Input Transformation Based Zero-Knowledge Argument System for Arbitrary Circuits with Practical Efficiency

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Abstract. We introduce a new efficient transparent interactive zero-knowledge argument system that is based on the new input transformation concept which we will introduce in this paper. The core of this concept is a mechanism that converts input parameters into a format that can be processed directly by the circuit so that the circuit output can be verified through direct computation of the circuit or business logic per se. In our protocol, we convert circuit inputs in Pedersen commitment form to linear polynomials in integer form so the verifiers can use standard integer operations to compute and verify the circuit output.
This direct computation mechanism replaces the constraint system often found in popular zero-knowledge protocols, and eliminates the need of using a front-end encoder to translate NP relation R to some zero-knowledge friendly representation $\hat{R}$ (such as the R1CS constraint system) before the relation can be converted to a proof system, making our protocol easy to implement and much easier to use compared to protocols using a constraint system.
Specifically, when running a circuit of 2^{20} sequential multiplication gates with 960 input bits (30 input integers) on a single CPU thread, the prover runtime of our protocol is 5.5 seconds, the verifier runtime is 26 milliseconds, and the communication cost is approximately 64 kilobytes. This result shows a significant improvement in verifier runtime by more than one order of magnitude over the state of the art while keeping the prover runtime and communication cost competitive with the state of the art.

1 Introduction

Ever since the discoveries of interactive proofs (IPs) [20] and probabilistically checkable proofs (PCPs) [5] [4] [3] [2] in the late last century, there has been a tremendous amount of research in the area of proof systems. More recently, the rise of Blockchain and Web3 has finally triggered real-world deployments of zero knowledge systems.

Popular zero knowledge systems are often divided into two phases: the first part, a "front-end" encoder converting a specification of an NP-relation R into a "zero-knowledge friendly" representation $\hat{R}$ (e.g. rank-1 constraint system); and then another "back-end" system converting $\hat{R}$ to a zero-knowledge proof system for R. The encoder based two-phased design has accelerated the development of zero knowledge system applications, but it has added cost of running the encoder to translate circuit logic to constraint system form.

Due to expensive computation during setup time of earlier SNARKs (Succinct Non-Interactive Argument of Knowledge), it has become a significant interest to have the structured reference string (SRS) be constructible in a "universal and updatable" fashion, meaning that the same SRS can be used for statements about all circuits of a certain bounded size. The first universal SNARK was in Groth et al [21], and Maller et al. improved the SRS size from quadratic to linear in Sonic [24]. More recently developed protocols such as PLONK [18], MARLIN [15] are universal fully-succinct SNARK with significantly improved prover runtime compared to the fully-succinct Sonic. However, there are still two drawbacks with these SNARKs: first, many of these universal succinct SNARKs systems require trusted setup; second, the prover run-time of these protocols are very expensive, which is prohibitive for many applications.

Protocols belong to the Goldwasser, Kalai, and Rothblum (GKR) class such as Hyrax [28], Virgo [33]; MPC-in-the-head class of Kushilevitz, Ostrovsky, and Sahai such as ZKBoo [19] and Ligero/Ligero++

[1] [8] offer efficient prover runtime that are at least one order of magnitude more efficient than pairing based SNARKs, and many of these protocols do not require trusted setups. However, these protocols are largely ignored by the industry (e.g. blockchain community) due to expensive verifier runtime and high communication cost (hundreds of KBs) than fully succinct protocols such as STARK [7], PLONK, MARLIN, and Supersonic [14]. Furthermore, state-of-the-art GKR protocols generally has additional dependency on circuit depth where protocol complexity increases and performance significantly degrades as the circuit depth gets longer, making them less attractive to the industry where complex business logics (e.g. inputs are floating point numbers) are expected on smart contracts.

Memory efficient privacy-free garbled circuits [23] [17] [22] and Vector Oblivious Linear Evaluation (VOLE) protocols [11] [25] [13] [12] [32] [30] [6] [31] generally offer better prover performance. However, their verifier runtimes are just as expensive as their prover runtime and cannot be trivially made non-interactive, and their communication cost is easily many orders of magnitude more expensive than other approaches.

NIZKs such as SpartanNIZK [26] and later Lakonia [27] seems to offer a much more balanced approach, where it offers efficient prover runtime (6-18 seconds single thread) and competitive communication cost for large circuits (2^{20} constraints) while not being layer dependent. However, the downside of these protocols is that their verifier performance is still expensive, usually in the 400+ ms range on a single thread CPU.

Our aim is to create a new transparent zero knowledge protocol that is extremely easy to construct for developers, while able to handle complex circuits s.t. prover runtime and communication cost comparable to that of the state-of-the-art, and that shows significant improvement in verifier runtime over that of the current state-of-the-art.

1.1 Summary of Contributions

Our approach is to design a new protocol that allows verifiers to efficiently validate circuit outputs by directly examining circuit inputs and the circuit design without going through some intermediate translation phase. We believe such approach would allow developers to code business rules exactly as they are and provide cost savings by minimizing the cost of translating business/circuit language to that of a constraint system.

In the past, Cramer and Damgård [16] first introduced a mechanism where input commitments are directly used in validating each multiplication gate. However, such an approach requires validation to be performed on each multiplication gate and therefore introduces a large communication overhead and requires both provers and verifiers to perform expensive operations in $\mathbb{G}$ for every multiplication gate. Although this approach, combined with some clever design, is adopted in some more recent protocols such as Hyrax [28], we still found such approach has some inherent inefficiencies and cannot be used to get the desired result we are seeking.

In our protocol, we start by transforming each committed input parameter to a circuit to some integer value in linear polynomial form, where verifiers can perform arithmetic operations (e.g., addition and multiplication) on them like they do on normal integers. Since field operations are cheap, the verifier can perform this step with high efficiency.

For a simple circuit ${a_1}^d + {a_2}^d + {a_3}^d = r$, circuit inputs are a_1, a_2, a_3 and the circuit output is r. In our protocol, inputs a_1, a_2, a_3 and output r are committed by the prover using Pedersen commitment. The prover then provides the transformed inputs a_1, a_2, a_3 in linear polynomial form a'_1, a'_2, a'_3 in $\mathbb{Z}_p$ s.t. $a'_1 = a_1 + X\alpha_1 \in \mathbb{Z}_p$ (α_i is its blinding key). Since the transformed inputs are in $\mathbb{F}$, the verifier can plug these values directly to the circuit to compute the output e.g. ${a_1}^d + {a_2}^d + {a_3}^d = o$. The circuit output $o \in \mathbb{Z}_p$ is a polynomial evaluated at point x s.t. $f(x) = o$. Since the degree of a polynomial will increase after it is multiplied with another polynomial, the degree of the circuit output polynomial is $d + 1$. The constant term of this polynomial is the circuit output r and all other coefficients are blinding values. If the prover can prove 1) it knows all coefficients of the output polynomial (e.g. using a polynomial commitment) 2) all input transformations are legit, then we say the proof is legit.

The output polynomial in the example above has degree of $d+1$ because transformed inputs (linear polynomials) are of degree of 1, multiplying them d times will get a polynomial with degree of $d+1$. So if the circuit is something like ${a_1}^2 + {a_2}^2 + {a_3}^2 +, ..., + {a_t}^2 = r$, the degree of the output polynomial is $3 = d + 1$ ($d = 2$) regardless of the value of t. Throughout our paper, we use symbol m_p to denote d, where m_p here represents the maximum number of multiplications included in any path that leads to the circuit output.

This input transformation and direct computation approach of our protocol does not require a "front end" encoder to compile business logic relation R into some zero-knowledge friendly representation $\hat{R}$. This construct makes our protocol relatively easy to implement and also makes it easier for end developers to apply zero knowledge design to real world applications.

For a deep circuit (m_p value is large), the prover runtime of the base version of our protocol (Protocol 1) is dominated by $O({m_p}^2 + m_p + l)$ field operations and $O(m_p + m_p^{1/2} + l)$ group exponentiations, where m_p stands for the total number of multiplication gates included in the path that contains most multiplications and l stands for the number of inputs to a circuit; the verifier runtime is dominated by $O(n + m_p^{1/2} + l)$ field operations and $O({m_p}^{1/2} + l)$ group exponentiations; and the communication cost is dominated by $O({m_p}^{1/2} + 1)$ group elements and $O({m_p}^{1/2} + l)$ field elements.

On the other hand, if the circuit is shallow (e.g. for a circuit with n addition operations and n multiplication operations: $r = \sum_{i=1}^{n} a \cdot b$ where r is the circuit output and a, b are circuit inputs, we have $m_p = 1$), the prover work would be dominated by $O(n)$ field addition operations which is very cheap.

Our protocol is specifically efficient for proving complex business logics where circuit depth is high. For example, inputs are floating point numbers and the business logic requires the circuit to perform a lot of floating point multiplications/divisions operations, a very likely scenario in the real world.

We introduce our protocol in an interactive setting where all verifier challenges are random field elements. In practice, we assume the Fiat-Shamir heuristic is applied to our protocol to obtain a non-interactive zero-knowledge argument in the random oracle model.

2 Preliminaries

2.1 Assumption

Definition 1. (Discrete Logarithmic Relation) For all PPT adversaries $\mathcal{A}$ and for all $n \geq 2$ there exists a negligible function $\boldsymbol{negl}(\lambda)$ s.t.

$$Pr\left[\begin{array}{c} \mathbb{G} = Setup(1^\lambda),\ g_0, ..., g_{n-1} \xleftarrow{\$} \mathbb{G} \\ a_0, ..., a_{n-1} \in \mathbb{Z}_p \leftarrow \mathcal{A}(g_0, ..., g_{n-1}) \end{array} \middle|\ \exists\, a_i \neq 0 \wedge \prod_{i=0}^{n-1} g_i^{a_i} = 1 \right] \leq \boldsymbol{negl}(\lambda)$$

The Discrete Logarithmic Relation assumption states that an adversary can't find a non-trivial relation between the randomly chosen group elements $g_0, ..., g_{n-1} \in \mathbb{G}^n$, and that $\prod_{i=0}^{n-1} g_i^{a_i} = 1$ is a non-trivial discrete log relation among $g_0, ..., g_{n-1}$.

2.2 Zero-Knowledge Argument of Knowledge

Interactive arguments are interactive proofs in which security holds only against computationally bounded provers. In an interactive argument of knowledge for a relation $\mathcal{R}$, a prover convinces a verifier that it knows a witness w for a statement x s.t. $(x, w) \in \mathcal{R}$ without revealing the witness itself to the verifier. When we say knowledge of an argument, we imply that the argument has witness-extended emulation.

Definition 2. (Interactive Argument) Let's say $(\mathcal{P}, \mathcal{V})$ denotes a pair of PPT interactive algorithms and $\boldsymbol{Setup}$ denotes a non-interactive setup algorithm that outputs public parameters pp given a security parameter λ that both $\mathcal{P}$ and $\mathcal{V}$ have access to. Let $\langle \mathcal{P}(pp, x, w), \mathcal{V}(pp, x) \rangle$ denote the output of $\mathcal{V}$ on input x after its interaction with $\mathcal{P}$, who has knowledge of witness w. The triple $(\boldsymbol{Setup}, \mathcal{P}, \mathcal{V})$ is called an argument for relation $\mathcal{R}$ if for all non-uniform PPT adversaries $\mathcal{A}$, the following properties hold:

- **Perfect Completeness**

$$Pr\left[\begin{array}{c|c} (pp, x, w) \notin \mathcal{R} \text{ or} & pp \leftarrow \boldsymbol{Setup}(1^\lambda) \\ \langle \mathcal{P}(pp, x, w), \mathcal{V}(pp, x) \rangle = 1 & (x, w) \leftarrow \mathcal{A}(pp) \end{array}\right] = 1$$

- **Computational Soundness**

$$Pr\left[\begin{array}{c|c} \forall w (pp, x, w) \notin \mathcal{R} \wedge & pp \leftarrow \boldsymbol{Setup}(1^\lambda) \\ \langle \mathcal{A}(pp, x, s), \mathcal{V}(pp, x) \rangle = 1 & (x, s) \leftarrow \mathcal{A}(pp) \end{array}\right] \leq \boldsymbol{negl}(\lambda)$$

- **Public Coin** All messages sent from $\mathcal{V}$ to $\mathcal{P}$ are chosen uniformly at random and independently of $\mathcal{P}$'s messages

Definition 3. (Computational Witness-Extended Emulation) Given a public-coin interactive argument tuple $(\mathbf{Setup}, \mathcal{P}, \mathcal{V})$ and arbitrary prover algorithm $\mathcal{P}^*$, let $\boldsymbol{Recorder}$ $(\mathcal{P}^*, pp, x, s)$ denote the message transcript between $\mathcal{P}^*$ and $\mathcal{V}$ on shared input x, initial prover state s, and pp generated by $\boldsymbol{Setup}$. Furthermore, let $\mathcal{E}$ $\boldsymbol{Recorder}$ $(\mathcal{P}, pp, x, s)$ denote a machine $\mathcal{E}$ with a transcript oracle for this interaction that can rewind to any round and run again with fresh verifier randomness. The tuple $(\boldsymbol{Setup}, \mathcal{P}, \mathcal{V})$ has computational witness-extended emulation if for every deterministic polynomial time $\mathcal{P}$ there exists an expected polynomial time emulator $\mathcal{E}$ such that for all non-uniform polynomial time adversaries $\mathcal{A}$ the following condition holds:

$$\left| Pr\left[\begin{array}{c|c} \mathcal{A}(tr) = 1 & pp \leftarrow Setup(1^\lambda) \\ & (x, s) \leftarrow \mathcal{A}(pp) \\ & tr \leftarrow \boldsymbol{Recorder}(\mathcal{P}^*, pp, x, s) \end{array}\right] - \right.$$

$$\left. Pr\left[\begin{array}{c|c} \mathcal{A}(tr) = 1 \wedge & pp \leftarrow Setup(1^\lambda) \\ tr \text{ accepting} \implies (x, w) \in \mathcal{R} & (x, s) \leftarrow \mathcal{A}(pp) \\ & (tr, w) \leftarrow \mathcal{E}^{\boldsymbol{Recorder}(\mathcal{P}^*, pp, x, s)}(pp, x) \end{array}\right] \right| \leq \boldsymbol{negl}(\lambda)$$

Definition 4. (Perfect Special Honest Verifier Zero Knowledge for Interactive Arguments) An interactive proof is $(\boldsymbol{Setup}, \mathcal{P}, \mathcal{V})$ is a perfect special honest verifier zero knowledge (PSHVZK) argument of knowledge for $\mathcal{R}$ if there exists a probabilistic polynomial time simulator $\mathcal{S}$ such that all pairs of interactive adversaries $\mathcal{A}_1, \mathcal{A}_2$ have the following property for every $(x, w, \sigma) \leftarrow \mathcal{A}_2(pp) \wedge (pp, x, w) \in \mathcal{R}$, where σ stands for verifier's public coin randomness for challenges

$$Pr\left[\begin{array}{c|c} \mathcal{A}_1(tr) = 1 & pp \leftarrow Setup(1^\lambda), \\ & tr \leftarrow \langle \mathcal{P}(pp, x, w), \mathcal{V}(pp, x) \rangle \end{array}\right] =$$

$$Pr\left[\begin{array}{c|c} \mathcal{A}_1(tr) = 1 & pp \leftarrow Setup(1^\lambda), \\ & tr \leftarrow \mathcal{S}(pp, x, \sigma) \end{array}\right]$$

Above property states that adversary chooses a distribution over statements x and witnesses w but is not able to distinguish between the simulated transcripts and the honestly generated transcripts for a valid statement/witnesses pair.

2.3 Polynomial Commitment Function

As in the case of other popular zero knowledge protocols that offer succinct proof size, our protocol uses a polynomial commitment evaluation protocol to construct most of our proof transcript. Our protocol uses a version of the polynomial commitment scheme defined by Bootle. et al. [9] Others have improved the square-root-based polynomial commitments by applying the inner product approach defined by Bunez et. al. and adding support for multilinear polynomials such as Hyrax [29] and Spartan [26]. Similar techniques may be used to improve our implementation in the future to reduce proof size in the expense of longer verifier runtime. The polynomial commitment function PolyCommitEval is defined as:

- $\boldsymbol{PolyCommitEval}(C, y, x; \vec{\tau}, \phi) \rightarrow boolean$ C is the committed polynomial in $\mathbb{G}$ where $\vec{\tau}$ are its coefficients and ϕ is its blinding key. The function returns a boolean value"åtrue" if the polynomial can be correctly evaluated at point x s.t. $y = f(x)$.

Assume the polynomial commitment scheme we use in this paper is the one defined by Bootle et al. [9]. In section 5, we will introduced a modified version of the polynomial commitment evaluation scheme defined by Bootle et al. that tailors to our need.

2.4 Zero Knowledge Proof of Discrete Logarithm

For a prover to prove it has the knowledge of a discrete logarithmic κ of some group element $s = h^{\kappa} \in \mathbb{G}$. We define the relation for this protocol as $\mathcal{R}_{PoD} = \{(h, s; \kappa) : s = g^{\kappa}\}$. We also define two functions $(\boldsymbol{ProveDL}, \boldsymbol{VerifyDL})$ for provers and verifiers to create and verify proof transcripts:

- $\boldsymbol{ProveDL}(g, \kappa) \rightarrow tr_{\kappa}$ generates proof transcript tr_{κ}, where κ is the witness.
- $\boldsymbol{VerifyDL}(g, s, tr_{\kappa}) \rightarrow b \in \{0, 1\}$ takes a proof transcript tr_{κ} and a pair of group elements with discrete log relation $(g, s \in \mathbb{G} \wedge s = h^{\kappa})$, and outputs $true$ if the knowledge of the relation is verified, $false$ otherwise.

In this paper, we assume the underlying implementation of the proof of discrete logarithm protocol is Schnorr's protocol. We know for a fact that Shnorr's protocol has perfect completeness, special honest verifier zero knowledge, and computational witness-extended emulation.

2.5 Notations

Let $\mathbb{G}$ denote any type of secure cyclic group of prime order p, and let $\mathbb{Z}_p$ denote an integer field modulo p. Group elements other then generators are denoted by capital letters. e.g., $C = u_1^{a_1} u_2^{a_2} ... u_n^{a_n} \in \mathbb{G}$ is a commitment commits to a vector $\vec{a}$ denoted by a capital letter, and $B \in \mathbb{G}$ is a random group element also denoted by a capital letter. For generators used as base points to compute other group elements in our protocol, such as $\vec{g}, h \in \mathbb{G}$, we use lower case letters to denote them. Greek letters are used to label hidden key values. e.g. υ is the blinding key for Pedersen commitment P on generator $h \in \mathbb{G}$ s.t. $P = g^a h^{\upsilon}$. Finally, we use standard vector notation $\vec{v}$ to denote vectors. i.e. $\vec{a} \in \mathbb{Z}_p^n$ is a list of n integers a_i for $i = \{1, 2, ..., n\}$.

We write $\mathcal{R} = \{(Public\ Inputs\ ; Witnesses) : Relation\}$ to denote the relation $\mathcal{R}$ using the specified public inputs and witnesses.

3 Input Transformation Based Zero Knowledge Argument

In this section, we will introduce the base version of our protocol, and then we will present the improved version of the baseline protocol in sections 4 and 5, which leverage number theoretic transform (NTT) to speed up polynomial multiplication operations.

For a high depth circuit with n multiplications, the prover work of the baseline version of our protocol is dominated by $O({m_p}^2+m_p+l)$ field operations and $O(m_p+m_p^{1/2}+l)$ group exponentiations. In section 4, we will introduce the full version of our protocol that leverages NTT to reduce the asymptotic prover runtime for field operations to $m_p \log m_p$.

The verifier work is dominated by $O(n + m_p^{1/2} + l)$ field operations and $O({m_p}^{1/2} + l)$ group exponentiations (we are using the univariate polynomial commitment scheme defined by Bootle et al. [9]). Although running $O(n)$ field operations is not technically sub-linear, it is still efficient even for large circuits with 2^{20} multiplication gates.

The communication cost is dominated by $O({m_p}^{1/2} + l)$ field elements and $O({m_p}^{1/2} + l)\mathbb{G}$ group elements

3.1 Zero Knowledge Argument Protocol Based on Input Transformation Concept

We first define the relation for the base version of our protocol. For l input parameters, let $\mathcal{C}_\mathbb{F}$ represent the set of arbitrary arithmetic circuits in $\mathbb{F}$, there exists a zero knowledge argument for the relation:

$$\begin{aligned}\{(g, h, R \in \mathbb{G}, \vec{P} \in \mathbb{G}^l, E_c \in \mathcal{C}_\mathbb{F} ; \vec{a}, \vec{\upsilon} \in \mathbb{Z}_p^l, r, \epsilon \in \mathbb{Z}_p) : \\ P_i = g^{a_i} h^{\upsilon_i} \; \forall_i \in [1, l] \; \wedge \; R = g^r h^\epsilon \; \wedge \; E_c(\vec{a}, \upsilon) = r, \epsilon\}\end{aligned} \tag{1}$$

The above relation states that each input parameter is a commitment P_i in $\mathbb{G}$, which hides input value a_i with blinding key υ_i. Plugging inputs into a circuit E_c will output a value r, which is committed using commitment $R \in \mathbb{G}$ with blinding key ϵ.

It is easy to see that we cannot take Pedersen commitments as inputs to a circuit because we cannot perform multiplication operation on them, so our protocol introduce the idea of input transformation. The main idea is to transform committed inputs in Perdersen commitment form in $\mathbb{G}$ to linear polynomials in $\mathbb{F}$ so that both the prover and the verifier can perform addition and multiplication operations just as they add and multiply integers in $\mathbb{F}$. For an input commitment P_i s.t. $P = g^{a_i} h^{\upsilon_i} \in \mathbb{G}$ where a_i is the input value and υ_i is its blinding key, we use the same value pair to create its corresponding integer value in linear polynomial form ${a_i}' \in \mathbb{Z}_p$:

$${a_i}' = a_i + X\upsilon_i \in \mathbb{Z}_p \tag{2}$$

Since input linear polynomials are just integer values, the verifier can easily perform arithmetic operations on them. The output value of a circuit is equivalent to a polynomial with degree $m_p + 1$ evaluated at point X. The constant term of the output polynomial is the expected circuit output value r and the coefficient of the degree 1 term is its blinding key ϵ.

We are now ready to expand the definition of relation 1 above with blinding keys used to hide inputs and outputs. We say witnesses $r, \epsilon, \vec{\tau}$ are coefficients of the polynomial representing the circuit output s.t. $\vec{\tau}$ are coefficients of terms of degree 2 to degree m_p+1, and these witnesses are also outputs computed from inputs $\vec{a}$ and their blinding keys $\vec{v}$ to a circuit E_c. The updated relation is as follows:

$$\begin{aligned}\{(g, h, R \in \mathbb{G}, \vec{u} \in \mathbb{G}^{m_p}, \vec{P} \in \mathbb{G}^l, E_c \in \mathcal{C}_\mathbb{F} ; \\ \vec{a}, \vec{v} \in \mathbb{Z}_p^l, r, \epsilon \in \mathbb{Z}_p, \vec{\tau} \in \mathbb{Z}_p^{m_p}) : \\ P_i = g^{a_i} h^{\upsilon_i} \forall_i \in [1, l] \wedge R = g^r h^\epsilon \wedge E_c(\vec{a}, \vec{v}) = r, \epsilon, \vec{\tau}\}\end{aligned} \tag{3}$$

In our protocol, the prover needs to generate transcripts to prove two things: 1, each input is correctly transformed; 2, the circuit output is correctly computed from transformed inputs.

To begin, we show how to prove each input transformation from P_i to a_i' is legit. The linear polynomial a_i' is obviously not binding to a_i as we can easily manipulate the value of a_i by altering its blinding key if the value of challenge X is known. e.g. $a_i' = (a_i+\delta)+x(\upsilon_i-\delta/x)$ (the "committed" value

a_i is altered to $a_i + \delta$). To combat this, verifiers use the following equation to confirm the mapping between $a_i{}'$ and P_i for some challenge x:

$$P_i/g^{a_i{}'} = \frac{g^{a_i}h^{\upsilon_i}}{g^{a_i+x\upsilon_i}} = (h/g^x)^{\upsilon_i} \tag{4}$$

If the prover can prove the knowledge of υ_i on generator $(h/g^x) \in \mathbb{G}$ using any proof of discrete log protocol, we have a concrete proof that the witnesses of $a_i{}'$ and P_i must match for some challenge x except for a negligible probability.

Before we move on, there are three issues in real world applications that we have to consider before we construct the base version of our protocol:

First, this Pedersen to linear polynomial transformation is not really zero knowledge. For example, if the prover commits to $P = g^a h^\upsilon$ and then sends $a' = a + x \cdot \upsilon$. An attacher can easily test whether P is a commitment to $a = 0$ (or any other value) by simply checking whether $P = h^{(0+x\cdot\upsilon)/x}$.

Second, since each committed P_i may be used multiple times as inputs to different circuits, an attacker can easily deduce the witness pair of P_i from two different challenges x_1, and x_2. (e.g. for $P_i = g^{a_i}h^{\upsilon_i}$, its linear polynomial form value from the first challenge x_1 would be $a_i + x_1\upsilon_i$, and from the second challenge x_2 would be $a_i + x_2\upsilon_i$, subtracting the two will get $\upsilon_i(x_1 - x_2)$ where an attacker can trivially extract the witness pair υ_i and a_i).

Third, for a circuit with l input parameters, it would be pretty inefficient to create l proof of knowledge transcripts for all l inputs, so we need some kind of batching mechanism to verify them in batch.

The first and second issues are non-issues in the the more complete versions of our protocol (Protocol 2 and 3) explained in section 4 and 5. Starting with Protocol 2, a' is moved to field q (instead of p) and its blinding key is an arbitrarily chosen $\alpha \in \mathbb{Z}_q$. However, for the seek of completeness, we will give a solution for protocol 1 here as well.

The prover gets around the first and second issues here by creating new blinding keys α_i for every linear polynomial $a_i{}'$ s.t. $i = \{1, ..., l\}$, and then commit to the differences between blinding key pairs κ protected by blinding key μ. To batch verify them, the verifier use a random challenge k to generate l challenges $\vec{k} = k^1, ..., k^l$ s.t.:

$$\kappa = \sum_{i=1}^{l}(\alpha_i - \upsilon_i)k^i \in \mathbb{Z}_p \tag{5}$$

$$PK_{\kappa\mu} = h^\kappa u^\mu \in \mathbb{G} \tag{6}$$

If the prover can prove the knowledge of κ, μ on generator $h, u \in \mathbb{G}$ using any proof of knowledge (proof of discrete logarithm) protocol, we can confirm that only the sum of products of blinding keys (exponent of h) are updated after performing equation 8 except for a negligible probability. To prove and verify the knowledge of two exponents on two generators of a committed value, we need to extend the functionality of proveDL and verifyDL defined earlier:

- $\boldsymbol{ProveEDL}(h, u, \kappa, \mu) \rightarrow tr_{\kappa\mu}$ generates proof transcript $tr_{\kappa\mu}$, where κ, μ are witnesses.
- $\boldsymbol{VerifyEDL}(h, u, PK_{\kappa\mu}, tr_{\kappa\mu}) \rightarrow b \in \{0, 1\}$ takes a proof transcript $tr_{\kappa\mu}$ and verify a pair of discrete log relation ($PK_{\kappa\mu} = h^\kappa u^\mu$), and outputs $true$ if the knowledge of these relation is verified, $false$ otherwise.

Neither of these two functions above will be used in the final version of our protocol, so we are just temporarily using them to get a secure the base version of our protocol. The implementation of the two functions above is trivial, one such example is the extended Shnorr protocol find in halo [10]. The key is to not allow attackers to retrieve h^κ and u^μ from proof transcripts. In protocol 1, the prover creates proof transcripts for the knowledge of κ, μ on generator $h, u \in \mathbb{G}$ as follows:

$$tr_{\kappa\mu} = ProveEDL(h, u, \kappa, \mu) \tag{7}$$

Since we are now using arbitrarily chosen $\vec{\alpha}$ as the blinding keys for $\vec{a}'$ and we want the verifier to verify all input mappings in a batch, the verifier adds the commitment to blinding key differences $PK_{\kappa\mu}$ back to the sum of products of input commitments:

$$P_t = (\prod_{i=1}^{l} P_i^{k^i}) \cdot PK_{\kappa\mu} \in \mathbb{G} \tag{8}$$

The prover use the new blinding keys $\vec{\alpha}$ to create linear polynomials such that $a_i{}' = a_i + x\alpha_i$ for all i. After challenge k is known, the prover can batch prove the mapping between committed values and their linear polynomial values by providing transcripts for tr_{α_t}.

$$\alpha_t = \sum_{i=1}^{l} (\alpha_i) k^i \in \mathbb{Z}_p \tag{9}$$

$$tr_{\alpha_t\mu} = ProveEDL((h/g^x), u, \alpha_t, \mu) \tag{10}$$

The verifier computes the sum of products of $\vec{a}$ and powers of k. With sum of products of both $\vec{P}, \vec{a'}$ (P_t, t) available, the verifier can trivially compute PK_{α_t} s.t.:

$$t = \sum_{i=1}^{l} a_i' k^i \in \mathbb{Z}_p \tag{11}$$

$$PK_{\alpha_t\mu} = P_t/g^t = (h/g^x)^{\alpha_t} u^{\mu} \tag{12}$$

If the prover can prove the knowledge of α_t on generator $(h/g^x) \in \mathbb{G}$ using any proof of knowledge protocol, we know that the mapping between $\vec{P}$ and $\vec{a'}$ is correct except for a negligible probability.

Next, we show the circuit output is corrected computed form transformed inputs $\vec{a}'$, the prover needs to prove it knows all coefficients of the output polynomial after finish running the circuit. For example, adding two input values in linear polynomial form is the same as adding two polynomials:

$$o = a_1{}' + a_2{}' = r + X \cdot \epsilon \tag{13}$$

Where $r = (a_1 + a_2)$ and the blinding key is $\epsilon = (\alpha_1 + \alpha_2)$. Likewise, multiplying two inputs $a_1{}', a_2{}'$ is the same as multiplying two polynomials:

$$o = a_1 \cdot a_2 = r + X \cdot \epsilon + X^2 \cdot \tau \tag{14}$$

Where $r = a_1 \cdot a_2$, $\epsilon = a_2\alpha_1 + a_1\alpha_2$, and $\tau = \alpha_1 \cdot \alpha_2$. We use the label "$o$" to represent the circuit output, which is equivalent to the output polynomial evaluated at a point X. The degree of the polynomial will increase after each multiplication operation, so the efficiency will drop as the maximum number of multiplications included in any path that leads to the circuit output (m_p) increases.

We also need the circuit output to be a linear polynomial $r' = r + X\epsilon$ maps to the commitment $R = g^r h^\epsilon$. However, the verifier instead gets a polynomial with degree $m_p + 1$ in the circuit output o unless there are no multiplication operations in the circuit.

To get the linear polynomial output we need, the verifier needs to subtract out all terms with degree higher than one. In the multiplication case above, the verifier needs to eliminate the term of degree 2, so the prover needs to commit to τ before the challenge x is known. When the challenge x is available, the prover sends y to the verifier and engage with the verifier to prove $f(x) = X^2\tau = y$. With $y = X^2\tau$ validated, the verifier can subtract y from o to get the output in linear polynomial form:

$$r' = o - y \tag{15}$$

We call y the "breaker" of our protocol, because it subtracts all noises (polynomial terms of degree higher than one) from the raw circuit output o. In practice, the prover and the verifier engages in a polynomial commitment protocol to confirm $f(x) = y$.

$$C = \prod_{i=1}^{m_p} u_i^{\tau_i} \in \mathbb{G} \tag{16}$$

In the final version of our protocol, the polynomial commitment evaluation logic broken apart and integrated with our protocol (protocol 3), so that we no longer need to call a polynomial commitment evaluation function as we do in Protocol 1 and Protocol 2 (see section 5).

We define two more functions for our protocol. function ***computeKeys*** is used by the prover to compute keys of a polynomial, and function ***computeCircuit*** is used by verifiers to compute the value of the result polynomial at evaluation point X:
1. function ***computeKeys***(circuit,"input values", "input keys") take input values $\vec{a}$ and keys $\vec{v}$ to compute $r, \epsilon, \vec{\tau}$ (coefficients of o) using the circuit provided to the protocol. function computeKeys uses function Multiply and function Add defined above to compute coefficients of o.
2. function ***computeCircuit***(circuit,"input values in linear polynomial form") trivially compute the result o from the inputs provided as they are integer values.

We don't waste space describing them in detail here since they are trivial to implement. With all the information available, we now formally introduce Protocol 1:

$$Input : (\vec{P} \in \mathbb{G}^l, \vec{u} \in \mathbb{G}^{m_p}, g, h \in \mathbb{G}, \vec{a}, \vec{v} \in \mathbb{Z}_p^l) \tag{17}$$

$$\mathcal{P}'s\ input : (\vec{P}, \vec{u}, g, h; \vec{a}, \vec{v}) \tag{18}$$

$$\mathcal{V}'s\ input : (\vec{P}, \vec{u}, g, h) \tag{19}$$

$$\mathcal{P}\ compute : \tag{20}$$

$$\alpha_i \xleftarrow{\$} \mathbb{Z}_p, \qquad i = \{1, ..., l\} \tag{21}$$

$$r, \varepsilon, \vec{\tau} = computeKeys(equation, \vec{a}, \vec{\alpha}) \tag{22}$$

$$R = g^r h^{\varepsilon} \in \mathbb{G} \tag{23}$$

$$tr_\epsilon = ((h/g^x), \epsilon) \tag{24}$$

$$C = \prod_{i=1}^{m_p} u_i^{\tau_i} \in \mathbb{G} \tag{25}$$

$$\mathcal{P} \rightarrow \mathcal{V} : C, R, tr_\epsilon \tag{26}$$

$$\mathcal{V}\ compute : \tag{27}$$

$$x \xleftarrow{\$} \mathbb{Z}_p \tag{28}$$

$$\mathcal{V} \rightarrow \mathcal{P} : x \tag{29}$$

$$\mathcal{P}\ compute : \tag{30}$$

$$a_i' = a_i + x\alpha_i \in \mathbb{Z}_p \qquad i = \{1, ..., l\} \tag{31}$$

$$y = \sum_{i}^{m_p} \tau_i \cdot x^{i+1} \in \mathbb{Z}_p \tag{32}$$

$$\mathcal{P} \rightarrow \mathcal{V} : \ \vec{a}', y \tag{33}$$

$$\mathcal{V}\ verify\ final\ output\ R : \tag{34}$$

$$o = computeCircuit(circuit, \vec{a}') \tag{35}$$

$$r' = o - y \in \mathbb{G} \tag{36}$$

$$PK_\epsilon = R/g^{r'} \in \mathbb{G} \tag{37}$$
if $PolyCommitEval(C, y, x; \vec{\tau})$ (38)
  then *continue* (39)
else *reject* (40)
if $VerifyDL((h/g^x), PK_\epsilon, tr_\epsilon)$ (41)
  then *continue* (42)
else *reject* (43)
$\mathcal{V}$ *compute* : (44)
$$k \xleftarrow{\$} \mathbb{Z}_p \tag{45}$$
$$\mathcal{V} \rightarrow \mathcal{P} : k \tag{46}$$
$\mathcal{P}$ *compute* : (47)
$$\alpha_t = \sum_{i=1}^{l} \alpha_i k^i \in \mathbb{Z}_p \tag{48}$$
$$tr_{\alpha_t \mu} = ProveEDL((h/g^x), u, \alpha_t, \mu) \tag{49}$$
$$\kappa = \sum_{i=1}^{l} (\alpha_i - \upsilon_i) k^i \in \mathbb{Z}_p \tag{50}$$
$$PK_{\kappa\mu} = h^\kappa u^\mu \in \mathbb{G} \tag{51}$$
$$tr_{\kappa\mu} = ProveEDL(h, u, \kappa, \mu) \tag{52}$$
$$\mathcal{P} \rightarrow \mathcal{V} : PK_{\kappa\mu}, tr_{\kappa\mu}, tr_{\alpha_t\mu} \tag{53}$$
$\mathcal{V}$ *verify inputs* : (54)
$$P_t = (\prod_{i=1}^{l} P_i^{k^i}) \cdot PK_{\kappa\mu} \in \mathbb{G} \tag{55}$$
$$t = \sum_{i=1}^{l} a_i' k^i \in \mathbb{Z}_p \tag{56}$$
$$PK_{\alpha_t\mu} = P_t/g^t \tag{57}$$
if $VerifyEDL(h, u, PK_{\kappa\mu}, tr_{\kappa\mu})$, (58)
and $VerifyEDL((h/g^x), u, PK_{\alpha_t\mu}, tr_{\alpha_t\mu})$ (59)
  then *accept* (60)
else *reject* (61)

Protocol 1

Theorem 1. *(Zero Knowledge Argument with Practical Efficiency). The proof system presented in this section has perfect completeness, perfect special honest verifier zero-knowledge, and computational witness extended emulation.*

The proof for Theorem 1 is presented in Appendix A.

The main idea of Protocol 1 is to convert commitments P_i to its linear polynomial form a_i' so that the verifier can just take linear polynomials as input values to the circuit and use standard integer operations to compute the circuit. The computation result o is a polynomial with $m_p + 1$ degree. By subtracting out all term with degree greater than one as in equation 15 explained, the verifier gets the circuit output in linear polynomial form that maps to commitment R.

4 Making Input Transformation Based Zero Knowledge Protocol Efficient with NTT

Protocol 1's prover isn't efficient because $O({m_p}^2)$ field operations in prover work can become expensive as m_p gets big. In this section, we introduce a mechanism that allows us to use the number theoretic transform (NTT) to cut prover's field operation work to $m_p \log m_p$.

4.1 Using Number Theoretic Transform to Improve Prover Performance in Field Operations

The objective of NTT is to multiply two polynomials such that the coefficients of the resultant polynomials are calculated under a particular modulo in $m_p \log m_p$, a major improvement over ${m_p}^2$ runtime in protocol 1. However, a major drawback of NTT is that it normally requires a prime modulo q of the form $q = r \cdot 2^k + 1$ to be the order of the group, where k and c are arbitrary constants. Since the order of widely used $\mathbb{G}$ in cryptography is usually not a prime with the aforementioned form, we need a mechanism to map linear polynomials with a prime modulo q that satisfies the aforementioned form to any group with prime order p. q is expected to be smaller than p because: 1) computation in p (e.g. polynomial commitment evaluation) won't overflow 2) lower communication cost. In our benchmark testing, we set q to a 61-bit prime number.

We redefine equation 2 s.t. a'_i and its blinding key α_i are now in field q instead of the larger field p.

$$a'_i = a_i + x\alpha_i \in \mathbb{Z}_q \qquad i = \{1, ..., l\} \tag{62}$$

We define a new blinding key $\omega_i \in \mathbb{Z}_p$ and mix that with blinding key υ_i in P_i and its matching α_i in a' to create S_i, T_i.

$$S_i = g^{\omega_i \cdot q} \in \mathbb{G} \qquad i = \{1, ..., l\} \tag{63}$$

$$T_i = g^{\upsilon_i - \alpha_i} \in \mathbb{G} \qquad i = \{1, ..., l\} \tag{64}$$

The prover then sends S_i, T_i for $i = \{1, .., l\}$ to the verifier. When the challenge $x \in \mathbb{Z}_q$ is available, the prover sends e_i s.t.:

$$e_i = ((x\alpha_i \bmod q) - x\alpha_i) \cdot x + \omega_i \cdot q \qquad i = \{1, ..., l\} \tag{65}$$

e_i is not in $\mathbb{Z}_p$, but it is a good idea to keep e_i smaller than p to keep the communication cost low. The idea here is that when we subtract e_i from a_i, we can subtract out the blinding modulo q element $(x\alpha_i \bmod q)$ from a'_i (e.g. $a'_i \cdot x - e_i = (a_i + x\alpha_i) \cdot x - \omega_i q$). The verifier can replace $x^2\alpha_i - \omega_i q$ part with the new blinding element $x^2\upsilon_i$ as the exponent of generator g by adding the previously committed values S_i, T_i.

$$g^{a'_i \cdot x - e_i} \cdot T_i^{x^2} \cdot S_i = (g^x)^{a_i + x\upsilon_i} \in \mathbb{G} \qquad i = \{1, ..., l\} \tag{66}$$

With $(g^x)^{a_i + x\upsilon_i}$ available, the verifier can trivially divide each P_i and taking their sum with powers of k to get PK_{υ_t}.

$$PK_{\upsilon_t} = \prod_{i=1}^{l} \left(\frac{P_i^x}{g^{a'_i \cdot x - e_i} \cdot T_i^{x^2} \cdot S_i} \right)^{k^i} \in \mathbb{G} \tag{67}$$

$PK_{\upsilon_t} = ((h/g)^x)^{\upsilon_t}$. The verifier can confirm the correctness of the transformation except with negligible probability if the prover can prove the knowledge of υ_t on generator $(h/g)^x \in \mathbb{G}$.

Finally, the verifier needs to make sure e_i doesn't alter the value of a_i. This can be done by taking the modulus q of e_i and checking if it returns 0. This is trivial to understand since a'_i is in $\mathbb{Z}_q$. If e_i is a multiple of q then it is obvious that it cannot alter the value of a_i.

$$\textbf{if } (e_i \bmod q) \stackrel{?}{=} 0, \textbf{ then } continue \tag{68}$$

e_i does not leak any information to the verifier either. This is because the first part of e_i: $((x\alpha_i \bmod q) - x\alpha_i) \cdot x$ is a multiple of q, which is also equivalent to $s \cdot q$ for some s. s value is perfectly hiding with the blinding term $w \cdot q$ in e_i.

We have so far skipped the overflow problem. If $a_i + (x\alpha_i \bmod q) > q$, then we will have an overflow problem in equation 66 67 when computing $a'_i \cdot x - e_i$. To get around this the prover simply needs to check if $a_i + (x\alpha_i) \bmod q$ overflows q, and subtracts $q \cdot x$ from e_i if that's the case.

$$\textbf{if } a_i + (x\alpha_i \bmod q) > q, \textbf{ then } e_i = e_i - q \cdot x \qquad i = \{1, ..., l\} \tag{69}$$

We now merge the NTT conversion code introduced in this section and formally define the efficient version of our protocol in Protocol 2.

$$Input : (\vec{P} \in \mathbb{G}^l, \vec{u} \in \mathbb{G}^{m_p}, g, h \in \mathbb{G}, \vec{a}, \vec{v} \in \mathbb{Z}_p^l) \tag{70}$$

$$\mathcal{P}'s\, input : (\vec{P}, \vec{u}, g, h; \vec{a}, \vec{v}) \tag{71}$$

$$\mathcal{V}'s\, input : (\vec{P}, \vec{u}, g, h) \tag{72}$$

$$\mathcal{P}\ compute : \tag{73}$$

$$\alpha_i \xleftarrow{\$} \mathbb{Z}_p, \qquad i = \{1, ..., l\} \tag{74}$$

$$\omega_i \xleftarrow{\$} \mathbb{Z}_p, \qquad i = \{1, ..., l\} \tag{75}$$

$$r, \varepsilon, \vec{\tau} = computeKeys(equation, \vec{a}, \vec{\alpha}) \in \mathbb{Z}_p^{m_p+2} \tag{76}$$

$$R = g^r h^{\varepsilon} \in \mathbb{G} \tag{77}$$

$$tr_{\epsilon} = ((h/g^x), \epsilon) \tag{78}$$

$$C = \prod_{i=1}^{m_p} g_i^{\tau_i} \in \mathbb{G} \tag{79}$$

$$S_i = g^{\omega_i \cdot q} \in \mathbb{G} \qquad i = \{1, ..., l\} \tag{80}$$

$$T_i = g^{v_i - \alpha_i} \in \mathbb{G} \qquad i = \{1, ..., l\} \tag{81}$$

$$\mathcal{P} \rightarrow \mathcal{V} : \vec{S}, \vec{T}, C, R, tr_{\epsilon} \tag{82}$$

$$\mathcal{V}\ compute : \tag{83}$$

$$x \xleftarrow{\$} \mathbb{Z}_p \tag{84}$$

$$\mathcal{V} \rightarrow \mathcal{P} : x \tag{85}$$

$$\mathcal{P}\ compute : \tag{86}$$

$$a'_i = a_i + x\alpha_i \in \mathbb{Z}_q \qquad i = \{1, ..., l\} \tag{87}$$

$$e_i = ((x\alpha_i \bmod q) - x\alpha_i)x + \omega_i q \in \mathbb{Z}_p \qquad i = \{1, ..., l\} \tag{88}$$

$$\textbf{if } a_i + (x\alpha_i \bmod q) > q, \textbf{ then } e_i = e_i - q \cdot x \qquad i = \{1, ..., l\} \tag{89}$$

$$y = \sum_{i}^{m_p} \tau_i \cdot x^{i+1} \in \mathbb{Z}_p \tag{90}$$

$$\mathcal{P} \rightarrow \mathcal{V} : \vec{e}, \vec{a}', y \tag{91}$$

$$\mathcal{V}\ verify\ final\ output : \tag{92}$$

$$o = computeEquation(equation, \vec{a}\,') \in \mathbb{Z}_p \tag{93}$$
$$r' = o - y \in \mathbb{Z}_p \quad // \; r + x \cdot \epsilon \tag{94}$$
$$PK_\epsilon = R/g^{r'} \in \mathbb{G} \quad //equal\ to\ (h/g^x)^\epsilon \tag{95}$$
$$\textbf{if}\ PolyCommitEval(C, y, x; \vec{\tau}),\ \textbf{then}\ continue \tag{96}$$
$$\textbf{else}\ reject \tag{97}$$
$$\textbf{if}\ VerifyDL((h/g^x), PK_\epsilon, tr_\epsilon),\ \textbf{then}\ continue \tag{98}$$
$$\textbf{else}\ reject \tag{99}$$
$$\mathcal{V}\ compute: \tag{100}$$
$$k \xleftarrow{\$} \mathbb{Z}_p \tag{101}$$
$$\mathcal{V} \rightarrow \mathcal{P} : k \tag{102}$$
$$\mathcal{P}\ compute: \tag{103}$$
$$\upsilon_t = \sum_{i=1}^{l} \upsilon_i k^i \in \mathbb{Z}_p \tag{104}$$
$$tr_{\upsilon_t} = ProveDL((h/g)^x, \upsilon_t) \tag{105}$$
$$\mathcal{P} \rightarrow \mathcal{V} : tr_{\upsilon_t} \tag{106}$$
$$\mathcal{V}\ verify\ inputs: \tag{107}$$
$$\textbf{if}\ (e_i \bmod q) \stackrel{?}{=} 0,\ \textbf{then}\ continue \qquad i = \{1, ..., l\} \tag{108}$$
$$\textbf{else}\ reject \tag{109}$$
$$PK_{\upsilon_t} = \prod_{i=1}^{l} \left(\frac{P_i^x}{g^{a_i' \cdot x - e_i} \cdot T_i^{x^2} \cdot S_i} \right)^{k^i} \in \mathbb{G} \tag{110}$$
$$\textbf{if}\ VerifyDL((h/g)^x, PK_{\upsilon_t}, tr_{\upsilon_t}),\ \textbf{then}\ accept \tag{111}$$
$$\textbf{else}\ reject \tag{112}$$

Protocol 2

Theorem 2. *(Efficient Zero Knowledge Protocol for Arbitrary Circuit with Practical Succinctness). The proof system presented in this section has perfect completeness, perfect special honest verifier zero-knowledge, and computational witness extended emulation.*

The proof for Theorem 2 is presented in Appendix B.

Since we are now evaluating the output polynomial at a smaller field q, the soundness error is increased due to existence of polynomial roots. The NTT acceptable prime we use in our implementation is $q = 1945555039024054273$, where $r = 27, k = 56, g = 5$ s.t. $q = r * 2^k + 1$.

4.2 The Asymptotic Cost of Protocol 2

The prover runtime of Protocol 2 is contributed by $O(m_p \log m_p + m_p + l)$ field operations and $O(m_p + {m_p}^{1/2} + l)$ group exponentiations; the verifier runtime is contributed by $O(n + {m_p}^{1/2} + l)$ field operations and $O({m_p}^{1/2} + l)$ group exponentiations; and the communication cost is contributed by $O({m_p}^{1/2} + l)$ group elements and $O({m_p}^{1/2} + l)$ field elements.

The worst possible scenario for our protocol is when we have a sequence of multiplications where both multiplier and multiplicand are the product of the previous multiplication operation (e.g. $((a^2)^2)^2$ is technically 3 multiplications, but it will result in $m_p = 2^3$), in such case m_p will grow exponentially. One way to tackle such problem is to break the circuit into segments of smaller sub-circuits so that m_p

value will be refreshed whenever it grows oversize, similar to the idea of bootstrapping in fully homomorphic encryption. One area worth investigate is whether we can batch process this bootstrapping technique so that we can use multiple breakers y to

5 Enhancing Efficiency for Circuits with High Depth

The efficiency of protocol 2 introduced in the last section degrades as the total number of multiplications in the path computing the output (m_p value) gets larger, a not-so-uncommon scenario for circuits with high depth. One potential way to get around this problem is to have m_b number of breakers so that the number of polynomial terms will never exceed b, where $b = \frac{m_p}{m_b}$ (instead of just one breaker at term m_p as in Protocol 2).

5.1 Batch Verification With Multiple Breakers

Each breaker y_i is an evaluation at point x for terms $x^2, ..., x^{b+1}$ of the polynomial accumulated thus far in the computation. Obviously, it is not efficient for us to commit and evaluate m number of polynomials, where m stands for the number of breakers $(y_1, ..., y_m)$ we use to evaluate a circuit.

Fortunately, we just need to make a simple modification to the polynomial commitment evaluation protocol defined by Bootle et al. to enable verifiers to evaluate m_b breakers all at once. We start by aligning each breaker y_i to the coefficients of the ith generator of vector commitments (columns of the $m_b \times b$ matrix).

$$\begin{matrix} u_1^{y_1'} \\ u_2^{y_2'} \\ u_3^{y_3'} \\ . \\ . \\ \mathrm{u}_{m_b}^{y_{m_b}'} \end{matrix} = \begin{pmatrix} u_1^{\tau_{1,1}} & u_1^{\tau_{1,2}} & . & . & u_1^{\tau_{1,b}} \\ u_2^{\tau_{2,1}} & u_2^{\tau_{2,2}} & . & . & u_2^{\tau_{2,b}} \\ u_3^{\tau_{3,1}} & u_3^{\tau_{3,2}} & . & . & u_3^{\tau_{3,b}} \\ . & . & . & . & . \\ . & . & . & . & . \\ u_{m_b}^{\tau_{m_b,1}} & u_{m_b}^{\tau_{m_b,2}} & . & . & u_{m_b}^{\tau_{m_b,b}} \end{pmatrix} \begin{pmatrix} x^2 \\ x^3 \\ x^4 \\ . \\ . \\ x^{b+1} \end{pmatrix}$$

Figure 1

Let $y_i = (y_i') \bmod q$, we can observe from figure 1 that each y_i' can be computed from the sum of products of exponents of u_i (e.g. $y_i' = \tau_{i,1}x^2 + \tau_{i,2}x^3 + ... + \tau_{i,b}x^{b+1}$).

In our protocol, the prover commits to the columns of the matrix in figure 1 just as that in Bootle et al.'s polynomial commitment evaluation scheme.

$$C_j = \prod_{i=1}^{m_b} u_i^{\tau_{i,j}} \qquad \text{for} \qquad j = \{1, ..., b\} \tag{113}$$

When the evaluation point x is known, the verifier computes the exponent of each C_j. If the equality below is true, all breakers are verified.

$$\prod_{i=1}^{m_b} u_i^{y_i'} \stackrel{?}{=} \prod_{j=1}^{b} C_j^{x^{j+1}} \tag{114}$$

Note that if each breaker y'_i is equal to the sum of products of b terms and each term is a product of $x^{j+1} \in \mathbb{Z}_q$ and $\tau_{i,j} \in \mathbb{Z}_q$, then the bit length of $|y'_i|$ can be expressed as $|y'_i| \approx 2 \cdot |q| + |b|$. The value of y'_i can also be expressed as $y'_i = y_i + z \cdot q$ for some z and $|z| \approx |q| + |b|$. However, passing raw y'_i values to the verifier may leak some information about the coefficients.

To cope with that, we make the prover commit to a blinding vector $\vec{\beta} \in \mathbb{Z}_p^m$ where $|m| = 2 \cdot |q| + |b|$, same as that of y'_i, and each y'_i is now computed as:

$$y'_i = \sum_{j=1}^{b} \tau_{i,j} x^j + \beta_i \qquad \text{for} \qquad i = \{1, ..., m_b\} \tag{115}$$

Note that the power of x in each term in the equation above is one degree lower than it needs to be, so to get y_i from y'_i the verifier needs computes:

$$y_i = (y'_i \cdot x) \bmod q \in \mathbb{Z}_q \qquad \text{for} \qquad i = \{1, ..., m_b\} \tag{116}$$

This implies the value of r' is now updated to $r'_i = r_i + x(\epsilon_i + \beta_i) \in \mathbb{Z}_q$, which is ok since r_i is not being altered and the prover can adjust the value of ϵ_i by adding β_i to it in the *computeKeys* function. The updated equality graph is shown in figure 2 below.

$$\begin{matrix} u_1^{y'_1} \\ u_2^{y'_2} \\ u_3^{y'_3} \\ . \\ . \\ u_{m_b}^{y'_{m_b}} \end{matrix} = \begin{pmatrix} u_1^{\tau_{1,1}} & u_1^{\tau_{1,2}} & . & . & u_1^{\tau_{1,b}} \\ u_2^{\tau_{2,1}} & u_2^{\tau_{2,2}} & . & . & u_2^{\tau_{2,b}} \\ u_3^{\tau_{3,1}} & u_3^{\tau_{3,2}} & . & . & u_3^{\tau_{3,b}} \\ . & . & . & . & . \\ . & . & . & . & . \\ u_{m_b}^{\tau_{m_b,1}} & u_{m_b}^{\tau_{m_b,2}} & . & . & u_{m_b}^{\tau_{m_b,b}} \end{pmatrix} \begin{pmatrix} x \\ x^2 \\ x^3 \\ . \\ . \\ x^b \end{pmatrix} \cdot \begin{pmatrix} u_1^{\beta_1} \\ u_2^{\beta_2} \\ u_3^{\beta_3} \\ . \\ . \\ u_{m_b}^{\beta_{m_b}} \end{pmatrix}$$

Figure 2

Let $B = \prod_{i=1}^{m_b} u_i^{\beta_i}$, the equality in figure 2 can also be expressed using the equation below:

$$\prod_{i=1}^{m_b} u_i^{y'_i} \stackrel{?}{=} \prod_{j=1}^{b} C_j^{x^j} \cdot B \tag{117}$$

If the commitments $\vec{C}, B$ and vector $\vec{y}$ satisfy the equation above, then we know breakers $\vec{y}$ are valid. The verifier applies equation 116 to each y'_i to get the actual breakers y_i used in computing o.

We are now ready to introduce Protocol 3, which replaces the generic polynomial commitment evaluation in Protocol 2 with the multi-breaker mechanism we introduced in this section.

$$Input : (\vec{P} \in \mathbb{G}^l, \vec{u} \in \mathbb{G}^{m_b}, g, h \in \mathbb{G}, \vec{a}, \vec{v} \in \mathbb{Z}_p^l) \tag{118}$$

$$\mathcal{P}'s\ input : (\vec{P}, \vec{u}, g, h; \vec{a}, \vec{v}) \tag{119}$$

$$\mathcal{V}'s\ input : (\vec{P}, \vec{u}, g, h) \tag{120}$$

$$\mathcal{P}\ compute : \tag{121}$$

$$\alpha_i \xleftarrow{\$} \mathbb{Z}_p, \qquad i = \{1, ..., l\} \tag{122}$$

$$\omega_i \xleftarrow{\$} \mathbb{Z}_p, \qquad i = \{1, ..., l\} \tag{123}$$

$$\beta_i \xleftarrow{\$} \mathbb{Z}_m, \qquad i = \{1, ..., m_b\} \tag{124}$$

$$r, \varepsilon, \vec{\tau} = computeKeys(equation, \vec{a}, \vec{\alpha}, \vec{\beta}) \in \mathbb{Z}_p^{m_p+2} \tag{125}$$

$$R = g^r h^{\varepsilon} \in \mathbb{G} \tag{126}$$

$$tr_\epsilon = ((h/g^x), \epsilon) \tag{127}$$

$$C_j = \prod_{i=1}^{m_b} u_i^{\tau_{i,j}} \in \mathbb{G} \qquad i = \{1, ..., b\} \tag{128}$$

$$S_i = g^{\omega_i \cdot q} \in \mathbb{G} \qquad i = \{1, ..., l\} \tag{129}$$

$$T_i = g^{v_i - \alpha_i} \in \mathbb{G} \qquad i = \{1, ..., l\} \tag{130}$$

$$B = \prod_{i=1}^{m_b} u_i^{\beta_i} \in \mathbb{G} \tag{131}$$

$$\mathcal{P} \rightarrow \mathcal{V} : \vec{S}, \vec{T}, \vec{C}, B, R, tr_\epsilon \tag{132}$$

$$\mathcal{V}\ compute: \tag{133}$$

$$x \xleftarrow{\$} \mathbb{Z}_p \tag{134}$$

$$\mathcal{V} \rightarrow \mathcal{P} : x \tag{135}$$

$$\mathcal{P}\ compute: \tag{136}$$

$$a_i' = a_i + x\alpha_i \in \mathbb{Z}_q \qquad i = \{1, ..., l\} \tag{137}$$

$$e_i = ((x\alpha_i \bmod q) - x\alpha_i)x + \omega_i q \in \mathbb{Z}_p \qquad i = \{1, ..., l\} \tag{138}$$

$$\textbf{if } a_i + (x\alpha_i \bmod q) > q, \textbf{ then } e_i = e_i - q \cdot x \qquad i = \{1, ..., l\} \tag{139}$$

$$y_i' = \sum_{j}^{b} \tau_{i,j} \cdot x^j + \beta_i \in \mathbb{Z}_p \qquad i = \{1, ..., m_b\} \tag{140}$$

$$\mathcal{P} \rightarrow \mathcal{V} : \vec{e}, \vec{a}\,', \vec{y'} \tag{141}$$

$$\mathcal{V}\ verify\ final\ output: \tag{142}$$

$$\textbf{if } (\prod_{i=1}^{m_b} u_i^{y_i'} \stackrel{?}{=} \prod_{j=1}^{b} C_j^{x^j} \cdot B) \textbf{ then } continue \tag{143}$$

$$\textbf{else } reject \tag{144}$$

$$\textbf{for } \quad i = 1, ..., m_b \quad \{ \tag{145}$$

$$o_i = computeEquation(equation, \vec{a}\,', r') \in \mathbb{Z}_p \tag{146}$$

$$y_i = (y_i' \cdot x) \bmod q \in \mathbb{Z}_q \tag{147}$$

$$r_i{}' = o_i - y_i \in \mathbb{Z}_p \tag{148}$$

$$r' = r_i{}' \in \mathbb{Z}_q \tag{149}$$

$$\} \tag{150}$$

$$PK_\epsilon = R/g^{r'} \in \mathbb{G} \quad //equal\ to\ (h/g^x)^\epsilon \tag{151}$$

$$\textbf{if } VerifyDL((h/g^x), PK_\epsilon, tr_\epsilon), \textbf{ then } continue \tag{152}$$

$$\textbf{else } reject \tag{153}$$

$$\mathcal{V}\ compute: \tag{154}$$

$$k \xleftarrow{\$} \mathbb{Z}_p \tag{155}$$

$$\mathcal{V} \rightarrow \mathcal{P} : k \tag{156}$$

$$\mathcal{P}\ compute: \tag{157}$$

$$v_t = \sum_{i=1}^{l} v_i k^i \in \mathbb{Z}_p \tag{158}$$

$$tr_{v_t} = ProveDL((h/g)^x, v_t) \tag{159}$$

$$\mathcal{P} \rightarrow \mathcal{V} : tr_{v_t} \tag{160}$$

$$\mathcal{V}\ verify\ inputs: \tag{161}$$

$$\textbf{if } (e_i \bmod q) \stackrel{?}{=} 0, \textbf{ then } continue \qquad\qquad i = \{1, ..., l\} \tag{162}$$

$$\textbf{else } reject \tag{163}$$

$$PK_{v_t} = \prod_{i=1}^{l} \left(\frac{P_i^x}{g^{a'_i \cdot x - e_i} \cdot T_i^{x^2} \cdot S_i} \right)^{k^i} \in \mathbb{G} \tag{164}$$

$$\textbf{if } VerifyDL((h/g)^x, PK_{v_t}, tr_{v_t}), \textbf{ then } accept \tag{165}$$

$$\textbf{else } reject \tag{166}$$

Protocol 3

Theorem 3. *(Efficient Zero Knowledge Argument for Arbitrary Circuits with Practical Efficiency). The proof system presented in this section has perfect completeness, perfect special honest verifier zero-knowledge, and computational witness extended emulation.*

The proof for Theorem 3 is presented in Appendix C.

From line 146 to 151 in protocol 3 we assumed our circuit is a linear circuit where there is only one path because it is easy to model. In practice, binary circuits generally have multiple "bit" paths that executes in parallel.

5.2 Booleanity Check and Bit Decomposition/Reposition

A common occurrence in proof systems is the need to enforce input data $a_i \in \{0, 1\}$ for some $i \in \{1, ..., l\}$. In practice, it is useful to decompose l full integer inputs into $l \cdot 32$ bits (assuming we use 32 bits to represent a full integer, like the int type in Java) in order to perform comparison operations on input data. If a committed value a_i is in $[0, 1]$, then its linear polynomial form a_i must have the following property:

$$(a'_i \cdot a'_i - a'_i) = \beta_1 x + \beta_2 x^2 \tag{167}$$

Where $\beta_2 = \alpha^2$, and $\beta_1 = \alpha$ when a_j is 1 and $\beta_1 = -\alpha$ when a_j is 0. To prove the correctness for all $a_i \in \{0, 1\}$, the prover commits to two polynomials K_1, K_2 s.t.

$$K_1 = u_1^{\beta_{1_1}} u_2^{\beta_{1_2}} ... u_l^{\beta_{1_l}} h^{\rho_1} \qquad and \qquad K_2 = u_1^{\beta_{2_1}} u_2^{\beta_{2_2}} ... u_l^{\beta_{2_l}} h^{\rho_2} \tag{168}$$

Where K_1 commits to coefficients on x term for $i \in \{1, ..., l \cdot 32\}$ and K_2 commits to coefficients on x^2 term for $i \in \{1, ..., l \cdot 32\}$. The prover sends K_1, K_2 to the verifier. When the challenge k is received, the prover sends the evaluation results y_1, y_2 to the verifier, and the verifier uses the polynomial commitment protocol to verify the correctness of y_1, y_2 at point k, and checks if the equality below is true:

$$y_1 \cdot x + y_2 \cdot x^2 = \sum_{j=1}^{l \cdot 32} (a'_i \cdot a'_i - a'_i) \cdot k^i \tag{169}$$

Once we know all linear polynomials maps to either 0 or 1, it is trivial to recompose the linear polynomial form of a full integer input a'_i from 32 decomposed bits $a'_{i,j}$ for $j = \{1, ..., 32\}$.

$$a'_i = \sum_{j=1}^{32} a'_{i,j} \cdot 2^j \tag{170}$$

In practice, we will conduct booleanity test on all $l \cdot 32$ bit values at once and then use equation 170 to convert them to l full integer values so that we can perform the "linear polynomial to Pedersen commitment" mapping test explained in the last two sections.

5.3 The Asymptotic Cost of Protocol 3

For a circuit with l input parameters s.t. each input parameter is composed of 32 bits, the prover runtime of the final version (Protocol 3) of our protocol is contributed by $O(m_p \log m_p^{1/2} + m_p + l + 32 \cdot l^{1/2})$ field operations (assuming additive operations in $\mathbb{F}$ are practically free) and $O(m_p + {m_p}^{1/2} + l + 32 \cdot l^{1/2})$ group exponentiations; the verifier runtime is contributed by $O(n + {m_p}^{1/2} + l + 32 \cdot l^{1/2})$ field operations and $O({m_p}^{1/2} + l + 32 \cdot l^{1/2})$ group exponentiations; and the communication cost is contributed by $O({m_p}^{1/2} + l + 32 \cdot l^{1/2})$ group elements and $O({m_p}^{1/2} + l + 32 \cdot l^{1/2})$ field elements.

6 Performance Comparison

We compare the performance of our protocol to some of the most popular transparent Zero Knowledge Protocols that open source codes are available. Our test runs are performed on Intel(R) Core(TM) i7-9750H CPU @ 2.60 Ghz. Only one core is being utilized, and all tests are run on a single CPU thread. Our test code is a non-interactive implementation (using Fiat-Shamir heuristic) of Protocol 3.

The baseline protocols we picked are Hyrax, Ligero, Aurora, and Spartan-NIZK. These protocols are chosen because they are the most representative of popular zero-knowledge protocols and can be verified with open source code. In particular, Aurora outperforms STARK in all key parameters (prover runtime, verifier runtime, proof size), and Spartan offers the most balanced performance across all performance parameters.

We didn't consider transparent protocols that highly depend on circuit depth such as GKR based protocols simply because they can't handle 2^{20} sequential multiplications. We also don't consider VOLE based protocols as they are only optimized for prover work. Other popular transparent schemes such as Bulletproofs are also not being considered because they have linear verifier runtime and therefore are not succinct.

Spartan++ and Lakonia are two more recent developments that we didn't include in our benchmark testing but are worth mentioning. The improvement of Spartan++ over SpartanNIZK is marginal, and the performance of Lakonia is largely comparable to that of SpartanNIZK (the prover performance of SpartanNIZK is approximately 3X more efficient, and the verifier performance is 1.5X more efficient than that of Lakonia, while Lakonia is 4X more efficient than SpartanNIZK in proof size).

We set the number of inputs to our protocol to 30 integers, and each input is represented by 32 bits so that there are a total of $30 \cdot 32 = 960$ input bits to the circuit. The circuit we use performs n sequential multiplications on l inputs, so we have $m_p = n$. Sequential multiplication is likely the worst case scenario of our protocol. If we run a shallow circuit where m_p number is small, the benchmark result will likely be significantly better.

The NTT acceptable prime number we picked for our benchmark testing is $q = 1945555039024054273$, a 61-bit number that implies the soundness error will be at most 2^{-51} for a circuit with 2^{20} sequential multiplications where $m_p = n$, more than enough in most real-life applications. If this is not enough, one can either pick a bigger NTT acceptable prime number or allow one interaction to make the verifier to send the challenge x to the prover.

To maximize the advantage of the NTT algorithm in computing sequential multiplications, we process each segment $(1, ..., m_b)$ of our circuit in binary tree format, such tuning may not be required in real-world applications since large circuits should have multiplication gates somewhat balanced out across layers.

For group operations, we use curve25519-dalek implementation, and Pippenger acceleration is applied to all sum-of-product group operations. For field operations, we use Montgomery algorithm to accelerate modular multiplications on the 61-bit NTT prime q.

Circuit size	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
Hyrax	1	2.8	9	36	117	486
Ligero	0.1	0.4	1.6	4	17	69
Aurora	0.5	1.6	6.5	27	116	485
SpartanNIZK	0.02	0.05	0.16	0.6	1.7	6.2
This Work($m_p = n$)	0.04	0.08	0.18	0.6	1.7	5.5

Table 1. Prover performance comparison (seconds)

Table 1 shows that as the circuit size gets bigger, the prover performance of our protocol is becoming increasingly more efficient than all of our baseline protocols, this is because the cost associated with the number of inputs to the circuit is fixed (960 bits) and its impact relative to the whole circuit cost gradually declines as the circuit size gets bigger.

From the chart we can observe that only SpartanNIZK offers comparable (only slightly worse) prover runtime performance to that of our protocol, but it is worth to note that this is not a fair comparison in our favor since we're comparing 2^{20} constraints in SpartanNIZK (provided in its test code) with the unlikely scenario of 2^{20} sequential multiplications in that of our protocol.

Circuit size	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
Hyrax	14	17	21	28	38	58
Ligero	546	1,076	2,100	5,788	10,527	19,828
Aurora	477	610	810	1,069	1,315	1,603
SpartanNIZK	9	12	15	21	30	48
This Work($m_p = n$)	12	14	18	23	36	65

Table 2. Proof size comparison (kilobytes)

Table 2 shows that the communication cost of our protocol dominates that of Ligero and Aurora, while largely comparable to SpartanNIZK and Hyrax. For higher input number counts, see Table 4 for more detail.

Table 1 and 2 shows that our protocol is largely comparable to the current state of art in prover runtime and communication cost. Table 3 demonstrates that our protocol achieves significant improvement by at least one order of magnitude in verifier runtime over all baseline protocols we are comparing against. Like that of communication cost, the verifier runtime of our protocol will grow when the number of inputs to the protocol grows.

Some may consider 30 integer inputs and 960 input bits to a circuit too small, so in table 4 we list performance benchmarks for different number of inputs (l) to a circuit with 2^{20} sequential multiplications.

Circuit size	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
Hyrax	206	253	331	594	1.6s	8.1s
Ligero	50	179	700	2s	7.5s	33s
Aurora	192	590	2s	7.2s	29.8s	118s
SpartanNIZK	7	11	17	36	103	387
This Work($m_p = n$)	8	9	10	13	18	26

Table 3. Verifier performance comparison (milliseconds)

Input bits ($l \cdot 32$)	Input Integers (l)	Prover time(s)	Verifier time(ms)	Proof size(kb)
960	30	5.5	26	65
1,280	40	5.5	27	69
1,600	50	5.5	27	73
1,920	60	5.5	28	77
2,240	70	5.5	29	81
2,560	80	5.5	30	85
2,880	90	5.6	30	88
3,200	100	5.6	31	93

Table 4. Performance comparison for different input numbers on circuits with 2^{20} sequential multiplications s.t. $m_p = n$

In table 4 we can observe that increases in prover runtime and verifier runtime are small as the input bits count approaching $3,200$. This is because the total input number is still small compared to the size of the circuit (2^{20} sequential multiplications). Communication cost gets impacted the most as the input count gets higher. This is because the prover have to send $l \cdot 32$ linear polynomials to the verifier. Technically speaking, more inputs usually implies lower circuit depth and less complex business logic.

Appendix

A. Proof for Theorem One

Proof. Perfect completeness follows from the fact that Protocol 1 is trivially complete. To prove perfect honest-verifier zero-knowledge, we define a simulator $\mathcal{S}$ to show that protocol 1 has perfect special honest verifier zero-knowledge for relation 3. $\mathcal{S}$ uses simulator $\mathcal{S}_S$ to simulate proof transcripts for proof of knowledge (or proof of discrete logarithm, which we know for a fact that it exists) protocols, and simulator $\mathcal{S}_p$ to simulate proof transcripts for polynomial commitment evaluation function PolyCommitEval.

Simulator $\mathcal{S}$ generates random group elements for C, R, proof of knowledge transcript tr_ϵ. After receiving challenge x from the verifier, the simulator generates l random integers to represent linear polynomials $\vec{a}'$ and one random integer to represent y and sends them to the verifier.

The verifier follows the protocol to compute PK_ϵ, then simulator $\mathcal{S}$ calls simulator $\mathcal{S}_S$ to interact with the verifier and generate all necessary transcripts to prove it knows the value of ϵ. This makes sense since we already know for a fact that schnorr and many other proof of knowledge protocols have perfect special honest verifier zero-knowledge. Similarly, the simulator $\mathcal{S}$ calls simulator $\mathcal{S}_P$ to simulate the transcripts for proving y is the evaluated value at point x for polynomial commitment C.

The simulator then simulates the transcripts to prove it knows α_t and κ. The simulator simply sends randomly generated $PK_{\kappa\mu}$ and random transcripts for $tr_{\kappa\mu}$ and $tr_{\alpha_t\mu}$, and calls simulator $\mathcal{S}_S$ to simulate transcripts needed to prove the knowledge of κ and α_t.

Simulator $\mathcal{S}$ chooses all proof elements and challenges according to the randomness supplied by the adversary from their respective domains or computes them directly as described in the protocol. Since all elements in proof transcripts are either independently randomly distributed or their relationship is fully defined by the verification equations, we can conclude that protocol 1 has perfect special honest verifier zero-knowledge.

To prove computational witness extended emulation, we construct an extractor $\mathcal{X}$, which uses extractor $\mathcal{X}$ to extract witnesses from proof of knowledge transcripts and extractor $\mathcal{X}$ to extract witnesses from polynomial commitments.

We validate the soundness of Protocol 1 in three steps. First, we show how to construct an extractor $\mathcal{X}$ for Protocol 1 s.t. on input $\vec{P} \in \mathbb{G}^l, R \in \mathbb{G}$, it either extracts witnesses $r, \epsilon, \vec{\tau}$ for relation 3, or discovers a non-trivial discrete logarithm relation among $g, h, \vec{u} \in \mathbb{G}$. Next, we show that the extractor $\mathcal{X}$ either extracts witnesses $\vec{a}, \vec{v}$ s.t. $\vec{v}$ maps to $\vec{\alpha}$ or discovers a non-trivial discrete logarithm relation among $g, h, u \in \mathbb{G}$. Finally, we validate the proof by checking if $r, \epsilon, \vec{\tau}$ can be computed from witnesses $\vec{a}, \vec{\alpha}$.

In step one, extractor $\mathcal{X}$ interacts with the prover in the same way as any verifier would and receives $C,\ R, tr_\epsilon$ from the prover. The extractor $\mathcal{X}$ then generates a challenge x_1 and forwards it to the prover. After receiving $\vec{a}'_1, y_1$, the extractor rewinds the prover and sends another challenge x_2 to retrieve $\vec{a}'_2, y_2$.

The extractor then follows the protocol and computes o and PK_ϵ, then calls extractor $\mathcal{X}_S$ to extract ϵ from tr_ϵ and PK_ϵ. With either x_1 or x_2, we can trivially retrieve r, ϵ from r' since $r' = r + x \cdot \epsilon$, and validate if $R = g^r h^\epsilon$. To validate if r, ϵ is correctly computed from the circuit, extractor $\mathcal{X}$ calls extractor $\mathcal{X}_P$ to retrieve set $\vec{\tau}$ from polynomial commitment C.

We have now retrieved witnesses $r, \epsilon, \vec{\tau}$ using the prover committed values C, R, tr_ϵ, and we know for a fact that o must also be computed from $\vec{a}, \vec{\alpha}$ and evaluation point x since:

$$o = r + \epsilon \cdot x + \sum_{i=1}^{n} \tau_i \cdot x^{i+1} \tag{171}$$

If the prover is honest, $r, \epsilon, \vec{\tau}$ must be computed by the prover from witnesses $\vec{a}, \vec{\alpha}$

So in the second step, we validate if witnesses of $\vec{a}'$ $(\vec{a}, \vec{\alpha})$ used in computing o maps to $\vec{a}, \vec{v}$ in $\vec{P}$ by checking if we can extract these witnesses. With $\vec{a}'_1$ and $\vec{a}'_2$ extractor $\mathcal{X}$ retrieved earlier, we can trivially retrieve $\vec{a}, \vec{\alpha}$ since for all $i = \{1, ..., l\}$ we have:

$$a'_{1_i} - a'_{2_i} = \alpha_i(x_1 - x_2)$$

We then extracts witnesses $\vec{a}, \vec{v}$ using $\vec{a}'$ and input commitments $\vec{P}$. The extractor first generates k_1 and then follows the protocol to get $PK_{\kappa\mu_1}, tr_{\kappa\mu_1}, PK_{\alpha_t\mu_1}, tr_{\alpha_t\mu_1}$ from the prover. The extractor then calls extractor $\mathcal{X}_S$ to retrieve κ_1 and α_{t1}. Rewind and repeat this procedure for another l times to retrieve $\kappa_2, ..., \kappa_{l+1}$ and $\alpha_{t2}, ..., \alpha_{tl+1}$ using evaluation points $k_2, ..., k_{l+1}$. (The extractor also retrieves blinding keys $\vec{\mu}$ in the process, but we don't use them here)

Through interpolation technique the extractor retrieves $(\alpha_i - \upsilon_i)$ and α_i for i in $\{1, ..., l\}$. With these information, we can now trivially compute $\vec{v}$ and verify if they can be mapped $\vec{P}$ s.t. $P_i = g^{a_i} h^{\upsilon_i}$ unless we found a non-trivial relationship among generators g, h.

In the last step, we must be able to re-compute witnesses $r, \epsilon, \vec{\tau}$ from $\vec{a}, \vec{\alpha}$ for equality 171 to be true except for a negligible probability or we found a non-trivial relationship among generators $g, h, \vec{u}$. We can therefore conclude Protocol 1 has computational witness extended emulation.

B. Proof for Theorem Two

Proof. Perfect completeness follows from the fact that Protocol 2 is trivially complete. To prove perfect honest-verifier zero-knowledge, we define a simulator $\mathcal{S}$ to show that protocol 2 has perfect special honest verifier zero-knowledge for relation 3. $\mathcal{S}$ uses simulator $\mathcal{S}_S$ to simulate proof transcripts for

proof of knowledge (discrete logarithm) protocols, and simulator $\mathcal{S}_p$ to simulate proof transcripts for polynomial commitment evaluation function PolyCommitEval.

The simulator $\mathcal{S}$ generates random group elements for $\vec{S}, \vec{T}, C, R$, proof of knowledge transcript tr_ϵ. After receiving challenge x from the verifier, the simulator generates l random integers to represent $\vec{e}$, l random integers to represent $\vec{a}'$, and one random integer to represent y and sends them to the verifier.

The simulator follows the protocol to compute o and PK_ϵ, then the simulator $\mathcal{S}$ calls simulator $\mathcal{S}_S$ to interact with the verifier and randomly generate all necessary transcripts to prove it knows the value of ϵ. This makes sense since we already know for a fact that schnorr and many other proof of knowledge protocols have perfect special honest verifier zero-knowledge. Similarly, the simulator $\mathcal{S}$ also calls simulator $\mathcal{S}_P$ to simulate transcripts for proving y is the evaluated value at point x for polynomial commitment C.

Next, simulator $\mathcal{S}$ simulates transcripts for proving the mapping from $\vec{a}'$ to $\vec{P}$. After challenge k is received from the prover, the simulator follows the protocol to compute PK_{υ_t}, then calls simulator $\mathcal{S}_S$ to simulate transcripts needed to prove knowledge of υ_t.

The simulator chooses all proof elements and challenges according to the randomness supplied by the adversary from their respective domains or computes them directly as described in the protocol. Since all elements in proof transcripts are either independently randomly distributed or their relationship is fully defined by the verification equations, we can conclude that protocol 2 is perfect special honest verifier zero-knowledge.

To prove computational witness extended emulation, we construct an extractor $\mathcal{X}$, which uses extractor $\mathcal{X}$ to extract witnesses from proof of knowledge transcripts and extractor $\mathcal{X}$ to extract witnesses from polynomial commitment C.

Like that of Protocol 1, we validate the soundness of Protocol 2 in three steps. First, we show how to construct an extractor $\mathcal{X}$ for Protocol 2 s.t. on input $\vec{P} \in \mathbb{G}^l, R \in \mathbb{G}$, it either extracts witnesses $r, \epsilon, \vec{\tau}$ for relation 3, or discovers a non-trivial discrete logarithm relation among $g, h, \vec{u} \in \mathbb{G}$. Next, we show that the extractor $\mathcal{X}$ either extracts witnesses $\vec{a}, \vec{v}$ s.t. $\vec{v}$ maps to $\vec{\alpha}$ or discovers a non-trivial discrete logarithm relation between $g, h \in \mathbb{G}$. Finally, we validate the proof by checking if $r, \epsilon, \vec{\tau}$ can be computed from witnesses $\vec{a}, \vec{\alpha}$.

In step one, the extractor $\mathcal{X}$ interacts with the prover in Protocol 2 and receives $\vec{S}$, $\vec{T}$, C, R, tr_ϵ from the prover. The extractor $\mathcal{X}$ then generates a challenge x_1 and forward it to the prover. After receiving $\vec{e}_1, \vec{a}'_1, y_1$, the extractor rewinds the prover and sends another challenge x_2 to receive $\vec{e}_2, \vec{a}'_2, y_2$.

The extractor then follows the protocol and calls extractor $\mathcal{X}_S$ to extract ϵ from tr_ϵ and PK_ϵ. With either x_1 or x_2, we can trivially retrieve r from r' since $r' = r + x \cdot \epsilon$ and validate $R = g^r h^\epsilon$. Likewise, the extractor $\mathcal{X}$ calls extractor $\mathcal{X}_P$ using either x_1, y_1 or x_2, y_2 pair to retrieve coefficient set $\vec{\tau}$ from polynomial commitment C. Like that of Protocol 1, we can compute o from r, ϵ at any evaluation point x as equality 171 states.

We have now retrieved witnesses $r, \epsilon, \vec{\tau}$ using transcripts C, R, tr_ϵ. If the prover is honest, $r, \epsilon, \vec{\tau}$ are coefficients computed by the prover from $\vec{a}, \vec{\alpha}$, where as $\vec{\alpha}$ maps to blinding keys $\vec{v}$.

In step two, we validate if witnesses of $\vec{a}'$ used in computing o maps to $\vec{a}, \vec{v}$ in $\vec{P}$ by checking if we can extract these witnesses. The extractor first generates k_1 and then follows the protocol to get $PK_{\upsilon_{t1}}, tr_{\upsilon_{t1}}$, then calls extractor $\mathcal{X}_S$ to retrieve υ_{t1} . The extractor then rewinds and repeats the above step l times to retrieve $\upsilon_{t2}, ..., \upsilon_{tl+1}$. Through interpolation the extractor retrieves witnesses υ_i for all i in $\{1, ..., l\}$. Dividing dividing P_i by h^{υ_i} we will get:

$$P_i / h^{\upsilon_i} = g^{a_i} \tag{172}$$

Using the two different challenges x_1, x_2 we mentioned earlier, the extractor gets $\vec{a}'_1$ and $\vec{a}'_2$ from the prover, which we can trivially retrieve $\vec{a}, \vec{\alpha}$ for all $i = \{1, ..., l\}$ since:

$$a'_{1_i} - a'_{2_i} = \alpha_i (x_1 - x_2)$$

If each υ_i maps to each α_i, then a_i must be the exponent of g in equality 173 or we found a non-trivial relationship among generators g, h.

In step three, we check that if we can re-compute witnesses $r, \epsilon, \vec{\tau}$ from $\vec{a}, \vec{\alpha}$. This must be true for equality 171 to be true except for a negligible probability or we found a non-trivial relationship among generators $g, h, \vec{u}$. We can therefore conclude Protocol 2 has computational witness extended emulation.

C. Proof for Theorem Three

Proof. Perfect completeness follows from the fact that Protocol 3 is trivially complete. To prove perfect honest-verifier zero-knowledge, we define a simulator $\mathcal{S}$ to show that protocol 3 has perfect special honest verifier zero-knowledge for relation 3. $\mathcal{S}$ uses simulator $\mathcal{S}_S$ to simulate proof transcripts for proof of knowledge (or proof of discrete logarithm) protocols.

The simulator $\mathcal{S}$ generates random group elements to represent $\vec{S}, \vec{T}, \vec{C}, B, R$, and the proof of knowledge transcript tr_ϵ. After receiving challenge x from the verifier, the simulator generates l random integers to represent $\vec{e}$, l random integers to represent $\vec{a}'$, and m_b random integers to represent breakers $\vec{y}$. The simulator sends them to the verifier.

The simulator follows the protocol to compute r' and PK_ϵ, then the simulator $\mathcal{S}$ calls the simulator $\mathcal{S}_S$ to interact with the verifier to randomly generate all the necessary transcripts to prove it knows the value of ϵ. This makes sense since we already know for a fact that schnorr and many other proof of discrete log protocols have perfect special honest verifier zero-knowledge.

Next, simulator $\mathcal{S}$ simulates transcripts for proving the mapping from $\vec{a}'$ to $\vec{P}$. After challenge k is received from the verifier, the simulator randomly generates tr_{υ_t} and then follows the protocol to compute PK_{υ_t}. In the final step, the simulator calls $\mathcal{S}$ the simulator $\mathcal{S}_S$ to simulate transcripts needed to prove knowledge of υ_t.

The simulator chooses all proof elements and challenges according to the randomness supplied by the adversary from their respective domains or computes them directly as described in the protocol. Since all elements in proof transcripts are either independently randomly distributed or their relationship is fully defined by the verification equations, we can conclude that protocol 3 is perfect special honest verifier zero-knowledge.

To prove computational witness extended emulation, we construct an extractor $\mathcal{X}$, which uses extractor $\mathcal{X}_S$ to extract witnesses from proof of knowledge transcripts.

Like that of Protocol 1 and 2, we validate the soundness of Protocol 3 in three steps. First, we show how to construct an extractor $\mathcal{X}$ for Protocol 3 s.t. on input $\vec{P} \in \mathbb{G}^l, R \in \mathbb{G}$, it either extracts witnesses $r, \epsilon, \vec{\tau}$ for relation 3, or discovers a non-trivial discrete logarithm relation among $g, h, \vec{u} \in \mathbb{G}$. Second, we show that the extractor $\mathcal{X}$ either extracts witnesses $\vec{a}, \vec{\upsilon}$ s.t. $\vec{\upsilon}$ maps to $\vec{\alpha}$ or discovers a non-trivial discrete logarithm relation between $g, h, \vec{u} \in \mathbb{G}$. Third, we complete validating the proof by checking if $r, \epsilon, \vec{\tau}$ can be computed from witnesses $\vec{a}, \vec{\alpha}$.

In the first step, the extractor $\mathcal{X}$ interacts with the prover in Protocol 3 and receives $\vec{S}$, $\vec{T}$, $\vec{C}$, B, R, tr_ϵ from the prover. The extractor $\mathcal{X}$ then generates at least $b+3$ challenges $\vec{x}$ and forwards them to the prover. After receiving $\vec{e}_1, \vec{a}'_1, \vec{y}_1$, the extractor rewinds and repeats this step $b+2$ times to receive $\vec{e}_2, ..., \vec{e}_{b+3}$, $\vec{a}'_2, ..., \vec{a}'_{b+3}$, and $\vec{y}_2, ..., \vec{y}_{b+3}$.

The extractor then follows the protocol and calls extractor $\mathcal{X}_S$ to extract ϵ from tr_ϵ and PK_ϵ. With any two challenges x_i, x_{i+1}, we can trivially retrieve r, ϵ since $r' = r + x \cdot \epsilon$, which must match the witness r, ϵ retrieved from $R = g^r h^\epsilon$ and PK_ϵ using extractor $\mathcal{X}_S$ except with a negligible probability or discover a non-trivial discrete log relation among generators $g, h \in \mathbb{G}$.

With challenges $x_1, ..., x_{b+3}$ and evaluation (breaker) sets $\vec{y}_1, ..., \vec{y}_{b+3}$, we apply Lagrange polynomial interpolation to retrieve witnesses $\vec{\tau}_1, ..., \vec{\tau}_{m_b}$ and $\vec{\beta}$, coefficients of commitments $\vec{C}, B$.

We have now retrieved witnesses $r, \epsilon, \vec{\tau}_1, ..., \vec{\tau}_{m_b}$ using transcripts $C, B,\ R, tr_\epsilon$. If the prover is honest, $r, \epsilon, \vec{\tau}_1, ..., \vec{\tau}_{m_b}$ are coefficients computed by the prover from $\vec{a}, \vec{\alpha}$, where as $\vec{\alpha}$ maps to blinding keys $\vec{v}$.

In the second step, we validate if witnesses $\vec{a}, \vec{\alpha}$ of $\vec{a}'$ used in computing o map to witnesses $\vec{a}, \vec{v}$ of $\vec{P}$ by checking if we can extract these witnesses and that $\vec{\alpha}$ map to $\vec{v}$. The extractor first generates k_1 and then follows the protocol to get $tr_{\upsilon_{t1}}, PK_{\upsilon_{t1}}$, then calls the extractor $\mathcal{X}_S$ to retrieve υ_{t1} . The extractor then rewinds and repeats this step l times to retrieve $\upsilon_{t2}, ..., \upsilon_{tl+1}$. Through interpolation, the extractor retrieves witnesses υ_i for all i in $\{1, ..., l\}$. Dividing dividing P_i by h^{υ_i} we will get:

$$P_i / h^{\upsilon_i} = g^{a_i} \tag{173}$$

Using any two different challenges x_i, x_{i+1} we mentioned earlier, the extractor gets $\vec{a}'_1$ and $\vec{a}'_2$ from the prover, which we can trivially retrieve $\vec{a}, \vec{\alpha}$ for all $i = \{1, ..., l\}$ since:

$$a'_{1_i} - a'_{2_i} = \alpha_i (x_1 - x_2)$$

If each υ_i maps to each α_i, then a_i must be the exponent of g in equality 173 or we found a non-trivial relationship among generators g, h.

In the final step, we validate the proof by checking if $r, \epsilon, \vec{\tau}$ can be computed from witnesses $\vec{a}, \vec{\alpha}$. This must be true for equality 171 to be true except for a negligible probability or we found a non-trivial relationship among generators $g, h, \vec{u}$. We can therefore conclude Protocol 3 has computational witness extended emulation.

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