Not Just Regular Decoding: Asymptotics and Improvements of Regular Syndrome Decoding Attacks

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Abstract. Cryptographic constructions often base security on structured problem variants to enhance efficiency or to enable advanced functionalities. This led to the introduction of the Regular Syndrome Decoding (RSD) problem, which guarantees that a solution to the Syndrome Decoding (SD) problem follows a particular block-wise structure. Despite recent attacks exploiting that structure by Briaud and Øygarden (Eurocrypt '23) and Carozza, Couteau and Joux (CCJ, Eurocrypt '23), many questions about the impact of the regular structure on the problem hardness remain open.
In this work we initiate a systematic study of the hardness of the RSD problem starting from its asymptotics. We classify different parameter regimes revealing large regimes for which RSD instances are solvable in polynomial time and on the other hand regimes that lead to particularly hard instances. Against previous perceptions, we show that a classification solely based on the uniqueness of the solution is not sufficient for isolating the worst case parameters. Further we provide an in-depth comparison between SD and RSD in terms of reducibility and computational complexity, identifying regimes in which RSD instances are actually *harder* to solve.
We provide the first asymptotic analysis of the CCJ algorithm, establishing its worst case decoding complexity as $2^{0.141n}$. We then introduce *regular-ISD* algorithms by showing how to tailor the whole machinery of advanced Information Set Decoding (ISD) techniques from attacking SD to the RSD setting. The fastest regular-ISD algorithm improves the worst case decoding complexity significantly to $2^{0.112n}$. Eventually, we show that also with respect to suggested parameters regular-ISD outperforms previous approaches in most cases, reducing security levels by up to 40 bits.

Keywords: Hardness Classification, Information Set Decoding, Code-Based Cryptography

1 Introduction

The Syndrome Decoding Problem (SDP) is one of the most fundamental problems in coding theory, and as such finds frequent applications as security foundation in cryptographic constructions. Given the parity-check matrix $\mathbf{H}$ of a linear code, a vector (the syndrome) $\mathbf{s}$ and an integer w the SDP asks to recover a vector $\mathbf{e}$ of Hamming weight w satisfying $\mathbf{He} = \mathbf{s}$.

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In recent years, to enable the design and increase the efficiency of (advanced) cryptographic constructions based on SDP, such as signatures [CCJ23a], efficient MPC [HOSS18], Vector Oblivious Linear Evaluation (VOLE) [BCGI18], Pseudorandom Correlation Generators (PCGs) [BCG+19b, BCG+20] or correlated Oblivious Transfer (OT) [BCG+19a, YWL+20], often a structured version of the problem is considered, known as Regular Syndrome Decoding (RSD) problem. Initially introduced in the context of the FSB hash function [AFS05], in this variant, the error vector $\mathbf{e}$ is known to be regular, i.e., it consists of w consecutive, equally sized chunks, each of weight exactly one.

Intuition suggests that the introduction of such regular structure decreases the problem hardness, but such intuition might not hold universally true. For example in the related LPN (or LWE) setting [HKL+12, LPR10], even after years of study, attacks on structured ring-variants are essentially the same as on their non-structured counterparts. A first attempt at the translation of concepts used in Information Set Decoding (ISD) attacks against SD to the regular case was given in the security analysis of [HOSS18]. However, the authors eventually concluded that those attacks even if tailored to the RSD setting obtain about the same complexity as direct SD attacks. Generally, the security analysis of RSD-based constructions has predominantly been performed in an adhoc manner and supported the assumption that the most efficient attacks on RSD remain the same as those on SD. A more focused study was then performed in [LWYY22], which, however, also finds standard ISD attacks to perform best for most of the suggested parameters.

Recently then, Briaud and Øygarden [BØ23] showed that the regular structure allows to model the problem as a quadratic system of equations, which in turn allows the application of algebraic solvers. They show that for many of the parameters found in the literature this approach yields the best running time. At the same time Carozza, Couteau and Joux (CCJ) [CCJ23a] designed a new signature scheme exploiting the regular structure of the RSD solution to obtain reduced signature sizes. In the full version of their work [CCJ23b] the authors give a first hardness classification of the RSDP in comparison to the SDP: Based on the uniqueness of the solution the authors identify three regimes in which they find either the SDP or the RSDP to be harder, while in the third regime both problems seem incomparable. Additionally, the authors designed a new algorithm based on enumerating the searched error vector in a meet-in-the-middle fashion. The authors state that this approach for many parameters obtains large improvements over the state of the art.

Those recent works made big steps towards classifying the hardness of RSD by showing that algorithms can indeed be tailored to exploit the regular structure. However, especially from a theoretical perspective the effect of the regular structure on the problem hardness remains largely unclear. Naturally, there arise questions about the equivalence of both problems, which were briefly touched in [CCJ23a], or about the amount of structure that can be induced until the problem hardness collapses. Further, in terms of computational hardness, Information Set Decoding algorithms are usually the best strategy to solve the SDP. However, so far it is not known to which extend those can be tailored to exploit the regular structure in case of RSD.

Our Contribution In this work we initiate the systematic study of the hardness of the regular syndrome decoding problem. We start this investigation from a theoretical viewpoint that allows us to isolate instances for which RSD is hard and for which its hardness collapses. This first asymptotic treatment then forms the foundation for a more comprehensive comparison between the hardness of SD and RSD for different parameter regimes. In the second part of this work we then show that against previous perceptions [HOSS18, CCJ23a] essentially all advanced techniques from ISD algorithms in the non-regular case, can be tailored to the RSD setting resulting in new *regular-ISD* algorithms, outperforming the state-of-the-art in the asymptotics as well as in concrete numbers.

Towards a rigorous hardness classification. We first present our results on the general hardness of the RSD problem. In particular, we show that there exist large regimes of parameters for which RSD instances are solvable in polynomial time. One of these regimes is a direct consequence of the introduced structure: The known block-wise structure of the solution gives rise to additional linear equations decreasing the dimension of the underlying code. Once the codes dimension is small enough, one can solve for $\mathbf{e}$ in polynomial time via basic linear algebra. Note that this is in direct contrast to the SD problem, which is known to be exponentially hard for any parameters that yield a unique solution, i.e., parameters below the Gilbert-Varshamov (GV) bound.

This motivates the question for which parameters RSD remains hard and, in particular, for which parameters it reaches the worst case. Carozza et al. [CCJ23a] identified an *RSD uniqueness-bound* similar to the Gilbert-Varshamov bound in the SD case and gave a classification of RSD's hardness based on this bound. In this context we show that the uniqueness bound alone unlike the GV-bound in the SD case, does not allow to classify the hardest instances in the RSD case. We then study how to isolate worst case instances and are able to identify a regime of parameters which includes the worst case instances, with respect to the proposed *regular-ISD* algorithms.

Furthermore, in our classification we identify large regimes in which RSD instances are actually *harder* to solve than SD instances with respect to the best algorithms. This is in contrast to the intuitive perception that exploiting the regular structure by switching from SD to RSD instances in cryptographic constructions inevitably leads to a decreased security level. Moreover, when comparing the complexity of the best algorithms on SD and RSD for their respective worst case parameters, we find that RSD is harder for all code rates larger than (about) $\frac{1}{2}$ and that RSD instances obtain a far larger absolute complexity. This effect is illustrated in Fig. 1 in which we compare regular-ISD performance on RSD worst case instances against ISD performance on SD worst case instances.

Eventually, we show that the complexity of SD and RSD with respect to the best algorithms converges for small w. More, precisely as soon as w is sublinear in the code length the complexity for solving both problems differs only by second order terms.

The regular-ISD approach. The second main contribution of this work is the design of *regular-ISD* algorithms. In all algorithms we first encode the regularity into the parity-check matrix, which essentially results in w additional rows. These linear equations have also been exploited in the algebraic attack by Briaud and Øygarden [BØ23] and in the enumeration routine by Carozza, Couteau and Joux [CCJ23a]. However, we show that the particular structure of these equations requires additional care in the analysis and, actually, prevents the use of some of those equations within the CCJ algorithm.

We provide the first asymptotic analysis of the CCJ algorithm which establishes an asymptotic baseline for algorithms solving the RSD problem on worst case instances with a

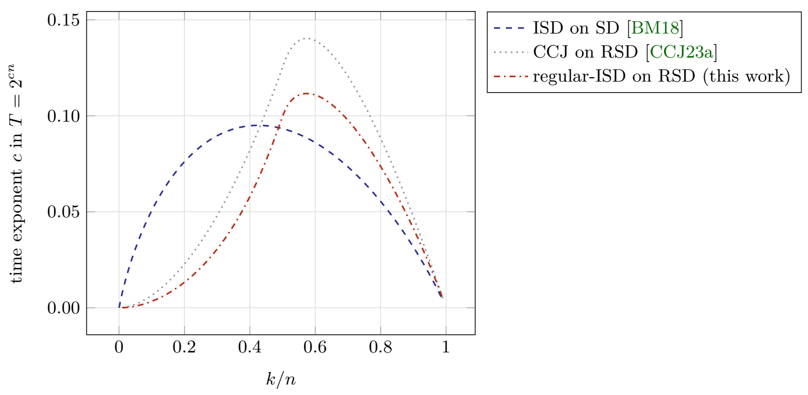


Fig. 1: Comparison of running time of best ISD algorithm on *SD* worst case instances and best regular-ISD algorithm (Section 4.4) or resp. CCJ algorithm on *RSD* worst case instances.

complexity of $2^{0.141n}$. We are then able to drastically improve this initial complexity to $2^{0.112n}$ by tailoring the whole machinery of ISD techniques to the regular setting. A comparison between the complexity of the CCJ algorithm and the regular-ISD approach for all rates is illustrated in Fig. 1.

ISD algorithms usually apply a permutation to the columns of the parity-check matrix to obtain a permuted instance $\mathbf{HP}$ with solution $\mathbf{P}^{-1}\mathbf{e}$ and then hope for a specific weight distribution on $\mathbf{P}^{-1}\mathbf{e}$. However, as observed in [CCJ23a] random permutations destroy the regular structure on the solution and might, hence, prevent further exploitation. We therefore restrict to a specific set of *regular permutations*, similar to a technique described in [HOSS18], which exploits the regular structure to enhance the success probability of obtaining the desired weight distribution. Moreover, any such regular permutation maintains a certain regular structure on the permuted solution, which we then exploit in order to improve the commonly applied enumeration subroutine of ISD procedures. We show that the use of regular permutations enables to leverage all linear equations encoding the regularity of the solution, regardless of their specific structure. Eventually, we show how to incorporate the most advanced concepts of *representations* [MMT11, BJMM12] and nearest-neighbor search [MO15, BM18] into regular-ISD procedures.

We also provide a concrete cost analysis of the regular-ISD techniques as well as the CCJ algorithm and evaluate their complexity on parameter sets provided in the literature. In Table 4 we give the complexity of the different approaches on selected parameter sets. As can be observed regular-ISD obtains large significant bit-security reductions of up-to 40 bits, showing that most of the depicted sets fall below the targeted security threshold. However, generally, we find that constructions based on RSD followed a (over-) conservative parameter selection procedure, such that even considering regular-ISD, most parameter sets still obtain comfortable margins (of up to 270 bits).

Outline. In Section 2 we cover necessary basics on coding theory, the SD and the RSD problem. Subsequently, in Section 3 we provide a systematic classification of the hardness of RSD, in particular in comparison to its non-regular counterpart. In Section 4 we present the

source	(n, k, w)	previous best	regular-ISD
[HOSS18]	$(1280, 860, 80)$	132	114
[LWYY22]	$(2^{10}, 652, 57)$	90	76
	$(2^{10}, 652, 106)$	129	113
	$(2^{12}, 1589, 172)$	135	109
	$(2^{14}, 3482, 338)$	135	118
	$(2^{16}, 7391, 667)$	139	126
[CCJ23a]	$(1842, 825, 307)$	193	153

Table 1: Bit security for selected instances considering regular-ISD in comparison to previous best approaches.

regular-ISD algorithms and provide their asymptotic analysis. At the end of that section we also give the analysis of the CCJ algorithm and an asymptotic comparison to previous approaches. Eventually, in Section 5, we present results on the performance of regular-ISD algorithms on suggested parameters.

Artifacts. We provide all used source code, including proof of concept implementations of the regular-ISD algorithms, numerical optimizations for asymptotic exponents as well as implementations of the concrete complexity formulas in the git repository `github.com/Memphisd/Regular-ISD`.

2 Preliminaries

We denote by $\mathbb{F}_2$ the binary finite field. Vectors (resp., matrices) are indicated with bold lowercase (resp., uppercase) letters. Vectors are viewed interchangeably as rows and columns. The Hamming weight of a vector $\mathbf{x}$ corresponds to the number of its non-null entries and is denoted $|\mathbf{x}|$. For a set of coordinates I we denote by $\mathbf{x}_I$ the projection of $\mathbf{x}$ on the coordinates indexed by I. For an integer a we let $[a] := \{1, \dots, a\}$.

By a linear code $\mathscr{C} \subseteq \mathbb{F}_2^n$, we refer to a linear subspace of the ambient space $\mathbb{F}_2^n$. Typical code parameters are the *length* n, the *dimension* $k = \kappa n$, for $\kappa \in [0; 1]$ being the *code rate*, and the co-dimension $n - k = (1 - \kappa)n$ (also called redundancy). Any linear code can be compactly represented by a *parity-check matrix*, that is, a full rank $\mathbf{H} \in \mathbb{F}_2^{(n-k)\times n}$ which serves as a basis for the null space of the code: a vector $\mathbf{c} \in \mathbb{F}_2^n$ is a codeword if and only if $\mathbf{Hc} = \mathbf{0}$. Linear codes are invariant under changes of basis, i.e., $\mathbf{H}$ and $\mathbf{SH}$, with $\mathbf{S} \in \mathbb{F}_2^{(n-k)\times(n-k)}$ being non singular, are parity-check matrices for the same code.

We say that a set $J \subseteq \{1, \cdots, n\}$ of size k is an *information set* for a code $\mathscr{C}$ if any two distinct codewords $\mathbf{c}, \mathbf{c}' \in \mathscr{C}$ are different in at least one of the coordinates indexed by J. This is equivalent to the columns of $\mathbf{H}$ which are not indexed by J forming a full rank $(n - k) \times (n - k)$ matrix.

For any vector $\mathbf{e} \in \mathbb{F}_2^n$ which is not a codeword, we have $\mathbf{He} = \mathbf{s} \neq \mathbf{0}$. The vector $\mathbf{s} \in \mathbb{F}_q^{n-k}$ is called a *syndrome*. The same syndrome correspond to multiple vectors: for any non null $\mathbf{c} \in \mathscr{C}$, $\mathbf{e}$ and $\mathbf{e} + \mathbf{c}$ have the same syndrome. Decoding a given syndrome into an arbitrary vector in $\mathbb{F}_2^n$ can be done by basic linear algebra. However, the problem becomes hard if the vector is required to have low Hamming weight, leading to the Syndrome Decoding Problem (SDP).

Definition 2.1 (Syndrome Decoding (SD)). *Let $k, n, w \in \mathbb{N}$ such that $k, w \leq n$. Given $\mathbf{H} \in \mathbb{F}_2^{(n-k)\times n}$ and $\mathbf{s} \in \mathbb{F}_2^{n-k}$, find a vector $\mathbf{e} \in \mathbb{F}_2^n$ of Hamming weight w such that $\mathbf{He} = \mathbf{s}$. We refer to the SDP problem with parameters (n, k, w) as $\mathcal{SDP}(n, k, w)$.*

Motivated by cryptographic constructions we always assume the existence of at least one solution, while the expected amount of solutions is given by

$$S = \frac{\text{Number of weight-}w\text{ vectors}}{\text{Number of syndromes}} = \binom{n}{w} 2^{-(n-k)}.$$

The best solvers for $\mathcal{SDP}(n, k, w)$ are ISD algorithms. The first algorithm of this class is due to Prange [Pra62] and finds a solutions to the $\mathcal{SDP}(n, k, w)$ in time

$$T = \tilde{\mathcal{O}}\left(\frac{\binom{n}{k}}{S \cdot \binom{n-k}{w}}\right). \tag{1}$$

The hardest instances of SDP are those where w is chosen as the maximum value which still leads to a unique solution. This is the case if w is equal to the Gilbert-Varshamov (GV) distance, which asymptotically gives $w = \omega n$, where $\omega = h^{-1}(1-\kappa)$ and $h(x) := -x\log_2(x) - (1-x)\log_2(1-x)$ is the binary entropy function.

Regular Syndrome Decoding. For a vector $\mathbf{e} \in \mathbb{F}_2^n$ we say it is *b-regular of weight $w \leq \frac{n}{b}$* if it can be written as $\mathbf{e} = \left(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{\frac{n}{b}}\right)$, where each $\mathbf{e}_i$ has length b and Hamming weight *at most* one. Furthermore, we simply call a weight-w vector $\mathbf{e}$ regular if $b = \frac{n}{w}$, i.e., each $\mathbf{e}_i$ is of Hamming weight *exactly* one.

Requiring that the solution to SDP is regular, one obtains the Regular Syndrome Decoding Problem (RSDP).

Definition 2.2 (Regular Syndrome Decoding (RSD)). *Let $k, n, w \in \mathbb{N}$ such that $k, w \leq \frac{n}{2}$. Given $\mathbf{H} \in \mathbb{F}_2^{(n-k)\times n}$ and $\mathbf{s} \in \mathbb{F}_2^{n-k}$, find a* regular *vector $\mathbf{e} \in \mathbb{F}_2^n$ of Hamming weight w such that $\mathbf{He} = \mathbf{s}$. We refer to the RSD problem with parameters (n, k, w) as $\mathcal{RSD}(n, k, w)$.*

The constraint $w \leq \frac{n}{2}$ is implied by the fact that the solution has to be regular.

3 Hardness Classification

In this section we present results on the general hardness of the RSD problem and draw direct comparison to the SD problem. We start by recalling and formalizing the required conditions to expect uniqueness of the solution in the RSD case.

3.1 Uniqueness bound

The uniqueness bound for RSD marks, analogous to the GV bound in the SD case, the transition to instances with multiple solutions. Any set of parameters satisfying this bound is expected to have at most one solution. The bound reads

$$\left(\frac{n}{w}\right)^w \leq 2^{n-k},$$

where the left side specifies the amount of regular vectors of weight w, i.e., the search space size, while the probability that a random element of the search space satisfies all parity equations is $2^{-(n-k)}$, leading to the term on the right.

Note that since $k = \kappa n$, $w = \omega n$ and, hence, $n/w = 1/\omega$, the above bound can be rewritten as $-\omega \log_2(\omega) \leq 1 - \kappa$. Therefore, once κ is known, one can determine the maximum value of ω for which uniqueness of the RSD solution (statistically) holds.

Observe that $-\omega \log_2(\omega)$ reaches its maximum at $\omega = e^{-1}$ with a maximum value of $(e \cdot \ln 2)^{-1} \approx 0.5307$. This implies that for any code rate $\kappa \leq 1 - (e \cdot \ln 2)^{-1} \approx 0.4693$, the inequality is satisfied for any value of ω. We summarize this in the following proposition.

Proposition 3.1 (Uniqueness bound for RSD). *Let $k = \kappa n$, $w = \omega n$. We expect a unique solution to $\mathcal{RSD}(n, k, w)$ if*

$$-\omega \log_2(\omega) \leq 1 - \kappa.$$

For $\kappa < 1 - (e \cdot \ln 2)^{-1}$, the inequality is satisfied for any choice of $0 < \omega \leq 0.5$.

The fact that for sufficiently small code rate, uniqueness is guaranteed regardless of ω is a remarkable difference to the GV bound in the SD case. Later we show that, also with respect to the hardness classification of instances, the RSD uniqueness bound and the GV bound in the SD case behave differently. Indeed, while the GV bound classifies the hardest instances, at least with respect to known algorithms worst RSD parameters do not always match the uniqueness bound.

3.2 Instances Solvable in Polynomial Time

In this section, we identify two large regimes of parameters for which RSD is solvable in polynomial time. One of those regimes is, analogous to the SD case, related to a large amount of existing solutions; instead, the other regime is specific to the regular structure of the RSD problem. Let us start with the RSD specific regime.

Encoding Regularity Note that a solution $\mathbf{e}$ to the RSD problem is a concatenation of w unit vectors of length $b := n/w$. Therefore, for each $i \in \{1, \cdots, w\}$, we can include the parity-check equation $\langle \mathbf{h}_i, \mathbf{e} \rangle = 1$ where

$$\mathbf{h}_i = \big(\underbrace{0, 0, \cdots, 0}_{(i-1)b}, \underbrace{1, 1, \cdots, 1}_{b}, \underbrace{0, 0, \cdots, 0}_{(w-i)b}\big).$$

Considering all these additional equations, we obtain a new parity-check matrix $\mathbf{H}'$ with $n-k+w$ rows and n columns, structured as

$$\mathbf{H}' = \left(\begin{array}{cccc|c} \multicolumn{4}{c}{\mathbf{H}} & \\ \hline 1\,1\cdots 1 & & & & \\ & 1\,1\cdots 1 & & & w \\ & & \ddots & & \\ & & & 1\,1\cdots 1 & \\ \hline b & b & & b & \end{array}\right) \tag{2}$$

Analogously, we update the syndrome as $\mathbf{s}' = (\mathbf{s}, 1, 1, \cdots, 1) \in \mathbb{F}_2^{n-k+w}$. This results in a new RSD instance $(\mathbf{H}', \mathbf{s}')$ where the parity-check matrix $\mathbf{H}'$ corresponds to a code of smaller dimension $k' = k - w$ (so, smaller rate $\kappa' = \kappa - w/n$), while maintaining the same co-dimension $n-k$ and same solution $\mathbf{e}$, i.e., it still holds that $\mathbf{H}'\mathbf{e} = \mathbf{s}'$.

This encoding of the regular structure leads to a polynomial regime whenever the new parity check matrix $\mathbf{H}'$ contains more than n rows.

Theorem 3.1. *Whenever $w \geq k$, $\mathcal{RSD}(n,k,w)$ is solvable in polynomial time with high probability.*

Proof. After encoding the regularity, the RSD solution is a solution to a linear system containing $n-k+w$ equations and n unknowns, represented by the parity check matrix and the syndrome $(\mathbf{H}', \mathbf{s}')$. Whenever $n-k+w \geq n$, which is equivalent to $w \geq k$, the system contains more equations than unknowns.

Note that for random $\mathbf{H}'$ such a system is solvable in polynomial time with high probability. Therefore, consider the case of $w = k$, i.e., $\mathbf{H}'$ being a square matrix. The corresponding system is solvable in polynomial time whenever $\mathbf{H}'$ has rank at least $n - \varepsilon$ with $\varepsilon = \mathcal{O}(\log n)$. Fulman and Goldstein [FG15, Eq. (1)] have shown that this happens with high probability.

The statement of the theorem follows by observing that for our precise choice of $\mathbf{H}'$ this probability is even higher. Indeed, we have $\mathbf{H}' = \begin{pmatrix}\mathbf{H}\\ \mathbf{B}\end{pmatrix}$, where the $w \times n$ matrix $\mathbf{B}$ contains the extra parity-check equations and $\mathbf{H}$ is a random $(n-k) \times n$ matrix. Note that $\mathbf{H}'$ is not of full rank if either $\mathbf{B}$ or $\mathbf{H}$ contain linearly dependent rows or if the space generated by the rows of $\mathbf{H}$ intersects with the space generated by the rows of $\mathbf{B}$.

However, for our choice of $\mathbf{H}'$, unlike the random case, all rows of $\mathbf{B}$ are linearly independent by construction. $\square$

Remark 3.1 (Full Rank $\mathbf{H}'$). In case when $\mathbf{H}'$ is square, asymptotically, for the considered case of $w = \Theta(n)$, the matrix behaves as a random matrix with respect to invertibility. Indeed, the additional w parity-check equations are linearly independent, that is, generate a space $\mathscr{B}$ with dimension w. So, $\mathbf{H}'$ has full rank whenever the rows of $\mathbf{H}$ have full rank r and generate a space which intersects trivially with $\mathscr{B}$. This happens with probability

$$\left(\frac{2^n - 2^w}{2^n}\right)\left(\frac{2^n - 2^{w+1}}{2^n}\right)\cdots\left(\frac{2^n - 2^{w+n-k-1}}{2^n}\right) = \prod_{i=1}^{n-w} 1 - 2^{-i} \xrightarrow{n\to\infty} 0.2887,$$

which converges to the same limit as for random $\mathbf{H}'$.

Amount of Solutions Intuitively and similar to the SD case, RSD becomes easy whenever there are too many solutions. We specify this regime of parameters in the following theorem.

Theorem 3.2. *Whenever $\frac{k}{n} \geq \frac{1}{2}$ and $w > n-k$, $\mathcal{RSD}(n,k,w)$ is solvable in expected polynomial time.*

Proof. Note that the expected amount of solutions to a random RSD instance is $S = \frac{\left(\frac{n}{w}\right)^w}{2^{n-k}}$. Further, the later introduced Theorem 4.1 states the expected running time for solving $\mathcal{RSD}(n,k,w)$ (up to polynomial factors) as

$$T = \frac{\left(1 - \frac{k-w}{n}\right)^{-w}}{S}.$$

Therefore as long as $T \leq 1$ which is equivalent to

$$\left(1 - \frac{k-w}{n}\right)^w \geq \frac{2^{n-k}}{\left(\frac{n}{w}\right)^w}, \tag{3}$$

RSD is solvable in expected polynomial time. Using $w \geq n-k$ we lower bound the left hand side of Eq. (3) as

$$\left(1 - \frac{k-w}{n}\right)^w \geq \left(1 - \frac{2k-n}{n}\right)^{n-k} = \left(2(1 - \frac{k}{n})\right)^{n-k}, \tag{4}$$

as long as $2k - n \geq 0$, which is equivalent to $\frac{k}{n} \geq \frac{1}{2}$ as stated in the theorem. Now using $w \geq n-k$, we upper bound the right hand side of Eq. (3) as

$$\frac{2^{n-k}}{\left(\frac{n}{w}\right)^w} \leq \frac{2^{n-k}}{\left(\frac{n}{n-k}\right)^{n-k}} = \left(2(1 - \frac{k}{n})\right)^{n-k}. \tag{5}$$

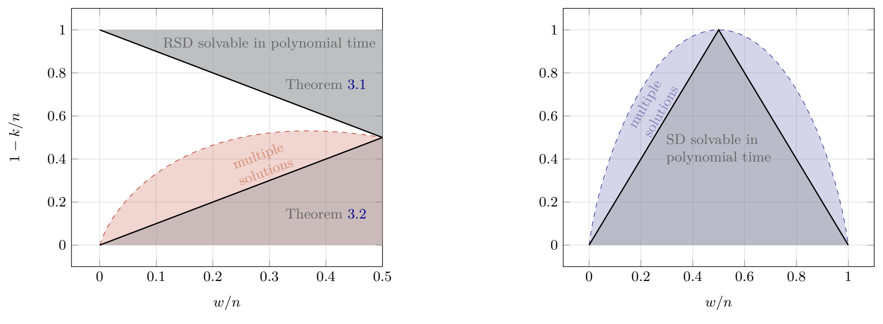
Finally, observe that (4)≥(5) , which is trivially fulfilled, implies Eq. (3). $\square$

Comparison to the Non-Regular Case In the case of SD instances over $\mathbb{F}_2$ it is well known that those are solvable in polynomial time whenever

$$\frac{n-k}{2} \leq w \leq \frac{n+k}{2}.$$

The corresponding algorithm simply solves the system restricted to the first $(n-k) \times (n-k)$ submatrix of $\mathbf{H}$. This solution has expected weight $(n-k)/2$. Note that arbitrary columns from the last k columns of $\mathbf{H}$ can be added to the syndrome prior to solving the system in order to increase the solution weight up to a maximum of $(n-k)/2 + k = (n+k)/2$.

In Fig. 2 we compare the regimes of parameters for which instances are solvable in polynomial time in the RSD and the SD case. The figure visualizes that Theorem 3.2 relates to a similar regime as the polynomial instances regime in the SD case, enclosed completely in the region of parameters that give rise to many solutions. In contrast, Theorem 3.1 relates to the whole regime of parameters lying in the upper gray shaded triangle. Note that for all included instances the solution is guaranteed to be unique. This shows that a hardness classification solely based on the uniqueness of the solution is not sufficient in the RSD case. Moreover, this raises the question about the worst case parameters in term of weight for RSD instances.



(a) RSD parameter regimes (b) SD parameter regimes

Fig. 2: Parameter regimes for RSD and SD, where gray shaded area marks instances solvable in polynomial time, while colored shaded area marks instances where multiple solutions exist. Dashed lines depict uniqueness bound and GV bound respectively.

3.3 Worst Case Instances

In the following we answer the question for the worst case weight of RSD instances. That is, given a dimension $k = \kappa n$ of a linear code, which $w^* = \omega^* n$ leads to the hardest RSD instance $\mathcal{RSD}(n, k, w^*)$.

From the SD case it is known that the hardest instances are those matching the GV bound. However, for RSD we know from Proposition 3.1 that not for every choice of κ a corresponding ω matching the uniqness bound exists.

Worst Case for SD Instances Note that there is no proof that SD instances matching the GV bound form indeed the worst case. However, there is an intuitive argument, which is further supported by all known algorithms reaching their worst case running time for those instances.

As long as the weight stays below the GV bound the solution is unique. Now, for increasing weight, the search space grows exponentially in n, rendering the problem more difficult. On the other hand if the weight exceeds the GV bound there exist exponentially many solutions, which makes the problem easier. Therefore it is assumed that the hardest instances are those whose weight matches the GV bound.

The Case of RSD In the RSD case the same argument fails, as even if the solution is unique, increasing ω gives rise to more equations encoding the regularity, which may render the problem easier.

In fact, we find that the worst case weight for the regular-ISD algorithms presented in Section 4 grows initially (almost) linear with the dimension as $\omega^* \approx c \cdot \kappa$, with $c \approx 1/2$. This trend continues until $c \cdot \kappa$ exceeds the uniqueness bound, from where ω^* is equal to the uniqueness bound. We visualize the worst case weight ω_{low} for the most basic regular-ISD algorithm in Fig. 3.

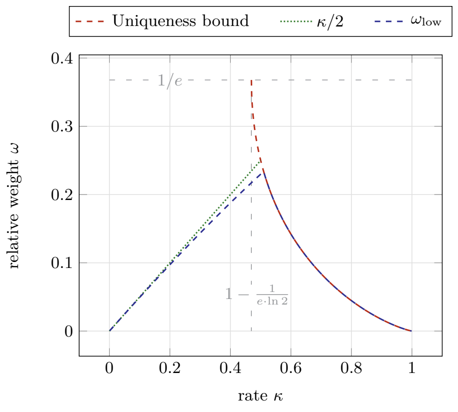


Fig. 3: Worst case weight ω_{low} of most basic regular-ISD algorithm (see Algorithm 1 and Theorem 4.1). From $\kappa \geq 0.51$ onwards ω_{low} matches the uniqueness bound.

Algorithm dependence. Moreover, we find that the precise weight that leads to the worst case depends on the considered algorithm. This behavior is similar to the *worst case rate* in the SD case, where the exact rate which maximizes the running time is algorithm dependent. Usually, it lies around $1/2$, but seems to be decreasing for more advanced procedures. Most recent improvements reach their worst case running time for a rate as low as $\kappa^* = 0.42$ [BM18, CDMHT22, Ess22].

We observe a similar behavior for regular-ISD algorithms where the *worst case weight* seems to approach $\kappa/2$ for more advanced procedures in the regime $\kappa \leq 1/2$. The most advanced regular-ISD algorithms almost exactly reach $\omega^* = \kappa/2$ in that regime. Note that the translation of this behavior from the worst case *rate* in the SD case to the worst case *weight* in the RSD case is likely to be explained by the regularity encoding equations. Those set the rate in direct dependence to the weight as the new dimension after encoding regularity is $k' = k - w$.

Worst case weight. Let $UB(\kappa)$ denote the weight $\omega \leq e^{-1}$ matching the RSD uniqueness bound if such an ω exists. We find that the worst case relative weight ω^* for RSD instances for all considered algorithms satisfies

$$\omega_{\text{low}} \leq \omega^* \leq \min\left(\frac{\kappa}{2} + \varepsilon, UB(\kappa)\right),$$

for a very small constant ε. Remember that for $\kappa \geq 0.51$ it holds that $\omega_{\text{low}} = UB(\kappa)$, implying $\omega^* = UB(\kappa)$. Generally, considering $\varepsilon = 0$ the right side of the inequality gives a well approximation of the worst case weight of all studied algorithms. This weight is also visualized in Fig. 3.

Note that this shows that even for the regime where there exist parameters matching the uniqueness bound, this bound does not suffice to classify the worst case weight. For example, for a rate of $\kappa = 0.47$, we find that the worst case weight is always smaller than $\omega^* \leq 0.235$ which is far from the uniqueness bound which would imply a weight of roughly 0.349.

3.4 Hardness Comparison to the Non-Regular Case

In this section we compare the hardness of RSD and SD directly. The goal is to identify those parameter regimes in which SD instances are strictly harder to solve than RSD instances and vice versa, if such exist. Also for regimes that do not allow such a strict separation we investigate which problem is harder with respect to known algorithms. In this case we consider for RSD the regular-ISD algorithms presented in the subsequent Section 4.

Classification Based on the Amount of Solutions A first step towards a rigorous hardness classification of RSD was made recently in [CCJ23a], where the authors identify three regimes in dependence on the amount of existing solutions. For each regime the authors argue which problem is harder, identifying one regime in which SD is harder, one in which RSD is harder and one in which both problems are incomparable. However, a proper definition of *problem A being harder than problem B* is missing. For some regimes the authors argue about the existence of polynomial reductions while for others they argue about possible algorithmic speedups. In the following we recall those regimes and give a slightly different categorization.

We compare both problems in those regimes with respect to reducibility as well as the asymptotic complexity of known algorithms to solve those problems.

Unique solutions. This first regime corresponds to parameters which yield a unique solution to both problems, i.e., to a weight below the GV bound.

Note that RSD always polynomially reduces to SD. This reduction permutes the columns of the parity-check matrix to obtain a randomly distributed solution, i.e., it calls the SD solver on input $(\mathbf{HP}, \mathbf{s})$ with solution $\mathbf{P}^{-1}\mathbf{e}$, where $\mathbf{P}$ is a random permutation matrix. On the other hand, there is no polynomial reduction from SD to RSD known in this regime.

Further, since the RSD problem allows to encode the regularity and, as we show in Section 4, allows for many other speedups, algorithms that exploit RSD specifics are strictly faster than those solving plain SD in this regime.

Multiple Solutions. This regime corresponds to parameters which yield multiple solutions to RSD as well as SD. Precisely, this regime corresponds to weights exceeding the RSD uniqueness bound and is visualized as the red shaded area in Fig. 2a. Note that in this regime for any given SD instance there exist regular solutions. Therefore one can apply any RSD solver to the given SD instance finding one of those valid regular solutions. This implies that SD polynomially reduces to RSD in this regime.

However, this reduction works vice-versa as for any RSD instance there exist non-regular solutions and, further, there still exist a reduction from RSD to SD regardless of the amount of solutions. Therefore in terms of polynomial reducibility RSD and SD are equivalent in this regime.

Due to the different amount of solutions to the problems and the effect of the reduction on those amounts, it is not immediately possible to derive a strict separation on the asymptotic complexity of algorithms solving both problems in this regime. Nevertheless, we find that for weights exceeding the uniqueness bound either both problems are solvable in polynomial time, i.e., they are equally hard, or RSD is indeed harder with respect to the best known algorithms.

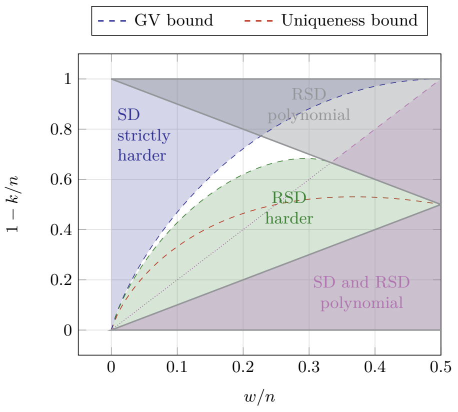


Fig. 4: Hardness comparison between RSD and SD with respect to known algorithms.

Unique RSD but multiple SD solutions The third regime corresponds to the case where SD has multiple solutions while the RSD solution is still unique. That means, the weight of those instances exceeds the GV bound but still lies below the uniqueness bound. In this regime RSD reduces to SD via the outlined reduction but not vice versa.

In terms of asymptotic complexity of algorithms solving both problems again a strict separation is not possible. Similar to the regime where both problems have multiple solutions the effect of those can not be quantified directly. We find that this regime contains parameters that lead to harder SD instances as well as harder RSD instances with respect to known algorithms.

Classification Based on Known Algorithms In the following we compare the hardness of RSD and SD based on the asymptotic complexity of known algorithms.

We illustrate in Fig. 4 the different parameter regions and their corresponding hardness classification. We mark the region in which RSD is solvable in polynomial time (see Theorems 3.1 and 3.2) as a gray shaded area framed by a solid gray line. The GV bound and the RSD uniqueness bound are depicted as blue and red dashed lines. The blue shaded area corresponds to weights below the GV bound, in which SD is strictly harder than RSD. The area below the uniqueness bound marks the regime of parameters that yield multiple solutions to both problems. The purple region corresponds to the area in which also SD is solvable in polynomial time.

Finally, the green shaded region marks the regime in which we find RSD to be harder than SD. Here the area extending the purple region to a triangle, separated by the dotted line includes parameters for which SD is solvable in polynomial time, while RSD instances remain exponentially hard. The rest of the green area corresponds to parameters where both problems are exponentially hard and we find RSD instances to be harder than SD instances based on known algorithms. For this classification we compare the running time of the basic ISD algorithm by Prange in the SD case (see Eq. (1)) against the running time of the permutation-based regular-ISD algorithm (see Algorithm 1 and Theorem 4.1). Note that, we also performed this comparison for the most advanced versions of ISD [BM18] and regular-ISD (Algorithm 3) but the picture remains almost the same with a slight increase of the green area. This indicates that the RSD case benefits slightly more from advanced techniques, as an enlarging of the green area corresponds to instances that are closer to the SD worst case (the GV bound) and further away from the RSD worst case, which is for most of the parameters in this region the RSD uniqueness bound.

Note that the white unfilled area between the green and blue area corresponds to parameters where we find SD to be harder with respect to known algorithms.

Comparison for Worst Case Parameters. Fig. 4 shows that there are regimes in which RSD is harder than SD if comparing the problems with the same parameterization. However, in cryptographic constructions parameters are usually chosen considering the individual problem properties. Schemes based on SD would craft parameter sets that are specifically well suited for SD and analogously for RSD. Therefore we compare in following the hardness of SD and RSD for their respective worst case instances. Again, this comparison is with respect to known algorithms, where in Fig. 5 we compare the running time of the fastest ISD algorithm in the SD case by Both-May against the best regular-ISD algorithm (Algorithm 3). We observe that for small rates SD instances following worst case parameters are generally

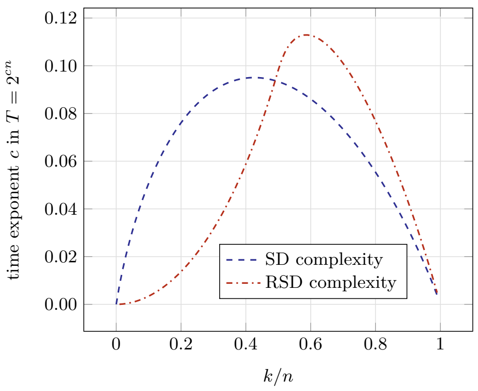


Fig. 5: Comparison of complexity of Both-May algorithm [BM18] on SD instances and regular-ISD algorithm (Algorithm 3) on RSD instances, where instances match respective worst case parameters.

harder with respect to known algorithms. However, interestingly, for all rates $\kappa > 0.49$ RSD instances following worst case parameters are harder to solve.

Comparison for Small Error Weight What is not captured visually in Fig. 4 is that the quotient of the time complexity to solve the respective SD and RSD instances $T_{\mathrm{SD}}/T_{\mathrm{RSD}}$ approaches 1 for $w/n \to 0$. Note that this does not contradict the "SD strictly harder" regime, as for any constant w we find $T_{\mathrm{SD}}/T_{\mathrm{RSD}} > 1$. But still, it is an indication that the smaller the solution weight, the closer both problems become in terms of computational complexity. In fact, for w constant, any random permutation of the columns of $\mathbf{H}$ corresponds to a regular solution with inverse polynomial probability. In turn this yields a polynomial reduction from SD to RSD. However, since both problems are solvable in polynomial time if w is a constant the reduction is not very meaningful.

In the following we show that with respect to ISD and regular-ISD algorithms both problems are solvable in same asymptotic running time as long as $w = o(n)$ is sublinear in n. Therefore note that in the SD case for $w = o(n)$, a result by Canto-Torres and Sendrier [TS16] proves that all known ISD algorithms obtain the same asymptotic running time of

$$T_{\mathrm{SD}} = \left(1 - \frac{k}{n}\right)^{-(w+o(w))}$$

.

On the other hand Theorem 4.1 states the running time (for any w) of the permutation-based regular-ISD algorithm as

$$T_{\mathrm{RSD}} = \tilde{\mathcal{O}}\left(\left(1 - \frac{k-w}{n}\right)^{-w}\right)$$

. Let $k = \kappa n$ with constant κ and $w = o(n)$ sublinear, then [3]

$$\frac{T_{\mathrm{SD}}}{T_{\mathrm{RSD}}} = \frac{\left(1 - \frac{k-w}{n}\right)^w}{\left(1 - \frac{k}{n}\right)^w} = \left(1 + \frac{w}{n(1-\kappa)}\right)^w = 2^{w \cdot \left(\frac{w}{n(1-\kappa)\ln(2)} + o\left(\frac{w}{n}\right)\right)} = 2^{o(w)}. \tag{6}$$

This shows that both complexities differ only by second order terms.

4 Regular Information Set Decoding

In this section we describe the regular-ISD algorithms that exploit the regular structure of the RSD solution and provide their asymptotic analysis.

The most advanced algorithms following the regular-ISD approach incorporate most modern techniques, including permutations, enumeration, representations and nearest-neighbor search tailored to the regular setting. For didactic reasons we provide an incremental description embedding one technique at a time, following a similar evolution as the ISD literature.

In the analysis, we ignore all rounding issues by performing computations with non-integer values if necessary. We show in Section 4.6 that these rounding issues affect the asymptotic complexity only very mildly.

4.1 Permutation-Based Regular-ISD

This first algorithm aims to find an error-free information set of the solution $\mathbf{e}$ by random selection or permutation of coordinates.

[3] From Taylor's expansion, for $x = o(1)$, we get $\log_2(1+x) = \frac{x}{\ln(2)} + \mathcal{O}(x^2) = \frac{x}{\ln(2)} + o(x)$. In our case, since $w = o(n)$, we get $w/n = o(1)$.

Error-free information sets. An *error-free* set for $\mathbf{e}$ corresponds to a set J such that $\mathbf{e}_J = \mathbf{0}$. Knowledge of such a set reveals the error vector $\mathbf{e}$ if it is an information set. Therefore, let $\mathbf{P}_J \in \mathbb{F}_2^{n\times n}$ describe any permutation matrix that permutes $\mathbf{e}_J$ to the back of $\mathbf{P}_J\mathbf{e}$, i.e., $\mathbf{P}_J\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_J) \in \mathbb{F}_2^{n-k} \times \mathbb{F}_2^k$.[4] In that case we have

$$\mathbf{He} = (\mathbf{HP}_J^{-1})(\mathbf{P}_J\mathbf{e}) = (\mathbf{HP}_J^{-1})(\mathbf{e}_1, \mathbf{e}_J) = \mathbf{H}_1\mathbf{e}_1 + \mathbf{H}_2\mathbf{e}_J = \mathbf{H}_1\mathbf{e}_1 = \mathbf{s}, \tag{7}$$

where we write $(\mathbf{HP}_J^{-1}) = (\mathbf{H}_1\mathbf{H}_2)$ with $\mathbf{H}_1 \in \mathbb{F}_2^{(n-k)\times(n-k)}$ and $\mathbf{H}_2 \in \mathbb{F}_2^{k\times(n-k)}$. Note that $\mathbf{H}_2\mathbf{e}_J = \mathbf{0}$ since $\mathbf{e}_J = \mathbf{0}$, as J represents an *error-free* information set. Now, as long as $\mathbf{H}_1$ is invertible we can compute $\mathbf{e}_1 = \mathbf{H}_1^{-1}\mathbf{s}$ and the solution as $\mathbf{e} = \mathbf{P}_J^{-1}(\mathbf{e}_1, 0^k)$. Note that $\mathbf{H}_1$ being invertible is the exact definition of J being an information set, and it happens for random $\mathbf{H}$ with constant probability over the choice of J.

The early ISD algorithm by Prange finds the error-free information set by sampling random permutations $\mathbf{P}$, computing $\mathbf{e}_1' = \mathbf{H}_1^{-1}\mathbf{s}$ and checking if $|\mathbf{e}_1'| = w$, in which case it outputs the solution $\mathbf{P}^{-1}(\mathbf{e}_1', 0^k)$, otherwise it starts over with a new permutation.

Regular Permutations. However, random permutations disregard the regular structure of the error and might lead to a decrease in the probability that $\mathbf{H}_1$ is invertible. Therefore, note that the equations added due to regularity are sparse and any permutation $\mathbf{P}$ that permutes all columns belonging to a single block to $\mathbf{H}_2$, i.e., to the back of the matrix, leads to an $\mathbf{H}_1$ containing zero rows, making it non-invertible.

In order to circumvent this problem and to take advantage of the regular structure of the solution, we modify how permutations are sampled. Namely, we ensure that for every permutation the matrix $\mathbf{H}_2$ is formed by selecting $v = \frac{k'}{w}$ columns from each block. Further, to enable later improvements the v columns taken from each block form again a consecutive block in $\mathbf{H}_2$. More formally, we define a v-regular permutation as follows.

Definition 4.1 (Regular Permutation). *Let $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_w)$ be a regular vector of weight w. For an integer v and a permutation matrix $\mathbf{P}$ let*

$$\mathbf{Pe} = (\mathbf{e}_1', \dots, \mathbf{e}_w', \mathbf{e}_1'', \dots, \mathbf{e}_w''),$$

with $\mathbf{e}_i' \in \mathbb{F}_2^{\frac{n}{b}-v}$ and $\mathbf{e}_i'' \in \mathbb{F}_2^v$. We call $\mathbf{P}$ a v-regular permutation if each $\mathbf{e}_i'$ and each $\mathbf{e}_i''$ is formed only by coordinates from $\mathbf{e}_i$.

The algorithm now samples permutations uniformly from the set of v-regular permutations until an error-free information set is found. The pseudocode of the overall algorithm is given in Algorithm 1.

Theorem 4.1 (Permutation-based Regular-ISD). *Algorithm 1 solves $\mathcal{RSD}(n, k, w)$ in expected time and memory*

$$T = \tilde{\mathcal{O}}\left(\frac{\left(1 - \frac{k-w}{n}\right)^{-w}}{S}\right) \quad \text{and} \quad M = \tilde{\mathcal{O}}\left(1\right),$$

where S is the amount of existing solutions.

[4] Recall that J being an information set also implies $|J| = k$.

Algorithm 1: Permutation-based Regular-ISD

```
Input  : Parity-check matrix $\mathbf{H}' \in \mathbb{F}_2^{(n-k)\times n}$, syndrome $\mathbf{s}' \in \mathbb{F}_2^{n-k}$, weight $w$
Output : Regular vector $\mathbf{e} \in \mathbb{F}_2^n$ of weight $w$, such that $\mathbf{He} = \mathbf{s}$
1 Let $k' := k - w$ , $v := k'/w$
2 Obtain $\mathbf{H} \in \mathbb{F}_2^{(n-k')\times n}, \mathbf{s} \in \mathbb{F}_2^{n-k'}$ from $(\mathbf{H}', \mathbf{s}')$ by encoding regularity
3 repeat
4     Sample random $v$-regular permutation matrix $\mathbf{P}$
5     $(\mathbf{H}_1, \mathbf{H}_2) \leftarrow \mathbf{HP}$
6     if $\mathbf{H}_1$ is non-singular then
7         $\mathbf{e}_1 \leftarrow \mathbf{H}_1^{-1}\mathbf{s}$
8         if $|\mathbf{e}_1| = w$ and $\mathbf{e}_1$ is regular then
9             return $\mathbf{P}^{-1}(\mathbf{e}_1, 0^{k'})$
```

Proof. The correctness follows from the previous argumentation on error-free information sets. Therefore note that as long as $v = k'/w < n/w - 1$ which is equivalent to $k/n \leq 1$ and hence strictly fulfilled, there always exist v-regular permutations that yield $\mathbf{Pe} = (\mathbf{e}_1, 0^{k'})$.

The algorithms complexity is the time per iteration divided by the success probability per iteration. First note, that the time spend in each iteration is polynomial, and hence, subsumed in our use of the Landau notation.

An iteration is successful if the chosen permutation distributes the whole support of $\mathbf{e}$ to the first $n - k'$ coordinates of $\mathbf{Pe}$ and if $\mathbf{H}_1$ is invertible. Note that $\mathbf{H}_1$, due to our restriction to regular permutations has a similar structure as $\mathbf{H}'$ from Eq. (2). Therefore, it is invertible with constant probability as detailed in Remark 3.1. Further, for $b = n/w$ the probability of the permutation distributing the weight as desired is

$$q := \Pr\left[\mathbf{Pe} = (\mathbf{e}_1, 0^{k'})\right] = \left(\frac{\binom{b-1}{v-1}}{\binom{b}{v}}\right)^w = \left(1 - \frac{v}{b}\right)^w = \left(1 - \frac{k'}{n}\right)^w. \tag{8}$$

Therefore note that the permutation has to distribute the single non-zero entry per block to the $\mathbf{H}_1$ part, which happens with probability $\frac{\binom{b-1}{v-1}}{\binom{b}{v}} = 1 - \frac{v}{b}$ per block, while there are w blocks. Note that in case of S existing solutions the probability of the permutation distributing the weight as desired for at least one of the solutions is about $q \cdot S$.

Therefore the running time of Algorithm 1 is

$$T = \tilde{\mathcal{O}}\left(\frac{\left(1 - \frac{k-w}{n}\right)^{-w}}{S}\right),$$

since $k' := k - w$. All stored objects are matrices of size polynomial in n, therefore the memory complexity is polynomial, i.e., $M = \tilde{\mathcal{O}}(1)$. □

4.2 Enumeration-Based Regular-ISD

We now extend Algorithm 1 by a meet-in-the-middle enumeration procedure that further exploits the regular structure of the error. Therefore we allow for a certain weight p, later to be optimized, within the coordinates of the information set J, i.e., we search for a set J such that $|\mathbf{e}_J| = p$.

Finiasz-Sendrier modelling We follow a modeling by Finiasz and Sendier [FS09] that increases the size of the set J to $|J| = k + \ell$ with ℓ being another optimization parameter. Let $\mathbf{P}_J$ again be a permutation matrix such that $\mathbf{P}_J\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_J) \in \mathbb{F}_2^{n-k-\ell} \times \mathbb{F}_2^{k+\ell}$, now with $|\mathbf{e}_J| = p$. Further let $\mathbf{Q} \in \mathbb{F}_2^{(n-k)\times(n-k)}$ be a matrix such that

$$\mathbf{Q}\mathbf{H}\mathbf{P}_J^{-1} = \begin{pmatrix} \mathbf{I}_{n-k-\ell} & \mathbf{H}_1 \\ \mathbf{0} & \mathbf{H}_2 \end{pmatrix}, \text{ with } \mathbf{H}_1 \in \mathbb{F}_2^{(n-k-\ell)\times(k+\ell)} \text{ and } \mathbf{H}_2 \in \mathbb{F}_2^{\ell\times(k+\ell)}.$$

This modeling allows to re-write the syndrome equation as

$$\mathbf{Q}\mathbf{H}\mathbf{e} = \mathbf{Q}\mathbf{H}\mathbf{P}_J^{-1}\mathbf{P}_J\mathbf{e} = (\mathbf{e}_1 + \mathbf{H}_1\mathbf{e}_J, \mathbf{H}_2\mathbf{e}_J) = (\mathbf{s}_1, \mathbf{s}_2) = \mathbf{Q}\mathbf{s}, \tag{9}$$

with $\mathbf{s}_1 \in \mathbb{F}_2^{n-k-\ell}$ and $\mathbf{s}_2 \in \mathbb{F}_2^{\ell}$.

This implies that once such a set J is known the solution can be found by enumerating all possible values for $\mathbf{e}_J$, i.e., all vectors $\mathbf{x} \in \mathbb{F}_2^{k+\ell}$ of weight p that satisfy $\mathbf{H}_2\mathbf{x} = \mathbf{s}_2$. Among those values there must be one $\mathbf{x} = \mathbf{e}_J$ for which $\mathbf{H}_1\mathbf{x} + \mathbf{s}_2 = \mathbf{e}_1$. Even if $\mathbf{e}_1$ is not known, this $\mathbf{x}$ can be efficiently determined by checking if $|\mathbf{H}_1\mathbf{x} + \mathbf{s}_2| = |\mathbf{e}_1| = w - p$. Eventually the solution can again be reconstructed as $\mathbf{P}_J^{-1}(\mathbf{e}_1, \mathbf{e}_J)$.

Regular Enumeration. For finding a suitable permutation that encodes such a set J we again use the sampling of regular permutations as in Algorithm 1. Recall, that this ensures that any permutation $\mathbf{P}$ permutes exactly $v := (k' + \ell)/w$ columns of each block to a consecutive block within the last $k' + \ell$ columns of $\mathbf{H}\mathbf{P}^{-1}$. Note that such permutations $\mathbf{P}$ reflect the regular structure to $\mathbf{P}\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) \in \mathbb{F}_2^{n-k'-\ell} \times \mathbb{F}_2^{k'+\ell}$ in the sense that $\mathbf{e}_2$ consists of w blocks of length v each having either weight 0 or weight 1. Put differently, if $\mathbf{P}$ encodes such a set J, $\mathbf{e}_2$ is v-regular of weight p (see Definition 4.1). Similarly, in that case $\mathbf{e}_1$ is $(n/w - v)$-regular of weight $w - p$.

We enumerate possible candidates for $\mathbf{e}_J$ in a meet-in-the-middle fashion taking into account the implied regularity. Therefore we write $\mathbf{e}_J = (\mathbf{z}_1, \mathbf{z}_2)$, with $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{F}_2^{(k'+\ell)/2}$, allowing to rewrite $\mathbf{H}_2\mathbf{e}_J = \mathbf{s}_2$ as

$$\mathbf{H}_2'\mathbf{z}_1 = \mathbf{H}_2''\mathbf{z}_2 + \mathbf{s}_2, \tag{10}$$

where $\mathbf{H}_2 = (\mathbf{H}_2', \mathbf{H}_2'')$. We then enumerate all possible $\mathbf{z}_i$ in lists L_i, where we only consider those of certain regularity. The full procedure is detailed in pseudocode in Algorithm 2.

Analysis of Algorithm 2. The correctness of the algorithm follows again from the previous argumentation. Therefore observe that once the permutation encodes a set J, $\mathbf{z}_1, \mathbf{z}_2$ from Eq. (10) form v-regular vectors of length- $(k' + \ell)/2$ and weight $p/2$, which are exhaustively enumerated in the lists L_1, L_2.

The expected running time of the algorithm is, analogous to Algorithm 1, the time spend in each iteration divided by the success probability per iteration. The time per iteration is dominated by the construction of the three lists L_1, L_2 and L. Lists L_1 and L_2 enumerate all v-regular vectors $\mathbf{z}_1, \mathbf{z}_2$ of length $(k'+\ell)/2$ and weight $p/2$, where $v := (k'+\ell)/w$. This implies

Algorithm 2: Enumeration-based Regular-ISD

```
Parameters : $\ell \leq n-k+w$ and $p \leq w$
Input      : Parity-check matrix $\mathbf{H}' \in \mathbb{F}_2^{(n-k)\times n}$, syndrome $\mathbf{s}' \in \mathbb{F}_2^{n-k}$
Output     : Regular vector $\mathbf{e} \in \mathbb{F}_2^n$ of weight $w$, such that $\mathbf{He} = \mathbf{s}$
1  let $k' := k - w$, $v := (k'+\ell)/w$
2  Obtain $\mathbf{H} \in \mathbb{F}_2^{(n-k')\times n}, \mathbf{s} \in \mathbb{F}_2^{n-k'}$ from $(\mathbf{H}', \mathbf{s}')$ by encoding regularity
3  repeat
       // Permutation and Gaussian Elimination
4      Sample a random $v$-regular permutation matrix $\mathbf{P}$
5      $\begin{pmatrix} \mathbf{I}_{n-k'-\ell} & \mathbf{H}_1 \\ \mathbf{0} & \mathbf{H}_2 \end{pmatrix} \leftarrow \mathbf{QHP}^{-1}$ , with $\mathbf{H}_1 \in \mathbb{F}_2^{(n-k'-\ell)\times(k'+\ell)}$, $\mathbf{H}_2 \in \mathbb{F}_2^{\ell\times(k'+\ell)}$
6      $(\mathbf{s}_1, \mathbf{s}_2) \leftarrow \mathbf{Qs}$ with $\mathbf{s}_1 \in \mathbb{F}_2^{n-k'-\ell}$ and $\mathbf{s}_2 \in \mathbb{F}_2^{\ell}$
       // Enumeration
7      $L_i \leftarrow \{\mathbf{z}_i \mid \mathbf{z}_i \in \mathbb{F}_2^{(k'+\ell)/2}$ is $v$-regular of weight $p/2\}$, $i = 1, 2$
8      $L \ \leftarrow \{(\mathbf{z}_1, \mathbf{z}_2) \in L_1 \times L_2 \mid \mathbf{H}_2(\mathbf{z}_1, \mathbf{z}_2) = \mathbf{s}_2\}$
9      for $\mathbf{e}_2 \in L$ do
10         $\mathbf{e}_1 \leftarrow \mathbf{H}_1\mathbf{e}_2 + \mathbf{s}_1$
11         if $|\mathbf{e}_1| = w$ and $\mathbf{e}_1$ is regular then
12             return $\mathbf{P}^{-1}(\mathbf{e}_1, \mathbf{e}_2)$
```

that $|L_1| = |L_2| = \binom{w/2}{p/2} v^{p/2}$. The list L then contains all pairs of elements $(\mathbf{z}_1, \mathbf{z}_2) \in L_1 \times L_2$ that satisfy Eq. (10). This list can be constructed in time linear in the involved list sizes, leading to a time per iteration of $T_{\text{it}} = \tilde{\mathcal{O}}\left(\max(|L_1|, |L_2|, |L|\right)$. The expected size of L is $\mathbb{E}[|L|] = \frac{|L_1 \times L_2|}{2^\ell}$.

An iteration is successful whenever the permutation distributes the weight on $\mathbf{Pe} = (\mathbf{e}_1, \mathbf{z}_1, \mathbf{z}_2)$ such that both $\mathbf{z}_1$ and $\mathbf{z}_2$ are of weight $p/2$. This happens with probability

$$q = \Pr\left[|\mathbf{z}_1| = |\mathbf{z}_2| = p/2\right] = \binom{w/2}{p/2}^2 \left(\frac{v}{b}\right)^p \left(1 - \frac{v}{b}\right)^{w-p}. \tag{11}$$

Therefore note that the probability of a block of length v in $\mathbf{z}_1$, resp. $\mathbf{z}_2$, being of weight one is $\frac{\binom{b-1}{v-1}}{\binom{b}{v}} = \frac{v}{b}$ and correspondingly it is of weight zero with probability $\left(1 - \frac{v}{b}\right)$. Now there are $\binom{w/2}{p/2}$ ways how the $p/2$ weight-1 blocks can be distributed among the $w/2$ blocks of $\mathbf{z}_1$ or $\mathbf{z}_2$, respectively.

In total this leads to a running time of

$$T = T_{\text{it}}/q = \tilde{\mathcal{O}}\left(\frac{\max\left(\binom{w/2}{p/2}v^{p/2}, \binom{w}{p}v^p/2^\ell\right)}{\binom{w}{p}\left(\frac{v}{b}\right)^p\left(1-\frac{v}{b}\right)^{w-p}}\right).$$

Note that here we use the fact that $\binom{w/2}{p/2}^2 = \tilde{\Theta}\left(\binom{w}{p}\right)$. The memory complexity is $M = \tilde{\mathcal{O}}\left(|L_1|\right) = \tilde{\mathcal{O}}\left(\binom{w/2}{p/2}v^{p/2}\right)$, as elements of $|L|$ can be checked on the fly for leading to a solution.

4.3 Representation-based Regular-ISD

Contrary to previous assumptions [CCJ23a], we show in this section how the enumeration procedure can be further improved by the use of a technique known as *representations.*

Representations. While we assume a certain familiarity of the reader with the representation technique[5], let us briefly recall its main idea in the non-regular case. A vector $\mathbf{e} \in \mathbb{F}_2^n$ of weight-p satisfying $\mathbf{He} = \mathbf{s}$ is split in the sum of two vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}_2^n$ of weight $p_{\mathbf{x}} := p/2 + \varepsilon_{\mathbf{x}}$ for a small $\varepsilon_{\mathbf{x}}$ that has to be optimized. Therefore we write $\mathbf{e} = \mathbf{x}_1 + \mathbf{x}_2$. Note that there are multiple choices for the $\mathbf{x}_1, \mathbf{x}_2$ in that equation, called *representations* of $\mathbf{e}$. Precisely, there are

$$R_{\mathbf{x}} = \binom{p}{p/2}\binom{n-p}{\varepsilon_{\mathbf{x}}}$$

many such representations. The technique builds on the observation that finding one of those representations reveals $\mathbf{e}$ and, hence, aims at enumerating a $1/R_{\mathbf{x}}$-fraction of all representations. This is achieved by only considering those representations which satisfy

$$\mathbf{Hx}_1 = \mathbf{r}_{\mathbf{x}} \quad \text{and} \quad \mathbf{Hx}_2 = \mathbf{s}_{[\ell_{\mathbf{x}}]} + \mathbf{r}_{\mathbf{x}}, \tag{12}$$

where $\ell_{\mathbf{x}} := \log R_{\mathbf{x}}$ and $\mathbf{r}_{\mathbf{x}} \in \mathbb{F}_2^{\ell_{\mathbf{x}}}$ is chosen arbitrarily. Note that the probability for any representation satisfying Eq. (12) is $\Pr\left[\mathbf{Hx}_1 = \mathbf{r}_{\mathbf{x}}\right] = 2^{-\ell_{\mathbf{x}}}$, as $\mathbf{Hx}_1 = \mathbf{r}_{\mathbf{x}}$ together with $\mathbf{He} = \mathbf{s}$ implies the second part of the equation.

While the values for $\mathbf{x}_1$ and $\mathbf{x}_2$ could now, as in the previous section, be enumerated in a meet-in-the-middle fashion based on the identities from Eq. (12), the technique becomes most effective if applied recursively. Therefore, the $\mathbf{x}_i$ are split again into the sum of vectors $\mathbf{y}_j$. More precisely, we write $\mathbf{x}_1 = \mathbf{y}_1 + \mathbf{y}_2$ and $\mathbf{x}_2 = \mathbf{y}_3 + \mathbf{y}_4$, where the $\mathbf{y}_i \in \mathbb{F}_2^n$ are of weight $p_{\mathbf{y}} = p_{\mathbf{x}}/2 + \varepsilon_{\mathbf{y}}$ for a $\varepsilon_{\mathbf{y}}$ that has to be optimized. Again each such $\mathbf{x}_1$ and $\mathbf{x}_2$ have $R_{\mathbf{y}} = \binom{p_{\mathbf{x}}}{p_{\mathbf{x}}/2}\binom{n-p_{\mathbf{x}}}{\varepsilon_{\mathbf{y}}}$ representations as $(\mathbf{y}_1, \mathbf{y}_2)$ and $(\mathbf{y}_3, \mathbf{y}_4)$ respectively. Therefore again constraints on the exact form of the matrix-vector product $\mathbf{Hy}_i$ are introduced to only enumerate a $1/R_{\mathbf{y}}$ fraction of those representations, still ensuring all $\mathbf{x}_i$ are constructed. Precisely, letting $\ell_{\mathbf{y}} := \log R_{\mathbf{y}}$ and $\mathbf{r}_{\mathbf{y}} \in \mathbb{F}_2^{\ell_{\mathbf{y}}}$ the algorithm enforces

$$\mathbf{Hy}_1 = \mathbf{r}_{\mathbf{y}} \quad \text{and} \quad \mathbf{Hy}_2 = (\mathbf{r}_{\mathbf{x}})_{[\ell_{\mathbf{y}}]} + \mathbf{r}_{\mathbf{y}} \qquad \text{, as well as} \tag{13}$$

$$\mathbf{Hy}_3 = \mathbf{r}_{\mathbf{y}} \quad \text{and} \quad \mathbf{Hy}_4 = (\mathbf{s} + \mathbf{r}_{\mathbf{x}})_{[\ell_{\mathbf{y}}]} + \mathbf{r}_{\mathbf{y}}. \tag{14}$$

Analogously to the first application of the technique, for any representation $(\mathbf{y}_1, \mathbf{y}_2)$ Eq. (13) is satisfied with probability $2^{-\ell_{\mathbf{y}}}$ giving the desired fraction of representations, as the second part of each equation is implied by the first part in conjunction with Eq. (12). Analogously, the same holds for a representation $(\mathbf{y}_3, \mathbf{y}_4)$ and Eq. (14).

[5] For an introduction we refer to [HJ10, MMT11].

Regular Representations. The representation enhanced algorithm for the regular case follows a similar initial construction as Algorithm 2. As such, it uses the same modelling by Finiasz-Sendrier and samples the same kind of regular permutations. The adaptation lies in the enumeration procedure, i.e., the way we enumerate $\mathbf{e}_J$. Recall that $\mathbf{e}_J$ is a v-regular vector of length $k' + \ell$ and weight p, with $v := (k' + \ell)/w$, satisfying the identity $\mathbf{He}_J = \mathbf{s}_2$. In order to embed representations into the regular enumeration we split $\mathbf{e}_J = \mathbf{x}_1 + \mathbf{x}_2$ in the sum of two v-regular vectors of same length $k' + \ell$ but weight $p_\mathbf{x} = p/2 + \varepsilon_\mathbf{x}$. Therefore we maintain the regular structure in the addends $\mathbf{x}_i$, which reduces the search space for the enumeration, while still obtaining multiple representations. Note that the number of representations in such a modelling becomes

$$R_\mathbf{x} = \binom{p}{p/2}\binom{w-p}{\varepsilon_\mathbf{x}} \cdot v^{\varepsilon_\mathbf{x}}. \tag{15}$$

As in the non-regular case the first binomial coefficient counts the possibilities how the p weight-1 blocks can be distributed equally over the addends $\mathbf{x}_1$ and $\mathbf{x}_2$. The second coefficient counts the possibilities to position the remaining $\varepsilon_\mathbf{x}$ non-zero blocks in which the weight cancels. Note that each canceling block offers v possibilities to place its weight, which leads to the final factor. We apply the technique recursively, splitting $\mathbf{x}_1 = \mathbf{y}_1 + \mathbf{y}_2$ and $\mathbf{x}_2 = \mathbf{y}_3 + \mathbf{y}_4$, where we let $\mathbf{y}_i$ be v-regular vectors of length $k' + \ell$ and weight $p_\mathbf{y} = p_\mathbf{x}/2 + \varepsilon_\mathbf{y}$. The vectors $\mathbf{y}_i$ are then enumerated in a meet-in-the-middle fashion based on the identities from Eqs. (13) and (14). A pseudocode description of the full procedure is given in Algorithm 3.

Analysis of Algorithm 3. The correctness of the procedure follows from the correctness of Algorithm 2 and the argumentation above. Therefore observe that for the enumeration procedure we enforce the necessary constraints $\mathbf{r}_\mathbf{x}, \mathbf{r}_\mathbf{y}$ of correct length $\ell_\mathbf{x} := \log R_\mathbf{x}, \ell_\mathbf{y} := \log R_\mathbf{y}$, where

$$R_\mathbf{y} = \binom{p_\mathbf{x}}{p_\mathbf{x}/2}\binom{w-p_\mathbf{x}}{\varepsilon_\mathbf{y}} \cdot v^{\varepsilon_\mathbf{y}}, \tag{16}$$

analogous to Eq. (15). Hence, the enumeration recovers all candidates for $\mathbf{e}_J$ in the final lists, i.e., all v-regular vectors of weight p satisfying $\mathbf{He}_J = \mathbf{s}_2$, implying that $\mathbf{e}_J$ is found once a correct permutation is encountered.

The running time is the number of iterations divided by the success probability of each iteration. An iteration is successful if the permutation distributes the weight as desired, which we already saw happens with probability q (see Eq. (11)).

The cost per iteration is dominated by the enumeration procedure, that is the construction of the lists. In the initial lists L_i, $i = 1, \ldots, 8$ all v-regular vectors of length $(k' + \ell)/2$ and weight $p_\mathbf{y}/2$ are enumerated. Therefore those lists are of size $|L_i| = \binom{w/2}{p_\mathbf{y}/2} v^{p_\mathbf{y}/2}$. The lists $L_{\mathbf{y}_j}$ are constructed from pairs from lists L_{2j-1}, L_{2j} by enforcing a constraint on $\ell_\mathbf{y}$ coordinates and is therefore of expected size $|L_{\mathbf{y}_j}| = |L_i|^2/2^{\ell_\mathbf{y}}$. The construction of this list is linear in the size of the L_i and the list $L_{\mathbf{y}_j}$ itself, hence it can be performed in time

$$T_\mathbf{y} = \tilde{\mathcal{O}}\left(\max(|L_i|, |L_{\mathbf{y}_i}|)\right).$$

Lists $L_{\mathbf{x}_j}$ are similarly constructed from pairs of the lists $L_{\mathbf{y}_{2j-1}}, L_{\mathbf{y}_{2j}}$ by enforcing a constraint on $\ell_\mathbf{x}$ coordinates. However, note that since every such pair satisfies Eq. (13) (resp. Eq. (14)) it already satisfies the first (resp. second) part of Eq. (12) on $\ell_\mathbf{y}$ coordinates. Hence only a

Algorithm 3: Representation-based Regular-ISD

```
Parameters : $\ell \leq n-k+w$, $p \leq w$, $\varepsilon_{\mathbf{x}} \leq k-w+\ell-p$ and $\varepsilon_{\mathbf{y}} \leq k-w+\ell-p/2-\varepsilon_{\mathbf{x}}$
Input      : Parity-check matrix $\mathbf{H}' \in \mathbb{F}_2^{(n-k)\times n}$, syndrome $\mathbf{s}' \in \mathbb{F}_2^{n-k}$
Output     : Regular vector $\mathbf{e} \in \mathbb{F}_2^n$ of weight $w$, such that $\mathbf{He} = \mathbf{s}$
1  let $k' := k-w$, $v := (k'+\ell)/w$, $p_{\mathbf{x}} := p/2 + \varepsilon_{\mathbf{x}}$ and $p_{\mathbf{y}} := p_{\mathbf{x}}/2 + \varepsilon_{\mathbf{y}}$
2  let $\ell_{\mathbf{x}} = \log R_{\mathbf{x}}$ and $\ell_{\mathbf{y}} = \log R_{\mathbf{y}}$ for $R_{\mathbf{x}}, R_{\mathbf{y}}$ as in Eqs. (15) and (16)
3  Obtain $\mathbf{H} \in \mathbb{F}_2^{(n-k')\times n}, \mathbf{s} \in \mathbb{F}_2^{n-k'}$ from $(\mathbf{H}', \mathbf{s}')$ by encoding regularity
4  repeat
       // Permutation and Gaussian Elimination
5      Sample a regular permutation matrix $\mathbf{P} \in R_v$
6      $\begin{pmatrix} \mathbf{I}_{n-k'-\ell} & \mathbf{H}_1 \\ \mathbf{0} & \mathbf{H}_2 \end{pmatrix} \leftarrow \mathbf{QHP}^{-1}$ , with $\mathbf{H}_1 \in \mathbb{F}_2^{(n-k'-\ell)\times(k'+\ell)}$, $\mathbf{H}_2 \in \mathbb{F}_2^{\ell\times(k'+\ell)}$
7      $(\mathbf{s}_1, \mathbf{s}_2) \leftarrow \mathbf{Qs}$ with $\mathbf{s}_1 \in \mathbb{F}_2^{n-k-\ell}$ and $\mathbf{s}_2 \in \mathbb{F}_2^{\ell}$
       // Enumeration
8      sample $\mathbf{r}_{\mathbf{x}} \in \mathbb{F}_2^{\ell_{\mathbf{x}}}$,$\mathbf{r}_{\mathbf{y}} \in \mathbb{F}_2^{\ell_{\mathbf{y}}}$
9      $L_i \leftarrow \{\mathbf{z}_i \mid \mathbf{z}_i \in \mathbb{F}_2^{(k'+\ell)/2}$ is $v$-regular of weight $p_{\mathbf{y}}/2\}$, $i = 1, \ldots, 8$
10     let $\mathbf{r}_{\mathbf{y}_1} = \mathbf{r}_{\mathbf{y}_3} := \mathbf{r}_{\mathbf{y}}$, $\mathbf{r}_{\mathbf{y}_2} = (\mathbf{r}_{\mathbf{x}})_{[\ell_{\mathbf{y}}]} + \mathbf{r}_{\mathbf{y}}$ and $\mathbf{r}_{\mathbf{y}_4} = (\mathbf{s}_2 + \mathbf{r}_{\mathbf{x}})_{[\ell_{\mathbf{y}}]} + \mathbf{r}_{\mathbf{y}}$
11     $L_{\mathbf{y}_i} \leftarrow \{(\mathbf{z}_{2i-1}, \mathbf{z}_{2i}) \in L_{2i-1} \times L_{2i} \mid \mathbf{H}_2(\mathbf{z}_{2i-1}, \mathbf{z}_{2i}) = \mathbf{r}_{\mathbf{y}_i}\}$, $i = 1,2,3,4$
12     let $\mathbf{r}_{\mathbf{x}_1} := \mathbf{r}_{\mathbf{x}}$ and $\mathbf{r}_{\mathbf{x}_2} = (\mathbf{s}_2)_{[\ell_{\mathbf{x}}]} + \mathbf{r}_{\mathbf{x}}$
13     $L_{\mathbf{x}_i} \leftarrow \left\{\mathbf{y}_{2i-1} + \mathbf{y}_{2i} \mid \mathbf{y}_j \in L_{\mathbf{y}_j} \wedge \mathbf{H}_2(\mathbf{y}_{2i-1} + \mathbf{z}_{2i}) = \mathbf{r}_{\mathbf{y}_i}\right\}$, $i = 1,2$
14     $L_{\mathbf{e}_J} \leftarrow \left\{\mathbf{x}_1 + \mathbf{x}_2 \mid \mathbf{x}_j \in L_{\mathbf{x}_j} \wedge \mathbf{H}_2(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{s}_2\right\}$
15     for $\mathbf{e}_J \in L$ do
16         $\mathbf{e}_1 \leftarrow \mathbf{H}_1\mathbf{e}_J + \mathbf{s}_1$
17         if $|\mathbf{e}_1| = w$ and $\mathbf{e}_1$ is regular then
18             return $\mathbf{P}^{-1}(\mathbf{e}_1, \mathbf{e}_J)$
```

new constraint on $\ell_{\mathbf{x}} - \ell_{\mathbf{y}}$ coordinates is introduced. The construction of list $L_{\mathbf{x}_j}$ therefore comes at a cost of

$$T_{\mathbf{x}} = \tilde{\mathcal{O}}\left(\max(|L_{\mathbf{y}_i}|, |L_{\mathbf{y}_i}|^2/2^{\ell_{\mathbf{x}}-\ell_{\mathbf{y}}})\right).$$

Observe that not all the $|L_{\mathbf{y}_i}|^2/2^{\ell_{\mathbf{x}}-\ell_{\mathbf{y}}}$ constructed elements satisfying the corresponding part of Eq. (12), have the correct number of non-zero blocks $p_{\mathbf{x}}$. This happens in the case when not exactly $\varepsilon_{\mathbf{x}}$ blocks cancel in the addition of elements from the lists $L_{\mathbf{y}_{2j-1}}, L_{\mathbf{y}_{2j}}$. We then discard elements that do not form v-regular vectors of weight $p_{\mathbf{x}}$. Due to the choice of constraint-sizes, which ensures that each v regular vector of weight $p_{\mathbf{x}}$ satisfying Eq. (12) is constructed, after discarding ill-formed elements the lists $L_{\mathbf{x}_j}$ are of size $|L_{\mathbf{x}_j}| = \binom{w}{p_{\mathbf{x}}} v^{p_{\mathbf{x}}}/2^{\ell_{\mathbf{x}}}$

Eventually, the final list $L_{\mathbf{e}_J}$ can analogously to list $L_{\mathbf{x}_j}$ be constructed in time

$$T_{\mathbf{e}_J} = \tilde{\mathcal{O}}\left(\max(|L_{\mathbf{x}_j}|, |L_{\mathbf{x}_j}|^2/2^{\ell-\ell_x}\right).$$

Therefore note that any sum $\mathbf{e}'_J$ of elements from lists $L_{\mathbf{x}_1}, L_{\mathbf{x}_2}$ already satisfies the equation $\mathbf{H}_2\mathbf{e}'_J = \mathbf{s}_2$ on $\ell_\mathbf{x}$ out of ℓ coordinates (compare to Eq. (12)).

The time complexity of the full algorithm is therefore given by

$$\begin{aligned} T &= \tilde{\mathcal{O}}\left(q^{-1} \cdot \max(T_\mathbf{y}, T_\mathbf{x}, T_{\mathbf{e}_J})\right) \\ &= \tilde{\mathcal{O}}\left(\frac{\max\left(\binom{w/2}{p_\mathbf{y}/2} v^{p_\mathbf{y}/2}, \binom{w}{p_\mathbf{y}} v_\mathbf{y}^{p}/2^{\ell_\mathbf{y}}, \binom{w}{p_\mathbf{y}}^2 v^{2p_\mathbf{y}}/2^{\ell_\mathbf{x}+\ell_\mathbf{y}}, \binom{w}{p_\mathbf{x}}^2 v^{2p_\mathbf{x}}/2^{\ell+\ell_\mathbf{x}}\right)}{\binom{w}{p}\left(\frac{v}{b}\right)^p\left(1-\frac{v}{b}\right)^{w-p}}\right), \end{aligned}$$

while the memory complexity is equal to the sizes of $L_i, L_\mathbf{y}, L_\mathbf{x}$, i.e., $M = \tilde{\mathcal{O}}\left(\max(|L_i|, |L_\mathbf{y}|, |L_\mathbf{x}|)\right)$.

Since for ISD algorithms in the SD context the optimal recursion depth of the representation technique differs, we also computed the complexity increasing the depth by one. However, this variant did not allow to obtain further improvements.

4.4 Nearest-Neighbor-Based Regular-ISD

Finally, we achieve a further improvement over Algorithm 3 by leveraging nearest-neighbor search techniques similar to [MO15].

Nearest-Neighbor-based ISD. Therefore note, that so far Algorithm 3 exploits only the relation $\mathbf{H}_2\mathbf{e}_J = \mathbf{s}_2$ to find the solution, i.e., the second part of Eq. (9). May and Ozerov observed, that the first part of this equation, namely $\mathbf{H}_1\mathbf{e}_J + \mathbf{e}_1 = \mathbf{s}_1$, can be interpreted as a nearest-neighbor identity and in turn be exploited in the candidate search for $\mathbf{e}_J$. Therefore note that if $\mathbf{e}_J = \mathbf{x}_1 + \mathbf{x}_2$, then the equation can be rewritten as

$$\mathbf{H}_1\mathbf{x}_1 = \mathbf{H}_1\mathbf{x}_2 + \mathbf{s}_1 + \mathbf{e}_1.$$

In other words $\mathbf{H}_1\mathbf{x}_1$ is equal to $\mathbf{H}_1\mathbf{x}_2 + \mathbf{s}_1$ up to the addition of $\mathbf{e}_1$. Now, since $\mathbf{e}_1$ is known to have low Hamming weight those values are actually close.

Therefore, once two lists $L_{\mathbf{x}_1}, L_{\mathbf{x}_2}$ with candidates for $\mathbf{x}_1, \mathbf{x}_2$ are constructed one can apply a nearest neighbor search to find those elements $\mathbf{x}'_1, \mathbf{x}'_2$ for which it holds that $\mathbf{H}_1\mathbf{x}'_1 \approx \mathbf{H}_1\mathbf{x}'_2 + \mathbf{s}_1$.

Integrating Nearest-Neighbor search into Algorithm 3. We embed the nearest-neighbor search in the same way. Therefore we exchange line 14 in Algorithm 3 with an application of a nearest neighbor search routine that computes $L_{\mathbf{e}_J}$ as

$$L_{\mathbf{e}_J} = \{\mathbf{x}_1 + \mathbf{x}_2 \colon \mathbf{x}_i \in L_{\mathbf{x}_i} \wedge |\mathbf{H}_1(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{s}_2| = w - p\}.$$

Note that we do not require the identity $\mathbf{H}_2\mathbf{e}_J = \mathbf{s}_2$ in the construction of $L_{\mathbf{e}_J}$ anymore. Therefore, we set $\ell = \ell_\mathbf{x}$, which ensures that the identity is already satisfied on all coordinates for any sum of elements from $L_{\mathbf{x}_1}, L_{\mathbf{x}_2}$. The only change in the complexity of the adapted algorithm in comparison to Algorithm 3 is with regards to the time to construct $L_{\mathbf{e}_J}$, which now becomes

$$T_{\mathbf{e}_J} = \tilde{\mathcal{O}}\left(\max(|L_{\mathbf{x}_j}|, \mathcal{N}(|L_{\mathbf{x}_1}|, n-k-\ell, w-p))\right).$$

Here $\mathcal{N}(|L_{\mathbf{x}_1}|, n-k'-\ell, w-p)$ describes the complexity of the nearest neighbor routine by May-Ozerov [MO15, Theorem 1] (or see [EKZ21, Theorem 1] for a generalization). In our case this routine finds all pairs of elements from the two lists of size $|L_{\mathbf{x}_1}|$ containing length-$(n-k'-\ell)$ vectors that sum to weight-$(w-p)$ vectors.

Remark 4.1 (Regular Nearest-Neighbor Search). We note that the difference between $\mathbf{H}_1\mathbf{x}_1$ and $\mathbf{H}_1\mathbf{x}_2 + \mathbf{s}_1$ is $\mathbf{e}_1$, i.e., a $(n-k'-\ell)/w$-regular vector of weight $w-p$. So far we do not exploit this structure in the nearest neighbor search. Internally, the May-Ozerov nearest neighbor search already applies a permutation to both lists that ensures a certain regular distribution of the difference. More precisely, the algorithm ensures that the difference $\mathbf{e}_1$ is formed by blocks of length v of same weight, where the precise choice of v depends on the analysis, but a common choice is $\log^2(k'+\ell)$. Note that enforcing such a distribution comes at a subexponential runtime overhead subsumed in the asymptotic notation (since lists are of exponential size). How to further exploit the regular structure when v is constant to obtain an asymptotic speedup is non-obvious. However, the second order terms in the complexity of the May-Ozerov algorithm can certainly be reduced if the difference is guaranteed to be regular.

4.5 Asymptotic Complexity Comparison

In this section we compare the theoretical runtime exponent of the different improvements against the state-of-the-art. Therefore note, that the asymptotic complexity of all presented algorithms can be expressed as $2^{c(\kappa,\omega)n}$, where c is a constant depending on the constant rate κ and the constant relative weight ω, where $k = \kappa n$ and $w = \omega n$.

Obtaining the runtime exponent. We obtain the constant c by a standard procedure in the context of ISD algorithms. That is by straightforward application of the well-known approximation $\binom{a}{b} = \Theta\left(2^{ah(b/a)}\right)$, to the previously stated runtime formulas, where h denotes the binary entropy function. Technically, the function c also depends on the optimization parameters, such as ℓ and p in the case of Algorithm 2. We then model those parameters to be linear in n, i.e. $\ell = c_\ell n$ and $p = c_p n$, where c_ℓ, c_p are constants. For given κ, ω, we then numerically minimize $c(\kappa,\omega)$ by the choice of c_ℓ, c_p. A python script performing the necessary numerical optimization is available at `github.com/Memphisd/Regular-ISD`.

State-of-the-art. Recently, Briaud and Øygarden [BØ23] presented an algorithm modelling the RSD problem as a multivariate-system. While their algorithm outperforms other approaches for certain, concrete parameters, its asymptotic scaling is unclear. We therefore compare against this approach in Section 5 where we consider concrete complexities. At the same time Carozza, Couteau and Joux [CCJ23a] presented an algorithm that is based on an enumeration approach. In the following, we briefly recall this algorithm and establish its asymptotic complexity.

Asymptotic of the Carozza-Couteau-Joux Algorithm The CCJ algorithm follows a pure enumeration strategy. Therefore, the algorithm relies on a similar technique as the Finiasz-Sendier modelling. The solution $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) \in \mathbb{F}_2^{n-k-\ell} \times \mathbb{F}_2^{k+\ell}$ is split in two parts, for some optimization parameter ℓ. Analogously to Eq. (9) this allows to rewrite the syndrome equation as

$$\mathbf{He} = \begin{pmatrix} \mathbf{I}_{n-k-\ell} & \mathbf{H}_1 \\ \mathbf{0} & \mathbf{H}_2 \end{pmatrix} (\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{H}_1\mathbf{e}_2 + \mathbf{e}_1, \mathbf{H}_2\mathbf{e}_2 = (\mathbf{s}_1, \mathbf{s}_2) = \mathbf{s}. \tag{17}$$

The algorithm then enumerates all candidates $\mathbf{x}$ for $\mathbf{e}_2$ exploiting its regularity. Among those candidates it recovers the one $\mathbf{x} = \mathbf{e}_2$ that satisfies $|\mathbf{H}_1\mathbf{x} + \mathbf{s}_1| = |\mathbf{e}_1| = w(1 - \frac{k+\ell}{n})$.

The authors of [CCJ23a] suggest to add the regularity encoding equations prior to the modelling from Eq. (17). However, we note that adding equations that have support only on their last $k+\ell$ coordinates does not improve the enumeration procedure. This is because the enumeration on those coordinates is already restricted to regular vectors which satisfy those equations by construction. More precisely, since adding equations decreases the dimension of the code, we need to ensure that

$$n - (k + \ell - i) \leq i \cdot b = i \cdot \frac{n}{w} \quad \Leftrightarrow \quad i \leq \frac{n-k}{n/w - 1} = \left(1 - \frac{w}{n}\right)^{-1} \left(1 - \frac{k+\ell}{n}\right) w.$$

This inequality is equivalent to all added vectors having support only on their first $n-k-\ell+i$ coordinates, i.e., the part which is not enumerated. The updated parity check matrix therefore corresponds to a code with dimension $\tilde{k} = k - \left(1 - \frac{w}{n}\right)^{-1} \left(1 - \frac{k+\ell}{n}\right) w$.

For the enumeration the algorithm uses a meet-in-the-middle strategy, enumerating n/w-regular vectors $\mathbf{e}_2', \mathbf{e}_2''$ of length $(\tilde{k}+\ell)/2$, where $\mathbf{e}_2 = (\mathbf{e}_2', \mathbf{e}_2'')$. Therefore the enumerated lists are of size $L = \left(\frac{n}{w}\right)^{\frac{w(\tilde{k}+\ell)}{2n}}$. Similar to the Finiasz-Sendrier modelling the algorithm then finds all pairs between the lists that satisfy

$$\mathbf{H}_2(\mathbf{e}_2', \mathbf{0}) = \mathbf{H}_2(\mathbf{0}, \mathbf{e}_2'') + \mathbf{s}_2.$$

On expectation there are $L^2/2^\ell$ pairs satisfying the above identity, leading to a running time of

$$T_{\text{CCJ}} = \tilde{\mathcal{O}}\left(\max\left(L, L^2/2^\ell\right)\right),$$

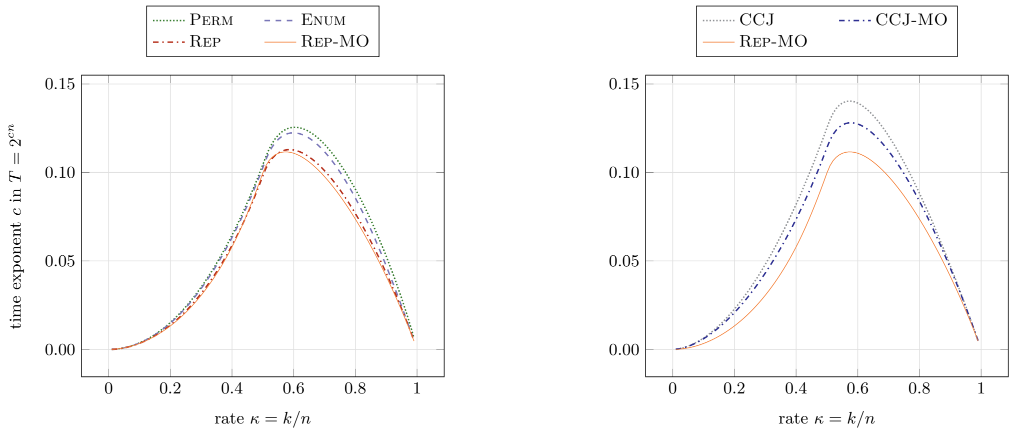
while the memory complexity is linear in the list size, i.e., $M_{\text{CCJ}} = \tilde{\mathcal{O}}(L)$.

Improvement by Nearest-Neighbors. Carozza, Couteau and Joux further applied the idea of nearest neighbor search in the spirit of [MO15]. In particular, this corresponds to the choice of $\ell = 0$ while using only the nearest neighbor identity $\mathbf{H}_1\mathbf{e}_2 = \mathbf{e}_1 + \mathbf{s}$ from the modelling in Eq. (17) for the subsequent matching of lists. In the parameter selection of their signature scheme they assume this regular nearest-neighbor search can be performed at no cost. While this leads to conservative parameters it does not yield a constructive algorithm. In the following we use the May-Ozerov nearest-neighbor search to perform this matching. Even though we do not exploit the regular structure, we obtain improvements over the pure enumeration variant. Note that the equations that can be added in this case are $k - \tilde{k}$ as before, now with the choice $\ell = 0$. Further, the nearest-neighbor search comes at a cost of $N := \mathcal{N}\left(L, n - \tilde{k}, \left(1 - \frac{\tilde{k}}{n}\right) w\right)$ (compare to Section 4.4), which results in a runtime of

$$T_{\text{NN}} = \tilde{\mathcal{O}}\left(\max\left(L, N\right)\right). \tag{18}$$

Algorithm	κ	ω	$c_T(\kappa,\omega)$	$c_M(\kappa,\omega)$
CCJ	0.57	0.1659	0.1404	0.1404
CCJ-MO	0.58	0.1575	0.1281	0.1054
Perm	0.60	0.1421	0.1256	**0.0000**
Enum	0.60	0.1421	0.1225	0.0287
Rep	0.59	0.1496	0.1130	0.0714
Rep-MO	0.57	0.1659	**0.1117**	0.0852

Table 2: Maximum runtime and corresponding memory exponents for different RSD algorithms. Runtime resp. memory are of the form $2^{c_T(\kappa,\omega)n}$ and and $2^{c_M(\kappa,\omega)n}$. Maximum time is obtained for stated κ and ω.



(a) Comparison of regular-ISD from Section 4.

(b) Comparison against other algorithms.

Fig. 6: Comparison of worst case time complexity exponents for RSD algorithms.

Asymptotic Comparison In the following we compare the worst case decoding complexity of the different regular-ISD algorithms from Section 4 against the state-of-the-art. The worst case complexity corresponds to the runtime exponent maximized over all rates for the respective worst case weight ω^*, i.e., $\max_\kappa c(\kappa, \omega^*)$. In the comparison we refer to the regular-ISD algorithms as Perm (Section 4.1), Enum (Section 4.2), Rep (Section 4.3) and Rep-MO (Section 4.4), where Rep-MO is the nearest neighbor enhanced variant of Rep. Further with CCJ we refer to the Carozza-Couteau-Joux algorithm [CCJ23a] and by CCJ-MO to the nearest-neighbor enhanced variant of the CCJ algorithm instantiated via the May-Ozerov nearest-neighbor search routine. In Table 2 we compare the obtained worst case time complexity exponents (and the corresponding memory) for the different algorithms.

We find that CCJ-MO imporoves on CCJ in both, time and memory. Overall the regular-ISD algorithms from Section 4 obtain the best time complexities. Notably, Perm offers a polynomial memory instantiation while outperforming non-regular-ISD approaches. Among regular-ISD approaches we observe a similar behavior as for ISD algorithms in the SD case, where advanced algorithms obtain the time improvements by spending higher amounts of memory. However, even the fastest regular-ISD instantiation uses much less memory than the CCJ-style algorithms.

In Fig. 6 we compare the time complexity exponent for all possible choices of the rate (for the respective worst case weight). On the left we compare the regular-ISD algorithms. Interestingly for rates $0.45 \leq \kappa \leq 0.55$ Rep outperforms Rep-MO. Since this is in contrast to similar nearest-neighbor search improvements in the SD case, it is a strong indication that the used nearest-neighbor routine is suboptimal for the regular case. We pose it as a further reasearch direction to investigate advanced procedures when the distance is known to be regular.

On the right, i.e. in Fig. 6b, we compare the CCJ-style algorithms against Rep-MO.

4.6 The Asymptotic Effect of Rounding Issues

In the previous section, for simplicity we ignored all rounding issues. Precisely, we performed computations with a block size of $b = n/w$ and we assumed that the (information) set J is formed by the same amount of $v = \frac{|J|}{w}$ coordinates from each block. However, since $|J|$ and w are linear in n and, hence, b and v are constant, this actually requires $\frac{n}{w}$ and $\frac{|J|}{w}$ to be integers.

As both constants are raised to a power linear in n, compare e.g. to Eq. (8), the rounding might affect the asymptotics. In the following we show that a tweak of the analysis, that works entirely with integer values affects the asymptotic runtime exponent for Perm (see Algorithm 1 and Theorem 4.1) only marginally.

Obtaining integer block size. First, for given integers $k = \kappa n$ and corresponding worst case weight $w^* = \omega^* n$ we obtain a valid $\mathcal{RSD}(\tilde{n}, \tilde{k}, w^*)$ instance with integer block size and unique solution by letting

$$b = \left\lceil \frac{1}{\omega} \right\rceil, \quad \tilde{n} = w^* \cdot b \quad \text{and} \quad \tilde{k} = \lceil \kappa \cdot \tilde{n} \rfloor. \tag{19}$$

The new instance obtains rate $\tilde{k}$ very close to κ but has a slightly lower relative weight $\frac{w^*}{n'} \leq \omega^*$, guaranteeing a unique solution. Overall, we expect this to decrease the obtained complexity exponent slightly as the relative weight does not match the worst case anymore.

Sampling the information set. In case w does not divide k, $v = \frac{k'}{w} = \frac{k}{w} - 1$ is not an integer and, hence, the information set J cannot be formed by the same amount of coordinates from all blocks. In such cases, we select $\left\lfloor \frac{k'}{w} \right\rfloor$ coordinates from some blocks, while $\left\lceil \frac{k'}{w} \right\rceil$ from others. Consequently the success probability of sampling an error-free information set J changes.

Let w_f and w_c denote the number of blocks from which we select $\left\lfloor \frac{k'}{w} \right\rfloor$ and $\left\lceil \frac{k'}{w} \right\rceil$ coordinates, respectively. Since our selection involves all blocks, and the total amount of selected coordinates needs to add to $|J| = k'$, we obtain the relations

$$w_\mathrm{f} + w_\mathrm{c} = w \quad \text{and} \quad w_\mathrm{f} \left\lfloor \frac{k'}{w} \right\rfloor + w_\mathrm{c} \left\lceil \frac{k'}{w} \right\rceil = k',$$

which together imply

$$w_\mathrm{f} = w - \left(k' - w \left\lfloor \frac{k'}{w} \right\rfloor \right) \quad \text{and} \quad w_\mathrm{c} = k' - w \left\lfloor \frac{k'}{w} \right\rfloor.$$

The modified success probability then becomes (compare to Eq. (8))

$$\tilde{q} = \left(1 - \frac{\lfloor k'/w \rfloor}{b}\right)^{w_{\mathrm{f}}} \left(1 - \frac{\lceil k'/w \rceil}{b}\right)^{w_{\mathrm{c}}}.$$

This leads to a modified time complexity of $\tilde{T} = \tilde{\mathcal{O}}\left(\frac{1}{\tilde{q}}\right) = 2^{\tilde{c}n}$. Let $\left\lfloor \frac{k'}{w} \right\rfloor = \frac{k'}{w} - \varepsilon$, for some $\varepsilon \in [0; 1]$ and, hence, $\left\lceil \frac{k'}{w} \right\rceil = \frac{k'}{w} - \varepsilon + 1$. Then we can rewrite $\tilde{q}$ as follows

$$\begin{aligned}\tilde{q} &= \left(1 - \frac{k'/w - \varepsilon}{b}\right)^{w - \left(k' - w\left(\frac{k'}{w} - \varepsilon\right)\right)} \left(1 - \frac{k'/w - \varepsilon + 1}{b}\right)^{k' - w\left(\frac{k'}{w} - \varepsilon\right)} \\ &= \left(1 - \kappa + (1+\varepsilon)\omega\right)^{w(1-\varepsilon)} \left(1 - \kappa + \varepsilon\omega\right)^{\varepsilon w}.\end{aligned}$$

Hence, it follows that

$$\tilde{c} = \frac{1}{n}\log_2(1/\tilde{q}_1) = \omega(1-\varepsilon)\log_2\left(1 - \kappa + (1+\varepsilon)\omega\right) + \varepsilon\omega\log_2(1 - \kappa + \varepsilon\omega), \qquad (20)$$

The joint effect on the runtime exponent. We now compare the exponent obtained in the theoretical analysis and the exponent obtained after resolving rounding issues. The theoretical exponent can be obtained from Theorem 4.1 as

$$c := \frac{1}{n}\log_2 T = \omega\log_2(1 - \kappa + \omega).$$

For obtaining the latter, we first update the instance parameters to obtain integer blocksize and then consider the modified sampling of the information set. A formula for this exponent is obtained by substituting n, ω, κ in Eq. (20) by the parameters of the updated instance $\tilde{n}, \tilde{\omega} = w/\tilde{n}, \tilde{\kappa} = \tilde{k}/\tilde{n}$ from Eq. (19).

In Fig. 7a we report the relative difference $\tilde{c}/c - 1$ between both exponents. We observe that for low rates resolving rounding issues leads to an increase in time complexity while for larger rates the complexity *decreases*. This is due to the initial adaptation of the instance parameters. For small rates κ with correspondingly small worst case weight ω^* the initial rounding of the block size leads only to a small deviation from the worst case relative weight. For larger rates and larger ω^* the deviation becomes more significant. In those cases the decreased difficulty of the initial instance compensates for the lower success probability caused by the rounding on $v = k'/w$. The highest increase by about 2.44% is obtained for a rate of $\kappa = 0.46$

In Fig. 7b we visualize the absolute value of the obtained exponents. Since the rate which achieves the maximum relative difference does not align with the worst case rate, the influence on the worst case complexity of Perm is far smaller than 2.44%. In fact we obtain a new worst case complexity exponent marginally increased by an additive 0.00083 (0.66%) of $\tilde{c}^* = 0.1263$ obtained for rate $\kappa^* = 0.597$.

Similar, but more technical computations can be performed for the other versions Enum, Rep and Rep-MO. However, we do not expect the picture to change and leave it for future work to determine the exact deviation. Note that our proof concept implementation of Enum provided with the other source code matches the theoretical predictions closely, supporting similar insignificant deviations due to rounding.

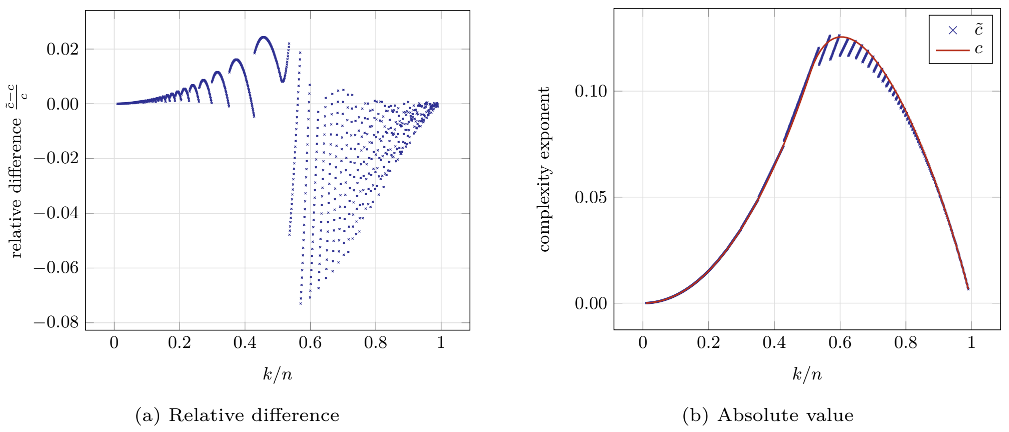


Fig. 7: Comparison of complexity exponent c obtained via Theorem 4.1 vs. $\tilde{c}$ incorporating rounding issues

5 Concrete Complexity of Regular-ISD

We now consider the algorithms introduced in the previous section and derive their concrete time complexity. For all algorithms, the time complexity is of the form T_{it}/q, where T_{it} is the cost per iteration and q is the success probability. Notice that closed form expressions for q have already been provided in Section 4. In order to derive estimates for the concrete cost per iteration, we consider that each Gaussian elimination (as well as Partial Gaussian elimination) takes time $(n-k')^2 n$ and that merging two lists L_1 and L_2 on an ℓ bit constraint takes time $|L_1| + |L_2| + \frac{|L_1 \times L_2|}{2^\ell}$. Further, we assume that all the auxiliary operations, as computing the lists elements or checking the weight after a merge, come at a computational overhead of n.[6] Taking these considerations into account, deriving concrete expressions for T_{it} becomes a simple exercise. For the sake of completeness, we report the full expressions in Appendix A.

We do not consider Rep-MO and CCJ-MO in the concrete comparison, since the May-Ozerov nearest-neighbor routine is known to inherit large polynomial overheads [MO15, EKZ21].

Bit-Security Estimates for RSD Parameters In the following we provide a discussion on the performance of the different algorithms on the parameters suggested in the literature. We compare CCJ [CCJ23a], the recently proposed algebraic attacks by Briaud and Øygarden [BØ23], as well as classical ISD attacks not tailored to the regular case against the regular-ISD approach from Section 4. Note that Briaud and Øygarden do not provide an estimation script to estimate arbitrary instances. We therefore include their estimates only for those parameters provided in their original work. For the estimation of classical ISD attacks, if not provided in the work suggesting the parameters, we rely on the *CryptographicEstimators*

[6] More precise estimates would make assumptions on the specific implementation of the algorithm and make the analysis more involved for an insignificant difference.

(n,k,w)	previous approaches			regular-ISD		
	ISD [LWYY22]	Algebraic [BØ23]	CCJ [CCJ23a]	Perm	Enum	Rep
				Section 4		
$(2^{10},652,106)$	164	146	129	133	115	**113**
$(2^{12},1589,172)$	135	136	160	131	110	**109**
$(2^{14},3482,338)$	135	140	204	140	**118**	**118**
$(2^{16},7391,667)$	139	141	249	149	**126**	127
$(2^{18},15336,1312)$	144	**123**	274	150	126	128
$(2^{20},32771,2467)$	148	**126**	335	164	138	141
$(2^{22},64770,4788)$	149	**103**	360	165	140	143
$(2^{10},652,57)$	94	101	90	94	77	**76**
$(2^{12},1589,98)$	86	103	115	96	**78**	**78**
$(2^{14},3482,198)$	91	106	143	103	**84**	85
$(2^{16},7391,389)$	96	108	171	110	**90**	92
$(2^{18},15336,760)$	101	104	192	114	**93**	97
$(2^{20},32771,1419)$	106	**98**	216	119	**98**	103
$(2^{22},64770,2735)$	108	**103**	233	123	**103**	108

Table 3: Comparison of RSD solvers on large weight (top) and small weight (bottom) instances from [LWYY22].

library [EVZB23], which incorporates an extension of the *syndrome decoding estimator* by Esser and Bellini [EB22].

The literature suggests a big variety of different parameters. While we provide a discussion about all of them we display in the tables sometimes only a selection of them for clarity. However, we provide the estimates of all suggested parameters together with our source code at `github.com/Memphisd/Regular-ISD`.

For the concrete parameters, whenever the code length n is not a multiple of w, the solution is known to be of the form $\mathbf{e} = (\mathbf{e}', 0^{n-w\cdot b})$ for $b = \lfloor n/w \rfloor$, where $\mathbf{e}'$ is a regular vector of length $w \cdot b$ and weight w. The known zeros, as well as the corresponding columns of the parity check-matrix can be safely discarded. This affects both the code length (which becomes wb) and dimension (which becomes $wb - (n-k)$), hence, the co-dimension remains unchanged with $n-k$.

Parameters Suggested in [LWYY22] In Table 3 we provide the estimated bit complexity of the different approaches on parameters suggested in the context of PCG constructions. Notice that the code parameters on the top and bottom of the table are the same, but the values of w are different; therefore, we distinguish between *large weight* (top) and *small weight* (bottom) instances.

We observe that, for almost all instances, regular-ISD either outperforms previous approaches or obtains a similar running time. The only instances for which this is not the case are the very low rate instances with higher weight (bottom of top half), for which the algebraic attack obtains the best performance. In general, regular-ISD obtains the highest gains if rate and relative weight are moderately large (see first rows of top and bottom halves). In case rate and relative weight decrease, we find that the performance of regular-ISD and standard ISD get closer (see last rows of top and bottom halves). This effect can be especially

source	(n,k,w)	previous approaches ISD [EB22]	CCJ [CCJ23a]	regular-ISD Section 4 Perm	Enum	Rep
[HOSS18, Table 1] (GMW-style)	$(245760, 245460, 15)$	127	**124**	179	127	140
	$(40960, 40660, 20)$	**126**	**126**	172	**126**	129
	$(7680, 7380, 30)$	**127**	131	166	132	132
	$(1280, 860, 80)$	134	132	137	117	**114**
[YWL$^+$20, Table 5]	$(609728, 36288, 1269)$	147	343	164	**140**	143
	$(10805248, 589760, 1319)$	**155**	492	176	157	164
[CCJ23a, pp. 560]	$(1842, 825, 307)$	289	193	178	156	**153**

Table 4: Comparison of different approaches on selected instances from the literature.

observed in the bottom half of Table 3 which includes the *small weight* instances. Recall, that we have shown in Section 3.4 that the complexities of Perm and general ISD algorithms converge for small relative weight. The bit-security results seem to further indicate that, as in the SD case [TS16], enumeration improvements become less effective for small weight, since also the complexities of Enum and Rep converge to those of ISD.

Parameters Suggested in [HOSS18] The authors of that work propose 32 different parameter sets for two different MPC protocols, all targeting 128-bit security.

The first twelve sets relate to the first MPC protocol (labeled GMW-style in [HOSS18]). Among those are seven parameters with high rate $\kappa \geq 0.96$ and small relative weight $\omega \leq 0.004$. For those regular-ISD is roughly on par with either CCJ or standard ISD. However, we find that some of those parameter sets even when considering only ISD do only roughly match the claimed security, which we show in Table 4. For the remaining four parameter sets with high rate we find huge margins of more than 110 bits. The parameters with rate $\kappa < 0.96$ use larger (but still small) weight $\omega < 0.016$. For those sets regular-ISD obtains slight improvements, but still those sets enjoy a comfortable margin of at least 10 bits. A notable exception is the parameter set $(1280, 860, 80)$, which is the only set using moderate relative weight of $\omega = 80/1280 = 0.0625$. For this set regular-ISD lowers the bit-security from 132 to 115 and therefore below the security target of 128-bit.

The next twenty suggested parameters relate to the second MPC protocol, labeled BMR-style in [HOSS18]. Most of the parameters again use high rate, low weight with some exceptions. Especially, for those exceptions regular-ISD improves on previous techniques, lowering estimates by up to 11 bits. However, we find that still all sets but two incorporate huge margins of 60 to 270 bits. The two outliers obtain margins of 18 and 39 bit.

Parameters Suggested in [BCG$^+$19a] The authors suggest in that work sixteen different parameter sets for a correlated OT application, eight aiming at 80-bit security and eight aiming for 128-bit. All parameter sets follow rate $\kappa \in \{\frac{1}{2}, \frac{3}{4}\}$, in combination with very low relative weight $\omega \leq 0.0063$. Due to the low relative weights, we find that regular-ISD and ISD algorithms perform roughly on par. All parameters satisfy the security goals, while for 80-bit instances we find margins between 8 to 29 bits and for 128-bit instances we find margins ranging from 10 to 42 bits.

Parameters Suggested in [YWL+20] This work uses two parameter sets within a correlated OT construction. For the set used within the setup, regular-ISD lowers the bit-security estimate from 147 to 140 bit, while for the second parameter set classic ISD leads to a security estimate of 155 bit as shown in Table 4.

Parameters Suggested in [CCJ23a] The authors use a single parameter set for their MPC-in-the-Head based signature construction. The regular-ISD approach lowers the bit-security estimate notably by 40 bits from 193 to 153 bits (see Table 4). Originally, this parameter set aims at 128-bit of security. However, the authors followed a conservative parameter selection in which they considered the running time of CCJ-MO assuming the nearest neighbor search routine comes at no additional cost (corresponding to $N = L$ in Eq. (18)). Note that while here this led to conservative parameters, in general this is not guaranteed. We find several parameters for which regular-ISD significantly outperforms such an "optimistic" version of Rep-MO.

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A Concrete Time Complexity Formulas

In this section we provide for completeness the concrete runtime formulas for the regular-ISD algorithms Perm, Enum and Rep, as well as the formula for CCJ.

Permutation-based Regular-ISD The most time consuming operation of one iteration of Algorithm 1 is the Gaussian elimination. All the other operations (e.g., applying the permutation and checking the weight of $\mathbf{e}_1$) are less costly and can, hence, be neglected, so that $T_{\text{it}} = n(n-k')^2$.

Enumeration-based Regular ISD In each iteration, the algorithm performs one partial Gaussian elimination and creates two lists with size $|L_1| = |L_2| = \binom{w/2}{p/2} v^{p/2}$, with $v := \frac{k'+\ell}{2}$ and $k' = k - w$. The lists are then merged into L, with average size $|L| = |L_1|^2 2^{-\ell}$. Putting everything together, we have that the cost of each iteration is

$$\begin{aligned} T_{\text{it}} &= n\left((n-k')^2 + \left(|L_1| + |L_2| + |L|\right)\right) \\ &= n\left((n-k')^2 + \cdot \binom{w/2}{p/2} v^{p/2} \cdot \left(2 + \binom{w/2}{p/2} v^{p/2} 2^{-\ell}\right)\right). \end{aligned}$$

The optimal time complexity is achieved when $|L| \approx |L_1|$, which implies

$$\ell \approx \log |L_1| = \log_2\left(\binom{w/2}{p/2}\right) + \frac{p}{2}\log_2\left(\frac{k'+\ell}{2}\right).$$

Representation-based ISD Again we have $v = \frac{k'+\ell}{w}$. For each level, we recall the number of lists and their average sizes.

- *Creating the initial lists*: the algorithm starts with 8 lists L_i, each with the same size
$$|L_1| = \binom{w/2}{p_{\mathbf{y}}/2} v^{p_{\mathbf{y}}/2}.$$
- *Obtaining $L_{\mathbf{y}_i}$, $i = 1, \cdots, 4$*: each list $L_{\mathbf{y}_i}$ is obtained as the merge of two of the initial lists, searching for collisions in $\ell_{\mathbf{y}}$ coordinates. The average number of collisions, which also corresponds to the average size of each $L_{\mathbf{y}_i}$, is
$$|L_{\mathbf{y}_i}| = |L_1|^2 2^{-\ell_{\mathbf{y}}} = \binom{w/2}{p_{\mathbf{y}}/2}^2 v^{p_{\mathbf{y}}} 2^{-\ell_{\mathbf{y}}}.$$
- *Obtaining $L_{\mathbf{x}_i}$, $i = 1, 2$*: each of these lists is the merge of a pair of lists $L_{\mathbf{y}_i}$. The average number of collisions is
$$N_{\mathbf{y}} = |L_{\mathbf{y}_1}|^2 2^{-(\ell_{\mathbf{x}} - \ell_{\mathbf{y}})} = |L_1|^4 2^{-2\ell_{\mathbf{y}}} 2^{-(\ell_{\mathbf{x}} - \ell_{\mathbf{y}})} = \binom{w/2}{p_{\mathbf{y}}/2}^4 v^{2p_{\mathbf{y}}} 2^{-(\ell_{\mathbf{x}} + \ell_{\mathbf{y}})}.$$
After collisions are filtered for the desired weight $p_{\mathbf{x}}$, the expected number of elements in each list $L_{\mathbf{x}_i}$ is
$$|L_{\mathbf{x}_i}| = \left(\binom{w/2}{p_{\mathbf{x}}/2} v^{p_{\mathbf{x}}/2}\right)^2 \cdot 2^{-\ell_{\mathbf{x}}} = \binom{w/2}{p_{\mathbf{x}}/2}^2 v^{p_{\mathbf{x}}} \cdot 2^{-\ell_{\mathbf{x}}}.$$
Here we take into account that only elements balanced elements, i.e., elements with $p_{\mathbf{x}}/2$ non-zero blocks on the first and second half of their coordinates can be constructed. This fact was previously disregarded as it is subsumed in the Landau notation.
- *Obtaining $L_{\mathbf{e}_J}$*: the expected number of produced collisions is
$$N_{\mathbf{x}} = |L_{\mathbf{x}_i}|^2 2^{-(\ell - \ell_{\mathbf{x}})} = \binom{w/2}{p_{\mathbf{x}}/2}^4 v^{2p_{\mathbf{x}}} 2^{-(\ell + \ell_{\mathbf{x}})}.$$

Putting everything together, we get
$$T_{\text{it}} = n(n-k')^2 + n \cdot \big(8|L_1| + 4|L_{\mathbf{y}_1}| + 2N_{\mathbf{y}} + 2|L_{\mathbf{x}_1}| + N_{\mathbf{x}}\big).$$
Substituting each of the above terms, we get
$$T_{\text{it}} = n(n-k')^2 + n \cdot \Bigg(8\binom{w/2}{p_{\mathbf{y}}/2} v^{p_{\mathbf{y}}/2} + 4\binom{w/2}{p_{\mathbf{y}}/2}^2 v^{p_{\mathbf{y}}} 2^{-\ell_{\mathbf{y}}} + 2\binom{w/2}{p_{\mathbf{y}}/2}^4 v^{2p_{\mathbf{y}}} 2^{-(\ell_{\mathbf{x}} + \ell_{\mathbf{y}})}$$
$$+ 2\binom{w/2}{p_{\mathbf{x}}/2}^2 2^{-\ell_{\mathbf{x}}} + \binom{w/2}{p_{\mathbf{x}}/2}^4 v^{2p_{\mathbf{x}}} 2^{-(\ell + \ell_{\mathbf{x}})}\Bigg)$$

Recall that $p_{\mathbf{x}} = p/2 + \varepsilon_{\mathbf{x}}$ and $p_{\mathbf{y}} = p_{\mathbf{x}}/2 + \varepsilon_{\mathbf{y}}$. Furthermore, $\ell_{\mathbf{x}} \approx \log R_{\mathbf{x}}$ and $\ell_{\mathbf{y}} \approx \log R_{\mathbf{y}}$, where[7]
$$R_{\mathbf{x}} = \binom{p/2}{p/4}^2 \binom{(w-p)/2}{\varepsilon_{\mathbf{x}}/2}^2 v^{\varepsilon_{\mathbf{x}}} \quad \text{and} \quad R_{\mathbf{y}} = \binom{p_{\mathbf{x}}/2}{p_{\mathbf{x}}/4}^2 \binom{(w-p_{\mathbf{x}})/2}{\varepsilon_{\mathbf{y}}/2}^2 v^{\varepsilon_{\mathbf{y}}}.$$

[7] Here we again account for the fact that we can construct only balanced elements, due to the meet-in-the-middle enumeration.

CCJ algorithm In this algorithm, lists have size $|L_1| = |L_2| = \left(\frac{n}{w}\right)^{\frac{w(\tilde{k}+\ell)}{2n}}$, where $\tilde{k} = k - \left(1 - \frac{w}{n}\right)^{-1} \left(1 - \frac{k+\ell}{n}\right) w$. The average number of collisions is $|L| = |L_1| \cdot |L_2| \cdot 2^{-\ell} = \left(\frac{n}{w}\right)^{\frac{w(\tilde{k}+\ell)}{n}} 2^{-\ell}$, so that the cost of one iteration is

$$T_{\text{it}} = n(n - \tilde{k})^2 + n \cdot \left(\frac{n}{w}\right)^{\frac{w(\tilde{k}+\ell)}{2n}} \cdot \left(2 + \left(\frac{n}{w}\right)^{\frac{w(\tilde{k}+\ell)}{2n}} 2^{-\ell}\right).$$