Sigma Protocols from Verifiable Secret Sharing and Their Applications

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Abstract. Sigma protocols are one of the most common and efficient zero-knowledge proofs (ZKPs). Over the decades, a large number of Sigma protocols are proposed, yet few works pay attention to the common design principal. In this work, we propose a generic framework of Sigma protocols for algebraic statements from verifiable secret sharing (VSS) schemes. Our framework provides a general and unified approach to understanding Sigma protocols. It not only neatly explains the classic protocols such as Schnorr, Guillou–Quisquater and Okamoto protocols, but also leads to new Sigma protocols that were not previously known.

Furthermore, we show an application of our framework in designing ZKPs for composite statements, which contain both algebraic and non-algebraic statements. We give a generic construction of non-interactive ZKPs for composite statements by combining Sigma protocols from VSS and ZKPs following MPC-in-the-head paradigm in a seamless way via a technique of *witness sharing reusing*. Our construction has advantages of requiring no "glue" proofs for combining algebraic and non-algebraic statements. By instantiating our construction using Ligero++ (Bhadauria et al., CCS 2020) and designing an associated Sigma protocol from VSS, we obtain a concrete ZKP for composite statements which achieves a tradeoff between running time and proof size, thus resolving the open problem left by Backes et al. (PKC 2019).

Keywords: Sigma protocols · Verifiable secret sharing · Composite statements · MPC-in-the-head.

1 Introduction

Zero-knowledge proofs (ZKPs), introduced by Goldwasser, Micali and Rackoff [GMR85], allow a prover to convince a verifier that a statement is true without revealing any further information. Goldreich, Micali, and Wigderson [GMW86] further showed that ZKP exists for every $\mathcal{NP}$ language, making it an extremely powerful tool in modern cryptography. Since its introduction in the mid 1980s, ZKPs have been used as an instrumental building block in a myriad of cryptographic protocols/schemes like identification protocols [FFS87], digital signatures [BCC+16, Sch91], CCA-secure public-key encryption [NY90, Sah99], anonymous credentials [CL01], voting [CF85], maliciously secure multi-party computation [GMW87], and privacy-preserving cryptocurrency [GK15, BCG+14].

In the realm of ZKPs[1], there are three types of statements. The first is algebraic statements, which are defined by relations over algebraic groups like prime-order groups and RSA-type groups, such as knowledge of discrete logarithm or modular root. The second is non-algebraic statements, which are expressed by arithmetic/boolean circuits, such as knowledge of preimage of SHA256 or knowledge of plaintext of AES encryption. The third is composite statements that mix algebraic and non-algebraic statements, e.g. the value w committed by Com also satisfies $C(w) = 1$, where the predicate C represents an arithmetic/boolean circuit. Below we briefly survey ZKPs for the three types of statements.

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[1] For the sake of convenience, we will not distinguish between computational and information-theoretic soundness, and thus refer to both proofs and arguments as "proofs".

ZKPs for non-algebraic statements. Since boolean/arithmetic circuits can describe arbitrary computations, ZKPs for non-algebraic statements are usually referred to as general-purpose. The last decade has seen tremendous progress in designing and implementing efficient general-purpose ZKPs (see [Tha22] for a comprehensive survey). These efforts can be roughly divided into four categories according to the underlying machinery.

The first is built upon probabilistic checkable proof (PCP). Following the seminal works of Kilian [Kil92] and Micali [Mic94] based on classical PCPs, recent works [AHIV17, BBHR18, BCR+19, ZXZS20, COS20, Set20] begin to build general-purpose ZKP from interactive variants of PCP, first in the model of interactive PCP [KR08] and then in the more general model of interactive oracle proofs [BCS16, RRR16]. ZKPs of this category have the advantages of not relying on public-key cryptography, not requiring trusted setup, and offering conjectured post-quantum security. The second is based on linear PCP, initiated by Ishai, Kushilevitz, and Ostrovsky [IKO07], and followed by a sequence of works [Gro10, Lip12, GGPR13, Gro16, MBKM19, CHM+20]. ZKPs of this category are featured with constant size proofs and fast verification, but they are quite slow on the prover side and require long and "toxic" common reference string. The third is based on inner product arguments. Initial work [Gro09] of this line has square root size proof and linear verification time. Followup works [BCC+16, BBB+18] managed to achieve logarithmic size proof, and the verification cost is finally reduced to logarithmic complexity [Lee21]. The fourth is based on garbled circuits. The original protocol due to Jawurek, Kerschbaum and Orlandi [JKO13] is secret-coin in nature. Recently, Cui and Zhang [CZ21] showed how to tweak the JKO protocol to public-coin. ZKPs of this category require linear prover time, proof size and verification time.

ZKPs for algebraic statements. Almost exclusively, the most common and efficient ZKPs for algebraic statements fall into a class known as Sigma protocols, introduced by Cramer [Cra96]. Let L be an $\mathcal{NP}$ language associated with relation R, i.e., $L = \{x \mid \exists w \text{ s.t. } R(x, w) = 1\}$. A Sigma ($\Sigma$) protocol for L is a 3-move public coin interactive proof system that allows a prover to convince a verifier that he knows a witness w of a public instance x without disclosing w. The Greek letter Σ visualizes the 3-move structure (commit, challenge and response). The prover sends an initial message a called a commitment to the verifier, the verifier replies with a uniformly and independently random chosen challenge e from some finite challenge space, and the prover answers with a response z as the final message. Finally, the verifier decides whether to accept or reject the statement based on the transcript (a, e, z).

Sigma protocols are very appealing due to many attractive properties. First, Sigma protocols are extremely efficient for algebraic statements. They yield short proof sizes, only require a constant number of public-key operations and do not need trusted common reference string generation. Although seemingly specific, Sigma protocols for algebraic statements cover a wide variety of tasks arise from practice such as proving the knowledge of discrete logarithm/modular root, a tuple is of the Diffie-Hellman type, an ElGamal/Paillier encryption is to a certain value, and many more. Second, Sigma protocols are closed under parallel composition, and thus it is possible to efficiently combine several simple Sigma protocols to prove compound statements. This further increases the usability of Sigma protocols. Third, the so-called special soundness make Sigma protocols easy to work with by providing a simple way to establish proof of knowledge property. Moreover, Sigma protocols can be made non-interactive using the Fiat-Shamir heuristic [FS86]. The above properties make Sigma protocols an incredibly powerful tool for various cryptographic tasks.

In contrast to the state of affairs of general-purpose ZKP, though Sigma protocols have been intensively studied in the last four decades, few attentions are paid to generic constructions. This is probably because that the design of Sigma protocols is relatively easier than that of general-purpose ZKPs. Sigma protocols in the literature such as the classic Schnorr [Sch91], Batching Schnorr [GLSY04], Guillou-Quisquater [GQ88], and Okamato protocol [Oka92] are ingenious but hand-crafted, and they came with a separate proof. It is curious to know whether there exists a common design principal.

ZKPs for composite statements. A composite statement is one that contains both algebraic and non-algebraic statements, e.g., x is a Pederden commitment to w such that $SHA256(w) = y$. As noted in [CGM16, AGM18, BHH+19], ZKPs for composite statements have various applications, such as proof of solvency for Bitcoin exchanges, anonymous credentials based on RSA and (EC-)DSA signatures, and 2PC with authenticated inputs.

To prove composite statements, a naïve approach is transforming composite statements into a single form, namely either algebraic or non-algebraic form, and then using only Sigma protocols or general-purpose ZKPs to prove it. In one direction, one could turn the non-algebraic statements expressed as a circuit into algebraic statements by expressing each gate of the circuit as an algebraic relation between input and output, and then use Sigma protocols to prove these relations. However, it would cost several public-key operations and group elements per gate, which is prohibitively expensive when the circuit is large. As noted by [AGM18], in case of hash functions and block-ciphers, it costs tens of thousands of exponentiations and group elements when proving the associated algebraic relations of the circuits. In the other direction, one could turn the algebraic statements into non-algebraic statements and then use general-purpose ZKPs to prove it. But this results in a substantial increase in the size of the statements. For example, the circuit for computing a single exponentiation could be of thousands or millions of gates depending on the group size. This in turn increases the overheads of both prover's/verifier's work and proof size. As mentioned before, the computation cost and proof size of all transparent general-purpose ZKPs grow with the circuit size. General-purpose ZKPs based on linear PCP offer efficient verification and constant proof size, while the prover's work is still heavy and they require a trusted setup.

A better approach, employed by most of prior works on this direction, is that: using Sigma protocols to prove the algebraic part, using off-the-shelf efficient general-purpose ZKPs to prove the non-algebraic part, then additionally designing customized protocols as a "glue" to link the two parts. "Glue" proofs play a crucial role in this approach. Without "glue" proofs, a cheating prover can easily generate proofs of the two parts using inconsistent witnesses (e.g., a cheating prover may give a proof π_1 for proving knowledge of w_1 such that $\mathsf{Com}(w_1) = x$ and a proof π_2 for proving knowledge of w_2 such that $\mathrm{SHA256}(w_2) = y$ where $w_1 \neq w_2$). The resulting proof systems will inherit the advantages and disadvantages of the underlying general-purpose ZKPs. For instance, [CGM16] presented two tailor-made "glue" proofs to link Sigma protocols with the JKO protocol [JKO13], yielding ZKPs which have a fast prover and verifier while they are private-coin inherently; [AGM18, CFQ19, ABC+22] each gave a generic construction of "glue" proofs to link Sigma protocols with ZKPs based on linear PCP, yielding proofs which are featured with constant size proofs and fast verification, but they are quite slow on the prover side and require a trusted setup; [BHH+19] customized two "glue" proofs to link Sigma protocols with the ZKBoo [GMO16]/ZKB++ [CDG+17] protocols, yielding transparent ZKPs which have a fast prover, but the proof size is linear in the circuit size.

However, this approach servers from two main drawbacks. First, "glue" proofs inevitably introduce additional overheads in both computation cost and proof size to enforce the witness consistency. Second, "glue" proofs must be tailored in a specific way to align with the general-purpose ZKPs, limiting the space of possible general-purpose ZKPs we can use. Particularly, "glue" proofs in [BHH+19] are tailored for the ZKBoo [GMO16]/ZKB++ [CDG+17] and they could not be applied to other similar proof systems like Ligero [AHIV17]/Ligero++ [BFH+20]. The authors left a more efficient and compact ZKP for composite statements using Ligero/Ligero++ as an open problem [2]. Therefore, an intriguing question is that whether the seemingly indispensable "glue" proofs are necessary when designing ZKPs for composite statements.

The above discussion motivates the main questions that we study in this paper:

Is there a generic framework of Sigma protocols? Can this framework help to give a generic construction of efficient ZKPs for composite statements without "glue" proofs?

1.1 Our Contributions

In this work, we positively answer the above two questions and summarize our contributions as below.

1.1.1 A Framework of Sigma Protocols for Algebraic Statements

We present a framework of Sigma protocols for algebraic statements from verifiable secret sharing (VSS) schemes. Our framework not only neatly explains existing classic Sigma protocols including the Schnorr,

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[2] Actually, it is hard to give a more efficient ZKP for composite statement using Ligero/Ligero++ than those using ZKBoo/ZKB++, since the former two protocols reduce the proof size, at the cost of increasing the computation.

Batching Schnorr, GQ, and Okamoto protocols, but also provides a unified paradigm of designing Sigma protocols for proving knowledge of openings of algebraic commitments.

MPC-in-the-head paradigm revisit. Ishai et al. [IKOS07] showed how to build general-purpose ZKPs by using MPC in a black-box manner. In a nutshell, to prove the knowledge of w such that $C(y, w) = 1$ for a circuit C and a value y, their construction proceeds as below: the prover simulate an execution of an n-party secure-computation protocol Π_f that evaluates the function $f_y(w_1, \ldots, w_n)$ which outputs "1" iff $C(y, w) = 1$ with $w = w_1 \oplus \cdots \oplus w_n$, and commit the views of the parties in the protocol. The verifier then picks and asks a random subset of those parties, and the prover opens the corresponding views. The verifier finally accepts if the opened views all output "1" and are consistent with each other. Their approach, known as MPC-in-the-head, presents a generic connection between ZKP and MPC, and gives rise to a rich line of transparent general-purpose ZKPs with continually improved performance, including ZKBoo [GMO16], ZKB++ [CDG$^+$17], KKW [KKW18], Ligero [AHIV17], Ligero++ [BFH$^+$20] and more, forming a promising subclass of general-purpose ZKPs based on PCP machinery. Interestingly, we find that the ZKPs from the MPC-in-the-head paradigm also follow the commit-challenge-response pattern. In light of this observation, the MPC-in-the-head paradigm actually gives a generic construction of Sigma protocols for non-algebraic statements. This suggests that when seeking for a generic framework of Sigma protocols for algebraic statements, one may start from some lite machinery than MPC.

VSS-in-the-head. An (n, t_p, t_f)-verifiable secret sharing (VSS) scheme allows a dealer to distribute a secret s among n participants, in such a way that no group of up to t_p participants could learn anything about s, any group of at least t_f participants could recover the secret, and the cheating behaviors of both the dealer and the participants can be detected. VSS is an essential building block employed for numerous MPC protocols with malicious players [GMW87, BGW88, CCD88, RB89]. Based on the above reasoning, VSS is arguably the right backbone of Sigma protocols for algebraic statements.

A REFINED DEFINITION OF VSS. In this work, we restrict ourselves to non-interactive VSS schemes. For simplicity, we will omit non-interactive hereafter when the context is clear. Before describing the framework, we first give a refined definition of VSS, which differs from the original definition proposed by Feldman [Fel87] in both syntax and security properties. In terms of syntax, there are two primary differences as below: (1) The secret is committed rather than being encrypted, such relaxation makes our definition more general; (2) The sharing algorithm is asked to additionally output *authentication information*, denoted by *aut*, which essentially commits to the sharing method (e.g., in the case of Feldman's VSS scheme, it is a vector of commitments to the polynomial's coefficients), and will later be used to check the validity of each share. This kind of information does not appear in the original definition. In terms of security properties, there are two differences as follows: (1) For correctness, the secrets recovered by different groups of participants are not stipulated to be consistent as in [Fel87], instead the recovered secrets are required to be an opening of the commitment. This property is crucial in this work and is actually met by many existing VSS schemes, but it has never been formally defined; (2) For privacy, we provide a simulation-based definition instead of a game-based one, making it more convenient to use in the context of ZKP and MPC. See Section 3.1 for the details of the refined definition.

SIGMA PROTOCOLS FROM VSS. Having settled on a satisfactory definition of VSS, we are ready to describe the framework of Sigma protocols for algebraic statements—"given a commitment x, prove the knowledge of an opening (s, r) such that $\mathsf{Com}(s; r) = x$". Our framework is built upon (n, t_p, t_f)-VSS schemes with respect to Com. Roughly speaking, in the commit phase, the prover shares the witness (s, r) into n pieces of shares $v_1, \ldots, v_n$ "in his head" and generates the associated authentication information *aut*, then sends *aut* to the verifier. In the challenge phase, the verifier picks a random subset I from the challenge space $[n]$, where $|I| \le t_p$, and acts as the set of participants in I to query their private shares. In the response phase, the prover answers with corresponding shares $(v_i)_{i \in I}$. Finally, the verifier decides to accept or reject the statement by checking whether each v_i is a valid share for participant P_i. For the security of the resulting Sigma protocols, the *special soundness* property follows from the *correctness* of VSS and the *special honest verifier zero-knowledge property* follows from the *privacy* of VSS.

The above framework from VSS encompasses almost all the classic Sigma protocols for proving knowledge of openings of algebraic commitments. As a concrete example, we show how to derive the celebrated Schnnor protocol from our framework. The start point is the Feldman's (n, t_p, t_f)-VSS scheme [Fel87] where $t_f = t_p + 1$: to distribute a secret $s \in \mathbb{F}_p$ among n participants $P_1, \ldots, P_n$, the dealer first computes a commitment $c = g^s$ to secret s, chooses a t_p-degree polynomial $f(x) = a_0 + a_1 x + \cdots + a_{t_p} x^{t_p}$ where $a_0, \ldots, a_{t_p-1} \xleftarrow{\text{R}} \mathbb{F}_p$ and $a_{t_p} = s$ (the coefficients of the polynomial could be viewed as the compact description of the sharing method), and sets the private share v_i for P_i as $f(i)$, then generates the commitment of the randomnesses as the authentication information, i.e., $aut = (c_0, \ldots, c_{t_p-1})$ where $c_j = g^{a_j}$ for $0 \leq j \leq t_p - 1$. The dealer then broadcasts c and aut, and sends v_i to P_i in private. Upon receiving the share, each participant checks the validity of the share with respect to c and aut, and rejects if it is invalid. The secret s can be recovered by pooling more than t_p valid shares. By setting the number of participants n to p (the size of the field $\mathbb{F}_p$), the privacy threshold t_p to 1, we immediately recover the classic Schnorr protocol. More examples can be found in Section 4.

1.1.2 A Framework of ZKPs for Composite Statements

To demonstrate the usefulness of our framework, we show its application in designing ZKPs for composite statements. Among various types of composite statements, *commit-and-prove*, i.e., a committed value w satisfies a circuit C, is the most common one. According to [CGM16, BHH+19], it is an building block for some other types. Therefore, we restrict ourselves to the *commit-and-prove* type.

In this work, we show that by reusing the witness sharing process, Sigma protocols from VSS and ZKPs following MPC-in-the-head paradigm can be combined seamlessly, yielding a generic construction of ZKPs for composite statements without "glue" proofs. Our generic construction enjoys two benefits: (i) eliminating the cost introduced by "glue" proofs; (ii) expanding the space of possible general-purpose ZKPs that we can use.

Enforcing consistency via witness sharing reusing. As mentioned before, ZKPs from MPC bear strong resemblance with Sigma protocols, as both of them follow the same commit-challenge-response pattern. This implies that ZKPs from MPC might be easily coupled with Sigma protocols from VSS to prove composite statements. However, if we combine them as the mainstream approach, "glue" proofs are still necessary. A key observation is that ZKPs from MPC and Sigma protocols from VSS not only follow the same pattern but also share a common *witness sharing procedure*: at the very beginning, the provers share the witnesses into n shares in their heads; in the challenge phase, the verifiers ask to reveal a subset of witness shares; in the response phase, the provers reply with corresponding shares (albeit in ZKPs from MPC, the shares are included as a part of parties' views), and finally the verifiers use the received shares to check the verification equations. This suggests that when combining Sigma protocols from VSS and ZKPs from MPC to prove composite statements, the *witness sharing procedure* of them are able to be reused. More precisely, the prover shares the witness only once, the verifier picks and asks only one challenge I, and then the prover responds with only one subset of shares, whereas the verifier accepts if and only if the shares pass verifications of both algebraic and non-algebraic parts. Such "reusing" enforces the prover to use a consistent witness without any additional "glue" proofs. From the perspective of security proof, one can construct an extractor Ext of ZKPs for composite statements by invoking extractors Ext_Σ of Sigma protocols from VSS and $\mathsf{Ext}_{\mathsf{ZKP}}$ of ZKPs from MPC as subroutines, both of which run the same recovering algorithm on the same input, and thus output the same witness satisfying both algebraic and non-algebraic statements. When implementing the above high-level idea, we encounter the following two main technical obstacles.

A GENERALIZATION OF MPC-IN-THE-HEAD PARADIGM. One obstacle comes from the MPC-in-the-Head Paradigm. Recall that the secret sharing mechanism in the original ZKPs from MPC [IKOS07] sticks to the XOR-based secret sharing (SS) schemes, which is a special case of $(n, n-1, n)$-SS, making it hard to interact with (n, t_p, t_f)-VSS schemes. To address this issue, we generalize the MPC-in-the-head paradigm by extending the XOR-based SS scheme to the (n, t_p, t_f)-SS schemes. Specifically, in the commit phase the prover P shares the witness w into n shares $w_1, \ldots, w_n$ by running SS.Share(w), which does not fix to picking

n shares satisfying $w = w_1 \oplus \cdots \oplus w_n$. This, in turn, requires an MPC protocol Π that evaluates n-party function f satisfying $f_y(w_1, \ldots, w_n) = 1$ iff $C(y, \cdot) = 1$ on input $w = \mathsf{SS.Recover}(w_1, \ldots, w_n)$. The proof of knowledge property is not explicitly given in [IKOS07]. In this work, we rigorously prove this property, which is crucial for our construction of ZKPs for composite statements and might be of independent interest.

Separable VSS Schemes. The other obstacle is that the relationship between VSS and SS is unclear, making it difficult to reuse the common witness sharing procedure. To overcome this obstruction, we introduce a mild property called *separability* for VSS which has been satisfied by many existing VSS schemes. Roughly speaking, we say a VSS scheme satisfies *separability* if its procedure of generating shares $(v_1, \ldots, v_n)$ and authentication information *aut* could be separated. Particularly, we say a VSS scheme aligns with an SS scheme if it generates the shares as per this SS scheme. Such delicate dissection allows us to distill the common secret sharing mechanism used in Sigma protocols from VSS and ZKPs from MPC, paving the way to implement the *witness sharing reusing* technique.

An efficient instantiation. We instantiate above framework of ZKPs for composite statements by choosing Ligero++ [BFH+20] as the underlying general-purpose ZKPs and designing a Sigma protocol from VSS which aligns with the SS component underlying Ligero++. The resulting protocol requires no trusted setup and no "glue" proofs, and achieves a tradeoff between proof size and running time. Concretely, the proof size is polylogarithmic to the circuit size and the number of expensive public-key operations required by prover and verifier is independent of the circuit size. See Section 6.2 for a detailed efficiency analysis.

Table 1 shows a brief comparison between closely related works. Among them, the protocols in [BHH+19] and [BBB+18] are the only two that are both public-coin and transparent. Compared to [BHH+19], our instantiation achieves asymptotically smaller proof size, thus settling the open problem in [BHH+19]: whether a more compact ZKP for composite statements can be constructed by using Ligero/Ligero++. Though the work [BBB+18] also proposed a proof system that achieves succinct proof size, the prover's work is still expensive. As noted in [Tha22, Section 19.3.2], for circuits with small size, $O(|C| \log(|C|))$ field operations are likely to be faster than $O(|C|)$ group operations. Thus, our instantiation is likely to have better prover performance when the circuit size is small.

Table 1: Comparisons among ZKPs for composite statements

Protocols	Public-coin	Transparent	Prover time	Verifier time	Proof size
[CGM16] Constr.1	✗	✓	$O(|w|)$ pub $O(|C|)$ sym	$O(|w|)$ pub $O(|C|)$ sym	$O(|C| + |w|)$
[CGM16] Constr.2	✗	✓	$O(\lambda)$ pub $O(|C| + |w|\lambda)$ sym	$O(\lambda)$ pub $O(|C| + |w|\lambda)$ sym	$O(|C| + |w|\lambda)$
[BBB+18]	✓	✓	$O(|C|)$ pub	$O(\frac{|C|}{\log(|C|)})$ pub	$O(\log(|C|))$
[AGM18]	✓	✗	$O(|C| + \lambda)$ pub	$O(|w| + \lambda)$ pub	$O(1)$
[BHH+19]	✓	✓	$O((|w| + \lambda)$ pub $O(|C| \cdot \lambda)$ sym	$O((|w| + \lambda)$ pub $O(|C| \cdot \lambda)$ sym	$O(|C|\lambda + |w|)$
[CFQ19] LegoAC1	✓	✗	$O(|C|)$ pub $O(|C| \log(|C|))$ sym	$O(|w|)$ pub	$O(1)$
[ABC+22]	✓	✗	$O(|C| + |w|)$ pub	$O(|w|)$ pub	$O(\log(|w|))$
This work	✓	✓	$O(\lambda)$ pub $O(|C| \log(|C|))$ sym	$O(\frac{(|w|+\lambda)^2}{\log(|w|+\lambda)})$ pub $O(|C|)$ sym	$O(\mathsf{polylog}(|C|) + \lambda)$

We use pub to indicate a public-key operation, sym to a symmetric-key operation. We denote by $|C|$ the circuit size, by $|w|$ the witness length, by λ the security parameter.

1.2 Related Work

Sigma protocols. The notion was first proposed by Cramer [Cra96] as an abstraction of Schnorr protocol [Sch91] for proving knowledge of discrete logarithm and Guillou-Quisquater protocol [GQ88] for proving knowledge of modular root. Since its introduction, Sigma protocols have received much attention due to their simplicity and high efficiency, and a great deal of works have focused on improving the efficiency or extending the functionality of Sigma protocols. For example, Beullens [Beu20] introduced a new notion called *sigma protocols with helper*, referring to the Sigma protocols where the prover and the verifier are assisted by a trusted third party, and further improved the efficiency of several Sigma protocols using the new notion. Cramer, Damgård and Schoenmakers (CDS) [CDS94] applied the secret sharing technique to construct proofs of partial knowledge, i.e., given n statements $x_1, \ldots, x_n$, convincing the verifier that the prover knows a witness w for at least one of the statements. Our framework seems like a dual construction of theirs. In their construction, the prover shares the challenge e rather than the witness w, while in ours, the prover shares the witness w instead of the challenge e. In any case, we both showed that (verifiable) secret sharing is an important technique for constructing Sigma protocols. Abe et al. [AAB+20] then improved the CDS technique by letting the prover hash the shares before using them as challenges, resulting in several significant benefits. Abe et al. [AAB+21] also introduced a model of monotone computation called *acyclicity program* (ACP), and proposed an alternative method for proving partial knowledge based on the ACP.

However, few works study the common design principal of Sigma protocols. To our knowledge, [Mau15] is the only work on this direction. In [Mau15], Maurer proposed a template for building Sigma protocols for algebraic statements that can be captured by preimage of a group homomorphism. Despite a large number of classic Sigma protocols can be explained by this template, it still has deficiencies in generality and utility. First, Maurer's template is tied to group homomorphism, and is less flexible cause it imposes fixed formats on three move messages. For instance, it fails to encompass the variant Schnorr and the batching Schnorr protocol introduced in [GLSY04] where the initial message is not computed using the same homomorphism as the statement. Second, Maurer's template does not establish connection between Sigma protocols and other cryptographic primitives. The shed light on the machinery of Sigma protocols is still unclear.

ZKPs for composite statements. This line of research started with the work of Chase et al. [CGM16]. They gave two efficient ZKPs for proving composite statement, of which the number of expensive public-key operations is independent of the size of the circuit C. However, both of the two constructions are based on the general-purpose ZKPs from garbled circuits proposed by [JKO13], which makes the protocols interactive inherently. Agrawal et al. [AGM18] further presented non-interactive protocols, which use the QAP-based succinct non-interactive arguments of knowledge (SNARK) to prove the non-algebraic part of the statement. Their protocols take advantage of having a small proof size and fast verification time, while require a trusted setup for generating the structured common reference string (CRS). Backes et al. [BHH+19] presented non-interactive protocols which require no trusted setup, and have efficient prover and verifier running time. However, their protocol makes use of the ZKBoo [GMO16]/ZKB++ [CDG+17] protocols which follow the MPC-in-the-head paradigm to prove the non-algebraic statement, thus resulting in a large proof size that is linear to $|C|$. Campanelli et al. [CFQ19] proposed a framework of ZKPs for composite statements utilizing pairing-based general-purpose ZKPs, achieving succinct proof size while all the instantiations they given require a trusted setup. Among these instantiations, the one reported in Table 1 has the shortest proof size and the most efficient verifier. Recently, Aranha et al. [ABC+22] proposed a general method of compiling Algebraic Holographic Proofs into ZKPs for composite statements, whose proof size is logarithmic to the number of commitments in the statements while also requiring a trusted setup. See Table 1 for a brief comparison between these works.

All the works above used customized "glue" proofs for proving consistency, which severely limit the space of general-purpose ZKPs that can be used and also causes additional overheads in both computation and communication. Aside from the works mentioned above, general-purpose ZKPs based on inner product arguments, such as [BBB+18, HKR19] are able to be combined with Pedersen commitments without any "glue" proofs. However, the algebraic parts of them are fixed to Pedersen commitments with some certain constraints, making the construction semi-generic. For example, the algebraic part of [BBB+18] is fixed to $|\mathbf{w}|$ Pedersen commitments, each of which commits to an entry of the witness $\mathbf{w}$. Once the algebraic part changes

to a single vector commitment to **w**, an additional "glue" proof is required. What's more, the prover's work is still expensive, since the number of public-key operations required by the prover is linear to the circuit size. To our knowledge, there is no generic construction of ZKPs for composite statements that is without "glue" proofs.

2 Preliminaries

Notations. For an integer n, we use $[n]$ to denote the set $\{1, \ldots, n\}$. For a set X and integer t, we use $|X|$ to denote the size of X, use X_t to indicate the set consisting of all t-sized subsets of X, and use $x \xleftarrow{\text{R}} X$ to denote sampling x uniformly at random from X. We use the abbreviation PPT to indicate probabilistic polynomial-time. We denote a negligible function in λ by $\mathsf{negl}(\lambda)$.

2.1 Commitment Schemes

We first recall the definition of commitment schemes.

Definition 1 (Commitment Schemes). *A commitment scheme is a triple of polynomial time algorithms as below:*

- $\mathsf{Setup}(1^\lambda)$*: on input a security parameter λ, outputs the public commitment key pp, which includes the descriptions of the message space M, randomness space R, and commitment space C.*
- $\mathsf{Com}(m; r)$*: on input a message $m \in M$ and a randomness $r \in R$, outputs a commitment c.*
- $\mathsf{Verify}(c, m, r)$*: on input a commitment $c \in C$, a message $m \in M$ and a randomness $r \in R$, outputs "1" if $\mathsf{Com}(m; r) = c$ and "0" otherwise.*

Additionally, we require the following properties of a commitment scheme.

Correctness. *For any $pp \leftarrow \mathsf{Setup}(1^\lambda)$, any $m \in M$ and any $r \in R$, it holds that $\mathsf{Verify}(\mathsf{Com}(m; r), m, r) = 1$.*
Hiding. *A commitment $\mathsf{Com}(m; r)$ should reveal no information about m. Formally, it is computationally (resp. statistically) hiding if for any PPT (resp. unbounded) adversary $\mathcal{A}$, it holds that:*

$$\Pr\left[b' = b : \begin{array}{l} pp \leftarrow \mathsf{Setup}(1^\lambda); \\ (m_0, m_1) \leftarrow \mathcal{A}(pp); \\ b \xleftarrow{\text{R}} \{0, 1\}, r \xleftarrow{\text{R}} R, c \leftarrow \mathsf{Com}(m_b; r); \\ b' \leftarrow \mathcal{A}(c); \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda).$$

For commitment schemes with deterministic Com algorithm, namely the randomness is null, we consider a weaker security notion called one-way hiding, which can be defined similarly as above. Roughly speaking, we say a commitment scheme is one-way hiding if the adversary only takes a negligible probability to open a randomly chosen commitment.
Binding. *A commitment can not be opened to two different messages. Formally, it is computationally (resp. statistically) binding if for any PPT (resp. unbounded) adversary $\mathcal{A}$, it holds that:*

$$\Pr\left[\begin{array}{c} m_0 \neq m_1 \wedge \\ \mathsf{Com}(m_0; r_0) = \mathsf{Com}(m_1; r_1) \end{array} : \begin{array}{l} pp \leftarrow \mathsf{Setup}(1^\lambda); \\ (m_0, r_0, m_1, r_1) \leftarrow \mathcal{A}(pp); \end{array}\right] \leq \mathsf{negl}(\lambda).$$

2.2 Sigma Protocols

Let L be an $\mathcal{NP}$ language and R be the associated binary relation. We say an instance x lies in L if and only if there exists a witness w such that $(x, w) \in \mathsf{R}$. Consider following three-move interaction between two PPT algorithms P and V: (1) Commit: P sends an initial message to V; (2) Challenge: V sends a challenge e to P; (3) Response: P replies with a response z. A formal definition of Sigma protocols is presented as below.

Definition 2 (Sigma Protocols). *A Sigma protocol for a relation* R *is a three-move public-coin protocol with above communication pattern and satisfies the following three properties:*

Completeness. *If P and V follow the protocol on input x and private input w to P where $(x, w) \in$ R, then V always accepts the transcript.*

*n-**Special soundness.** There exists a PPT extraction algorithm* Ext *that on input any instance x and any n accepting transcripts $(a, e_1, z_1), \ldots, (a, e_n, z_n)$ for x where all e_i's are distinct, outputs a witness w for x.*

Special honest-verifier zero-knowledge (SHVZK). *There exists a PPT simulator* Sim *that on input any instance x and any challenge e, generates a transcript (a, e, z) such that the triple is distributed identically to an accepting transcript generated by a real protocol run between the honest $P(x, w)$ and $V(x)$.*

Lemma 1 ([ACK21, GMO16]). *Let n be a positive integer bounded by a polynomial and $\langle P, V \rangle$ be a Sigma protocol with n-special soundness. If the verifier V samples the challenge uniformly at random from the challenge space C, then $\langle P, V \rangle$ is knowledge sound with knowledge error bounded by $(n-1)/|C|$.*

2.3 Secure Multiparty Computation

A multiparty computation (MPC) protocol allows n parties $P_1, \ldots, P_n$ to jointly compute an n-party function f over their inputs while maintaining the privacy of their inputs. For a set of parties $I \subseteq [n]$, we denote by f_I the outputs of parties in I after the joint computation of f. Let $view_i$ be the view of P_i during the execution of an MPC protocol, including its private input, randomness and the received messages. Below, we recall some important definitions and lemmas of MPC protocols from [IKOS07] .

Definition 3 (Consistent Views). *We say a pair of views $(view_i, view_j)$ are consistent, with respect to the protocol Π and some public input x, if the outgoing messages implicit in $view_i$, x are identical to the incoming messages reported in $view_j$ and vice versa.*

Lemma 2 (Local vs. Global Consistency). *Let Π be an n-party protocol, x be a public input and $view_1, \ldots, view_n$ be n (possible incorrect) views. Then all pairs of views are consistent with respect to Π and x if and only if there exists an honest execution of Π with public input x (and some choice of private inputs and random inputs).*

In the semi-honest model, the security of an MPC protocol can be divided into the following two requirements.

Definition 4 (Correctness). *An MPC protocol Π realizes an n-party functionality $f(x, w_1, \ldots, w_n)$ with perfect correctness, if for all inputs $x, w_1, \ldots, w_n$, the probability that the outputs of some players are different from the output of f is 0.*

Definition 5 (t-privacy). *Let $1 \leq t < n$. We say an MPC protocol Π realizes an n-party functionality f with perfect t-privacy, if there exists a PPT simulator* Sim *such that for any inputs $x, w_1, \ldots, w_n$, and any set of parties $I \subset [n]$ where $|I| \leq t$, the joint view of parties in I is distributed identically to* Sim$(I, x, (w_i)_{i \in I}, f_I(x, w_1, \ldots, w_n))$.

2.4 (Verifiable) Secret Sharing

A secret sharing (SS) scheme [Sha79] among a dealer and n participants $P_1, \ldots, P_n$ consists of two phases, called *Sharing* and *Reconstruction*. In the *Sharing* phase, the dealer shares a secret s (either a single value or a vector) among n participants, in such a way that no unauthorized subsets of participants can learn anything about the secret, while any authorized subsets of participants can recover the secret in the *Reconstruction* phase. The formal definition is as below.

Definition 6 (Secret Sharing). *A secret sharing scheme consists of three polynomial-time algorithms as follows:*

- Setup(1^λ): *on input a security parameter λ, outputs the system parameters pp, including descriptions of secret space M, share space S, the number of participants n, the privacy threshold t_p and the fault-tolerance threshold t_f, where all the three parameters n, t_p and t_f are positive integers and hold that $n \geq t_f > t_p$.*
- Share(s): *on input the secret $s \in M$, outputs n shares $(s_i)_{i \in [n]} \in S^n$.*
- Recover($I, (s_i)_{i \in I}$): *on input a set of participants $I \subseteq [n]$ and a vector of shares $(s_i)_{i \in I}$ where $s_i \in S$, outputs a secret $s \in M$ or a special reject symbol $\perp$ denoting failure.*

An SS scheme should satisfy the following two properties:

t_f-***Correctness.*** *In Reconstruction phase, any group of at least t_f participants can recover the secret. Formally, for any pp $\leftarrow$ Setup(1^λ) where pp include the fault-tolerance threshold t_f, any secret $s \in M$, any $(s_i)_{i \in [n]} \leftarrow$ Share(s) and any subset $I \subseteq [n]$ where $|I| \geq t_f$, it holds that Recover($I, (s_i)_{i \in I}$) = s.*

t_p-***Privacy.*** *In Sharing phase, the joint view of at most t_p participants reveals nothing about the secret. Formally, for any pp $\leftarrow$ Setup(1^λ) where pp include the privacy threshold t_p, any $s \in M$ and any set $I \subset [n]$ where $|I| \leq t_p$, there exists a simulator Sim such that the distributions of the outputs of Sim(I) and $(s_i)_{i \in I}$ that generated by a real execution of Share(s) are identical.*

Verifiable secret sharing. Note that a secret sharing scheme only considers semi-honest dealer and participants, while in many applications, a scheme which is able to prevent malicious behaviours from them is needed. Thereby, Chor et al. [CGMA85] put forward a stronger notion called verifiable secret sharing (VSS) schemes, where each participant is able to check the validity of the received share, such that the behavior of delivering invalid shares will be detected. Feldman [Fel87] further introduced the concept of non-interactive VSS schemes, where each participant could check the validity of his own share without interaction between other participants.

3 A Framework of Sigma Protocols From VSS

3.1 A Refined Definition of VSS Schemes

Before describing the framework, we first give a refined definition of VSS, adapted from the definition in [Fel87].

Definition 7 (Verifiable Secret Sharing). *A verifiable secret sharing scheme consists of following four algorithms:*

- Setup(1^λ): *on input the security parameter λ, outputs system parameters pp, including descriptions of secret space M, share space S, randomness space R (if there is any), commitment space C, the number of participants n, the privacy threshold t_p and the fault-tolerance threshold t_f, where all the three parameters n, t_p and t_f are positive integers and hold that $n \geq t_f > t_p$.*
- Share(s): *on input a secret $s \in M$, outputs a commitment $c \in C$, n shares $(v_i)_{i \in [n]} \in S^n$ and the authentication information aut. For ease of exposition, we describe the process by two algorithms:*

$$c \leftarrow \mathsf{Com}(s; r),$$
$$((v_i)_{i \in [n]}, aut) \leftarrow \mathsf{Share}^*(s, r),$$

where the randomness r could be null in some settings.
- Check(i, v_i, c, aut): *on input P_i's index i and share v_i, a commitment c and the authentication information aut, outputs "1" iff v_i is valid for P_i w.r.t. c and aut; outputs "0", otherwise.*
- Recover($I, (v_i)_{i \in I}$): *on input a set of participants $I \subseteq [n]$ and a vector of shares $(v_i)_{i \in I}$ where $v_i \in S$, outputs a secret $s \in M$ and a randomness $r \in R$ (if there is any), or a special reject symbol $\perp$ denoting failure.*

A VSS scheme should satisfy following three properties:

Acceptance. *If the dealer honestly shares the secret, then all honest participants who receive correct shares will output "accept" in the end of Sharing phase. Formally, for any $pp \leftarrow \mathsf{Setup}(1^\lambda)$, $s \in M$, $(c, (v_i)_{i \in [n]}, aut) \leftarrow \mathsf{Share}(s)$, it holds that $\mathsf{Check}(i, v_i, c, aut) = 1$ for all $1 \leq i \leq n$.*

t_f***-Correctness.*** *Any group with at least t_f honest participants who output "accept" at the end of Sharing phase can recover a secret via algorithm $\mathsf{Recover}$ and the reconstructed secret must be an opening of the public commitment. Formally, for any $pp \leftarrow \mathsf{Setup}(1^\lambda)$ where pp include the fault-tolerance threshold t_f, any $c \in C$, any aut and any vector of shares $(v_i)_{i \in I} \in S^{|I|}$ where $I \subseteq [n]$ and $|I| \geq t_f$, if for all $1 \leq j \leq m$, it holds that $\mathsf{Check}(i, v_i, c, aut) = 1$, then for $(s, r) \leftarrow \mathsf{Recover}(I, (v_i)_{i \in I})$, it satisfies $\mathsf{Com}(s; r) = c$.*

t_p***-Privacy.*** *The joint view of t_p or less participants reveals nothing about the secret except a commitment to it. Formally, for any $pp \leftarrow \mathsf{Setup}(1^\lambda)$ where pp include the privacy threshold t_p, any $s \in M$, any $c \leftarrow \mathsf{Com}(s; r)$ and any set $I \subset [n]$ where $|I| \leq t_p$, there exists a simulator Sim such that the distributions of the output of $\mathsf{Sim}(c, I)$ and $((v_i)_{i \in I}, aut)$ that generated by the real execution of $\mathsf{Share}^*(s, r)$ are identical.*

For notation convenience, we denote (n, t_p, t_f)-(V)SS by (verifiable) secret sharing schemes with number of participants n, privacy threshold t_p and fault-tolerance threshold t_f. Particularly, we say a verifiable secret sharing scheme VSS is with respect to a commitment scheme Com, if VSS.Share runs Com.Com as a subroutine to commit to the secret.

A dissection of Share^* algorithm. In conventional syntax of VSS, Share^* algorithm outputs all shares $(v_1, ..., v_n)$ in one shot, where n denotes the maximum number of possible participants. Such syntax is fine when n is polynomial in λ. But, it is problematic when n is superpolynomial[3] in λ becasue Share^* algorithm becomes inefficient. To fix this issue, we further dissect Share^* algorithm as below:

(i) $\mathsf{Share\text{-}in\text{-}Mind}(s, r)$: on input a secret s and a randomness r, outputs a compact description of the sharing method SH_{cpt} and the associated authentication information aut. Both of their sizes are no larger than $\mathsf{poly}(\lambda)$.

(ii) $\mathsf{Distribute}(s, r, SH_{\mathrm{cpt}}, i)$: on input the secret s, the randomness r, the compact description of the sharing method SH_{cpt} and an index i, generates share v_i for participant P_i as per the prefixed sharing method. This step is analogous to the private key extraction algorithm in identity-based cryptography, which generates the private keys for users on-the-fly upon request.

Evidently, our refined syntax can precisely captures all VSS schemes, while the conventional syntax is only suitable for VSS schemes with polynomial size n.

Flexible design of VSS. The VSS schemes can be designed in a flexible manner. For example, when the secret s is a vector, the commitment c could either be a single vector commitment committing to the multiple entries of the secret at once (e.g., using Pedersen vector commitment [Ped91, BBB+18]), or a vector of commitments committing to each entry of the secret (e.g., the VSS scheme in Section 4.2). Meanwhile, the shares v_i's could either be packed shares of the multiple entries of s (e.g., being generated by using packed Shamir's secret sharing scheme [FY92] as the VSS scheme in Section 4.2), or be a collection of separate shares of each entry of the secret. Moreover, the authentication information aut could be viewed as a commitment to the sharing procedure, which possibly are in the form of polynomial commitments, non-interactive zero-knowledge proofs or something else.

Comparing with the definition in [Fel87]. Our refined definition has several differences from the definition in [Fel87]. In terms of syntax, there are four differences:

1. In our definition, the secret s is committed via an algorithm Com rather than being encrypted as in Feldman's definition. This change makes our definition more general, as it allows for utilizing a broader range of cryptographic techniques.

2. Our definition incorporates the committing as a sub-process of sharing, while committing and sharing are treated as separate processes in Feldman's definition. This modification emphasizes the inherent connection between the committed value and the shared secret, rendering the definition more in line with the functionality of VSS.

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[3] The value n could even be exponential in security parameter λ (e.g. the size of a finite field).

3. The algorithm Share in our definition will additionally output authentication information *aut* (and a commitment *c* generated by its subroutine Com), while in Feldman's definition, the algorithm Share only outputs the shares. The information *aut* is crucial for participants to check the validity of their own shares.
4. The algorithm Recover in our definition will output the opening of a commitment, i.e., the secret *s* and the randomness *r* (if there is any), while in Feldman's definition, the algorithm Recover only outputs the secret *s*. Looking ahead, this modification is crucial for the security proof of our Sigma protocols framework.

In terms of security properties, there are two differences:

1. For the correctness, our definition does not stipulate that the secrets recovered by different groups of participants are consistent as in Feldman's definition, instead we require that the recovered secrets and randomness (if there is any) must be an opening of the commitment *c*. Looking ahead, this requirement is crucial for our application and in fact has been met by many existing VSS schemes, such as the Feldman's [Fel87] and Pedersen's VSS schemes [Ped91], but it has never been formally defined.
2. For privacy, we provide a simulation-based definition rather than a game-based one as in Feldman's definition. Such adoption makes our definition more convenient to use in the context of ZKP and MPC. In particular, the simulator Sim is given the commitment *c* as an auxiliary input, making the definition more general, as it allows the use of commitment schemes satisfying merely one-way hiding property.

3.2 The Framework of Sigma Protocols

Having settled a satisfactory definition of VSS, we are ready to describe our framework of Sigma protocols. Let Com = (Setup, Com, Verify) be an algebraic commitment scheme, and VSS = (Setup, Share, Check, Recover) be an (n, t_p, t_f)-VSS scheme w.r.t Com. The framework of Sigma protocols for relation $R_{Com} = \{(x; s, r) : Com(s; r) = x\}$ proceeds as below (see Figure 1 for a pictorial view).

– **Commit:** the prover P acts as the dealer running $((v_i)_{i \in [n]}, aut) \leftarrow VSS.Share^*(s, r)$ "in his head", and then sends the authentication information *aut* to the verifier V;
– **Challenge:** V chooses a random set of participants $I \subset [n]$ subject to $|I| = t_p$, and queries P for corresponding shares;
– **Response:** P replies with the shares $(v_i)_{i \in I}$.

Finally, V verifies whether $(v_i)_{i \in I}$ are valid shares for $(P_i)_{i \in I}$ w.r.t. *aut* and x, and outputs *accept* iff Check$(i, v_i, x, aut) = 1$ for all $i \in I$.

Theorem 1. *Suppose* VSS *is an (n, t_p, t_f)-VSS scheme where $t_f \log t_f = O(\log \lambda)$, then the protocol described in Figure 1 is a Sigma protocol for $\mathcal{NP}$ relation R_{Com} with $\left(\binom{t_f - 1}{t_p} + 1 \right)$-special soundness.*

Proof. We separately argue its completeness, special soundness and SHVZK.

Completeness. This follows readily from the acceptance property of the underlying VSS schemes.
Special Soundness. We argue this by constructing a PPT extractor Ext as below. For notation convenience, let $k = \binom{t_f - 1}{t_p} + 1$. Since $t_f \log t_f = O(\log \lambda)$, k is bounded by poly(λ). Given any k accepting transcripts $(aut, I_j, (v_i)_{i \in I_j})_{j \in [k]}$, where $|I_j| = t_p$ and $I_j \neq I_{j'}$ for all $j \neq j'$, first note that, there exist at least t_f distinct indices $i_1, \ldots, i_{t_f} \in [n]$ along with corresponding shares $v_{i_1}, \ldots, v_{i_{t_f}}$ (which are possibly not unique) subject to VSS.Check$(i_j, v_{i_j}, x, aut) = 1$ for all $j \in [t_f]$. This is because if not, then there must be a $(t_f - 1)$-sized set T, such that all I_j's are subsets of T. Since the total number of t_p-sized subsets of T is $\binom{t_f - 1}{t_p} < \binom{t_f - 1}{t_p} + 1$, there must exist two sets $I_j = I_{j'}$ where $j \neq j'$ by the pigeonhole principle. This contradicts to the hypothesis that $I_j \neq I_{j'}$ for all $j \neq j'$. Thus, Ext can extract a witness simply through running VSS.Recover on input $(i_j)_{j \in [t_f]}$, $(v_{i_j})_{j \in [t_f]}$ and taking the output (s, r) as its own output. By the correctness of VSS scheme, the reconstructed witness (s, r) must hold that Verify$(x, s, r) = 1$. This implies that the soundness error of the Sigma protocol in 1 is $\binom{t_f - 1}{t_p} / \binom{n}{t_p}$, which is no greater than $(t_f / n)^{t_p}$.
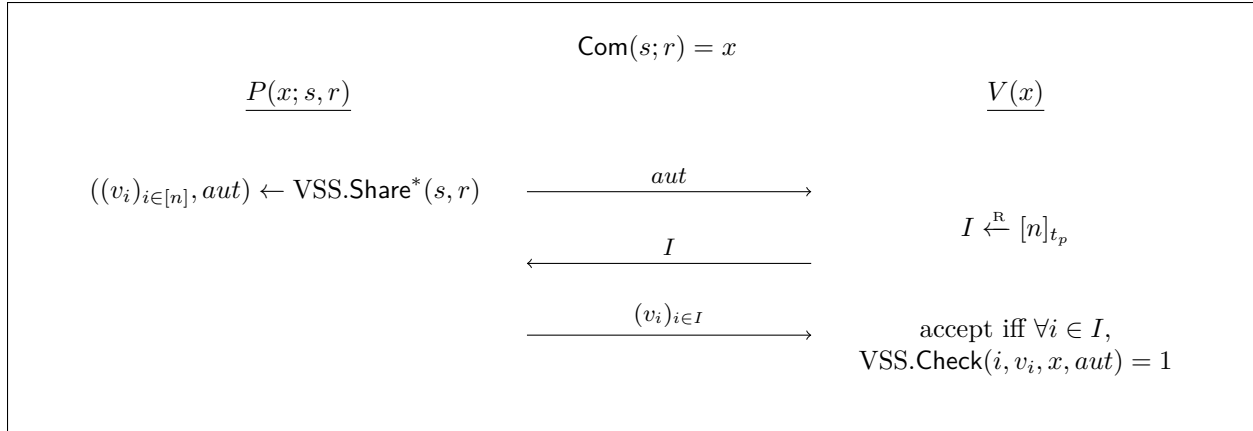
$$\mathsf{Com}(s; r) = x$$

$\underline{P(x; s, r)}$ $\underline{V(x)}$

$((v_i)_{i \in [n]}, aut) \leftarrow \mathsf{VSS.Share}^*(s, r)$ $\xrightarrow{\quad aut \quad}$

                                                    $I \xleftarrow{\mathrm{R}} [n]_{t_p}$

                    $\xleftarrow{\quad I \quad}$

                    $\xrightarrow{\quad (v_i)_{i \in I} \quad}$ accept iff $\forall i \in I$,

                                              $\mathsf{VSS.Check}(i, v_i, x, aut) = 1$

Fig. 1: A framework of Sigma protocols for algebraic commitments

SHVZK. We prove the SHVZK property by constructing a simulator Sim as below. Given the statement x and a challenge $I \in [n]_{t_p}$, the simulator Sim invokes the simulator of VSS scheme $\mathsf{Sim}_{\mathrm{VSS}}$ on input (x, I) and outputs the same as $\mathsf{Sim}_{\mathrm{VSS}}$ does, which includes the joint views of parties in I, namely the shares $(v_i)_{i \in I}$ and the authentication information aut. Based on the t_p-privacy of VSS scheme, the simulated transcript is distributed identically to real one.

A more detailed framework. In the light of the dissection of Share algorithm in Section 3.1, the framework of Sigma protocols from VSS could also be dissected. Specifically, in the *Commit* phase, P runs $(SH_{\mathrm{cpt}}, aut) \leftarrow \mathsf{VSS.Share\text{-}in\text{-}Mind}(s, r)$ and sends aut to V. In the *Challenge* phase, V chooses and sends random set $I \subset [n]$ as before. In the *Response* phase, P runs $v_i \leftarrow \mathsf{VSS.Distribute}(s, r, SH_{\mathrm{cpt}}, i)$ for all $i \in I$. In fact, this framework could yield more efficient Sigma protocols, since the prover only needs to compute the requested shares, not all the shares. Sigma protocols in Sections 4.2 to 4.4 all follow this framework.

Parameters selection. The three parameters n, t_p, t_f of the underlying VSS schemes could be any positive integers subject to $|\mathbb{F}| \geq n \geq t_f > t_p$ where $\mathbb{F}$ is the field to which parameters n, t_p, t_f belong. However, there are two caveats that warrant attention:

1. When n is superpolynomial in the security parameter λ, the Sigma protocols from such VSS schemes follow the detailed version of the framework. This is because, the underlying Share algorithm in this case must be dissected for efficiency reasons.
2. If the soundness error $(t_f/n)^{t_p}$ in a single execution of the protocol is not negligible in the security parameter λ, one should repeat the protocol in parallel to amplify soundness. To achieve soundness error of $2^{-\lambda}$, one should set the repetition number $\rho = \frac{\lambda}{t_p(\log n - \log t_f)}$.

Size of I. For the sake of simplicity, we set the size of I to t_p, which is equal to the privacy threshold of the VSS scheme. Actually, it is possible to set the size of I to be an arbitrary positive number k smaller than t_p, thus leading to Sigma protocols with $\left(\binom{t_f - 1}{k} + 1\right)$-special soundness. This can be proved similarly as in the proof of Theorem 1.

4 Instantiations of Our Framework

In this section, we demonstrate the generality of our framework by recovering the classic Schnorr [Sch91], Batching Schnorr [GLSY04], Okamoto [Oka92] and GQ [GQ88] protocols from corresponding VSS schemes.

4.1 Proof of Knowledge of A Discrete Logarithm

Let $\mathbb{G}$ be a cyclic group with generator g and prime order p, define $\mathsf{Com}(s) = g^s$. Given a commitment $x \in \mathbb{G}$, we show how to prove knowledge of s such that $g^s = x$. In Section 1.1.1, we have showed how to recover the classic Schnorr protocol from Feldman's VSS scheme. Below, we present another Sigma protocol from the following additive VSS scheme.

- $\mathsf{Setup}(1^\lambda)$: runs $(\mathbb{G}, p, g) \leftarrow \mathsf{GroupGen}(1^\lambda)$, sets the total number of participants $n \leq p$, the privacy threshold $t_p = n - 1$ and the fault-tolerance threshold $t_f = n$, outputs $pp = ((\mathbb{G}, p, g), n, t_p, t_f)$.
- $\mathsf{Share}(s)$: computes commitment $c = g^s$, picks $s_1, \ldots, s_n \xleftarrow{\text{R}} \mathbb{Z}_p$ subject to $s = \sum_{i=1}^{n} s_i \mod p$, then sets P_i's share $v_i = s_i$ and $aut = (c_1, \ldots, c_{n-1})$ where $c_i = g^{s_i}$ for $i \in [n-1]$, outputs the vector $(c, (v_i)_{i \in [n]}, aut)$.
- $\mathsf{Check}(i, v_i, c, aut)$: parses $aut = (c_1, \ldots, c_{n-1})$, if $i \in [1, n-1]$, then outputs "1" iff $g^{v_i} = c_i$ and outputs "0" otherwise; if $i = n$, then outputs "1" if $g^{v_i} = c/\prod_{j=1}^{n-1} c_j$ and "0" otherwise.
- $\mathsf{Recover}(I, (v_i)_{i \in I})$: outputs $s = \sum_{i \in I} v_i \mod p$.

Theorem 2. *The above VSS scheme satisfies acceptance, n-correctness and $(n-1)$-privacy properties.*

By plugging the above VSS scheme into our framework, we obtain a variant of Schnorr protocol for proving knowledge of a discrete logarithm (as depicted in Figure 2).
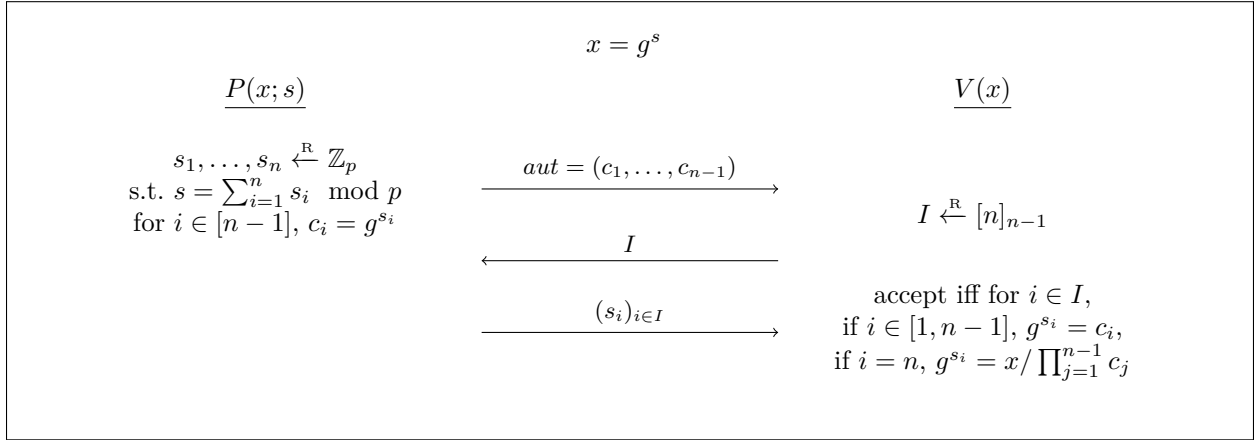

$$x = g^s$$

$\underline{P(x; s)}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\underline{V(x)}$

$s_1, \ldots, s_n \xleftarrow{\text{R}} \mathbb{Z}_p$
s.t. $s = \sum_{i=1}^{n} s_i \mod p$ $\qquad \xrightarrow{\quad aut = (c_1, \ldots, c_{n-1}) \quad}$
for $i \in [n-1]$, $c_i = g^{s_i}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $I \xleftarrow{\text{R}} [n]_{n-1}$

$\xleftarrow{\qquad\qquad I \qquad\qquad}$

$\qquad\qquad\qquad\qquad$ accept iff for $i \in I$,
$\xrightarrow{\qquad (s_i)_{i \in I} \qquad}$ if $i \in [1, n-1]$, $g^{s_i} = c_i$,
$\qquad\qquad\qquad\qquad$ if $i = n$, $g^{s_i} = x/\prod_{j=1}^{n-1} c_j$

Fig. 2: A Sigma protocol for proving knowledge of a discrete logarithm

4.2 Proof of Knowledge of Several Discrete Logarithms

Define $\mathsf{Com}(\mathbf{s}) = (g^{s_j})_{j \in \{1, \ldots, |\mathbf{s}|\}}$. Given a vector of commitments $\mathbf{x} = (x_j)_{j \in [\ell]}$, we show how to prove knowledge of $\mathbf{s} = (s_j)_{j \in [\ell]}$ such that $g^{s_j} = x_j$ for all $j \in [\ell]$. Consider following VSS scheme:

- $\mathsf{Setup}(1^\lambda)$: runs $(\mathbb{G}, p, g) \leftarrow \mathsf{GroupGen}(1^\lambda)$, picks a positive number $\ell \in \mathbb{Z}_p^*$, sets the total number of participants $n \leq p$ and the privacy threshold t_p and the fault-tolerance threshold $t_f = t_p + \ell$, outputs $pp = ((\mathbb{G}, p, g), n, t_p, t_f, \ell)$.
- $\mathsf{Share}(\mathbf{s})$: on input the secret $\mathbf{s} = (s_j)_{j \in [\ell]}$, runs following three algorithms and outputs $(\mathbf{c}, SH_{\mathrm{cpt}}, aut)$:
  - $\mathsf{Com}(\mathbf{s})$: computes $c_j = g^{s_j}$ for $j \in [\ell]$, outputs $\mathbf{c} = (c_j)_{j \in [\ell]} \in \mathbb{G}^\ell$;

- **Share-in-Mind($\mathbf{s}$):** selects $a_1, \ldots, a_{t_p} \xleftarrow{\text{R}} \mathbb{Z}_p^*$, defines a polynomial $A(x) = \sum_{j=1}^{t_p+\ell} a_j \cdot x^{j-1}$ where $a_{t_p+j} = s_j$ for all $j \in [\ell]$, sets $SH_{\text{cpt}} = (a_j)_{j \in [t_p]}$ and $aut = (\widetilde{c}_j)_{j \in [t_p]}$ where $\widetilde{c}_j = g^{a_j}$ for $j \in [t_p]$, outputs (SH_{cpt}, aut);
- **Distribute($\mathbf{s}, SH_{\text{cpt}}, i$):** parses $\mathbf{s} = (s_j)_{j \in [\ell]}$ and $SH_{\text{cpt}} = (a_j)_{j \in [t_p]}$, sets $a_{t_p+j} = s_j$ for $j \in [\ell]$, computes $v_i = \sum_{j=1}^{t_p+\ell} a_j \cdot i^{j-1} \mod p$, outputs v_i. (This algorithm is run upon request.)

– **Check($i, v_i, \mathbf{c}, aut$):** parses $\mathbf{c} = (c_j)_{j \in [\ell]}$ and $aut = (\widetilde{c}_j)_{j \in [t_p]}$, outputs "1" if it holds that $g^{v_i} = \left(\prod_{j=1}^{t_p} \widetilde{c}_j^{i^{j-1}} \right) \cdot \left(\prod_{j=1}^{\ell} c_j^{i^{t_p+j-1}} \right)$ and "0" otherwise.

– **Rec($I, (v_i)_{i \in I}$):** computes a polynomial $A(x)$ such that $A(i) = v_i$ for all $i \in I$, sets s_j be the $(t_p + j)$-th coefficient of A, outputs $(s_j)_{j \in [\ell]}$.

Theorem 3. *Above VSS scheme satisfies acceptance, $(t_p + \ell)$-correctness and t_p-privacy.*

By plugging the above VSS scheme into our framework, we obtain a Sigma protocol for proving knowledge of several discrete logarithms (as depicted in Figure 3). By setting parameters $n = p$ and $t_p = 1$, we recover the Batching Schnorr protocol [GLSY04].
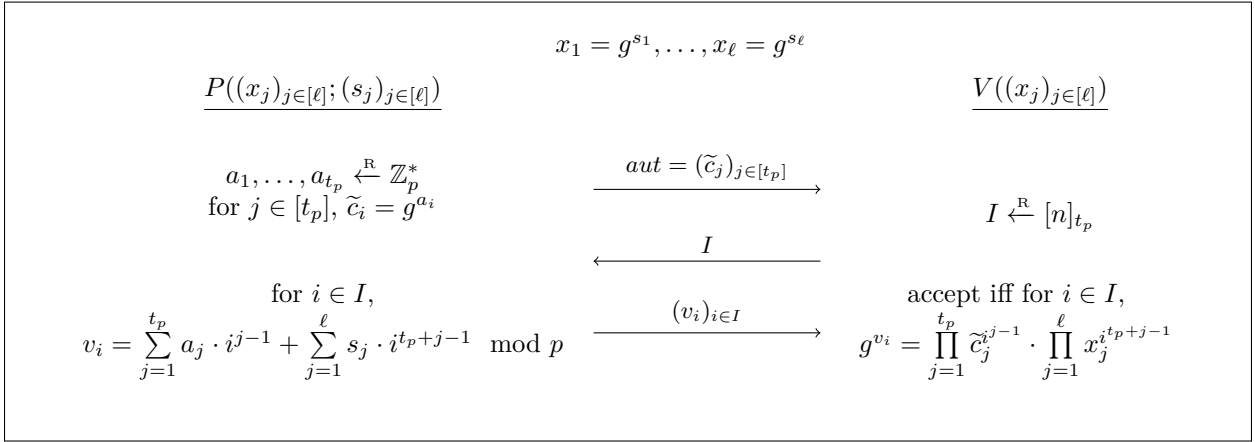

$$x_1 = g^{s_1}, \ldots, x_\ell = g^{s_\ell}$$

$\underline{P((x_j)_{j \in [\ell]}; (s_j)_{j \in [\ell]})}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{V((x_j)_{j \in [\ell]})}$

$a_1, \ldots, a_{t_p} \xleftarrow{\text{R}} \mathbb{Z}_p^*$

for $j \in [t_p]$, $\widetilde{c}_i = g^{a_i}$ $\qquad\qquad \xrightarrow{\quad aut = (\widetilde{c}_j)_{j \in [t_p]} \quad}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad I \xleftarrow{\text{R}} [n]_{t_p}$

$\qquad\qquad\qquad\qquad\qquad \xleftarrow{\qquad I \qquad}$

for $i \in I$, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ accept iff for $i \in I$,

$v_i = \sum_{j=1}^{t_p} a_j \cdot i^{j-1} + \sum_{j=1}^{\ell} s_j \cdot i^{t_p+j-1} \mod p$ $\xrightarrow{\quad (v_i)_{i \in I} \quad}$ $g^{v_i} = \prod_{j=1}^{t_p} \widetilde{c}_j^{i^{j-1}} \cdot \prod_{j=1}^{\ell} x_j^{i^{t_p+j-1}}$

Fig. 3: A Sigma protocol for proving knowledge of several discrete logarithms

4.3 Proof of Knowledge of A Representation

Define $\mathsf{Com}(s; r) = g^s h^r$ where g, h are two different generators of group $\mathbb{G}$. Given a commitment x, we show how to prove knowledge of (s, r) such that $g^s h^r = x$ from the Pedersen's VSS scheme [Ped91] as below:

– **Setup(1^λ):** runs $(\mathbb{G}, p, g, h) \leftarrow \mathsf{GroupGen}(1^\lambda)$, sets the total number of participants $n \leq p$, the privacy threshold t_p and the fault-tolerance threshold $t_f = t_p + 1$, outputs $pp = ((\mathbb{G}, p, g, h), n, t_p, t_f)$.

– **Share(s):** on input the secret s, runs following three algorithms and outputs $(c, SH_{\text{cpt}}, aut)$:
  - **Com($s; r$):** picks a random element $r \xleftarrow{\text{R}} \mathbb{Z}_p^*$, outputs $c = g^s h^r$;
  - **Share-in-Mind(s, r):** picks two random t_p-degree polynomials $A(x) = \sum_{i=0}^{t_p} a_i \cdot x^i$ and $B(x) = \sum_{i=0}^{t_p} b_i \cdot x^i$ subject to $a_{t_p} = s$ and $b_{t_p} = r$, computes $c_j = g^{a_j} h^{b_j}$ for $0 \leq j \leq t_p - 1$, sets $SH_{\text{cpt}} = (a_j, b_j)_{0 \leq j \leq t_p-1}$ and $aut = (c_j)_{0 \leq j \leq t_p-1}$, outputs (SH_{cpt}, aut);
  - **Distribute(s, r, SH_{cpt}, i):** parses $SH_{\text{cpt}} = (a_j, b_j)_{0 \leq j \leq t_p-1}$, sets $a_{t_p} = s$ and $b_{t_p} = r$, computes $s_i = \sum_{j=0}^{t_p} a_j \cdot i^j \mod p$ and $r_i = \sum_{j=0}^{t_p} b_j \cdot i^j \mod p$, outputs $v_i = (s_i, r_i)$. (This algorithm is run upon request.)

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- Check(i, v_i, c, aut): parses $v_i = (s_i, r_i)$ and $aut = (c_j)_{0 \leq j \leq t_p - 1}$, outputs "1" if $g^{s_i} h^{r_i} = c^{i^{t_p}} \cdot \prod_{j=0}^{t_p - 1} c_j^{i^j}$ and "0" otherwise.
- Recover($I, (v_i)_{i \in I}$): parses $v_i = (s_i, r_i)$, constructs two polynomials $A(x)$, $B(x)$ such that $A(i) = s_i$ and $B(i) = r_i$ for all $i \in I$, sets s be the coefficient of the t_p-degree term of A and r be that of B, outputs (s, r).

Theorem 4. *Pedersen's VSS scheme satisfies acceptance, $(t_p + 1)$-correctness and t_p-privacy.*

By plugging the above VSS scheme into our framework, we obtain a Sigma protocol for proving knowledge of a representation (as depicted in Figure 4). By setting parameters $n = p$ and $t_p = 1$, we recover the classic Okamoto protocol [Oka92].
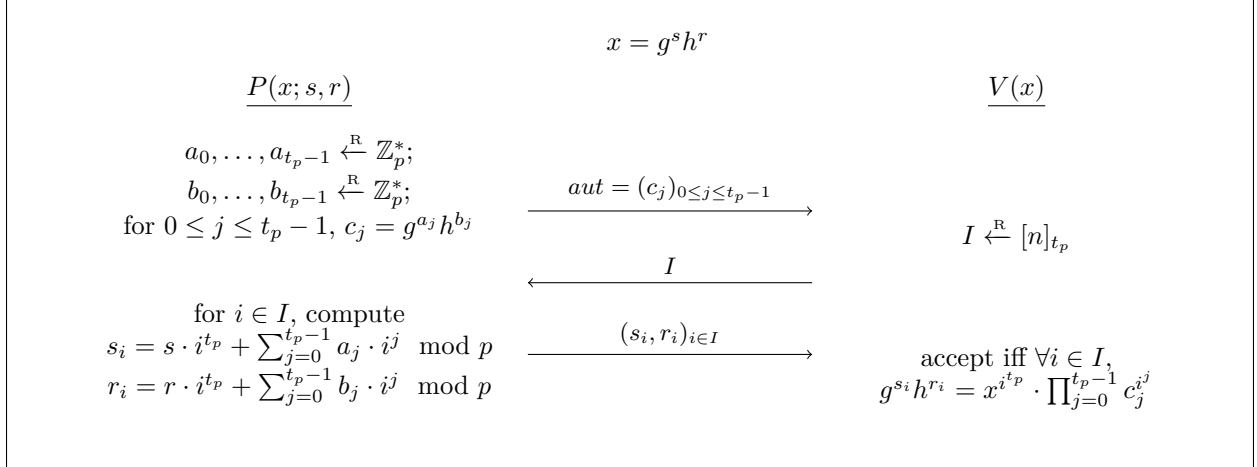

$$x = g^s h^r$$

$\underline{P(x; s, r)}$ $\underline{V(x)}$

$a_0, \ldots, a_{t_p - 1} \xleftarrow{\text{R}} \mathbb{Z}_p^*$;
$b_0, \ldots, b_{t_p - 1} \xleftarrow{\text{R}} \mathbb{Z}_p^*$;
for $0 \leq j \leq t_p - 1$, $c_j = g^{a_j} h^{b_j}$

$\xrightarrow{\quad aut = (c_j)_{0 \leq j \leq t_p - 1} \quad}$

$I \xleftarrow{\text{R}} [n]_{t_p}$

$\xleftarrow{\qquad\qquad I \qquad\qquad}$

for $i \in I$, compute
$s_i = s \cdot i^{t_p} + \sum_{j=0}^{t_p - 1} a_j \cdot i^j \mod p$
$r_i = r \cdot i^{t_p} + \sum_{j=0}^{t_p - 1} b_j \cdot i^j \mod p$

$\xrightarrow{\quad (s_i, r_i)_{i \in I} \quad}$

accept iff $\forall i \in I$,
$g^{s_i} h^{r_i} = x^{i^{t_p}} \cdot \prod_{j=0}^{t_p - 1} c_j^{i^j}$

Fig. 4: A Sigma protocol for proving knowledge of a representation

4.4 Proof of Knowledge of An eth Root

Let GenRSA be a PPT algorithm that on input security parameter λ, outputs an RSA public key (N, e), where e is prime. Given an $x \in \mathbb{Z}_N^*$, we show how to prove knowledge of s such that $x = s^e \mod N$ from following VSS scheme:

- Setup(1^λ): runs $(N; e) \leftarrow$ GenRSA(1^λ), where e is prime, sets the total number of participants $n \leq e$, and sets the privacy threshold $t_p = 1$ and the fault-tolerance threshold $t_f = 2$, outputs $pp = ((N, e), n, t_p, t_f)$.
- Share(s): on input a secret $s \in \mathbb{Z}_N^*$, runs following three algorithms and outputs $(c, SH_{\text{cpt}}, aut)$:
  - Com(s): computes the commitment $c = s^e \mod N$, outputs c;
  - Share-in-Mind(s): picks a random element $a \in \mathbb{Z}_N^*$, defines a function $f(x) = a \cdot s^x \mod N$, sets $SH_{\text{cpt}} = a$, computes $aut = a^e \mod N$, outputs (SH_{cpt}, aut);
  - Distribute(s, SH_{cpt}, i): parses $SH_{\text{cpt}} = a$, computes $s_i = a \cdot s^i \mod N$, outputs $v_i = s_i$. (This algorithm is run upon request.)
- Check(i, v_i, c, aut): outputs "1" if $v_i^e = aut \cdot c^i \mod N$ and "0" otherwise.
- Recover($I, (v_i)_{i \in I}, c$): if $|I| < 2$ outputs $\perp$; else, runs the extended Euclidean algorithm yields integers $\alpha, \beta \in \mathbb{Z}_N^*$ such that $\alpha \cdot e + \beta \cdot (i_2 - i_1) = 1$, outputs $s = c^\alpha (v_{i_2} / v_{i_1})^\beta \mod N$.

Theorem 5. *The above VSS scheme satisfies acceptance, 2-correctness and 1-privacy properties.*

By plugging the above VSS scheme into our framework, we obtain a Sigma protocol for proving knowledge of an e-th root (as depicted in Figure 5). By setting the parameter $n = e$, we recover the classic GQ protocol [GQ88].
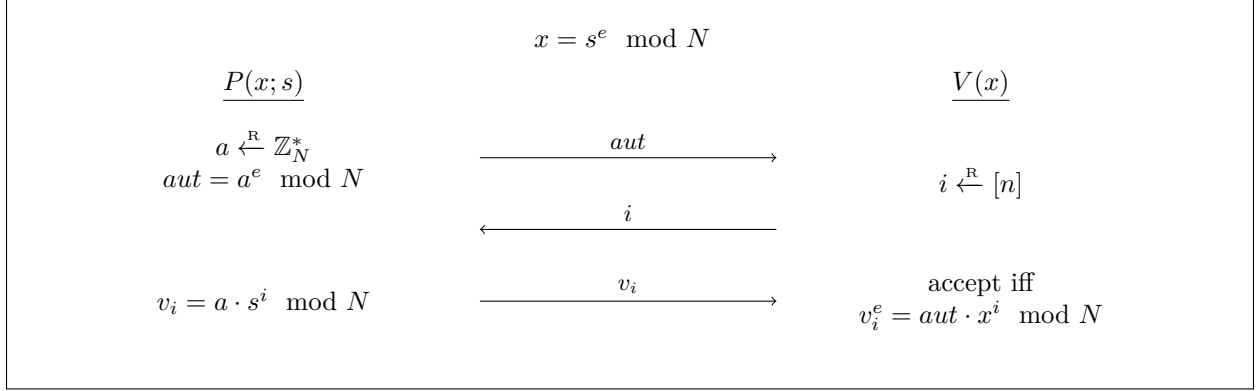
$$x = s^e \mod N$$

$\underline{P(x;s)}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\underline{V(x)}$

$a \xleftarrow{\text{R}} \mathbb{Z}_N^*$

$aut = a^e \mod N$ $\qquad\xrightarrow{\qquad aut \qquad}\qquad$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $i \xleftarrow{\text{R}} [n]$

$\qquad\xleftarrow{\qquad i \qquad}\qquad$

$v_i = a \cdot s^i \mod N$ $\qquad\xrightarrow{\qquad v_i \qquad}\qquad$ accept iff

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $v_i^e = aut \cdot x^i \mod N$

Fig. 5: A Sigma protocol for proving knowledge of an e-th root

5 A Framework of ZKPs for Composite Statements

In this section, we are going to show the application of the Sigma protocols from VSS in giving a generic construction of efficient ZKPs for composite statements. In this work, we focus on a common form of composite statement where given a commitment x and a value y, the prover wants to prove the knowledge of (s, r) such that $\mathsf{Com}(s; r) = x \land C(s) = y$, where C is an arithmetic/boolean circuit. In a nutshell, we use Sigma protocols from VSS to prove the algebraic parts, use ZK protocols from MPC to prove the non-algebraic parts, and enforce consistency between the witnesses used in two parts via *witness sharing reusing*.

5.1 A Generalization of MPC-in-the-Head Paradigm

Before designing the framework of ZKPs for composite statements, we first generalize the MPC-in-the-head paradigm introduced by Ishai et al. [IKOS07] via extending the XOR-based secret sharing scheme to an (n, t_p, t_f)-SS scheme. Precisely, to construct a ZK protocol for $\mathcal{NP}$ relation $\mathsf{R}_C = \{(y; s) : C(s) = y\}$ using MPC-in-the-head technique, we need three building blocks: a secret sharing scheme, an MPC protocol and a commitment scheme.

Let $\mathsf{SS} = (\mathsf{Setup}, \mathsf{Share}, \mathsf{Recover})$ be an $(n, t_p^{\text{ss}}, t_f)$-secret sharing scheme, $\widehat{\mathsf{Com}} = (\mathsf{Setup}, \mathsf{Com}, \mathsf{Verify})$ be a commitment scheme and Π_f be a t_p^{mpc}-private n-party protocol that realizes a n-party function f, where $f(y, s_1, \ldots, s_n) = 1$ if and only if $C(\mathsf{Recover}([n], (s_i)_{i \in [n]})) = y$, and integers $t_p^{\text{mpc}} < t_p^{\text{ss}}$. Then, ZK protocols following MPC-in-the-head paradigm proceeds as below (as depicted in Figure 6):

- **Commit:** the prover P shares the witness s into n shares $s_1, \ldots, s_n$ by running $\mathsf{SS}.\mathsf{Share}(s)$, then runs MPC protocol Π_f "in his head" with shares $s_1, \ldots, s_n$ as input of n virtual parties, then commits to each party's share s_i and view $view_i$ (without loss of generality, we separate P_i's input s_i from his view $view_i$ and concatenate them with notation $||$), and sends the n commitments to V;

- **Challenge:** V picks a random t_p^{mpc}-sized subset I of $[n]$ and sends it to P;

- **Response:** P opens corresponding commitments through revealing corresponding shares and views to V.

Finally, V outputs "accept" iff the three conditions listed hereunder hold:

1. the commitments are successfully opened;
2. all the outputs of participants in I are "1", which are determined by their inputs s_i and views $view_i$;
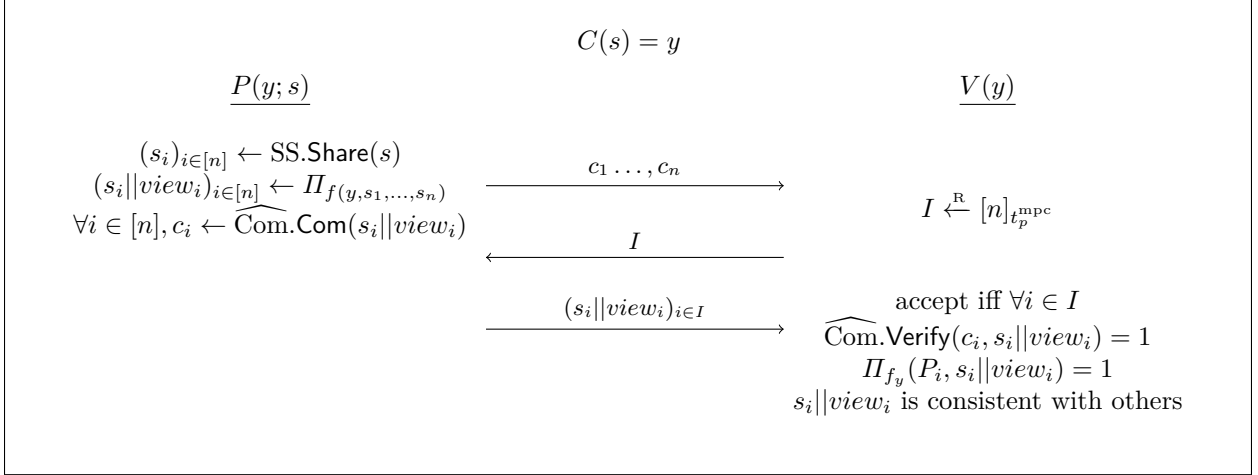3. all the opened views are consistent with each other with respect to y and Π_f.

Fig. 6: A ZKP from MPC-in-the-head paradigm

Theorem 6. *Let $n > 2$, $t_p^{ss} \geq t_p^{mpc}$, $t_p^{mpc} \cdot \log n = O(\log \lambda)$, and R_C, f be as above. Suppose SS is an (n, t_p^{ss}, t_f)-secret sharing scheme, the MPC protocol Π_f realizes the n-party functionality f with correctness and t_p^{mpc}-privacy and $\widehat{\mathsf{Com}}$ is a commitment scheme, then the protocol in Figure 6, is a Sigma protocol for relation R_C with $\left(\binom{n-2}{t_{mpc}^{p}} + 2\binom{n-2}{t_{mpc}^{p}-1} + 1\right)$-special soundness.*

Proof. We separately argue its completeness, special soundness and SHVZK.

Completeness. If $(y, s) \in R_C$ and the prover is honest, then by the correctness of $\widehat{\mathsf{Com}}$, the requested commitments are always opened successfully. By the correctness of the secret sharing scheme SS, the shares $s_1, \ldots, s_n$ hold that $\mathsf{Recover}([n], (s_i)_{i \in [n]}) = s$ and thus $f(y, s_1, \ldots, s_n) = 1$. Then, by the correctness of Π_f, the views $s_1||view_1, \ldots, s_n||view_n$ always have output "1". Besides, since the views are produced by an honest execution of the Π_f, by Lemma 2, they are all consistent with each other.

Special Soundness. For notation convenience, let $k = \binom{n-2}{t_{mpc}} + 2\binom{n-2}{t_{mpc}-1} + 1$. We argue the k-special soundness by constructing a PPT extractor Ext that can extract a witness s such that $(y, s) \in \mathsf{R}_C$, given any k accepting transcripts with the same initial message and different challenges. Since $t_p^{mpc} \cdot \log n = O(\log \lambda)$, k is bounded by $\mathsf{poly}(\lambda)$. In following argument, we assume for simplicity that both binding of commitment scheme and correctness of MPC protocol never fail.

Consider k accepting transcripts $((c_i)_{i \in [n]}, I_j, (s_i||view_i)_{i \in I_j})_{j \in [k]}$: first note that since the k sets $I_1, \ldots, I_k$ are distinct t_p^{mpc}-sized subset of $[n]$ and $k > \binom{n-1}{t_p^{mpc}}$, we have $\cup_{j=1}^{k} I_j = \{1, \ldots, n\}$, which implies that for all $1 \leq i \leq n$, P_i's view $s_i||view_i$ is revealed at least once. (This is because if not, namely if there exists at least one index $i \in [n]$ such that $i \notin \cup_{j=1}^{k} I_j$, then all I_j's are t_p^{mpc}-sized subsets of the set $[n] \setminus i$. Since the total number of such subsets is only $\binom{n-1}{t_{mpc}^{p}} < k$, there must exist two sets $I_j = I_{j'}$ where $j \neq j'$ by pigeonhole principle. This contradicts to the hypothesis that the k sets $(I_j)_{j \in [k]}$ are distinct.) Thanks to the binding property of $\widehat{\mathsf{Com}}$, for the same index i, all $s_i||view_i$'s revealed in different transcripts are identical. Thereby, the k transcripts provide unique n views $s_1||view_1, \ldots, s_n||view_n$ and the extractor Ext is able to compute a witness s through running the efficient algorithm $\mathsf{Recover}([n], (s_i)_{i \in [n]}) = s$. Below, we argue that s is a valid witness such that $C(s) = y$.

First note that for each accepting transcript $((c_i)_{i \in [n]}, I, (s_i||view_i)_{i \in I})$, it indicates that for each $i \in I$, the output of player P_i with respect to Π_f, and $s_i||view_i$ is "1", and $s_i||view_i$, $s_j||view_j$ are consistent for all $i, j \in I$. Then, we prove that in the n-tuple views $s_1||view_1, \ldots, s_n||view_n$ provided by the k accepting transcripts, all pairs of views $s_i||view_i$, $s_j||view_j$ are consistent with each other. Namely, for all $i, j \in [n]$,

there is a set $I \in \{I_1, \ldots, I_k\}$ such that $i \in I \wedge j \in I$. The reason is that, if there is a pair of $i', j' \in [n]$ which does not hold above, then for each set $I \in \{I_1, \ldots, I_k\}$, it must be one of the following cases:

- $i' \notin I \wedge j' \notin I$: there are $\binom{n-2}{t_p^{\mathrm{mpc}}}$ cases;
- $i' \in I \wedge j' \notin I$: there are $\binom{n-2}{t_p^{\mathrm{mpc}}-1}$ cases;
- $i' \notin I \wedge j' \in I$: there are $\binom{n-2}{t_p^{\mathrm{mpc}}-1}$ cases.

However, the total number of the above cases is only $\binom{n-2}{t_p^{\mathrm{mpc}}} + 2\binom{n-2}{t_p^{\mathrm{mpc}}-1}$, which is smaller than k. By the pigeonhole principle, there must be two identical sets in $I_1, \ldots, I_k$, which contradicts the hypothesis that all of them are distinct. Therefore, for all pairs $i, j \in [n]$, there must be a set $I \in \{I_1, \ldots, I_k\}$ such that $i \in I \wedge j \in I$ and thus all pairs of views $s_i||view_i$, $s_j||view_j$ are consistent with each other. By Lemma 2, the n-tuple views provided by the k accepting transcripts correspond to an honest execution of Π_f. Moreover, by the correctness of Π_f, in an honest execution of Π_f on input $s_1, \ldots, s_n$, the probability that the output of some player is different from the output of f is 0. Besides, as mentioned before, all the revealed views $s_i||view_i$'s are such that the output of P_i is "1". Thus, it holds that $f(y, s_1, \ldots, s_n) = 1$ and as the definition of f, we have $C(\mathsf{Recover}([n], (s_j)_{j \in [n]})) = C(s) = y$.

SHVZK. Let $\mathsf{Sim}_{\mathrm{SS}}$ and Sim_{Π_f} be the simulators for the underlying secret sharing scheme SS and MPC protocol Π_f, respectively. We prove SHVZK by constructing a PPT simulator $\mathsf{Sim}(y, I)$ where $I \subset [n]$ and $|I| = t_p^{\mathrm{mpc}}$, by invoking $\mathsf{Sim}_{\mathrm{SS}}$ and Sim_{Π_f} as below:

1. Run $\mathsf{Sim}_{\mathrm{SS}}$ on input I, receiving a vector of shares $(s_i)_{i \in I}$.
2. Run Sim_{Π_f} on input $(I, y, (s_i)_{i \in I}, 1)$, receiving t_p^{mpc} views $(s_i||view_i)_{i \in I}$.
3. For $i \in [n] \wedge i \notin I$, select random string $str_i \xleftarrow{\mathrm{R}} \{0,1\}^{|s||view|}$ and set $s_i||view_i = str_i$.
4. For $i \in [n]$, compute $\widehat{\mathsf{Com}}.\mathsf{Com}(s_i||view_i) \to c_i$.
5. Output $((c_i)_{i \in [n]}, I, (s_i||view_i)_{i \in I})$.

Then we show that the transcripts output by Sim are indistinguishable from transcripts of real executions of the protocol with an honest verifier via a sequence of hybrid transcripts as follows:

- Hybrid_0: Real transcript.
- Hybrid_1: Same as Hybrid_0, except that the simulator is given a random challenge $I \xleftarrow{\mathrm{R}} [n]_{t_p^{\mathrm{mpc}}}$ in advance, and for $i \in [n] \wedge i \notin I$, it selects random string $str_i \xleftarrow{\mathrm{R}} \{0,1\}^{|w||view|}$, and computes $c_i \leftarrow \widehat{\mathsf{Com}}.\mathsf{Com}(str_i)$.
- Hybrid_2: Same as Hybrid_1, except that the simulator runs Sim_{Π_f} on input $(I, y, (s_i)_{i \in I}, 1)$, obtaining the simulated views $(s_i||view_i)_{i \in I}$.
- Hybrid_3: Same as Hybrid_2, except that the simulator is not provided the witness and instead it runs $\mathsf{Sim}_{\mathrm{SS}}$ on input I and obtains a vector of shares $(s_i)_{i \in I}$.

Since the commitments $(c_i)_{i \in [n] \wedge i \notin I}$ are never opened, the indistinguishability of Hybrid_0 and Hybrid_1 follows directly from the hiding property of the commitment scheme. The indistinguishability of Hybrid_1 and Hybrid_2 follows from the t_p^{mpc}-privacy property of Π_f. Since $t_p^{\mathrm{mpc}} \leq t_p^{\mathrm{ss}}$, The indistinguishability of Hybrid_2 and Hybrid_3 follows straightforwardly from the t_p^{ss}-privacy property of the underlying secret sharing scheme SS.

5.2 Separable VSS Schemes

As discussed in Section 1, in order to combine Sigma protocols from VSS and ZK protocols from MPC seamlessly, we are interested in VSS schemes which satisfy a mild property called *Separability*. Since the parameter n in the MPC-in-the-head paradigm is bounded by $\mathsf{poly}(\lambda)$, we consider the separability of VSS schemes simply using the syntax in Definition 7. Informally, for a VSS scheme, we say it satisfies *Separability* if the following two conditions hold:

1. The algorithm $\mathsf{Share}^*(s, r)$ could be separated into two sub-algorithms, one for generating the shares $(v_i)_{i\in[n]}$ and the other for generating the authentication information aut. Particularly, the shares $(v_i)_{i\in[n]}$ are generated as per some secret sharing schemes and aut is generated by committing to the sharing method (i.e., the shares $(v_i)_{i\in[n]}$ in the syntax in Definition 7 or the compact description of the sharing method SH_{cpt} in the dissected version).

2. If the randomness r is not a dummy value, then each share v_i could be divided into two values s_i and r_i, where the former is a share of the secret s and the later is a share of the randomness r. That is, s and r are secret-shared separately.

Below, we formally define the *Separability* property.

Definition 8 (Separability). *For an (n, t_p, t_f)-VSS scheme VSS, we say it satisfies separability if there is an (n, t_p, t_f)-SS scheme SS and an algorithm AutGen such that the algorithms $\mathsf{VSS.Share}^*$ and $\mathsf{VSS.Recover}$ can be separated as below:*

$$\mathsf{VSS.Share}^*(s, r) : (s_i)_{i\in[n]} \leftarrow \mathsf{SS.Share}(s)$$
$$(r_i)_{i\in[n]} \leftarrow \mathsf{SS.Share}(r)$$
$$aut \leftarrow \mathsf{AutGen}((s_i, r_i)_{i\in[n]})$$
$$return \ ((s_i, r_i)_{i\in[n]}, aut)$$

$$\mathsf{VSS.Recover}(I, (v_i)_{i\in I}) : \forall 1 \leq j \leq |I|, \ \mathrm{parse} \ v_i = (s_i, r_i)$$
$$s \leftarrow \mathsf{SS.Recover}(I, (s_i)_{i\in[n]})$$
$$r \leftarrow \mathsf{SS.Recover}(I, (r_i)_{i\in[n]})$$
$$return \ (s, r)$$

If r is null, then only the s will be secret-shared and recovered.

Particularly, we say a VSS scheme aligns with an SS scheme if it generates the shares as per this SS scheme.

Remark 1. More generally, in such separable VSS schemes, the SS schemes used to share secret s and randomness r could be different in some settings.

5.3 Generic Construction of ZKPs for Composite Statements

Now, we proceed to describe the generic construction of ZKPs for composite statements. Formally, let Com be an algebraic commitment algorithm and C be an arbitrary circuit, we give a zero-knowledge proof for relation:

$$\mathsf{R}_{\mathrm{cs}} = \{(x, y; s, r) : \mathsf{Com}(s; r) = x \wedge C(s) = y\}.$$

Let Π_C^{MPC} be a Sigma protocol for $\{(y; s) : C(s) = y\}$ from MPC as depicted in Figure 6 and using building blocks: an $(n, t_p^{\mathrm{ss}}, t_f)$-SS scheme SS, a commitment scheme $\widehat{\mathsf{Com}}$, and a t_p^{mpc}-private n-party protocol Π_f. Let $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$ be a Sigma protocol for $\{(x; s, r) : \mathsf{Com}(s; r) = x\}$ following the framework as in Figure 1 and using building blocks: an $(n, t_p^{\mathrm{vss}}, t_f)$-VSS scheme VSS w.r.t. Com and SS. Below, we show how to obtain a ZK protocol $\Pi_{\mathrm{Com}, C}$ for composite statements through combining Π_C^{MPC} and $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$, which is also a Sigma protocol. The full protocol is presented in Figure 7 and the overlap between Π_C^{MPC} and $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$ are highlighted in rectangles.

– **Commit:** P proceeds as in $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$, running algorithm $((s_i, r_i)_{i\in[n]}, aut) \leftarrow \mathsf{VSS.Share}^*(s, r)$, which can be separated into three algorithms $(s_i)_{i\in[n]} \leftarrow \mathsf{SS.Share}(s)$, $(r_i)_{i\in[n]} \leftarrow \mathsf{SS.Share}(r)$ and $aut \leftarrow \mathsf{AutGen}((s_i, r_i)_{i\in[n]})$. Then, P proceeds as in Π_C^{MPC} while reusing the shares $(s_i)_{i\in[n]}$. Next, P sends $c_1, \ldots, c_n$ and aut to V.

- **Challenge:** V picks a random t_{mpc}-sized subset I of $[n]$ as in Π_C^{MPC}.

- **Response:** P responds with participants' inputs and views $(s_i||view_i)_{i \in I}$ as in Π_C^{MPC} and shares of randomness $(r_i)_{i \in I}$ as in $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$.

Finally, V outputs "accept" iff $(s_i||view_i)_{i \in I}$ pass the verification of Π_C^{MPC} and $(s_i, r_i)_{i \in I}$ pass the verification of $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$.
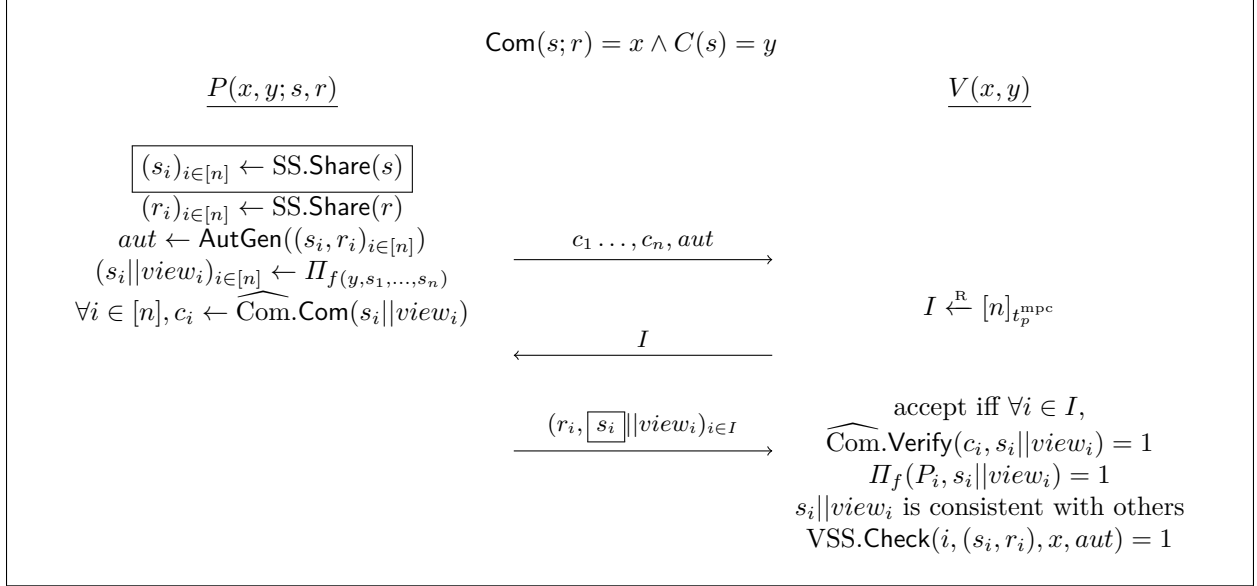

Fig. 7: A ZKP for composite statements

Theorem 7. *Let $n > 2$, $t_p^{\mathrm{ss}} \geq t_p^{\mathrm{mpc}}$, $t_p^{\mathrm{mpc}} \cdot \log n = O(\log \lambda)$. Suppose the protocol Π_C^{MPC} constructed as in Figure 6 using building blocks SS, $\widehat{\mathrm{Com}}$ and Π_f as above, is a Sigma protocol for relation $\{(y;s) : C(s) = y\}$ with $\left(\binom{n-2}{t_p^{\mathrm{mpc}}} + 2\binom{n-2}{t_p^{\mathrm{mpc}}-1} + 1\right)$-special soundness, protocol $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$ constructed as in Figure 1 using building block VSS which is with respect to Com and aligns with SS, is a Sigma protocol for relation $\{(x;s,r) : \mathrm{Com}(s;r) = x\}$, then the protocol $\Pi_{\mathrm{Com},C}$ constructed as in Figure 7 is a Sigma protocol for $\mathrm{R_{cs}}$ with $\left(\binom{n-2}{t_p^{\mathrm{mpc}}} + 2\binom{n-2}{t_p^{\mathrm{mpc}}-1} + 1\right)$-special soundness.*

Proof. We separately argue its completeness, special soundness and SHVZK.

Completeness. We first argue the correctness of protocol $\Pi_{\mathrm{Com},C}$. For an honestly generated transcript $((c_1, \ldots, c_n, aut), I, (r_i, s_i||view_i)_{i \in I})$, it is obvious that $((c_1, \ldots, c_n), I, (s_i||view_i)_{i \in I})$ is actually an honestly generated transcript of Π_C^{MPC}, thus by the correctness of Π_C^{MPC}, the verifier will pass the first three checks. And $(aut, I, (s_i, r_i)_{i \in I})$ is an honestly generated transcript of $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$, thus by the correctness of $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$ the verifier will pass the last check, and finally outputs "accept".

Special soundness. For notation convenience, let $k = \binom{n-2}{t_{\mathrm{mpc}}} + 2\binom{n-2}{t_{\mathrm{mpc}}-1} + 1$. We argue the k-special soundness by constructing a PPT extractor Ext that can extract a valid witness $w = (s, r)$, given any k accepting transcripts with the same initial message and different challenges. Since $t_p^{\mathrm{mpc}} \cdot \log n = O(\log \lambda)$, k is bounded by $\mathsf{poly}(\lambda)$. Likewise, we assume for simplicity that both binding of $\widehat{\mathrm{Com}}$ and correctness of Π_f never fail.

First note that, both Π_C^{MPC} and $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$ satisfy k-special soundness. The former has been proved in Theorem 6. As shown in the proof of Theorem 6, the PPT extractor Ext_C works as follows: on input k accepting transcripts as required, runs $\mathsf{SS.Recover}([n], (s_i)_{i\in[n]}) = s$ where $(s_i)_{i\in[n]}$ is uniquely determined by the k accepting transcripts, outputs a witness s such that $C(s) = y$. The latter can be proved similarly as the proof of Theorem 1. Since VSS aligns with SS, there exists a PPT extractor $\mathsf{Ext}_{\mathrm{Com}}$ proceeds as follows: on input k accepting transcripts as required, runs $\mathsf{SS.Recover}([n], (s_i)_{i\in[n]}) = s$ and $\mathsf{SS.Recover}([n], (r_i)_{i\in[n]}) = r$ where the shares $(s_i, r_i)_{i\in[n]}$ are subject to $\mathsf{VSS.Check}(i, (s_i, r_i), x, aut) = 1$ for all $i \in [n]$, and then outputs (s, r). By the correctness of VSS, it holds that $\mathsf{Com}(s; r) = x$.

Next, we show how the extractor Ext makes use of Ext_C and $\mathsf{Ext}_{\mathrm{Com}}$ to extract a witness $w = (s, r)$ such that $(x, y; s, r) \in \mathsf{R}_{\mathrm{cs}}$. Concretely, on input k accepting transcripts $((c_i)_{i\in[n]}, aut), I_j, (r_i, s_i||view_i)_{i\in I_j})_{j\in[k]}$, Ext works as below:

1. Run Ext_C on input $((c_i)_{i\in[n]}, I_j, (s_i||view_i)_{i\in I_j})_{j\in[k]}$ and obtain s;
2. Run $\mathsf{Ext}_{\mathrm{Com}}$ on input $(aut, I_j, (r_i, s_i)_{i\in I_j})_{j\in[k]}$ and obtain (s', r);
3. outputs (s, r).

Since both s and s' are output by the deterministic algorithm $\mathsf{SS.Recover}$ on the same input $([n], (s_i)_{i\in[n]})$ (where $(s_i)_{i\in[n]}$ is uniquely determined by the k accepting transcripts), we have $s = s'$. Furthermore, by the k-special soundness of Π_C^{MPC} and $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$, it holds that $\mathsf{Com}(s; r) = x \wedge C(s) = y$.

SHVZK. We prove this by constructing a PPT simulator Sim that on input any tuple $((x, y), I)$ where $|I| \leq t_p^{\mathrm{mpc}}$ could simulate the transcript of $\Pi_{\mathrm{Com}, C}$.

First note that, both Π_C^{MPC} and $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$ satisfy the SHVZK property. The former has been proved in Theorem 6. As shown in the proof of Theorem 6, the PPT simulator $\mathsf{Sim}_C(y, I)$ where $|I| \leq t_p^{\mathrm{mpc}}$ generates $(s_i)_{i\in I}$ essentially by running $(s_i)_{i\in I} \leftarrow \mathsf{Sim}_{\mathrm{SS}}(I)$. The latter has been proved in Theorem 1. As shown in the proof Theorem 1, the PPT simulator $\mathsf{Sim}_{\mathrm{Com}}(x, I)$ where $|I| \leq t_p^{\mathrm{vss}}$ simulates the transcripts essentially by running $((s_i, r_i)_{i\in I}, aut) \leftarrow \mathsf{Sim}_{\mathrm{VSS}}(x, I)$. Furthermore, since VSS aligns with SS, we have $t_p^{\mathrm{vss}} = t_p^{\mathrm{ss}}$ and the simulator $\mathsf{Sim}_{\mathrm{VSS}}(x, I)$ generates $(s_i)_{i\in I}$ by running $(s_i)_{i\in I} \leftarrow \mathsf{Sim}_{\mathrm{SS}}(I)$ as well.

Next, we show how Sim makes use of Sim_C and $\mathsf{Sim}_{\mathrm{Com}}$ to simulate a transcript for $\Pi_{\mathrm{Com}, C}$. Concretely, on input $((x, y), I)$ where $|I| \leq t_p^{\mathrm{mpc}} \leq t_p^{\mathrm{ss}}$, it works as below:

1. Run $\mathsf{Sim}_C(y, I)$ and obtain $((c_i)_{i\in[n]}, I, (s_i||view_i)_{i\in I})$;
2. Run $\mathsf{Sim}_{\mathrm{Com}}(x, I)$, and obtain $(aut, I, (s_i, r_i)_{i\in I})$, where $(s_i)_{i\in I}$ is a reuse of $(s_i)_{i\in I}$ that generated in Step 1, rather than a new one generated by running $\mathsf{Sim}_{\mathrm{SS}}(I)$ a second time;
3. Output $(((c_i)_{i\in[n]}, aut), I, (r_i, s_i||view_i)_{i\in I})$.

By the SHVZK property of Π_C^{MPC} and $\Pi_{\mathrm{Com}}^{\mathrm{VSS}}$, it is straightforward that the distribution of Sim's output is indistinguishable with a real transcription of $\Pi_{\mathrm{Com}, C}$.

Remark 2 (Key element required for combining). In order to get better efficiency, some practical protocols in the MPC-in-the-head paradigm slightly deviate from the template in Section 5.1, depending on the concrete MPC protocols they used. For example, the KKW protocol [KKW18] utilizes an MPC protocol designed in the preprocessing model and the Ligero [AHIV17]/Ligero++ [BFH+20] protocols make use of a particular type of MPC protocols in the malicious model. Nevertheless, they all retain the secret sharing procedure (though different secret sharing schemes are employed), which is the key element that is required for combining with our Sigma protocols framework in Section 3.2.

6 An Instantiation of ZKP for Composite Statements

In this section, we give a ZK protocol for composite statements by instantiating the underlying MPC-in-the-head protocol with Ligero++ [BFH+20]. Let $\mathbb{F}_p$ be a large prime field and $C : \mathbb{F}_p^m \to \mathbb{F}_p$ be an arithmetic circuit. We show how to prove following composite statements: given a vector of Pedersen commitments $\mathbf{x} = (x_1, \ldots, x_m)$, the prover wants to convince the verifier that he knows the witness $(\mathbf{s}, \mathbf{r}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m$ such that $C(\mathbf{s}) = 1 \wedge x_i = g^{s_i} h^{r_i}$ for $1 \leq i \leq m$.

As we have noticed, in order to construct a ZK protocol for composite statements using Ligero++, the key point is giving a VSS scheme that aligns with the SS scheme used by Ligero++, and then constructing a Sigma protocol from it.

6.1 Review of Ligero++

We briefly recall the Ligero++ protocol and analyze the SS scheme it uses. (Notably, Ligero++ uses the same SS scheme as Ligero [AHIV17].) At a high level, to prove knowledge of $\mathbf{s} = (s_i)_{i \in [m]} \in \mathbb{F}_p^m$ such that $C(\mathbf{s}) = 1$, the Ligero++ prover first generates an extended witness which contains the circuit input $\mathbf{s}$ and the outputs of $|C|$ gates, then arranges the extended witness in a matrix of size $\frac{C}{\mathsf{polylog}|C|} \times \mathsf{polylog}|C|$ (where the first m entries are $(s_i)_{i \in [m]}$) and encodes each row using Reed-Solomon (RS) Code. The verifier challenges the prover to reveal the linear combinations of the rows of the codeword matrix, and checks its consistency through invoking inner-product argument (IPA) protocols on $\widetilde{t}$ randomly picked columns. As mentioned in [BFH+20], to remain zero-knowledge during the consistency check, it is desirable to either utilize zero-knowledge IPA protocols or make the encoding randomized. For further consideration, we use a randomized RS encoding to ensure zero knowledge. The formal definition of RS code is presented below.

Definition 9 (Reed-Solomon Code). *For positive integers n, k, a finite field $\mathbb{F}$, and a vector $\eta = (\eta_1, \ldots, \eta_n)$ of distinct elements of $\mathbb{F}$, the code $\mathsf{RS}_{\mathbb{F},n,k,\eta}$ is the $[n, k, n-k+1]$ linear code over $\mathbb{F}$ that consists of all n-tuples $(P(\eta_1), \ldots, P(\eta_n))$ where P is a polynomial of degree $< k$ over $\mathbb{F}$.*

Definition 10 (Encoded message). *Let $L = \mathsf{RS}_{\mathbb{F},n,k,\eta}$ be an RS code and $\zeta = (\zeta_1, \ldots, \zeta_\ell)$ be a vector of distinct elements of $\mathbb{F}$ for $\ell \leq k$. For a codeword $u = (u_1, \ldots, u_n) \in L$, we say it encodes (or rather, can be decodes to) the message $(P_u(\zeta_1), \ldots, P_u(\zeta_\ell))$, where P_u is the polynomial (of degree $< k$) corresponding to u.*

Encoding & Sharing. We can simply make the RS code $\mathsf{RS}_{\mathbb{F},n,k,\eta}$ randomized via increasing the degree of polynomials by $\widetilde{t}$ where $\widetilde{t} < k$, and it is evident that the randomized RS code $\mathsf{RS}_{\mathbb{F},n,k,\eta}$ can be viewed as the (variant) packed Shamir's SS scheme [FY92] with number of participants n, privacy threshold $t_p = \widetilde{t}$ and the fault-tolerance $t_f = k$. That is, encoding a message is equivalent to sharing the message: to encode (resp., share) a message $(s_i)_{i \in [\ell]}$ using randomized $\mathsf{RS}_{\mathbb{F},n,k,\eta}$ (resp., packed Shamir's SS scheme), one first selects $\widetilde{t}$ random elements $\alpha_1, \ldots, \alpha_{\widetilde{t}} \in \mathbb{F}$ where $\ell + \widetilde{t} = k$ and generates a polynomial $P(x)$ with degree $< \ell + \widetilde{t}$ such that $P(\zeta_i) = s_i$ for all $i \in [\ell]$ and $P(\zeta_{\ell+i}) = \alpha_i$ for all $i \in [\widetilde{t}]$, then sets the codeword (resp., shares) to be $(P(\eta_1), \ldots, P(\eta_n))$. Therefore, the codeword matrix aforementioned is also the shares matrix.

Modifications to Ligero++. As mentioned before, the Ligero++ protocol does not strictly conform to the generalized MPC-in-the-head paradigm in Section 5.1, due to the different MPC model it used. There are two main differences that could pose challenges in combining Ligero++ with Sigma protocols. First, the witness to be shared is an expanded version that encompasses the input of circuit and the outputs of all circuit gates, rather than only the input itself, making the shares opened later be an expanded version as well. Second, the $\widetilde{t}$ random columns of shares matrix will not be opened directly due to the invocation of IPA protocols, causing obstructions of reusing witness shares. Fortunately, both of them can be overcame with a few modifications to Ligero++: dividing the shares matrix into two vertically concatenated sub-matrices and handling them differently when in the consistency check. Specifically, the two sub-matrices and their respective handling methods are as follows:

- The first sub-matrix is the first m/ℓ rows of the shares matrix (WLOG., we assume $m = c \cdot \ell$ for some integer $c > 0$), which in fact is the shares of circuit input $\mathbf{s}$. When in the consistency check, the prover opens its $\widetilde{t}$ entries directly to the verifier and the verifier computes the inner product of these entries with random vectors directly. Thereby, the shares of circuit input $\mathbf{s}$ could be reused later. Since the encoding is randomized, the openings leak nothing about the witness.
- The second sub-matrix is the remaining rows of the shares matrix, which are the shares of outputs of gates. When in the consistency check, the prover inputs its $\widetilde{t}$ entries on IPA protocols as originally while the inner product checked in IPA protocols should be modified according to the opened entries of the first sub-matrix.

By doing so, the shares of inputs $\mathbf{s}$ and shares of gates' outputs are separated. Moreover, it makes witness shares reusing available while maintaining the advantage of utilizing IPA technique.

6.2 A Sigma Protocol for Pedersen Commitments

Having specified the SS scheme that Ligero++ employs, we are now ready to present a VSS scheme that aligns with this SS scheme and later give a corresponding Sigma protocol from it.

The following notation will be useful below: for a field $\mathbb{F}$, an integer m and a vector $\mathbf{a} = (a_i)_{i \in [m]} \in \mathbb{F}^m$, we denote by $\mathbf{V}(\mathbf{a})$ the $m \times m$ Vandermonde matrix $(v_{i,j})_{i,j \in [m]}$ where $v_{i,j} = a_i^{j-1}$ for all $i, j \in [m]$ and denote by $\mathbf{V}(\mathbf{a})^{-1}$ the inverse of this Vandermonde matrix.

Since the parameter n in the SS scheme used by Ligero++ is bounded by $\mathsf{poly}(\lambda)$, we describe the VSS scheme simply using the syntax in Definition 7. The VSS scheme consists following four algorithms:

- $\mathsf{Setup}(1^\lambda)$: runs $(\mathbb{G}, p, g, h) \leftarrow \mathsf{GroupGen}(1^\lambda)$, sets the total number of participants n, the privacy threshold t_p, the fault-tolerance threshold threshold $t_f = \ell + t_p$, picks two disjoint vectors $\boldsymbol{\zeta} = (\zeta_j)_{j \in [\ell + t_p]} \in \mathbb{F}_p^{\ell + t_p}$ and $\boldsymbol{\eta} = (\eta_i)_{i \in [n]} \in \mathbb{F}_p^n$, and both $\boldsymbol{\zeta}, \boldsymbol{\eta}$ contain distinct elements, outputs $pp = ((\mathbb{G}, p, g, h), n, t_p, t_f, \ell, \boldsymbol{\zeta}, \boldsymbol{\eta})$.
- $\mathsf{Share}(\mathbf{s})$: on input a vector of secret $\mathbf{s} = (s_j)_{j \in [\ell]} \in \mathbb{F}_p^\ell$, runs following two algorithms and outputs $(\mathbf{c}, (v_i)_{i \in [n]}, aut)$:
  - $\mathsf{Com}(\mathbf{s}; \mathbf{r})$: selects a vector of randomness $\mathbf{r} = (r_j)_{j \in [\ell]} \xleftarrow{\mathrm{R}} \mathbb{F}_p^\ell$, outputs a vector of commitments $\mathbf{c} = (c_j)_{j \in [\ell]}$, where $c_j = g^{s_j} h^{r_j}$ for all $j \in [\ell]$.
  - $\mathsf{Share}^*(\mathbf{s}, \mathbf{r})$: chooses two random vectors $(\alpha_j)_{j \in [t_p]}, (\beta_j)_{j \in [t_p]} \xleftarrow{\mathrm{R}} \mathbb{F}_p^{t_p}$, interpolates two polynomials $A(x)$ and $B(x)$ such that

$$
\begin{aligned}
&\forall 1 \le j \le \ell,\ A(\zeta_j) = s_j,\ B(\zeta_j) = r_j; \\
&\forall \ell + 1 \le j \le \ell + t_p,\ A(\zeta_j) = \alpha_{j-\ell},\ B(\zeta_j) = \beta_{j-\ell},
\end{aligned}
\tag{1}
$$

    outputs shares $(v_i)_{i \in [n]}$ where $v_i = (A(\eta_i), B(\eta_i))$ for all $i \in [n]$ and $aut = (\widetilde{c}_j)_{j \in [t_p]}$ where $\widetilde{c}_j = g^{\alpha_j} h^{\beta_j}$ for all $j \in [t_p]$.
- $\mathsf{Check}(i, v_i, \mathbf{c}, aut)$: parses $v_i = (v_{i1}, v_{i2})$ and $aut = (\widetilde{c}_j)_{j \in [t_p]}$, computes $h_k = \left(\prod_{j=1}^{\ell} c_j^{\delta_{k,j}} \right) \cdot \left(\prod_{j=1}^{t_p} \widetilde{c}_j^{\delta_{k,\ell+j}} \right)$ for $k \in [\ell + t_p]$, where the matrix $(\delta_{k,j})_{1 \le k, j \le \ell + t_p}$ is equal to $\mathbf{V}(\boldsymbol{\zeta})^{-1}$, outputs "1" if $g^{v_{i1}} h^{v_{i2}} = \prod_{k=1}^{\ell + t_p} h_k^{\eta_i^{k-1}}$ and "0" otherwise.
- $\mathsf{Recover}(I, (v_i)_{i \in I})$: parses $v_i = (v_{i1}, v_{i2})$, uses Lagrange Interpolation to compute polynomials $A(x)$ and $B(x)$ such that $A(\eta_i) = v_{i1}$ and $B(\eta_i) = v_{i2}$ for all $i \in I$, outputs $(\mathbf{s}, \mathbf{r})$ where $(s_j, r_j) = (A(\zeta_j), B(\zeta_j))$ for $j \in [\ell]$.

Theorem 8. *The VSS scheme described above satisfies acceptance, $(\ell + t_p)$-correctness and t_p-privacy.*

By plugging the above VSS scheme into the framework in Section 1.1.1, we obtain a Sigma protocol (as depicted in Figure 8) for proving knowledge of openings of several Pedersen commitments.

Parameters selection. In order to combine with Ligero++, some of the public parameters of above VSS scheme, including p, n, t_p, ℓ, $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$, should be in line with that of Ligero++. Since Ligero++ performs interpolation and evaluation using fast Fourier transform (FFT), above VSS scheme should be implemented using elliptic curves whose scalar fields $\mathbb{F}_p$ are FFT-friendly. One can refer to [AHG22] for a suitable elliptic curve.

Security analysis. Based on Lemma 1, Theorem 1 and Theorem 8, it is straightforward that the protocol in Figure 8 is a Sigma protocol with soundness error $\binom{t_f - 1}{t_p} / \binom{n}{t_p}$. When setting $n = c \cdot t_f$ for some constant $c \ge 1$, we must set $t_p = \lambda / \log c$ to achieve a soundness error of $2^{-\lambda}$ without repetition. Since $\binom{t_f - 1}{t_p} / \binom{n}{t_p}$ is smaller than the soundness error of Ligero++, the soundness error of ZK protocols for composite statements, obtained by combining Sigma protocols in Figure 8 and Ligero++, is dominated by the soundness error of Ligero++.

$$x_1 = g^{s_1}h^{r_1}, \ldots, x_\ell = g^{s_\ell}h^{r_\ell}$$

$$\underline{P((x_j)_{j\in[\ell]}; (s_j, r_j)_{j\in[\ell]})} \qquad\qquad\qquad\qquad\qquad \underline{V((x_j)_{j\in[\ell]})}$$

$(\alpha_j)_{j\in[t_p]}, (\beta_j)_{j\in[t_p]} \overset{\text{R}}{\leftarrow} \mathbb{F}_p^{t_p}$

compute $A(x)$, $B(x)$ as in (1)

$\forall\, i \in [n]$: $s_i = A(\eta_i)$, $r_i = B(\eta_i)$

$\forall\, j \in [t_p]$, $\widetilde{c}_j = g^{\alpha_j}h^{\beta_j}$

$$\xrightarrow{\quad aut = (\widetilde{c}_j)_{j\in[t_p]} \quad}$$

$$\xleftarrow{\qquad\quad I \qquad\quad} \qquad I \overset{\text{R}}{\leftarrow} [n]_{t_p}$$

$$\xrightarrow{\quad (s_i, r_i)_{i\in I} \quad} \qquad (\delta_{k,j})_{1\le k,j\le \ell+t_p} = \mathbf{V}(\boldsymbol{\zeta})^{-1}$$

$$\forall k \in [\ell + t_p],\ h_k = \prod_{j=1}^{\ell} x_j^{\delta_{k,j}} \cdot \prod_{j=\ell+1}^{\ell+t_p} \widetilde{c}_{j-\ell}^{\delta_{k,j}}$$

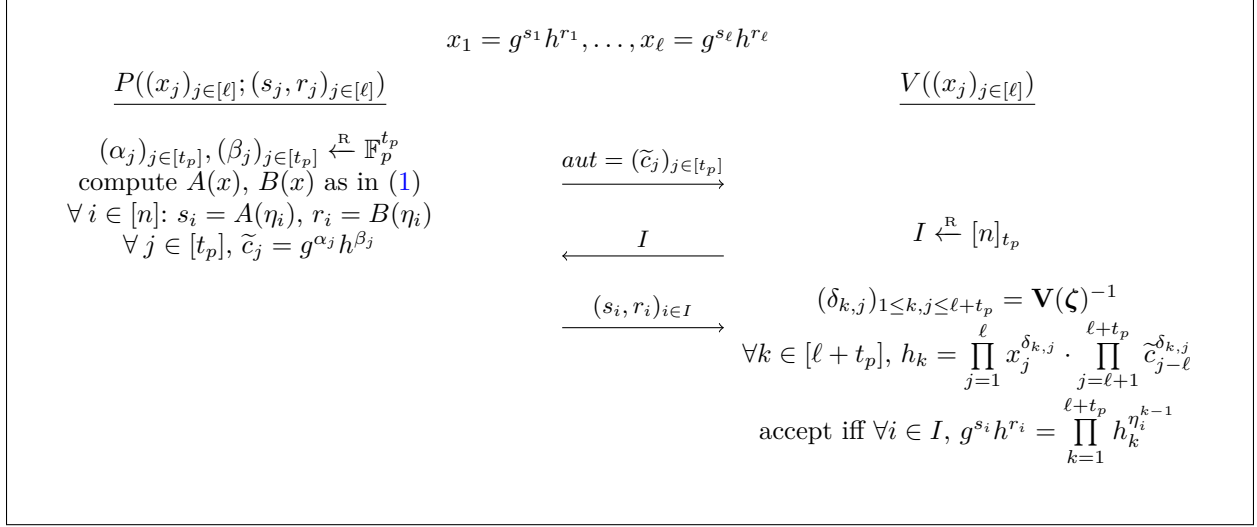$$\text{accept iff } \forall i \in I,\ g^{s_i}h^{r_i} = \prod_{k=1}^{\ell+t_p} h_k^{\eta_i^{k-1}}$$

Fig. 8: A Sigma protocol for Pedersen commitments

Efficiency analysis. Let λ be the security parameter, $\ell_{\mathbb{G}}$ be the length of a group element, $\ell_{\mathbb{F}}$ be the length of a field element and $\ell = |\mathbf{x}|$ be the number of commitments in the statement. Fix parameters n, t_p, t_f where $n = c \cdot t_f$ for some constant $c \ge 1$ and $t_f = \ell + t_p$. Then, the proof size is $t_p \cdot \ell_{\mathbb{G}} + 2t_p \cdot \ell_{\mathbb{F}})$, which asymptotically is $O(\lambda)$. The prover's work includes the computations of c_k's, which need $O(t_p)$ group operations; interpolation and evaluation of polynomials, which need $O((\ell + t_p) \cdot \log(\ell + t_p))$ field operations by using FFT. The verifier's work includes the computations of matrix $(\delta_{k,j})$, which need $O((\ell + t_p)^2)$ field operations; the computations of h_k's, which need $O(\ell + t_p)$ multi-exponentiations of size $\ell + t_p$; and the computations in the verification equations, which need $O(t_p)$ multi-exponentiations of size $\ell + t_p$. (Pippenger's [Pip80] algorithm could be used to accelerate the computations of multi-exponentiations.)

Having given the Sigma protocol for Pedersen commitments, it is not difficult to combine it with the Ligero++ protocol and get a ZK protocol for composite statements, following the method in Section 5.3, and we omit the details in this paper. The efficiency of the final ZK protocol reported in Table 1 is obtained by directly summing the costs of Ligero++ and above Sigma protocol. Since the underlying SS components are identical in Ligero and Ligero++, the Sigma protocol could also be combined with Ligero seamlessly by choosing appropriate parameters. This will lead to a faster prover while a larger proof size.

7 Conclusion

Sigma protocols are the most efficient ZKPs for proving knowledge of openings of algebraic commitments, which are defined as relations over algebraic groups. They have now become an important building block for a variety of cryptosystems. In this work, we presented a framework of Sigma protocols for algebraic statements from verifiable secret sharing schemes. This framework neatly explains the design principal underlying those classic Sigma protocols, including the Schnorr, Batching Schnorr, GQ and Okamoto protocol. In addition, it gives a generic construction of Sigma protocols for proving knowledge of algebraic commitments, thus being able to lead to new Sigma protocols that were not previously known. Furthermore, we also showed its application in designing ZKPs for composite statements. By using the *witness sharing reusing* technique, we combined the Sigma protocols from VSS and general-purpose ZKPs following MPC-in-the-head paradigm seamlessly, yielding a generic construction of ZKPs for composite statements which enjoys the advantages of requiring no "glue" proofs. Through instantiating the underlying general-purpose ZKPs with Ligero++

and tailoring a corresponding Sigma protocol, we obtain a concrete ZKP for composite statements, which achieves a tradeoff between running time and proof size, thus resolving the open problem left by Backes et al. (PKC 2019).

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A Missing Proofs

A.1 A Proof for Theorem 2

Proof. We separately argue the three properties.

Acceptance. For all honestly generated $(v_i)_{i \in [n]}$, c and $aut = (c_1, \ldots, c_{n-1})$, it always holds that $g^{v_i} = c_i$ for all $i \in [n-1]$ and $g^{v_n} = g^{s_n} = g^{s - \sum_{i=1}^{n-1} s_i} = g^s / \prod_{j=1}^{n-1} g^{s_j} = c / \prod_{j=1}^{n-1} c_j$.

n-Correctness. For any commitment c, $aut = (c_1, \ldots, c_{n-1})$, participants $I = \{1, \ldots, n\}$ and shares $(v_i)_{i \in [n]}$, if it holds that $\mathsf{Check}(i, v_i, c, aut) = 1$ for all $1 \le i \le n$ (namely $g^{v_i} = c_i$ for all $1 \le i \le n-1$ and $g^{v_i} = c / \prod_{j=1}^{n-1} c_j$ for $i = n$), then it is evident that $g^{\sum_{i=1}^{n} v_i} = \prod_{i=1}^{n} g^{v_i} = c$. Thus, $s = \sum_{i=1}^{n} v_i$ is an opening for c.

$(n-1)$-privacy. We argue this by constructing the simulator $\mathsf{Sim}(c, I)$. In the two different cases, it works as follows:

1. Case I: $n \notin I$. Without loss of generality, we assume $I = \{1, \ldots, n-1\}$. Sim simulates the shares for $(P_i)_{i \in I}$, by picking $s_1, \ldots, s_{n-1} \xleftarrow{\text{R}} \mathbb{Z}_p$, and setting $v_i = s_i$. Then compute $c_i = g^{s_i}$ and set the authentication information $aut = (c_1, \ldots, c_{n-1})$. Output $((v_i)_{i \in [n-1]}, aut)$.
2. Case II: $n \in I$. Without loss of generality, we assume $I = \{2, \ldots, n\}$. Sim simulates the shares for $(P_i)_{i \in I}$, by picking $(s_i)_{i \in [2,n]} \xleftarrow{\text{R}} \mathbb{Z}_p$, and setting $v_i = s_i$. Then compute $c_i = g^{s_i}$ for $i \in [2, n]$ and $c_1 = c / (\prod_{j=2}^{n} c_j)$, set the authentication information $aut = (c_1, \ldots, c_{n-1})$. Output $((v_i)_{i \in [2,n]}, aut)$.

It is direct that the distributions of the outputs of simulator $\mathsf{Sim}(c, I)$ are identical to that of the outputs of algorithm $\mathsf{Share}^*(s)$ for participants in I.

A.2 A Proof for Theorem 3

Proof. We separately argue the three properties.

Acceptance. For all honestly generated v_i, $\mathbf{c} = (c_j)_{j \in [\ell]}$ and $aut = (\widetilde{c}_j)_{j \in [t_p]}$, it always holds that $g^{v_i} = g^{A(i)} = g^{\sum_{j=1}^{t_p+\ell} a_j \cdot i^{j-1}} = \prod_{j=1}^{t_p+\ell} g^{a_j \cdot i^{j-1}} = \left(\prod_{j=1}^{t_p} \widetilde{c}_j^{i^{j-1}} \right) \cdot \left(\prod_{j=1}^{\ell} c_j^{i^{t_p+j-1}} \right)$.

$(\boldsymbol{t_p + \ell})$**-Correctness**. For any set of participants $I \subset [n]$ where $|I| = m \geq t_p + \ell$ and corresponding shares $(v_i)_{i \in I}$, if it holds that $\mathsf{Check}(i, v_i, c, aut) = 1$ for all $i \in I$, then we have $g^{v_i} = \left(\prod_{j=1}^{t_p} \widetilde{c}_j^{i^{j-1}} \right) \cdot \left(\prod_{j=1}^{\ell} c_j^{i^{t_p+j-1}} \right)$ for all $i \in I$. Let $A(x) = \sum_{k=1}^{m} a_k \cdot x^{k-1}$ be the polynomial that interpolates the points $(i, v_i)_{i \in I}$. Note that, the degree of $A(x)$ is at most $m - 1$. Then, it holds that $a_{t_p+j} = \sum_{i=1}^{m} \delta_{t_p+j,i} \cdot v_i \mod p$ for $j \in [\ell]$, where the vector $(\delta_{t_p+j,i})_{i \in [m]}$ is the $(t_p + j)$-th row of the matrix $\mathbf{V}(1, \ldots, m)^{-1}$. Thus, for $j \in [\ell]$, we have

$$
\begin{aligned}
g^{a_{t_p+j}} &= g^{\sum_{i=1}^{m} \delta_{t_p+j,i} \cdot v_i} \\
&= \prod_{i=1}^{m} (g^{v_i})^{\delta_{t_p+j,i}} \\
&= \prod_{i=1}^{m} \left(\left(\prod_{k=1}^{t_p} \widetilde{c}_k^{i^{k-1}} \right) \cdot \left(\prod_{k=1}^{\ell} c_k^{i^{t_p+k-1}} \right) \right)^{\delta_{t_p+j,i}} \\
&= \left(\prod_{k=1}^{t_p} \widetilde{c}_k^{\sum_{i=1}^{m} i^{k-1} \cdot \delta_{t_p+j,i}} \right) \cdot \left(\prod_{k=1}^{\ell} c_k^{\sum_{i=1}^{m} i^{t_p+k-1} \cdot \delta_{t_p+j,i}} \right) \\
&= c_j
\end{aligned}
$$

$\boldsymbol{t_p}$**-privacy**. We argue this by constructing the simulator $\mathsf{Sim}(\mathbf{c}, I)$. Without loss of generality, we assume $I = [t_p]$. The simulator $\mathsf{Sim}(\mathbf{c}, I)$ proceeds as follows:

1. Simulate the shares for $P_1, \ldots, P_{t_p}$, by picking $v_1, \ldots, v_{t_p} \xleftarrow{\mathrm{R}} \mathbb{Z}_p$;
2. Compute the authentication information $aut = (\widetilde{c}_j)_{j \in [t_p]}$ based on the algorithm Check. Concretely, set $\widetilde{c}_j = \prod_{i=1}^{t_p} (g^{v_i} / \prod_{k=1}^{\ell} c_k^{i^{t_p+k-1}})^{\gamma_{j,i}}$ for $j \in [t_p]$, where the $t_p \times t_p$ matrix $(\gamma_{j,i})_{1 \leq j,i \leq t_p}$ is the inverse of the Vandermonde matrix $\mathbf{V}(1, \ldots, t_p)$:
3. Output $((v_i)_{i \in [t_p]}, aut)$.

It is direct that the outputs of simulator $\mathsf{Sim}(\mathbf{c}, I)$ are distributed identically to the outputs of algorithm $\mathsf{Share}^*(\mathbf{s})$ for participants in I.

A.3 A Proof for Theorem 4

Proof. We separately argue the three properties.

Acceptance. For all honestly generated $v_i = (s_i, r_i)$, c and $aut = (c_j)_{0 \geq j \geq t_p-1}$, it holds that $g^{s_i} h^{r_i} = g^{\sum_{j=0}^{t_p} a_j \cdot i^j} h^{\sum_{j=0}^{t_p} b_j \cdot i^j} = \prod_{j=0}^{t_p} g^{a_j \cdot i^j} h^{b_j \cdot i^j} = \prod_{j=0}^{t_p-1} c_j^{i^j} \cdot c^{i^{t_p}}$.

$(\boldsymbol{t_p + 1})$**-Correctness**. For any set of participants $I \subset [n]$ where $|I| = m \geq t_p + 1$ and corresponding shares $(v_i)_{i \in I}$ where $v_i = (s_i, r_i)$, if it holds that $\mathsf{Check}(i, v_i, \mathbf{c}, aut) = 1$ for all $i \in I$, then we have $g^{s_i} h^{r_i} = \prod_{j=0}^{t_p-1} c_j^{i^j} \cdot c^{i^{t_p}}$ for all $i \in I$. Let $A(x) = \sum_{j=0}^{m-1} a_j \cdot x^j$ and $B(x) = \sum_{j=0}^{m-1} b_j \cdot x^j$ be the two polynomials that interpolate two sets of points $(i, s_i)_{i \in I}$ and $(i, r_i)_{i \in I}$ respectively. Note that, both the degrees of $A(x)$ and

$B(x)$ are at most $m-1$. Then, it holds that $a_{t_p} = \sum_{i=1}^{m} \delta_{t_p+1,i} \cdot s_i \mod p$ and $b_{t_p} = \sum_{i=1}^{m} \delta_{t_p+1,i} \cdot r_i \mod p$, where the vector $(\delta_{t_p+1,i})_{i\in[m]}$ is the (t_p+1)-th row of the matrix $\mathbf{V}(1,\ldots,m)^{-1}$. Therefore, we have

$$
\begin{aligned}
g^{a_{t_p}} h^{b_{t_p}} &= g^{\sum_{i=1}^{m} \delta_{t_p+1,i}\cdot s_i} h^{\sum_{i=1}^{m} \delta_{t_p+1,i}\cdot r_i} \\
&= \prod_{i=1}^{m}(g^{s_i} h^{r_i})^{\delta_{t_p+1,i}} \\
&= \prod_{i=1}^{m}(\prod_{j=0}^{t_p-1} c_j^{i^j} \cdot c^{i^{t_p}})^{\delta_{t_p+1,i}} \\
&= \prod_{j=0}^{t_p-1} c_j^{\sum_{i=1}^{m} i^j \cdot \delta_{t_p+1,i}} \cdot c^{\sum_{i=1}^{m} i^{t_p} \cdot \delta_{t_p+1,i}} \\
&= c
\end{aligned}
$$

t_p-**privacy**. We argue this by constructing the simulator $\mathsf{Sim}(c, I)$. Without loss of generality, we assume $I = \{1, \ldots, t_p\}$. On input a commitment c and a set I, the simulator $\mathsf{Sim}(c, I)$ proceeds as follows:

1. Simulate the private shares for participants in I, by picking $s_1, \ldots, s_{t_p}, r_1, \ldots, r_{t_p} \xleftarrow{\text{R}} \mathbb{Z}_p$ and setting $v_i = (s_i, r_i)$;
2. Compute the authentication information $aut = (c_j)_{0 \geq j \geq t_p}$ based on the algorithm Check. Concretely, set $c_j = \prod_{i=1}^{t_p}(g^{s_i} \cdot h^{r_i}/c^{i^{t_p}})^{\gamma_{j,i}}$, where the $t_p \times t_p$ matrix $(\gamma_{j,i})_{1\leq j+1, i\leq t_p}$ is the inverse of Vandermonde matrix $\mathbf{v}(1, \ldots, t_p)$.
3. Output $((v_i)_{i\in I}, aut)$.

It is direct that outputs of simulator $\mathsf{Sim}(c, I)$ are distributed identically to the outputs of algorithm $\mathsf{Share}^*(s, r)$ for $(P_i)_{i\in I}$.

A.4 A Proof for Theorem 5

Proof. We separately argue the three properties.

Acceptance. For all honestly generated v_i, c and aut, it always holds that $v_i^e = (a \cdot s^i)^e = a^e \cdot (s^e)^i = aut \cdot c^i \mod N$.

2-Correctness. For any set of participants $I = \{i_j\}_{j\in|I|} \subset [n]$ where $|I| \geq 2$ and corresponding shares $(v_i)_{i\in I}$, if it holds that $\mathsf{Check}(i, v_i, c, aut) = 1$ for $i \in I$, then we have $v_i^e = aut \cdot c^i$ for $i \in I$. Therefore, the secret s output by Rec holds that $s^e = (c^\alpha (v_{i_2}/v_{i_1})^\beta)^e = c^{\alpha \cdot e} c^{(i_2-i_1)\beta} = c$.

1-privacy. we argue this by constructing the simulator $\mathsf{Sim}(c, i)$, where $i \in [e]$. Concretely, the simulator $\mathsf{Sim}(c, i)$ proceeds as follows:

1. Simulate the share for P_i, by picking random $v_i \xleftarrow{\text{R}} \mathbb{Z}_N^*$;
2. Compute the authentication information $aut = v_i^e \cdot c^{-i}$.
3. Output (v_i, aut).

It is straightforward to check that the (v_i, aut) output by the simulator $\mathsf{Sim}(c, i)$ satisfies $\mathsf{Check}(i, v_i, c, aut) = 1$ and it is distributed identically to the output of algorithm $\mathsf{Share}^*(s)$ for P_i.

A.5 A Proof for Theorem 8

Proof. We separately argue the three properties.

Acceptance. For honestly generated $\mathbf{c} = (c_j)_{j\in[\ell]}$, $(v_i)_{i\in[n]}$ and $aut = (\widetilde{c}_j)_{j\in[t_p]}$, we show $\mathsf{Check}(i, v_i, \mathbf{c}, aut) = 1$ for all $i \in [n]$. First note that for honestly generated polynomials $A(x) = \sum_{k=1}^{\ell+t_p-1} a_k \cdot x^{k-1}$ and $B(x) = \sum_{k=1}^{\ell+t_p-1} b_k \cdot x^{k-1}$, their coefficients a_k's and b_k's hold that $a_k = \sum_{j=1}^{\ell} \delta_{k,j} \cdot s_j + \sum_{j=\ell+1}^{\ell+t_p} \delta_{k,j} \cdot \alpha_{j-\ell} \mod p$ and $b_k = \sum_{j=1}^{\ell} \delta_{k,j} \cdot r_j + \sum_{j=\ell+1}^{\ell+t_p} \delta_{k,j} \cdot \beta_{j-\ell} \mod p$, where the matrix $(\delta_{k,j})_{1\le k,j\le \ell+t_p}$ is equal to $\mathbf{V}(\zeta)^{-1}$. Moreover, if $\mathbf{c}$ and aut are computed honestly, we have $g^{a_k} h^{b_k} = \prod_{j=1}^{\ell} c_j^{\delta_{k,j}} \cdot \prod_{j=\ell+1}^{\ell+t_p} \widetilde{c}_{j-\ell}^{\delta_{k,j}} = h_k$ for all $k \in [\ell + t_p]$. Thus, for honestly generated v_i, parsed as (v_{i1}, v_{i2}), it holds that

$$
\begin{aligned}
g^{v_{i1}} h^{v_{i2}} &= g^{A(\eta_i)} h^{B(\eta_i)} \\
&= g^{\sum_{k=1}^{\ell+t_p} a_k \cdot \eta_i^{k-1}} h^{\sum_{k=1}^{\ell+t_p} b_k \cdot \eta_i^{k-1}} \\
&= \prod_{k=1}^{\ell+t_p} (g^{a_k} h^{b_k})^{\eta_i^{k-1}} \\
&= \prod_{k=1}^{\ell+t_p} h_k^{\eta_i^{k-1}}.
\end{aligned}
$$

$(\boldsymbol{\ell + t_p})$-**Correctness**. For any set of participants $I \subset [n]$ where $|I| = m \ge t_p + \ell$ and corresponding shares $(v_i)_{i\in I}$ where $v_i = (v_{i1}, v_{i2})$, if it holds that $\mathsf{Check}(i, v_i, c, aut) = 1$ for all $i \in I$, then we have $g^{v_{i1}} h^{v_{i2}} = \prod_{k=1}^{\ell+t_p} h_k^{\eta_i^{k-1}}$ and $h_k = \prod_{j=1}^{\ell} c_j^{\delta_{k,j}} \cdot \prod_{j=\ell+1}^{\ell+t_p} \widetilde{c}_{j-\ell}^{\delta_{k,j}}$, where the matrix $(\delta_{k,j})_{1\le k,j\le \ell+t_p}$ is equal to $\mathbf{V}(\zeta)^{-1}$.

Let $A(x) = \sum_{k=1}^{m} a_k \cdot x^{k-1}$ and $B(x) = \sum_{k=1}^{m} b_k \cdot x^{k-1}$ be the two polynomials that interpolate the points $(\eta_i, v_{i1})_{i\in I}$ and $(\eta_i, v_{i2})_{i\in I}$, respectively. Note that, both the degrees of $A(x)$ and $B(x)$ are at most $m - 1$. Then, for all $k \in [m]$, it holds that $a_k = \sum_{i=1}^{m} \gamma_{k,i} \cdot v_{i1} \mod p$ and $b_k = \sum_{i=1}^{m} \gamma_{k,i} \cdot v_{i2} \mod p$, where the matrix $(\gamma_{k,i})_{1\le k,j\le m}$ is equal to $\mathbf{V}((\eta_i)_{i\in I})^{-1}$. Therefore, for all $j \in [\ell]$, we have

$$
\begin{aligned}
g^{s_j} h^{r_j} &= g^{A(\zeta_j)} h^{B(\zeta_j)} \\
&= g^{\sum_{k=1}^{m} a_k \cdot \zeta_j^{k-1}} \cdot h^{\sum_{k=1}^{m} b_k \cdot \zeta_j^{k-1}} \\
&= \prod_{k=1}^{m} \left(g^{a_k} \cdot h^{b_k} \right)^{\zeta_j^{k-1}} \\
&= \prod_{k=1}^{m} \left(g^{\sum_{i=1}^{m} \gamma_{k,i} \cdot v_{i1}} \cdot h^{\sum_{i=1}^{m} \gamma_{k,i} \cdot v_{i2}} \right)^{\zeta_j^{k-1}} \\
&= \prod_{k=1}^{m} \left(\prod_{i=1}^{m} (g^{v_{i1}} h^{v_{i2}})^{\gamma_{k,i}} \right)^{\zeta_j^{k-1}} \\
&= \prod_{k=1}^{m} \left(\prod_{t=1}^{\ell+t_p} \left(h_t^{\sum_{i=1}^{m} \eta_i^{t-1} \cdot \gamma_{k,i}} \right) \right)^{\zeta_j^{k-1}} \\
&= \prod_{k=1}^{m} (h_k)^{\zeta_j^{k-1}} \\
&= \prod_{k=1}^{m} \left(\prod_{u=1}^{\ell} c_u^{\delta_{k,u}} \cdot \prod_{u=\ell+1}^{\ell+t_p} \widetilde{c}_{u-\ell}^{\delta_{k,u}} \right)^{\zeta_j^{k-1}} \\
&= c_j
\end{aligned}
$$

$\boldsymbol{t_p}$**-privacy**. we argue this by constructing a PPT simulator $\mathsf{Sim}((c_j)_{j\in\ell}, I)$ as below. Without loss of generality, we assume $I = \{1, \ldots, t_p\}$. On input the commitments $(c_j)_{j\in\ell} \in \mathbb{G}^\ell$ and the set I, the simulator Sim proceeds as follows:

1. Simulate the private shares for participants in I, by picking two random vectors $(v_{i1})_{i\in[t_p]}, (v_{i2})_{i\in[t_p]} \xleftarrow{\text{R}}$ $\mathbb{F}_p^{t_p}$ and setting $v_i = (v_{i1}, v_{i2})$;
2. Compute the $t_p \times (\ell + t_p)$ matrix $\mathbf{G}$ as below:

$$\mathbf{G} = \begin{bmatrix} 1 & \eta_1 & \cdots & \eta_1^{\ell+t_p-1} \\ 1 & \eta_2 & \cdots & \eta_2^{\ell+t_p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \eta_{t_p} & \cdots & \eta_{t_p}^{\ell+t_p-1} \end{bmatrix} \cdot \mathbf{V}(\zeta)^{-1},$$

   Let $\mathbf{G}_1$ be the matrix formed by the first ℓ columns of $\mathbf{G}$ and $\mathbf{G}_2$ be the matrix formed by the last t_p columns of $\mathbf{G}$.
3. Compute $aut = (\widetilde{c}_1, \ldots, \widetilde{c}_{t_p})$ on the basis of deterministic algorithm Check. Specifically, set $\widetilde{c}_k = \prod_{i=1}^{t_p}(g^{v_{i1}} h^{v_{i2}} / \prod_{j=1}^\ell c_j^{\gamma_{i,j}})^{v_{k,i}}$, where the $t_p \times \ell$ matrix $(\gamma_{i,j})_{i\in[t_p],j\in[\ell]}$ is equal to the matrix $\mathbf{G}_1$ and the $t_p \times t_p$ matrix $(v_{k,i})_{k,i\in[t_p]}$ is the inverse of the matrix $\mathbf{G}_2$:
4. Output $(v_1, \ldots, v_{t_p}, aut)$.

It is direct that the outputs of simulator Sim are distributed identically to the outputs of algorithm $\mathsf{Share}^*(\mathbf{s}, \mathbf{r})$ for participants in I.