group elements  $\mathbf{y} = (\mathcal{Y}_0, \dots, \mathcal{Y}_{M-1})^\top \in E^{M \times 1}$ , where  $\mathbf{y} = U_{M,N,q}\mathbf{x}$ . We next show how this can be done using  $M \cdot (N - (M + 1)/2)$  group additions.

Define group elements  $\mathcal{X}_k^{(i)}$  as in (7) and (8). Then by (9), we have  $\mathcal{Y}_i = \mathcal{X}_i^{(i)}$  for  $i = 0, \dots, M-1$ . These values can clearly be computed using

$$(N-1) + (N-2) + \cdots + (N-M) = M \cdot (N - (M + 1)/2)$$

group additions.

Theorem 6.2 was already proved in [HDSY16]. They also prove that if one augments  $U_{M,N,q}$  by adding the column  $(0, \dots, 0, 1)^\top$ , the resulting  $M \times (N + 1)$  matrix  $U'_{M,N,q}$  is also super-invertible. This result is easily seen to apply to any  $M \times N$  super-invertible matrix  $A$  with the property that the  $(M - 1) \times N$  submatrix consisting of the first  $M - 1$  rows of  $A$  is also super-invertible.

The augmented matrix  $U'_{M,N,q}$ , then, is an  $M \times (N + 1)$  super-invertible matrix that we can multiply on the right by a vector of group elements using just  $M \cdot (N - (M + 1)/2) + 1$  group additions. In our application to batch randomness extraction, we need an  $(n - 2t) \times (n - t)$  super-invertible matrix. We can therefore apply the construction  $U'_{M,N,q}$  with  $M = n - 2t$  and  $N = n - t - 1$ , and we get a cost of  $(n - 2t) \cdot (n/2 - 3/2) + 1$  group additions.

### 6.3 Better super-invertible matrices from hyper-invertible matrices

Let  $A \in \mathbb{Z}_q^{M \times N}$  be a matrix with  $M$  rows and  $N$  columns. The matrix  $A$  is called **hyper-invertible** if every square submatrix is invertible — that is, if for every  $k \in [\min\{M, N\}]$ , and every subset  $\mathcal{I} \subseteq [M]$  of size  $k$  and every subset  $\mathcal{J} \subseteq [N]$  of size  $k$ , the  $k \times k$  submatrix of  $A$  with rows indexed by  $\mathcal{I}$  and columns indexed by  $\mathcal{J}$  is invertible. The definition of hyper-invertible matrices and their application to multi-party computation comes from [BH08]. Such matrices also have been studied in the context of error-correcting codes, where they are called *super-regular matrices* — see, for example, [RL89].<sup>5</sup> In coding theory, one of the main features of such a matrix, as observed, for example, in [RL89], is that if  $I$  is the  $M \times M$  identity matrix, then  $A' := [I \mid A]$  is a generator matrix for a systematic MDS code. In the language of cryptographers,  $A'$  is a super-invertible matrix. One of the nice features of  $A'$  is that the cost of multiplying  $A'$  on the right by a vector of group elements is just  $M$  additions plus the cost of multiplying  $A$  on the right by a vector of group elements. This feature was noted in [BHK<sup>+</sup>23], and used to get a more efficient batch randomness extraction algorithm.

An interesting question to which do not know a very good answer is: under what conditions the symmetric Pascal matrix  $S_N$  is hyper-invertible mod  $q$  (i.e.,  $S_{N,N,q}$  is hyper-invertible)? An obvious necessary condition is that  $q \geq 2N$  (this is clear from the decomposition (6), as otherwise some entries of  $S_N$  are zero mod  $q$ ). We do know that  $S_N$  is (*strictly*) *totally positive*, meaning that every

<sup>5</sup> Some authors use the term super-regular to define a somewhat broader class of matrices. See, for example, [AN20].

square submatrix of  $S_N$  has positive determinant — this follows from the corresponding result for  $U_N$  (see [GV85]), along with the Cauchy-Binet identity and the fact that  $S_N = U_N^\top U_N$  (see [Fal01], who also outlines a direct proof of the total positivity of  $S_N$  based on bidiagonal decomposition [GP96]). Thus, we can trivially show that  $S_N$  is hyper-invertible mod  $q$  if every square submatrix of  $S_N$  has determinant less than  $q$ . A convenient fact about totally positive matrices is that we can bound the determinant of such a matrix by the product of its diagonal entries — this is a special case of Fischer’s inequality (see [Fal01]).<sup>6</sup> Because of the fact that entries of  $S_N$  only increase as we move down or to the right, we may conclude that every square submatrix of  $S_N$  has determinant at most

$$B_N := C_{2,1} \cdot C_{4,2} \cdot C_{2(N-1),N-1}.$$

By direct calculation, one can verify that  $B_N < 2^{237}$  for  $N \leq 17$ . This means that for  $N \leq 17$  and  $q > 2^{237}$ , the symmetric Pascal matrix  $S_N$  is hyper-invertible mod  $q$ . These ranges of  $N$  and  $q$  are already useful in a range of practical cryptographic applications.

Generalizing the above argument, the matrix  $S_{M,N,q}$  is hyper-invertible provided

$$\prod_{j=1}^{\min(M,N)} C_{M+N-2j,N-j} < q. \tag{10}$$

In any case, for those values of  $M$ ,  $N$ , and  $q$  for which  $S_{M,N,q}$  is hyper-invertible, if  $I$  is the  $M \times M$  identity matrix, the augmented  $M \times (M + N)$  matrix  $S'_{M,N,q} := [I \mid S_{M,N,q}]$  is super-invertible, and the cost of multiplying  $S'_{M,N,q}$  on the right by a vector of group elements is just  $M \cdot N$  group additions. In our application to batch randomness extraction, we need an  $(n - 2t) \times (n - t)$  super-invertible matrix. We can therefore apply the construction  $S'_{M,N,q}$  with  $M = n - 2t$  and  $N = t$ , and we get a cost of  $(n - 2t) \cdot t$  group additions.

## 7 Some details on the online phase of the signing protocol

### 7.1 Opening a shared secret: error correction and batching

Since our construction in the offline phase produces shared secrets without corresponding polynomial commitments, to the protocol to open a shared secret must rely on error correcting codes — specifically, Reed-Solomon codes.

The fact that  $n > 3t$  means that we can use the well-known technique of **online error correction**. The paper [CP17] contains a very nice description of this technique. We recall here a few of the details. Assume, for simplicity, that  $n = 3t + 1$  and that the secret shares are the values of a polynomial of degree at most  $t$  over  $\mathbb{Z}_q$ .

Suppose every party sends its share of a secret to a party  $P$ . Party  $P$  can wait for  $2t + 1$  shares. Upon receiving these, he can interpolate through the first  $t + 1$  shares to get a polynomial  $f$  of degree at most  $t$ , and then check that  $f$  is consistent with the remaining shares. If this check passes,  $P$  can be sure that  $f$  is correct and so the secret is  $f(0)$ . If this check fails,  $P$  can be sure that one of the  $2t + 1$  shares it has received is incorrect. On the one hand,  $P$  cannot be sure which share is incorrect; on the other hand,  $P$  can certainly afford to wait for another share from an honest party. The details of the protocol from this stage forward do not matter too much (it involves running a

<sup>6</sup> See <https://nhigham.com/2021/07/20/what-is-a-totally-nonnegative-matrix/> for a useful survey on totally positive matrices.

Reed-Solomon decoder a number of times). Eventually, not only will  $P$  recover the secret, it will also find out who was lying in the initial stage. Thus, while a corrupt party may get away with lying to  $P$  once, and forcing him to do extra work, he will never bother  $P$  again —  $P$  may safely ignore him from that point on. Thus, in a long run of the system, the cost of the initial stage, which is not much more expensive than a standard polynomial interpolation, is all that matters — the cost of the Reed-Solomon decoders does not matter at all.

Now suppose we want to publicly open a shared secret, so that every party broadcasts its share, and every party recovers the secret as above. The communication cost is  $O(n^2\lambda)$ , assuming  $q \approx 2^\lambda$ . Also, the computation costs  $O(n^2)$  arithmetic operations in  $\mathbb{Z}_q$ , assuming a naive algorithm for polynomial interpolation. Both of these costs can be reduced by a factor of  $O(n)$  by using a well-known “batching” technique, also described in [CP17]. To exploit this technique, we need to process batches of at least  $t + 1$  openings at a time. There is a penalty for this, however — opening such a batch of shared secrets now costs two rounds of communication, instead of 1.

Consider the online signing phase for the threshold Schnorr protocol such as the one obtained from our constructions here. In a heavily loaded system, it may make sense to exploit this batching technique. Indeed, in some settings, the extra communication latency may not matter too much. For example, for a distributed signing service driven by a blockchain, where both signing requests and results are placed on the blockchain, the extra latency incurred by this batched opening technique may be easily masked by the latency inherent in the blockchain.

## 7.2 Protecting against presignature attacks

As already mentioned, if presignatures are used directly, an adversary can carry out a subexponential time attack. See [Sho23] for details.

One mitigation strategy analyzed in [Sho23] (building on ideas from [BHK<sup>+</sup>23]) is to obtain a random  $\delta \in \mathbb{Z}_q$  from a Random Beacon after a signing request has been made. If the presignature to be used for that signing request ( $[r], \mathcal{R}$ ) (which was visible to the adversary before making the signing request), we “tweak” it by replacing it by the presignature ( $[r + \delta], \mathcal{R} + \delta\mathcal{G}$ ), and use this “tweaked” presignature to carry out the signing request.

As already mentioned above in Section 7.1, we may wish to anyway batch signing requests in some way. If we do this, we can actually use the same value  $\delta \in \mathbb{Z}_q$  and the same group element  $\delta\mathcal{G}$  to securely tweak all the presignatures in the batch (as briefly considered in [Sho23] as “batch re-randomization”). This allows us to amortize the cost of generating  $\delta$  and computing  $\delta\mathcal{G}$  over the size of the batch. We can easily implement the Random Beacon using a shared random value generated earlier by an AVSS protocol. So the cost of generating  $\delta$  is just one opening of a shared secret (see above in Section 7.1 for some details on this). The cost of computing the group element  $\delta\mathcal{G}$  is just one scalar/group multiplication. As the communication and computational complexity of these steps is amortized over the size of the batch, these costs can be safely ignored. However, one cost that cannot be amortized away is latency: the Random Beacon costs one round of latency. If we also implement the batching techniques in Section 7.1, this results in a latency of 3 rounds of communication to satisfy a signing request. As mentioned in Section 7.1, in some settings, such as a blockchain setting, the extra latency can be masked by other latencies.

Alternatively, if we want to minimize latency of signing requests at all costs, we can use the FROST-like mitigation analyzed in a general setting in [Sho23]. With this mitigation, we can derive  $\delta$  from a hash function applied to (among other things) the message to be signed. This eliminates the extra latency incurred by the Random Beacon, but it comes at a cost. To carry

out this mitigation, for one signature, we actually need two presignatures  $([r], \mathcal{R})$  and  $([s], \mathcal{S})$ , and the “tweaked” signature used to sign the message is  $([r + \delta s], \mathcal{R} + \delta \mathcal{S})$ . This means that in the offline phase, the amortized cost per signature will increase by a factor of 2, as we now need two presignatures for every signature. In addition, for every signature, we now have to do a scalar/group multiplication — a rather expensive one, as now the group element is not fixed. Indeed, our offline protocol is so efficient that now the computational cost of this one scalar/group multiplication may well be more than the more the amortized computational cost of the offline phase.

Perhaps a good compromise is a “hybrid” strategy, where we produce in the offline phase a large number of presignatures to be used with the Random Beacon strategy, and a smaller number to be used with the FROST strategy. When the system is lightly loaded, we can use the FROST strategy to generate signatures, and when heavily loaded, the Random Beacon strategy. This would guarantee that under light loads (or for select users willing to pay a higher price), the system has minimal latency, while under heavy loads, we increase latency to ensure good throughput is maintained. Although we have not carried it out in detail, it should be possible to extend the security analysis in [Sho23] to cover this hybrid strategy.

## 8 Comparison to SPRINT

Up until now, the state of the art in threshold Schnorr is the SPRINT protocol [BHK<sup>+</sup>23].

We first review the SPRINT protocol at a very high level. The SPRINT protocol has a number of parameters that can be tuned. One of these parameters is an “efficiency parameter” that allows one to trade off between efficiency and resilience. For now, we focus on the setting of this parameter that (like our protocol) maintains optimal resilience, that is, any number of parties up to  $t < n/3$  may be corrupt. For simplicity, we assume  $n = 3t + 1$  here. We also assume  $q \approx 2^\lambda$  and that elements of  $E$  can be encoded as  $2\lambda$ -bit strings (which corresponds to the standard affine representation for a point on an elliptic curve).

Like the threshold ECDSA protocol in [GS22], SPRINT makes use of the standard technique of implementing a AVSS protocol using “polynomial commitments”. Unlike [GS22], SPRINT uses so-called Feldman commitments, which are perfectly binding but not perfectly hiding. For a polynomial  $f = \sum_k f_k x^k \in \mathbb{Z}_q[x]$ , its commitment is  $\mathcal{F} := \sum_k \mathcal{F}_k x^k \in E[x]$ , where  $\mathcal{F}_k := f_k \mathcal{G} \in E$ . We also define the notation  $f\mathcal{G} := \mathcal{F}$ . The evaluation of such a polynomial commitment at a point  $\alpha \in \mathbb{Z}_q$  is defined as  $\mathcal{F}(\alpha) := \sum_k \alpha^k \mathcal{F}_k \in E$ . Clearly, if  $\mathcal{F} = f\mathcal{G}$ , then  $\mathcal{F}(\alpha) = f(\alpha)\mathcal{G}$ . Thus,  $\mathcal{F}(\alpha)$  is a commitment to  $f(\alpha)$ . For the AVSS, we assume evaluation points  $e_1, \dots, e_n$ , which should be distinct nonzero elements of  $\mathbb{Z}_q$ . In practice, one would likely use  $e_j = j$  for  $j \in [n]$ .

In their protocol, each party  $P_i$  runs an AVSS protocol acting as a dealer where all parties  $P_1, \dots, P_n$ , act as receivers. The dealer  $P_i$  chooses a random polynomial  $f_i \in \mathbb{Z}_q[x]$  of degree at most  $t$ , computes  $\mathcal{F}_i \leftarrow f_i \mathcal{G} \in E[x]$ , and broadcasts  $\mathcal{F}_i$  to all parties, and sends each party  $P_j$  its secret share  $v_{i,j} := f_i(e_j) \in \mathbb{Z}_q$  over a private channel. Each party  $P_j$  can verify its share  $v_{i,j} \in \mathbb{Z}_q$  by checking that  $v_{i,j} \mathcal{G} = \mathcal{F}_i(e_j)$ .

Then, the protocol agrees on a set of  $\mathcal{I} \subseteq \{1, \dots, n\}$  of  $n - t = 2t + 1$  dealers who have had their dealings sufficiently well-validated. [BHK<sup>+</sup>23] introduces a specific agreement protocol for this purpose. The details of this protocol are not significant here, except to point out that at the end of the protocol, we have a set  $\mathcal{J}$  of  $2t + 1$  parties, each of which is either a corrupt party or an honest party that holds shares of all sharings produced by the dealers in  $\mathcal{I}$ . This is a consequence of that fact that their AVSS protocol does satisfy a “completeness” property like ours.

These dealings are then combined using a super-invertible matrix  $W$ . To this end, the matrix  $W$  must be multiplied on the right by the column vector  $\{\mathcal{F}_i(0)\}_{i \in \mathcal{I}}$  to get  $t + 1$  ephemeral public keys needed to produce or verify a signature. To generate a signature, each party can easily generate signature share by a simple local computation and broadcast this to other parties. Parties can then use this data to compute signature shares, which are just linear combinations of their secret shares, and which are then “opened” to produce a signature. As a consequence of the fact that their AVSS protocol is incomplete, these signature cannot be “opened” using error-correcting codes, as in Section 7.1. Nevertheless, each party can collect  $t + 1$  such shares and optimistically interpolate and test if the resulting signature share is valid. On the pessimistic path, where this fails, a party must compute a polynomial commitment as a linear combination of the polynomial commitments  $\{\mathcal{F}\}_{i \in \mathcal{I}}$ , and use this to validate individual signature shares. We shall analyze only the optimistic path.

### 8.1 Communication complexity

To make an apples-to-apples comparison of the communication complexity, we estimate the amortized communication complexity per signature on the optimistic path, where communication complexity is defined as the sum, over all honest parties  $P$  and all parties  $Q$ , of the total number of bits that  $P$  sends to  $Q$  over a point-to-point channel. While SPRINT was designed and analyzed on top of a specific type of broadcast channel, no matter how this channel is implemented, broadcasting 1 bit to  $n$  parties will require 1 bit to be delivered to each on  $n$  parties, and so will contribute a term of  $n$  to the communication complexity as we have defined it, which ultimately corresponds to how communication bandwidth must be modeled as a finite resource in the real world. Indeed, assuming parties are located in different data centers, for each broadcast bit, each party must download that bit, and so if we just consider download bandwidth, this is the right measure.

In the offline phase, each party broadcasts  $t + 1$  group elements, and sends  $n$  scalars over point-to-point channels, leading to a communication complexity of

$$\approx 2n^2(t + 1)\lambda + n^2\lambda$$

per dealer. This excludes many overheads which we assume can be ignored by appropriate additional batching.<sup>7</sup> To get the amortized cost per signature, we multiply by  $n$  (the number of dealers) and divide by  $t + 1 \approx n/3$  (the size of the batch in the batch randomness extraction protocol). Thus, the amortized communication complexity per signature is  $\approx (2n^2 + 3n)\lambda$ . Their protocol is very symmetric, so in fact, each party transmits  $\approx (2n + 3)\lambda$  bits per signature.

In the online phase, each party broadcasts a signature share, which means communication complexity per signature of  $\approx n^2\lambda$ . Again, the protocol is very symmetric, so each party transmits  $\approx n\lambda$  bits per signature. For security reasons, signing requests are processed in batches of size  $t + 1$ .

Thus, both the offline and online phases of SPRINT has a communication complexity per signature of  $O(n^2\lambda)$  in both the offline and online phases. In contrast, our protocol has a communication complexity per signature of  $O(n\lambda)$  in both the offline and online phases, assuming signing requests are also processed in batches of size  $t + 1$ , as discussed in Section 7.1. Perhaps one could reduce the communication complexity of SPRINT’s offline phase by using the “compact polynomial commitment” scheme of [KZG10], but that would restrict the use of the protocol to pairing-friendly elliptic

<sup>7</sup> As presented, the SPRINT protocol becomes insecure if additional batching is allowed. This is discussed in [Sho23], as well as other approaches that could be used instead to avoid this problem.



curves  $E$ . Perhaps one could reduce the communication complexity of SPRINT’s online phase by using the same batching technique we discussed in [Section 7.1](#). However, it is not at all clear how to do this in an “optimistic” way without losing the ability to correctly identify a misbehaving party (this stems from the fact that their AVSS protocol is incomplete, which precludes the use of error correction).

For a more concrete comparison, recall that in our protocol, each party transmits 18 scalars and 9 group elements per signature in the offline phase. This translates to  $36\lambda$  bits. Compare this to  $(2n+3)\lambda$  bits for SPRINT. Thus, already for  $n > 16$ , our protocol will exhibit better communication complexity in the offline phase. As for the online phase, in our protocol, each party transmits 6 scalars per signature, which translates to  $6\lambda$  bits. Compare this to  $n\lambda$  for SPRINT. Thus, already for  $n > 6$ , our protocol will exhibit better communication complexity in the online phase.

*Example 8.1.* As we did above in [Example 5.1](#), consider the case  $n = 49 = 3 \cdot 16 + 1$ , considering the communication complexity for both the offline and online phase per signature, the communication complexity of SPRINT is  $150/42 \approx 4.7$  times that of our protocol.  $\square$

## 8.2 Computational complexity

Here, we focus on the running time of an individual party in the offline phase. We break the analysis up into two stages: Stage 1, which includes just the AVSS portion, and Stage 2, which includes just the batch randomness extraction. The offline phase of both the SPRINT protocol and ours can be decomposed into these two stages.

**Stage 1: AVSS** In this stage, we are just considering the AVSS protocol, but not the computation of the super-invertible-matrix/vector product. To state computational costs, in addition to the terms “full scalar/group multiplication”, “tiny scalar/group multiplication”, and “tiny scalar/scalar multiplication” introduced in [Section 2.1](#), we introduce the following term:

**Short scalar/group multiplication:** This is a scalar/group multiplication, where the scalars are small (at most  $n$ ) and public, and the group element is variable and public.

For SPRINT, each party as a dealer does one polynomial commitment computation and  $n$  local verifications of a share. One polynomial commitment computation costs  $t + 1$  full scalar/group multiplications. For a local verification of a share, a party  $P_j$  must evaluate a polynomial commitment  $\mathcal{F}$  at a point  $e_j$ , and must compute one full scalar/group multiplication. For the evaluation of a polynomial commitment, we may exploit the fact that the evaluation points  $e_i$  are small integers. In this setting, the best way to do this is to use Horner’s rule, which takes  $t$  short scalar/group multiplications and  $t$  group additions. Thus, the total running time of each party during this stage is dominated by the cost of  $(t + 1) + n$  full scalar/group multiplications, and  $nt$  short scalar/group multiplications, and  $nt$  group additions. We divide these numbers by  $t + 1 \approx n/3$  to get the amortized cost per signature:

- 4 full scalar/group multiplications,
- $n$  short scalar/group multiplications, and
- $n$  group additions.

Compare these to the corresponding numbers for our protocol, stated in [Section 2.1](#), which we repeat here for convenience. Let  $\rho = \lceil 3\sigma/n \rceil + 2$ , where  $\sigma$  is a statistical security parameter

(typically  $\sigma = 80$ ). This is the parameter used to define tiny scalar/group multiplications and tiny scalar/scalar multiplications.

- $1/(t + 1)$  full scalar/group multiplications,
- 3 tiny scalar/group multiplications, and
- $4n$  tiny scalar/scalar multiplications.

*Example 8.2.* As we did above in [Example 5.1](#), consider the case  $n = 49 = 3 \cdot 16 + 1$  and  $\lambda = 256$ . As we did in that example, we estimate the cost of a full scalar/group multiplications to be 64 group additions. For short scalar/group multiplications, let  $A(m)$  denote the length of the shortest addition-subtraction chain for  $m$  (which is the right metric when  $E$  is an elliptic curve, as subtraction is free). See <https://oeis.org/A128998> for a list of values of  $A(m)$  for  $m = 1, \dots, 87$ . In the range  $m = 1, \dots, 49$ , the maximum value for  $A(m)$  is 7, and so we use this as our estimate for the cost of a short scalar/group multiplication.

So the amortized cost per signature is

$$\approx 4 \cdot 64 + 49 \cdot 7 + 49 \approx 648$$

group additions per signature in this stage.

As we saw in [Example 5.1](#), with security parameter  $\sigma = 80$ , the cost of this stage of our protocol per signature is 7 group additions, plus the equivalent of 5 group additions to account for the cost of the tiny scalar/scalar multiplications. So we estimate the cost of this stage of our protocol to be the equivalent of 12 group additions per signature.

Thus, we conclude that our new protocol in this stage is about  $648/12 \approx 54$  times faster than SPRINT. Of course, this estimate must be taken with a grain of salt, as there are a number of built-in assumptions, and a number of algorithmic overheads (both for SPRINT and our new protocol) which we have not fully taken into account. That said, we believe that to a first approximation, this is a fair comparison that indicates that our new protocol potentially provides a significant performance improvement.  $\square$

**Stage 2: Batch randomness extraction** In this stage, the only thing we are interested in is the computational cost of multiplying an  $M \times N$  super-invertible matrix on the right by a column vector of group elements, where  $N = t + 1$  and  $N = 2t + 1$ . Both the SPRINT protocol and our protocol behave identically in this stage.

While [\[BHK<sup>+</sup>23\]](#) does consider some constructions of super-invertible matrices, these constructions will not yield very efficient algorithms for moderately sized  $n$  (up to a few hundred). Prior to this work, for such moderately sized  $n$ , the best approach known was to use a Vandermonde matrix, where each column consists of powers of a small evaluation. We could, for example, use evaluation points  $0, 1, \dots, 2t$ ; however, in the setting where group negation is essentially free (as in the case of an elliptic curve), a better approach would be to use the evaluation points  $0, \pm 1, \dots, \pm t$ . The cost then would be  $2t^2$  short scalar/group multiplications (with all scalars now in the range  $1, \dots, t$ ) and  $t(2t - 1) + 2t = t(2t + 1)$  group additions.

With our new construction for a super-invertible matrix, that cost becomes  $(t + 1) \cdot 2t$  group additions.

*Example 8.3.* Let us continue with our example with  $n = 49 = 3 \cdot 16 + 1$  and  $\lambda = 256$ . For the Vandermonde construction, we estimate the cost of computing  $1 \cdot \mathcal{G}_1, 2 \cdot \mathcal{G}_2, \dots, 16\mathcal{G}_{16}$  to be

$\sum_{m=1}^{16} A(m) = 54$ , where  $A(m)$  is defined as in [Example 8.2](#). Multiplying this by  $2t$ , the total cost of computing the  $2t^2$  short scalar/group multiplications is  $32 \cdot 54 = 1728$  additions. We perform an additional  $16 \cdot (2 \cdot 16 + 1) = 528$  group additions for a total of  $1728 + 528 = 2256$  group additions. We divide by 17 to get the amortized cost per signature, so about 133 group additions per signature.

With our new construction, the cost is  $17 \cdot 16 = 272$  group additions (here, we are using the construction  $S'_{17,16,q}$  in [Section 6.3](#)). We divide by 17 to get the amortized cost per signature, so 16 group additions per signature. So our new construction is  $133/16 \approx 8$  times faster.

Putting together the amortized cost per signature in both stages of the offline phase, SPRINT with the Vandermonde matrix construction costs  $648 + 133 \approx 781$  group additions, while our new protocol with the new matrix construction costs  $12 + 16 \approx 28$  group operations.<sup>8</sup>

So considering the entirety of the offline phase, our new protocol with the new matrix construction is  $781/28 \approx 28$  times faster than SPRINT with the old matrix construction.  $\square$

### 8.3 Packing secrets

One of SPRINT's innovations is to exploit “packed” secret sharing. This is an idea that goes back at least to [\[FY92\]](#). The idea is that instead of having a polynomial  $f \in \mathbb{Z}_q[x]$  encode a single secret as the value of  $f$  at the point 0, we instead have it encode several secrets, as the value of  $f$  at several points. Let  $p$  be a “packing” parameter. Recall that we are assuming evaluation points  $1, \dots, n$  for the secret shares. Then we can use the evaluation points  $-(p-1), \dots, -1, 0$  as the evaluation points for the packed secrets. Setting  $p > 1$  can lead to better amortized performance, as we discuss. However, this comes at a cost: the protocol is no longer optimally resistant. Indeed, if  $t$  is a bound on the number of corrupt parties, instead of requiring  $n > 3t$ , the SPRINT protocol requires that  $n > 3t + 2(p-1)$ .

The only change to Stage 1 of the offline phase of SPRINT is that now, in the AVSS protocol, instead of sharing polynomials of degree less than  $t+1$ , we need to share polynomials of degree less than

$$d := t + 1 + 2(p - 1).$$

The requirement that  $n > 3t + 2(p-1)$  is equivalent to  $n \geq d + 2t$ . For simplicity, let us assume that

$$n = d + 2t = 3t + 1 + 2(p - 1)$$

going forward.

We calculate the following cost per signature for Stage 1:

- $(1 + n/d)/p$  full scalar/group multiplications,
- $n/p$  short scalar/group multiplications, and
- $n/p$  group additions.

When Stage 1 completes, as before, we agree on a set  $\mathcal{I} \subseteq \{1, \dots, n\}$  of  $n-t$  dealers with corresponding polynomial commitments  $\{\mathcal{F}_i\}_{i \in \mathcal{I}}$ . Before proceeding to Stage 2, in need to additionally compute

$$\mathcal{F}_i(j) \quad \text{for all } i \in \mathcal{I} \text{ and } j \in \{-(p-1), \dots, -1, 0\}.$$

Let us call this Stage 1.5.

<sup>8</sup> This is a little bit higher than the cost reported in the Introduction, as we include here the cost of the tiny scalar/scalar multiplications, translated to an equivalent number of group operations.

We note that in [BHK<sup>+</sup>23], the authors also suggest to use a nonstandard polynomial commitment scheme, which instead of being  $f\mathcal{G}$  as we defined above, is the vector of group elements

$$f(-(p-1))\mathcal{G}, \dots, f(-1)\mathcal{G}, f(0)\mathcal{G}, f(1)\mathcal{G}, \dots, f(d-p)\mathcal{G}.$$

The point of this “optimization” is to ensure that the polynomial commitments themselves contain the group elements we need to compute in Stage 1.5, so that this step is free. However, this slows down Stage 1 significantly. Indeed, with this nonstandard polynomial commitment, for parties  $P_j$  with  $j > d-p$ , the cost of computing the share commitment  $f(j)\mathcal{G}$  from this nonstandard polynomial commitment will be  $d$  scalar/group multiplications, where the scalars are now full-sized elements of  $\mathbb{Z}_q$  (Lagrange interpolation coefficients). This is much greater than the cost of computing  $d-1$  short scalar/group multiplications and additions using standard polynomial commitments. In particular, any savings by obtained by packing will almost surely be more than offset by this extra cost.

A better approach is to use a standard polynomial commitment, and Stage 1.5 now costs  $(p-1)(n-t)(d-1)$  short scalar/group multiplications and group additions, where the scalars are now integers less than  $p$  in absolute value. To get the amortized cost per signature of doing this, we divide by  $p(n-2t)$ , to get an amortized cost of at most about  $2(d-1)$  short scalar/group multiplications and additions per signature.

For Stage 2, we now have to compute  $p$  matrix-vector products, where that matrix is an  $(n-2t) \times (n-t)$  super-invertible matrix, and the vectors are  $\{\mathcal{F}_i(j)\}_{i \in \mathcal{I}}$  for  $j = -(p-1), \dots, -1, 0$ , as calculated in Stage 1.5. Let us be generous and assume that we use our new matrix construction for batch randomness extraction. Assuming we use the construction  $U'_{n-2t, n-t-1, q}$  in Section 6.2, this contributes an amortized cost of about  $n/2 - 3/2$  group additions per signature (for each packed secret, we have to do one matrix-vector multiply, so the amortized cost of this operation is independent of  $p$ ). If the determinant bounds discussed in Section 6.3 allow, we can use the better construction  $S'_{n-2t, t, q}$  discussed there, which contributes an amortized cost of  $t$  group additions per signature.

*Example 8.4.* Let us continue our example with  $n = 49 = 3 \cdot 16 + 1$  and  $\lambda = 256$ .

Let us set the packing parameter  $p := 7$ . This means we can set  $t := 12$  so that  $n = 3t+1+2(p-1)$  as required, and  $d = 25$ . So compared to before, with no packing, the protocol can withstand only 12 corruptions, rather than 16. In Stage 1, the amortized cost per signature is about

$$(3 \cdot 64 + 49 \cdot 7 + 49)/7 \approx 84$$

group additions.

For Stage 1.5, the amortized cost per signature is at most  $2(d-1)$  short scalar/group multiplications and additions. With  $A(m)$  defined as in Example 8.2, we calculate this amortized cost as

$$\frac{(n-t)(d-1) \sum_{m=1}^{p-1} A(m)}{p(n-2t)} = \frac{37 \cdot 24 \cdot 11}{7 \cdot 25} \approx 56$$

group additions.

For Stage 2, we can in fact use the construction  $S'_{25, 12, q}$  in Section 6.3, since the bound (10) is satisfied. Therefore, the amortized cost is  $t = 12$  group additions.

So the total cost per signature in the offline phase is about  $84 + 56 + 12 = 152$  group additions.

Recall that we estimated the amortized cost per signature for our protocol in the offline phase at 28 group addition. So SPRINT with these parameters allows only 12 corruptions, while our allows

16, but SPRINT is still  $152/28 \approx 5.4$  times slower than ours (even after we also “gifted” SPRINT a better polynomial commitment strategy and a better super-invertible matrix construction).  $\square$

The above discussion focused on the impact of packing on the computational complexity of SPRINT in the offline phase. As for communication complexity, it reduces the amortized communication complexity in both the offline phase and online phases by a factor of  $p$ .

*Example 8.5.* Let us continue our example with  $n = 49 = 3 \cdot 16 + 1$  and  $\lambda = 256$ , but now with  $p = 7$ . We estimated that without packing SPRINT’s communication complexity was about 4.7 times that of ours. With packing, it is therefore  $\approx 4.7/7 \approx 0.67$  times that of ours.  $\square$

*Remark 8.1.* While we do not think it will be very useful in the context of threshold Schnorr, our GoAVSS protocol also generalizes to support packed secret sharing (see Section 5.4).  $\square$

## 8.4 Beyond SPRINT

As designed, SPRINT’s threshold Schnorr protocol does not use a batch GoAVSS protocol like ours (where each dealer shares many polynomials at a time). In fact, it cannot. The reason is that, as pointed out in [Sho23], SPRINT’s strategy for mitigating presignature attacks does not allow for this type of batch production of presignatures — there is a subexponential attack if this is done. However, as also pointed out in [Sho23], there are better mitigation strategies that could be used instead (see also Section 7.2 above). Given that fact, we might consider improving the performance of SPRINT’s GoAVSS protocol by making it batched.

To that end, in this section, we focus on the general problem of optimizing a “batched Feldman” GoAVSS protocol based on polynomial commitments. We focus exclusively on the computational cost to generate and validate dealings, and do not concern ourselves with issues of communication,<sup>9</sup> or the exact procedure used to ensure that sufficiently many receivers have validated a dealing (one could use techniques such as [BHK<sup>+</sup>23] or also [GS22]). However, we will assume that  $n - 2t$  honest parties must successfully validate a dealing for it to be accepted. The basic ideas we present here were already in sketched in Section 8.9 of [GS22], but we flesh them out here a bit more, and add some extra ideas.

Assume a dealer  $P_i$  is going to share a batch of polynomials  $f_{i,1}, \dots, f_{i,L} \in \mathbb{Z}_q[x]$ , each of degree at most  $t$ . As pointed out in [GS22], one standard trick that we can apply is that of “vector” polynomial commitments. Suppose we have group elements  $\mathcal{G}_1, \dots, \mathcal{G}_L$  (and assume that finding nontrivial linear relations between them is computationally as hard as computing discrete logarithms in  $E$ ). Then the vector polynomial commitment for these polynomials is just

$$\mathcal{F}_i = \sum_{\ell} f_{i,\ell} \mathcal{G}_{\ell} \in E[x].$$

A party  $P_j$  that is given corresponding putative shares  $v_{i,j,1}, \dots, v_{i,j,L}$  from party  $P_i$ , along with a vector commitment  $\mathcal{F}_i$ , can validate these shares by testing if

$$\sum_{\ell} v_{i,j,\ell} \mathcal{G}_{\ell} = \mathcal{F}_i(j). \tag{11}$$

---

<sup>9</sup> That said, the use of vector polynomial commitments, discussed below, can substantially reduce the communication complexity of the protocol.

However, in order for this protocol to be useful to us (to actually be a GoAVSS protocol), we have to augment the protocol so that the dealer  $P_i$  also publishes group elements

$$\mathcal{S}_{i,1} := f_{i,1}(0)\mathcal{G}, \dots, \mathcal{S}_{i,L} := f_{i,L}(0)\mathcal{G},$$

along with a non-interactive zero-knowledge proof that these group elements are consistent with  $\mathcal{F}_i$ . That is, if  $\mathcal{C}_i := \mathcal{F}_i(0)$ , the dealer must prove knowledge of  $s_{i,1}, \dots, s_{i,L} \in \mathbb{Z}_q$  such that  $\mathcal{C}_i = \sum_{\ell} s_{i,\ell}\mathcal{G}_{\ell}$  and

$$\mathcal{S}_{i,1} = s_{i,1}\mathcal{G}, \dots, \mathcal{S}_{i,L} = s_{i,L}\mathcal{G}.$$

A standard Sigma protocol for this runs as follows:

- Prover chooses  $s'_{i,\ell} \in \mathbb{Z}_q$  at random for  $\ell \in [L]$ , and sends

$$\mathcal{C}'_i := \sum_{\ell} s'_{i,\ell}\mathcal{G}_{\ell} \quad \text{and} \quad \{ \mathcal{S}'_{i,\ell} := s'_{i,\ell}\mathcal{G} \}_{\ell}$$

to the verifier.

- The verifier chooses  $e \in \mathbb{Z}_q$  at random and sends  $e$  to the prover.
- The prover responds to the verifier with

$$\{ z_{i,\ell} := s'_{i,\ell} + e \cdot s_{i,\ell} \}_{\ell}.$$

- The verifier then checks that

$$\sum_{\ell} z_{i,\ell}\mathcal{G}_{\ell} = \mathcal{C}'_i + e \cdot \mathcal{C}_i \tag{12}$$

and

$$z_{i,\ell}\mathcal{G} = \mathcal{S}'_{i,\ell} + e \cdot \mathcal{S}_{i,\ell} \quad (\text{for } \ell \in [L]). \tag{13}$$

Of course, we turn this into a non-interactive zero-knowledge proof using the Fiat-Shamir heuristic.

The verifier can combine the checks (11) and (12) into a single check by just checking a random linear combination of the two. So the cost now of this check is essentially one multi-scalar/group multiplication of length  $L$ , with full-sized private scalars.

Party  $P_j$  can easily reduce its cost of checking (13) by choosing  $r_1, \dots, r_L \in \mathbb{Z}_q$  at random, and then checking if

$$\left( \sum_{\ell} r_{\ell} z_{i,\ell} \right) \mathcal{G} = \sum_{\ell} r_{\ell} \mathcal{S}'_{i,\ell} + e \cdot \left( \sum_{\ell} r_{\ell} \mathcal{S}_{i,\ell} \right). \tag{14}$$

Moreover, the  $r_{\ell}$ 's may be small (and do not need to remain private). In fact, if we choose the  $r_{\ell}$ 's to be of length  $\rho$ , then we obtain the same security bound as in (3) in Theorem 5.1. This is because we are assuming that  $n - 2t$  honest parties must successfully validate a dealing for it to be accepted, so the same “distributed verification” argument made in Theorem 5.1 applies here as well.

With this, we can calculate the amortized cost per signature (or “processed sharing”) of this “batched Feldman” protocol as follows:

- F1)**  $2/(t + 1)$  full scalar/group multiplications (as a dealer, once for each  $\mathcal{S}_{\ell}$  and  $\mathcal{S}'_{\ell}$ ),
- F2)** 6 tiny scalar/group multiplications (as a receiver, for the check (14)), and
- F3)** 4 full\* scalar/group multiplications (1 as a dealer to compute the vector polynomial commitment, and 3 as a receiver to check the combination of (11) and (12)).

Here, we define a *full\* scalar/group multiplication* to be one where the scalars are full sized and private, but the resulting products are used as terms in a long summation (with fixed and public group elements).

We can compare these numbers to our new GoAVSS protocol:

- N1)**  $1/(t + 1)$  full scalar/group multiplications,
- N2)** 3 tiny scalar/group multiplications, and
- N3)**  $4n$  tiny scalar/scalar multiplications.

We see that the optimized batched Feldman protocol is twice as slow as our new GoAVSS protocol on the first two counts. How it ultimately compares depends on the relative cost of (F3) vs (N3).

*Example 8.6.* Let us compare using  $n = 49 = 3 \cdot 16 + 1$  and  $\lambda = 256$ . For the purposes of this comparison, we will assume the full\* scalar/group multiplications cost 32 group additions. This can be achieved, for example, using the bucket method in [Appendix B](#) with a window size of 8. However, we cannot actually use that algorithm directly, as it is not a constant time algorithm (as required here since the scalars are private); nevertheless, for the purposes of comparison, this is not an unreasonable estimate.

This leads to an estimate for (F3) above of  $4 \cdot 32 = 128$  group additions. Compare to the cost of (N3) above, which we estimated in [Example 5.1](#) to be equivalent to about 5 group additions.

Putting everything together, we find that this optimized batch Feldman is still

$$\approx (7 \cdot 2 + 128)/(7 + 5) \approx 12$$

slower than our new GoAVSS protocol.

We note that it may be possible to reduce the cost (F3) above even further, if each party  $P_i$  optimistically batches the corresponding checks from different dealers  $P_i$ . With luck,  $P_j$  can wait to collect data from all  $n$  dealers and perform all of its checks on those dealers in one batch. This would be difficult to implement easily in an asynchronous environment where some dealers may indeed be temporarily offline, if not fully malicious. But even if this works, the cost of (F3) would still be 1 full\* scalar/group multiplication (arising from the fact that  $P_j$ , itself acting as dealer, has to generate one vector polynomial commitment), and this optimized optimistic batch Feldman is still

$$\approx (7 \cdot 2 + 32)/(7 + 5) \approx 3.8$$

times slower than our new GoAVSS protocol.

But this is all really beside the point. All of this effort spent on optimizing the Feldman approach is aimed at reducing the cost of things that we can just simply avoid doing to begin with.  $\square$

## 9 Relation to [\[ABCP23\]](#)

The paper [\[ABCP23\]](#) presents (among other things) VSS, group-oriented VSS, and threshold Schnorr protocols. These protocols work exclusively in the *synchronous* communication model, rather than the asynchronous model considered here. While the goals and results of [\[ABCP23\]](#) are in many ways incomparable with our results, there is some overlap in techniques. We briefly discuss these here.



[ABCP23] begins with a synchronous VSS protocol that at its core is an application of the same statistical test used in the VSS protocol in [DN07] (and which is also the essentially the same statistical test presented in [BGR98] in a somewhat different context) — even though [ABCP23] takes a very different approach via Sigma protocols. The main technical differences are as follows.

- Unlike [DN07], [ABCP23] does not do any batching (i.e.,  $L = 1$  in our notation), and as such achieves significantly worse communication complexity than [DN07].
- Unlike [DN07], [ABCP23] makes use of computational assumptions, including a commitment scheme and a hash function which is modeled as a random oracle. The commitment scheme enables a simpler and more effective dispute resolution mechanism than [DN07]. The random oracle eliminates the need for a random beacon, saving on interaction.

We note that the paper [SS23] shows how [DN07] can be similarly enhanced by using commitments and/or a random oracle, although the focus of [SS23] is the asynchronous communication model. However, since [SS23] also incorporates the batching techniques of [DN07], it retains the better communication complexity of [DN07].

[ABCP23] also presents a synchronous, group-oriented VSS protocol. Besides our protocol, it is the only other group-oriented VSS protocol (synchronous or asynchronous) that we are aware of that is not based on polynomial commitments. At its core, this protocol is based on a statistical test that is essentially equivalent to that of our alternative protocol  $\Pi_{\text{GoAVSS2}}$  in Section 5.3 with degenerate parameters  $K = L = 1$  — even though [ABCP23] takes a very different approach via Sigma protocols. Because the parameters  $K = L = 1$  in  $\Pi_{\text{GoAVSS2}}$  are very far from optimal, this protocol achieves significantly worse communication and computational complexity than our main protocol  $\Pi_{\text{GoAVSS1}}$  (and, of course, only works in the synchronous model)

[ABCP23] also presents a synchronous threshold Schnorr protocol. We note that this protocol does not have a separate offline phase — the ephemeral DKG is only performed in response to a signing request. This means that this protocol does not have to mitigate against presignature attacks. It also means that the protocol cannot exploit batching. That is, it cannot exploit batching in the VSS protocol itself, nor can it exploit batched randomness extraction — both levels of batching are essential to getting a truly high-performance protocol. All of these factors together result in a threshold Schnorr protocol that only works in the synchronous communication model, and achieves significantly worse communication and computational complexity than our protocol (or SPRINT, for that matter). Per signature, if  $q \approx 2^\lambda$ , each party in their protocol (roughly speaking)

- transmits  $O(n^2\lambda)$  bits, and
- performs  $O(n\lambda)$  group additions,

whereas each party in SPRINT

- transmits  $O(n\lambda)$  bits, and
- performs  $O(\lambda + n \log n)$  group additions,

and each party in our protocol

- transmits  $O(\lambda)$  bits, and
- performs  $O(\lambda/n + n)$  group additions.<sup>10</sup>

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<sup>10</sup> All communication estimates are in terms of bits transmitted across a network consisting of point-to-point channels. Note that [ABCP23], like [BHK<sup>+</sup>23], measures broadcast communication complexity, but to make a fair and realistic



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comparison, we have translated everything to point-to-point communication complexity, as discussed above in Section 8.1). For all protocols, we have (somewhat simplistically) translated the computational cost of one (full sized) scalar/group multiplication to  $O(\lambda)$  group additions.

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## A Scalar/group multiplication algorithms

In this section, we discuss algorithms for scalar/group multiplication: given a generator  $\mathcal{G}$  for a group  $E$  of prime order  $q$ , along with a scalar  $x \in \mathbb{Z}_q$ , compute  $x\mathcal{G} \in E$ . More specifically, we assume that

- $\mathcal{G}$  is a public generator that remains fixed for many executions of the algorithm with different scalar values  $x$ , and so we may perform a precomputation that depends only on  $\mathcal{G}$ ;
- the scalar values  $x$  are secret, and so the algorithm should not leak any information about  $x$  via a “side channel”;
- $q$  is a  $\lambda$ -bit integer.

The problem of computing scalar/group products using precomputation has been extensively studied in both the cryptography and algorithms research communities. For example, in the cryptography research community, [BGMW92] and [LL94] are oft-cited papers that present algorithms for this problem. However, as pointed out by [BDLO12], these algorithms are essentially special cases of algorithms invented much earlier by Pippenger [Pip76]. In any case, with these algorithms, it is possible to perform a precomputation that produces a table of group elements, and using this table, we can then compute  $x\mathcal{G}$  for any given  $x \in \mathbb{Z}_q$  much faster than we can without the benefit of this precomputation. Asymptotically, with such a precomputed table, we can compute  $x\mathcal{G}$  using just  $(1 + o(1))\lambda/\log_2(\lambda)$  group additions.

Unfortunately, none of the papers above consider *side-channel attacks*, such as a timing attack [Koc96], which must be taken into account in the setting where  $x \in \mathbb{Z}_q$  must remain secret. To prevent such an attack, it is essential to use “constant time” algorithms — algorithms whose running time do not depend in any way on the secret input  $x$ . In any of the scalar/group multiplication algorithms discussed above that use a precomputed table, care must be taken to ensure that lookups into this table are constant time. Unfortunately, achieving this goal is very challenging on typical hardware, as evidenced by the CacheBleed attack [YGH16]. Because of this, library designers often take a very conservative approach to the lookup-table-based algorithms, which limits to a certain degree their applicability.

We describe here the approach taken in the `libsecp256k1` library (see <https://github.com/bitcoin-core/secp256k1>), and we assume this approach is used in our examples in this paper. This library implements algorithms for the elliptic curve `secp256k1` [Cer10]. For this curve,  $\lambda = 256$ . This library’s scalar/group multiplication algorithm works as follows. For a given scalar  $x \in \mathbb{Z}_q$ , we may write  $x$  in base 16 as  $\sum_{i=0}^{63} d_i 16^i$ , where each  $d_i \in \{0, \dots, 15\}$ . Then we have  $x\mathcal{G} = \sum_i d_i \mathcal{G}_i$ , where  $\mathcal{G}_i = 16^i \mathcal{G}$ . We precompute tables  $T_0, \dots, T_{63}$ , where each  $T_i$  is an array of size 16 and  $T_i[d] = d\mathcal{G}_i$  for  $d = 0, \dots, 15$ . Therefore, we can compute  $x\mathcal{G}$  as  $\sum_i T_i[d_i]$ , using just 64 table lookups and group additions (actually only 63 group additions are needed, but the algorithm in `libsecp256k1` uses 64 — see below). However, to make the table lookups constant time, for each  $i$ , the algorithm actually reads all 16 entries in  $T_i$ , performing 16 a constant-time conditional moves to store the value  $T_i[d_i]$ . This method of performing table lookups obviously incurs a nontrivial overhead. Based on benchmarks, we estimate that this overhead increases the overall running time of computing  $x\mathcal{G}$  by about 30%. However, using a better implementation, this overhead could be reduced to under 10% (see more details below in Appendix A.1).

We mention a couple of other details about `libsecp256k1`’s scalar/group multiplication algorithm. First, although the algorithm is designed to be constant time, just to be on the safe side, it actually computes  $x\mathcal{G}$  as  $(x + r)\mathcal{G} + r\mathcal{G}$ , where  $r \in \mathbb{Z}_q$  is generated at random but generally remains

fixed after that (this is why the group addition count for scalar/group multiplication is 64 rather than 63). Second, the algorithm takes additional steps to ensure that none of the table entries or any of the intermediate sums are the point at infinity on  $E$  — this avoids corner cases that are hard to deal with efficiently using a constant-time point addition algorithm. This is done very simply by defining  $T_i[d] = d\mathcal{G}_i + 2^iU$  for  $i = 0, \dots, 62$  and  $T_{63}[d] = d\mathcal{G}_{63} + (1 - 2^{63})U$ , where  $U$  is a group element of unknown discrete logarithm base  $\mathcal{G}$ .

### A.1 More details on libsecp256k1

We discuss here a few more details of the `libsecp256k1` library, which we downloaded from <https://github.com/bitcoin-core/secp256k1> and benchmarked using the provided benchmarking programs on a Macbook Pro with a 2.6 GHz 6-Core Intel Core i7-9750H (Coffee Lake) processor. For these benchmarks, we disabled the Intel Turbo Boost feature on the Macbook so that we get consistently reproducible results that we can easily translate into cycles. (This particular system supports a maximum Turbo Boost clock rate of 4.5 GHz, and we did run the same benchmarks with Turbo Boost enabled, and on a lightly loaded system saw proportionately faster results).

The `libsecp256k1` library represents points on the curve using either affine coordinates (that is, as a pair  $(X, Y)$ ) or Jacobian coordinates (that is, as  $(X, Y, Z)$ , which represents the point whose affine coordinates are  $(X/Z^2, Y/Z^3)$ ). It is generally preferred to work Jacobian coordinates where possible, as point addition is much faster — this is because no field inversions (which are quite expensive) are required to produce results in Jacobian coordinates.

Here are running times of some basic operations:

operation	semantics	constant?	ops	time ( $\mu s$ )
<code>group_double</code>	$J \leftarrow 2J$	yes	$3M + 4S$	0.18
<code>group_add_var</code>	$J \leftarrow J + J$	no	$12M + 4S$	0.44
<code>group_add_affine</code>	$J \leftarrow J + A$	yes	$7M + 5S$	0.35
<code>group_add_affine_var</code>	$J \leftarrow J + A$	no	$8M + 53$	0.32

In the column labeled “semantics”, we describe the operation using “ $J$ ” to indicate Jacobian coordinates, and “ $A$ ” to indicate affine coordinates. So “ $J \leftarrow J + A$ ” means to add a point in Jacobian coordinates and a point in affine coordinates, and store the result in Jacobian coordinates. The column labeled “constant?” reports whether the operation is constant time or not. The constant-time operation `group_add_affine`, requires that neither input is the point at infinity. The column labeled “ops” reports the number of field multiplications (“ $M$ ”) and the number of fields squarings (“ $S$ ”).

We described above `libsecp256k1`’s scalar/group multiplication algorithm, which takes as input a secret scalar  $x \in \mathbb{Z}_q$  and a fixed, public generator  $\mathcal{G} \in E$ , and computes  $x\mathcal{G}$  using 64 constant-time table lookups and group additions. The group elements in the table are stored using affine coordinates, while intermediate results are stored using Jacobian coordinates. So the group additions are performed using `group_add_affine`. The time to compute these additions is therefore estimated to be  $64 \cdot 0.35\mu s = 22.4\mu s$ . We also benchmarked the running time for the scalar/group multiplication algorithm to be  $29.2\mu s$ . This suggests that the overhead for the constant-time table lookups of about 30% ( $29.2/22.4 \approx 1.3$ ). In fact, we ran a number of experiments that confirm that this overhead is almost completely due to the constant-time table lookups. We also investigated some alternative strategies to implement the constant-time table lookups at lower cost. One strategy we implemented uses AVX2 instructions, which operate in a SIMD fashion on 256-bit registers. Using

this strategy, we benchmarked the running time for the scalar/group multiplication algorithm to be  $24.3\mu s$ , reducing the overhead to about 8% ( $24.3/22.4 \approx 1.08$ ).

## B Multi-scalar/group multiplication algorithms

In this section, we discuss algorithms for multi-scalar/group multiplication: given elements  $\mathcal{G}_1, \dots, \mathcal{G}_L$  in a group  $E$  of prime order  $q$ , and scalars  $x_1, \dots, x_L$ , compute  $x_1\mathcal{G}_1 + \dots + x_L\mathcal{G}_L$ . More specifically, we assume that

- the group elements  $\mathcal{G}_\ell$  are public but not fixed;
- the scalars  $x_\ell$  are public, random  $\rho$ -bit numbers, where  $\rho$  will typically be much smaller than the bit length of  $q$ ;
- the value of  $L$  significantly larger than  $\rho$ , typically in the range from several hundred to several thousand.

Under these assumptions, a very simple yet effective algorithm is a special case of a more general algorithm due to Pippenger [Pip76]. This simple algorithm is described very clearly and succinctly in Section 4 of [BDLO12], and is nowadays commonly called the “bucket method” — see, for example, the paper [LFG23] for many more details and variations. For completeness, we give a brief description here of a version of the bucket method that is adequate for our purposes.

This algorithm is parameterized by a window size  $w$ . We set  $k := \lceil \rho/w \rceil$  and write each scalar  $x_\ell$  in base  $2^w$  as

$$x_\ell = \sum_{i=0}^{k-1} d_{\ell,i} 2^{wi}, \quad (15)$$

with each digit  $d_{\ell,i} \in \{0, \dots, 2^w - 1\}$ . The algorithm then runs as follows:

```

 $\mathcal{P} \leftarrow \mathcal{O}$ 
for  $i = k - 1$  down to 0 do
  initialize “buckets”  $\mathcal{B}_d \leftarrow \mathcal{O} \in E$  for  $d = 1, \dots, 2^w - 1$ 
  for  $\ell = 1$  to  $L$  do: if  $d_{\ell,i} \neq 0$  then  $\mathcal{B}_{d_{\ell,i}} \leftarrow \mathcal{B}_{d_{\ell,i}} + \mathcal{G}_\ell$ 
   $\mathcal{S} \leftarrow \mathcal{T} \leftarrow \mathcal{O}$ 
  for  $d = 2^w - 1$  down to 1 do:  $\mathcal{S} \leftarrow \mathcal{S} + \mathcal{B}_d$ ,  $\mathcal{T} \leftarrow \mathcal{T} + \mathcal{S}$ 
  //  $\mathcal{T} = \sum_{\ell} d_{\ell,i} \mathcal{G}_\ell$ 
   $\mathcal{P} \leftarrow 2^w \mathcal{P} + \mathcal{T}$ 
return  $\mathcal{P}$ 

```

The algorithm has a computational cost of at most

$$k(L + 2^w) + (k - 1)w$$

group additions, and requires space for about  $2^w$  group elements.

If the cost of negation is negligible (as it is in the case of an elliptic curve), then using standard techniques, this cost further be reduced. For example, if we view the scalars as  $\rho$ -bit two’s complement signed integers, we can easily decompose each  $x_\ell$  as

$$x_\ell = \sum_{i=0}^{k-1} \tilde{d}_{\ell,i} 2^{wi}, \quad (16)$$

but with each digit  $\tilde{d}_{\ell,i} \in \{-2^{w-1}, \dots, 0, \dots, 2^{w-1}\}$ . If we assume that in (15) we have  $d_{\ell,i} \in \{0, \dots, 2^w - 1\}$  for  $i = 0, \dots, k-2$  and  $d_{\ell,k-1} \in \{-2^{w-1}, \dots, 0, 2^{w-1} - 1\}$ , then for each  $\ell \in [L]$ , we can easily compute the  $\tilde{d}_{\ell,i}$ 's as follows:

```

initialize  $\tilde{d}_{\ell,i} \leftarrow d_{\ell,i}$  for  $i = 0, \dots, k-1$ 
for  $i = 0$  to  $k-2$  do
  if  $d_{\ell,i} \geq 2^{w-1}$  then:  $\tilde{d}_{\ell,i} \leftarrow \tilde{d}_{\ell,i} - 2^w$ ,  $\tilde{d}_{\ell,i+1} \leftarrow \tilde{d}_{\ell,i+1} + 1$ 

```

This essentially cuts the number of buckets in half, which reduces the computational cost to at most

$$k(L + 2^{w-1}) + (k-1)w$$

group additions and reduces the space requirement to about for about  $2^{w-1}$  group elements.<sup>11</sup> If  $L \gg 2^w$ , this optimization has no practical effect. However, for a fixed  $w$ , this optimization allows for more flexible choice of  $L$ .

We stress that this algorithm is quite simple, and there are no hidden costs. For our application,  $\rho$  is very small — for example,  $\rho = 7$  in [Example 5.1](#), and in this case, we can just set the window size  $w := \rho$ . For larger values of  $\rho$ , the window size  $w$  should be chosen that the space requirement  $2^w$  (or  $2^{w-1}$ ) group elements is reasonable, and the term  $2^w$  (or  $2^{w-1}$ ) appearing in the computational cost estimate is significantly less than  $L$ . For the parameter sizes that are relevant to our application, a window size of 8 is probably quite reasonable (see [Example 5.2](#)).

Note that this algorithm is not a constant time algorithm, but this is acceptable, as all inputs are public.

---

<sup>11</sup> This algorithm for computing the signed digits is essentially the same as others appearing in the literature (see, for example, [LFG23]). However, starting with signed  $x_\ell$ 's simplifies the decomposition into signed digits, as it avoids “overflow” issues. However, in our GoAVSS protocol, this strategy possibly complicates the fast implementation of tiny scalar/scalar multiplications. However, there are practical ways to deal with this.