The One-Wayness of Jacobi Signatures

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Abstract. In this short note, we show that under a mild number-theoretic conjecture, recovering an integer from its Jacobi signature modulo $N = p^2q$, for primes p and q, is as hard as factoring N.

1 Introduction

In 1988, Damgård [5] proposed a pair of cryptographic pseudorandom generators, based on quadratic characters. For a fixed natural number N, he speculated that the function that maps $x \in \mathbb{Z}_N^*$ to the sequence of Jacobi symbols

$$\left[\left(\frac{x+1}{N}\right), \left(\frac{x+2}{N}\right), \ldots, \left(\frac{x+\ell}{N}\right) \right] \quad \in \{-1, 1\}^\ell,$$

for some $\ell \in \mathbb{N}$, is a pseudorandom generator. Following prior work [4], we refer to this sequence of Jacobi symbols as the *length-ℓ Jacobi signature of x modulo N*. Damgård also considered the case when the modulus is a prime p; in that case we replace Jacobi symbols with Legendre symbols and refer to the sequence as the *Legendre signature of x modulo p*.

He left as an open question whether is is possible to relate the task of breaking these pseudorandom generators to any other number-theoretic problem.

This work. In this short note, we consider Damgård's pseudorandom generator based on Jacobi symbols modulo $N = p^2q$, for primes p and q. We show that this function is a one-way function if:

- factoring integers of the form p^2q is hard, and
- if every number modulo p has a unique Legendre signature of length $\log^2(p)$.

Under a much stronger (and less plausible) number-theoretic assumption, we can show that finding collisions in Damgård's Jacobi pseudorandom generator is as hard as factoring.

Both results are based on the simple observation that Jacobi symbol of x modulo $N = p^2q$ is equal to the Legendre symbol of x modulo q. Thus, if we give an attacker the Jacobi signature of a secret value x modulo N, we reveal no information to the attacker about the Legendre signature of x modulo p.

If the attacker succeeds at inverting the Jacobi-signature function modulo N, we then get a value $x' \in \mathbb{Z}_N^*$ such that x and x' have the same Legendre signature modulo q. Under a standard number-theoretic conjecture on the uniqueness of Legendre signatures [4], this implies that $x = x' \bmod q$. At the same time,

since the attacker has no information about $x \bmod p^2$, it is extremely likely that $x \neq x' \bmod p^2$. In this case, the the greatest common divisor of $x - x'$ and the modulus N will yield a non-trivial factor of N.

Related work. Peralta and Okamoto [12] use Jacobi signatures modulo $N = p^2q$ to speed up the elliptic-curve factoring algorithm. In particular, they use Jacobi signatures modulo N to quickly search a list of integers $x_1, x_2, \ldots, x_k \in \mathbb{Z}_N^*$ for a pair whose difference has a non-trivial greatest common divisor with N. Several cryptosystems have also based their security on the hardness of factoring moduli of the form p^2q [7, 11].

Adleman and McCurley [1] discuss the problem of finding the smallest prime q whose Legendre symbols modulo the first ℓ primes matches a prescribed pattern in $\{-1, 1\}^\ell$. Solving this problem, they note, is as hard as factoring numbers of the form $N = p^2q$, provided that the signature length ℓ is long enough to uniquely identify the prime q. Adleman and McCurley's problem becomes easy if we ask only for some prime q (and not the smallest) that matches the given Legendre pattern.

Grassi et al. [9] propose using a variant of Damgård's construction as a pseudorandom function. For a fixed prime p, key $k \in \mathbb{Z}_p^*$, and input $x \in \mathbb{Z}_p^*$, the function's output is the Legendre symbol of $(k + x)$ modulo p. This function has a small arithmetic circuit over $\mathbb{F}_p$, which makes it useful in multiparty computation [2, 6, 9]. Several recent works have also studied the concrete hardness of the Legendre pseudorandom function [3, 10].

2 Preliminaries

Throughout this work, we write $\lambda \in \mathbb{N}$ to denote a security parameter. We say that an algorithm is efficient if it runs in probabilistic polynomial time in the length of its input. We say that a function $f(\lambda)$ is negligible if $f = o(\lambda^{-c})$ for all constants $c \in \mathbb{N}$; we denote this by writing $f = \mathrm{negl}(\lambda)$. To denote the greatest common divisor of natural numbers x and y, we write $\gcd(x, y)$. For a natural number λ, we let Primes_λ denote the set of λ-bit primes.

2.1 Legendre and Jacobi Signatures

We now recall the concept of a Legendre signature and a Jacobi signature.

Definition 2.1 (Jacobi and Legendre Signatures). For an integer N and $x \in \mathbb{Z}_N^*$, let $\left(\frac{x}{N}\right) \in \{-1, 1\}$ denote the Jacobi symbol of x modulo N. Then, for a positive integer N and signature length ℓ, we define the Jacobi-signature function $J_{N,\ell} \colon \mathbb{Z}_N^* \to \{-1, 1\}^\ell$ as the function

$$J_{N,\ell}(x) := \left[\left(\frac{x+1}{N}\right), \left(\frac{x+2}{N}\right), \ldots, \left(\frac{x+\ell}{N}\right) \right] \in \{-1, 1\}^\ell.$$

When p is a prime, we refer to the function $J_{p,\ell}$ as the "Legendre signature."

Fact 2.2 (Jacobi Signatures with $N = p^2q$). *For odd primes p, q and $N = p^2q$, for all $x \in \mathbb{Z}_N^*$ and $\ell \in \mathbb{Z}$, $J_{N,\ell}(x) = J_{q,\ell}(x)$.*

Proof. The statement follows because the Jacobi symbol is multiplicative and takes on values in $\{-1, 1\}$:

$$\left(\frac{x}{N}\right) = \left(\frac{x}{p}\right)^2 \left(\frac{x}{q}\right) = \left(\frac{x}{q}\right). \qquad \square$$

2.2 Standard Cryptographic Definitions

We recall a few standard cryptographic definitons.

Definition 2.3 (One-Way Function). For a family of functions $\mathcal{F} = \{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{N}}$, where each function $f \in \mathcal{F}_\lambda$ has the type $f\colon \mathcal{X}_\lambda \to \mathcal{Y}_\lambda$, define the *advantage of an algorithm $\mathcal{A}$ at breaking the one-wayness of $\mathcal{F}$* as:

$$\mathsf{OWFAdv}[\mathcal{A}, \mathcal{F}](\lambda) := \Pr\left[f(x) = f(x') : \begin{array}{c} f \xleftarrow{R} \mathcal{F}_\lambda, x \xleftarrow{R} \mathcal{X}_\lambda \\ x' \leftarrow \mathcal{A}(f, f(x)) \end{array}\right]$$

Definition 2.4 (Collision Resistance). For a family of functions $\mathcal{F} = \{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{N}}$, where each function $f \in \mathcal{F}_\lambda$ has the type $f\colon \mathcal{X}_\lambda \to \mathcal{Y}_\lambda$, define the *advantage of an algorithm $\mathcal{A}$ at breaking the collision resistance of $\mathcal{F}$* as:

$$\mathsf{CRHFAdv}[\mathcal{A}, \mathcal{F}](\lambda) := \Pr\left[f(x) = f(x') \text{ and } x \neq x' : \begin{array}{c} f \xleftarrow{R} \mathcal{F}_\lambda \\ (x, x') \leftarrow \mathcal{A}(f) \end{array}\right]$$

Definition 2.5 (Factoring $N = p^2q$). We define the advantage of an algorithm $\mathcal{A}$ at factoring integers of the form p^2q, for primes p and q, as

$$\mathsf{FactAdv}[\mathcal{A}](\lambda) := \Pr\left[1 < \gcd(t, N) < N : \begin{array}{c} p, q \xleftarrow{R} \mathsf{Primes}_\lambda. \\ t \leftarrow \mathcal{A}(p^2q) \end{array}\right]$$

3 One-Wayness of Jacobi Signatures

Our first result relies on a conjecture of Boneh and Lipton [4], which states that, for a fixed prime p, each value in $\mathbb{Z}_p^*$ has a unique Legendre signature of length $\lceil 2\log^2 p\rceil$:

Conjecture 3.1 (Boneh and Lipton [4]). *For all sufficiently large primes p, for all distinct $x, x' \in \mathbb{Z}_p^*$, and for $\ell = \lceil 2\log^2 p\rceil$, it holds that $J_{p,\ell}(x) \neq J_{p,\ell}(x')$.*

Our results also hold under a weaker conjecture, where the signature length is $\ell = \log^c(p)$, for any $c > 2$.

Under Conjecture 3.1, we can show that inverting the Jacobi-signature function modulo an integer $N = p^2q$, for primes p and q, is as hard as hard as factoring N, provided that the Jacobi-signature length is at least $\lceil 2\log^2 N\rceil$. Specifically, we define $\mathcal{J}_\lambda^{\mathsf{OWF}}$ to be

$$\mathcal{J}_\lambda^{\mathsf{OWF}} = \{J_{N,2\lambda^2} \mid p, q \xleftarrow{R} \mathsf{Primes}_\lambda; N \leftarrow p^2 \cdot q\}.$$

We then have:

Proposition 3.2 (One-Wayness of Jacobi Signatures). *Under Conjecture 3.1, for every efficient algorithm $\mathcal{A}$ that breaks the one-wayness of $\mathcal{J}^{\mathsf{OWF}} = \{\mathcal{J}_\lambda^{\mathsf{OWF}}\}_{\lambda\in\mathbb{N}}$ with advantage $\mathsf{OWFAdv}[\mathcal{A}, \mathcal{J}^{\mathsf{OWF}}](\lambda)$, there is an efficient algorithm $\mathcal{B}$ for factoring integers of the form p^2q, for primes p and q, with advantage $\mathsf{FactAdv}[\mathcal{B}](\lambda)$ where*

$$\mathsf{OWFAdv}[\mathcal{A}, \mathcal{J}^{\mathsf{OWF}}](\lambda) \leq \mathsf{FactAdv}[\mathcal{B}](\lambda) + \mathrm{negl}(\lambda).$$

Proof. Suppose there exists an efficient adversary $\mathcal{A}$ that breaks one-wayness of $\mathcal{J}^{\mathsf{OWF}}$ with advantage $\varepsilon = \mathsf{OWFAdv}[\mathcal{A}, \mathcal{J}^{\mathsf{OWF}}](\lambda)$. We construct an algorithm $\mathcal{B}$ for factoring integers of the form p^2q as follows:

- On input the modulus N, Algorithm $\mathcal{B}$ samples $x \xleftarrow{\mathrm{R}} \mathbb{Z}_N$ and computes $t = \gcd(x, N)$. If $t \neq 1$, then Algorithm $\mathcal{B}$ outputs t.
- If $\gcd(x, N) = 1$, then $x \in \mathbb{Z}_N^*$, so Algorithm $\mathcal{B}$ runs $x' \leftarrow \mathcal{A}(J_{N,\ell}, J_{N,\ell}(x))$ where $\ell = 2\lambda^2$ is the signature length.
- Algorithm $\mathcal{B}$ computes $t = \gcd(N, x - x')$.

To complete the proof, we analyze the advantage of algorithm $\mathcal{B}$:

- By definition, the challenger samples $N = p^2q$, where p and q are odd primes.
- Consider the initial value x that Algorithm $\mathcal{B}$ samples. If $\gcd(x, N) \neq 1$, then Algorithm $\mathcal{B}$ successfully factored N. If $\gcd(x, N) = 1$, then the distribution of x is uniform over $\mathbb{Z}_N^*$. By assumption, with probability at least ε, Algorithm $\mathcal{A}$ then outputs x' such that $J_{N,\ell}(x') = J_{N,\ell}(x)$.
- By Fact 2.2, $J_{N,\ell}(x') = J_{q,\ell}(x') = J_{q,\ell}(x) = J_{N,\ell}(x)$. By Conjecture 3.1, this means $x = x' \bmod q$.
- Next, consider the view of adversary $\mathcal{A}$. Again by Fact 2.2,

  $$J_{N,\ell}(x) = J_{q,\ell}(x) = J_{q,\ell}(x \bmod q).$$

  Since $J_{N,\ell}(x)$ is only a function of $x \bmod q$, we conclude via the Chinese Remainder Theorem that $J_{N,\ell}(x)$ information-theoretically hides the value of $x \bmod p^2$. This means the value of $x' \bmod p^2$ that Algorithm $\mathcal{B}$ chooses is independent of $x \bmod p^2$. Moreover, since the distribution of x is uniform over $\mathbb{Z}_N^*$, the value of $x \bmod p^2$ is uniform over $\mathbb{Z}_{p^2}^*$. Thus,

  $$\Pr[x = x' \bmod p^2] = \frac{1}{|\mathbb{Z}_{p^2}^*|} = \frac{1}{p(p-1)} = \mathrm{negl}(\lambda).$$

  Thus, with probability $1 - \mathrm{negl}(\lambda)$, it holds that $x \neq x' \bmod p^2$. If $x = x' \bmod q$ and $x \neq x' \bmod p^2$, then it follows that $\gcd(x - x', N) \in \{q, pq\}$ so algorithm $\mathcal{B}$ produces a non-trivial factor of N.

We conclude that algorithm $\mathcal{B}$ succeeds in factoring N with probability

$$\mathsf{FactAdv}[\mathcal{B}](\lambda) \geq \varepsilon - \mathrm{negl}(\lambda) = \mathsf{OWFAdv}[\mathcal{A}, \mathcal{J}_\lambda^{\mathsf{OWF}}](\lambda) - \mathrm{negl}(\lambda). \qquad \square$$

4 Collision Resistance of Jacobi Signatures

In this section, we show that if:

– factoring numbers of the form $N = p^2 q$, for primes p and q, is hard, and
– there exists a constant $k \in (2, 3)$ such that for most primes p, all Legendre signatures of length $\lceil k \log p \rceil$ are unique

then the Jacobi-signature function modulo N is collision resistant when the signature length is $\lceil \frac{k}{3} \log N \rceil$.

More precisely, our argument for collision resistance relies on the following number-theoretic assumption:

Assumption 4.1. *There exists a constant $k \in (2, 3)$ such that for a random λ-bit prime p, for all distinct $x, x' \in \mathbb{Z}_p^*$, and for $\ell = \lceil k \log p \rceil$, it holds that $J_{p,\ell}(x) \neq J_{p,\ell}(x')$, except with probability negligible in λ. More formally, we assume that for $\ell = \lceil k \log p \rceil$, there exists a negligible function* $\mathrm{negl}(\cdot)$ *such that for all $\lambda \in \mathbb{N}$,*

$$\Pr[\exists x \neq x' : J_{p,\ell}(x) = J_{p,\ell}(x') \mid p \leftarrow \mathsf{Primes}_\lambda] = \mathrm{negl}(\lambda).$$

This assumption differs from Conjecture 3.1 in two ways. In particular,

1. this assumption considers Legendre signatures of length $O(\log p)$ whereas Conjecture 3.1 considers Legendre signatures of length $\Omega(\log^2 p)$, and
2. this assumption is a statement about *a large fraction* of primes p, whereas Conjecture 3.1 is a statement about *all* large enough primes p.

We need the first modification since for the Jacobi-signature function $J_{N,\ell}$ to be compressing, the signature length ℓ must satisfy $\ell < \log N$. When $N = p^2 q$, this requires $k < 3$. For our argument to go through, we must argue about relatively *short* Legendre signatures. We consider values $k > 2$ to evade the birthday bound. Specifically, for a prime p, if we *heuristically* model the Jacobi signatures $J_{p,\ell}(x)$ for each $x \in \mathbb{Z}_p^*$ as uniform random strings drawn from $\{-1, 1\}^\ell$, then by the birthday bound, with constant probability, there will exist two distinct $x, x' \in \mathbb{Z}_p^*$ with a common Jacobi signature. However, if we consider signatures of length $\ell = (2+\varepsilon)\lceil \log p \rceil$ for any constant $\varepsilon > 0$ and again heuristically modeling the Jacobi signatures as uniform random strings, then the probability that there exist $x \neq x'$ with the same Jacobi signature is at most $p^2/p^{2+\varepsilon} = 1/p^\varepsilon = \mathrm{negl}(\lambda)$.

The second modification is also necessary, since the conclusion of the assumption does not hold for all primes p. That is, there are infinitely many primes p for which there exist pairs $x, x' \in \mathbb{Z}_p^*$ whose Legendre signatures of length $\lceil 100 \log p \rceil$ are identical. This follows from the fact that there are infinitely many primes p for which the least quadratic non-residue is $\Omega(\log p \log\log\log p)$ [8]. For such primes p, the Legendre signatures of the elements "1" and "2" will be identical, provided that the signature length is $O(\log p)$.

It is not at all obvious to us that Assumption 4.1 is true. That said, assumptions used in the cryptanalysis of the Legendre-signature-based cryptosystems [3] imply Assumption 4.1.

Collision resistant hash function from Jacobi signatures. We now give the main result of this section. Let $k \in (2, 3)$ be the constant from Assumption 4.1. On security parameter λ, let

$$\mathcal{J}_\lambda^{\mathsf{CRHF}} = \{J_{N,k\lambda} \mid p, q \xleftarrow{\mathrm{R}} \mathsf{Primes}_\lambda;\ N \leftarrow p^2 \cdot q\}$$

be the family of Jacobi-signature functions defined on number of the form $N = p^2q$. Notice that on modulus N, the signature length is $k\lambda = \lceil \frac{k}{3} \log N \rceil$. For this signature length, the Jacobi-signature function is compressing.

Claim 4.2 (Collision Resistance of Jacobi Signatures). *Under Assumption 4.1, for every efficient algorithm $\mathcal{A}$ that breaks the collision-resistance of the family of Jacobi-signature functions $\mathcal{J}^{\mathsf{CRHF}} = \{\mathcal{J}_\lambda^{\mathsf{CRHF}}\}_{\lambda\in\mathbb{N}}$ with advantage $\mathsf{CRHFAdv}[\mathcal{A}, \mathcal{J}^{\mathsf{CRHF}}](\lambda)$, there is an algorithm $\mathcal{B}$ for factoring integers of the form p^2q, for primes p and q, that achieves advantage $\mathsf{FactAdv}[\mathcal{B}](\lambda)$ where*

$$\mathsf{CRHFAdv}[\mathcal{A}, \mathcal{J}^{\mathsf{CRHF}}](\lambda) \leq \mathsf{FactAdv}[\mathcal{B}](\lambda) + \mathrm{negl}(\lambda).$$

Proof. Suppose there exists an efficient adversary $\mathcal{A}$ that breaks collision resistance of $\mathcal{J}^{\mathsf{CRHF}}$ with advantage $\varepsilon = \mathsf{CRHFAdv}[\mathcal{A}, \mathcal{J}^{\mathsf{CRHF}}](\lambda)$. We use Algorithm $\mathcal{A}$ to construct Algorithm $\mathcal{B}$ of the claim. Algorithm $\mathcal{B}$, on input $N = p^2q$, runs the collision finder $(x, x') \leftarrow \mathcal{A}(J_{N,\ell})$ where $\ell = k\lambda$, and outputs $\gcd(N, x - x')$. We analyze Algorithm $\mathcal{B}$'s advantage:

- Whenever Algorithm $\mathcal{A}$ outputs a valid collision in $J_{N,\ell}$, we have $J_{N,\ell}(x) = J_{N,\ell}(x')$ and $x \neq x' \bmod N$.
- Since N is of the form p^2q, by Fact 2.2, a collision in the Jacobi signature modulo N implies a collision in the Legendre signature modulo q: $J_{q,\ell}(x) = J_{q,\ell}(x')$.
- By Assumption 4.1, if $J_{q,\ell}(x) = J_{q,\ell}(x')$, then

  $$x = x' \bmod q \quad \Longrightarrow \quad (x - x') = 0 \bmod q,$$

  except with probability negligible in λ.
- However, since $x \neq x' \bmod N$, it must be that

  $$x \neq x' \bmod p^2 \quad \Longrightarrow \quad (x - x') \neq 0 \bmod p^2.$$

  Therefore $(x - x')$ is a multiple of q and not a multiple of p^2. This means $\gcd(x - x', N) \in \{q, pq\}$, and Algorithm $\mathcal{B}$ obtains a factor of N with advantage

  $$\mathsf{FactAdv}[\mathcal{B}](\lambda) \geq \varepsilon - \mathrm{negl}(\lambda) = \mathsf{CRHFAdv}[\mathcal{A}, \mathcal{J}^{\mathsf{CRHF}}] - \mathrm{negl}(\lambda). \qquad \square$$

5 Open Problems

This note shows a new connection between the hardness of inverting Jacobi sequences and factoring. One potential next step would be to show that *distinguishing* a Jacobi sequence from random is as hard as a more traditional number-theoretic problem (e.g., quadratic residuosity). Another question is whether it is possible to remove our results' reliance on number-theoretic conjectures, or to show hardness under the assumption that factoring integers of the form $p \cdot q$, for primes p and q, is intractable.

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