ON THE NOTION OF CARRIES OF NUMBERS $2^n - 1$ AND SCHOLZ CONJECTURE

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Abstract. Applying the pothole method on the factors of numbers of the form $2^n - 1$, we prove that if $2^n - 1$ has carries of degree at most

$$\kappa(2^n-1) = \frac{1}{2(1+c)}\lfloor \frac{\log n}{\log 2}\rfloor - 1$$

for $c > 0$ fixed, then the inequality

$$\iota(2^n-1) \leq n - 1 + (1+\frac{1}{1+c})\lfloor \frac{\log n}{\log 2}\rfloor$$

holds for all $n \in \mathbb{N}$ with $n \geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$. In general, we show that all numbers of the form $2^n - 1$ with carries of degree

$$\kappa(2^n-1) := (\frac{1}{1+f(n)})\lfloor \frac{\log n}{\log 2}\rfloor - 1$$

with $f(n) = o(\log n)$ and $f(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ for $n \geq 4$ then the inequality

$$\iota(2^n-1) \leq n - 1 + (1+\frac{2}{1+f(n)})\lfloor \frac{\log n}{\log 2}\rfloor$$

holds.

1. Introduction

An addition chain producing $n \geq 3$, roughly speaking, is a sequence of numbers of the form $1, 2, s_3, s_4, \ldots, s_{k-1}, s_k = n$ where each term is the sum of two earlier terms- not necessarily distinct - in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The length of the chain is determined by the number of entries in the sequence excluding the mandatory first term 1, since it is the only term which cannot be expressed as the sum of two previous terms in the sequence. There are numerous addition chains that result in a fixed number n; In other words, it is always possible to construct as many addition chains producing a fixed number positive integer n as n grows in magnitude. The shortest among these possible chains producing n is regarded as the optimal or the shortest addition chain producing n. There is currently no efficient method for getting the shortest addition yielding a given number, thus reducing an addition chain might be a difficult task, thereby making addition chain theory a fascinating subject to study. By letting $\iota(n)$ denotes the length of the shortest addition chain producing n, then Arnold Scholz conjectured and alfred Braurer proved the following inequalities

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Theorem 1.1 (Braurer)**.** *The inequality*

$$m + 1 \leq \iota(n) \leq 2m$$

for $2^m + 1 \leq n \leq 2^{m+1}$ *holds for* $m \geq 1$*.*

Conjecture 1.1 (Scholz)**.** The inequality

$$\iota(2^n - 1) \leq n - 1 + \iota(n)$$

holds for all $n \geq 2$.

It has been shown computationally by Neill Clift, that the conjecture holds for all $n \leq 5784688$ and in fact it is an equality for all exponents $n \leq 64$ [2]. Alfred Brauer proved the Scholz conjecture for the star addition chain, a special type of addition chain where each term in the sequence obtained by summing uses the immediately subsequent number in the chain. By denoting with $\iota^*(n)$ as the length of the shortest star addition chain producing n, it is shown that (See [1])

Theorem 1.2. *The inequality*

$$\iota^*(2^n - 1) \leq n - 1 + \iota^*(n)$$

holds for all $n \geq 2$*.*

In relation to Conjecture 1.1, Arnold Scholz postulated that Conjecture 1.1 can be improved in general. In particular, Alfred Braurer [1] proved the inequality

$$\iota(n) < \frac{\log n}{\log 2}\left(1 + \frac{1}{\log \log n} + \frac{2 \log 2}{(\log n)^{1 - \log 2}}\right)$$

for $2^m \leq n < 2^{m+1}$ for all sufficiently large n.
Quite a particular special cases of the conjecture has also be studied by many authors in the past. For instance, it is shown in [4] that the scholz conjecture holds for all numbers of the form $2^n - 1$ with $n = 2^q$ and $n = 2^s(2^q + 1)$ for $s, q \geq 0$. If we let $\nu(n)$ denotes the number of $1'$s in the binary expansion of n for $m = 2^n - 1$, then it is shown in [3] that the Scholz conjecture holds in the case $\nu(n) = 5$.

In this paper, we combine the factor method and the newly introduced "fill in the pothole" method with the notion of carries to study the shortest or the optimal addition chains producing numbers of the form $2^n - 1$ and the Scholz conjecture. Given any number of the form $2^n - 1$, we obtain the general decomposition

$$2^n - 1 = (2^{\lfloor \frac{n}{2} \rfloor} - 1)(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \frac{(1 - (-1)^n)}{2}(2^{n - (1 - (-1)^n)\frac{1}{2}})$$

which eventually yield the following decomposition $2^n - 1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1)$ in the case $n \equiv 0 \pmod 2$ and

$$2^n - 1 = (2^{\frac{n-1}{2}} - 1)(2^{\frac{n-1}{2}} + 1) + 2^{n-1}$$

in the case $n \equiv 1 \pmod 2$. We iterate this decomposition up to a certain desired frequency and apply the factor method on all the factors obtained from this decomposition. We then apply the pothole method to obtain a bound for the shortest addition chain producing the only factor of form $2^v - 1$. The length of the shortest addition chains of numbers of the form 2^v+1 is easy to construct, by first constructing the shortest addition chain producing 2^v, adding the first term of the chain to the last term and adjoining to the chain. That is, the chain

$$1, 2, 2^2, \cdots, 2^v, 2^v + 1$$

is the shortest addition chain producing 2^v+1 of length $\iota(2^v+1) = \iota(2^v)+1 = v+1$. Earlier on, the author combined the method of **filling the potholes** and the factor method to prove the improved inequality

Theorem 1.3.

$$\iota(2^n - 1) \leq n + 1 - \sum_{j=1}^{\lfloor \frac{\log n}{\log 2} \rfloor} \xi(n, j) + 3\lfloor \frac{\log n}{\log 2} \rfloor$$

for all $n \in \mathbb{N}$ with $n \geq 4$ for $0 \leq \xi(n, j) < 1$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

The following definitions and elementary properties of addition chains and the shortest addition chain producing n are worth noting parsing the proof of the inequalities in the sequel.

Definition 1.4. Let $n \geq 3$, then by the addition chain of length $k - 1$ producing n, we mean the sequence of positive integers

$$1, 2, \ldots, s_{k-1}, s_k$$

where each term s_j $(j \geq 3)$ in the sequence is the sum of two earlier terms in the sequence, with the corresponding sequence of partition

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

with $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the i^{th} **generator** of the chain for $2 \leq i \leq k$. We call a_i the **determiners** and r_i the **regulator** of the i^{th} generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2 \leq i \leq k$. The **determiners** are the terms produced by summing of previous terms, whereas the **regulators** are chosen from previous terms in the sequence.

Definition 1.5. Let the sequence $1, 2, \ldots, s_{k-1}, s_k = n$ be an addition chain producing n with the corresponding sequence of partition

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then, we call the sub-sequence (s_{j_m}) for $1 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a **sub-addition** chain of the addition chain producing n. We say it is **complete** sub-addition chain of the addition chain producing n if it contains exactly the first t terms of the addition chain. Otherwise we say it is an **incomplete** sub-addition chain.

Lemma 1.6. *Let $\iota(n)$ denotes the length of the shortest addition chain producing n. Then we have the inequality*

$$\lfloor \frac{\log n}{\log 2} \rfloor \leq \iota(n).$$

Proof. The proof of this Lemma can be found in [1]. □

Lemma 1.7. *Let $\iota(n)$ denotes the length of the shortest addition chain producing n. If $a,b \in \mathbb{N}$ then*
$$\iota(ab) \leq \iota(a) + \iota(b).$$

Proof. The proof of this Lemma can be found in [1]. □

2. The notion of carries

We devote this section to the study of the notion of carries and its number theoretic properties. It turns out that this notion plays an important role in controlling the length of an addition for numbers of the form $2^n - 1$. Short addition chains with small carries almost satisfy the Scholz conjecture . We launch the following languages.

Definition 2.1. Consider the decomposition
$$2^n - 1 = (2^{\lfloor \frac{n}{2} \rfloor} - 1)(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \frac{(1-(-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}})$$
for $n \geq 2$. Then the non-zero remainder
$$\eta(2^n - 1) := \frac{(1-(-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}})$$
is the level one carry of $2^n - 1$. We say that $2^n - 1$ is free of level one carries if $\eta(2^n - 1) = 0$. By letting
$$m = \lfloor \frac{n}{2} \rfloor$$
then we obtain the decomposition
$$2^m - 1 = (2^{\lfloor \frac{m}{2} \rfloor} - 1)(2^{\lfloor \frac{m}{2} \rfloor} + 1) + \frac{(1-(-1)^m)}{2}(2^{m-(1-(-1)^m)\frac{1}{2}})$$
and we denote the carry with
$$\eta(2^m - 1) = \frac{(1-(-1)^m)}{2}(2^{m-(1-(-1)^m)\frac{1}{2}})$$
and we say it is the level two carry of $2^n - 1$ if $\eta(2^m - 1) \neq 0$. In general, we denote the level k carry of $2^n - 1$ as the remainder
$$\eta(2^r - 1) = \frac{(1-(-1)^r)}{2}(2^{r-(1-(-1)^r)\frac{1}{2}})$$
with
$$r = \lfloor \frac{n}{2^k} \rfloor.$$
We say that $2^n - 1$ is free of level k carries if $\eta(2^r - 1) = 0$. The number of non-zero levels of carry of $2^n - 1$ for all $1 \leq k \leq \lfloor \frac{\log n}{\log 2} \rfloor$ is the **degree** of carry of $2^n - 1$.

Proposition 2.1. *The number $2^n - 1 \quad (n \geq 2)$ is free of level one carry if and only if $n \equiv 0 \pmod 2$.*

Proof. Suppose that $2^n - 1$ is free of level one carry, then

$$\eta(2^n - 1) = \frac{(1 - (-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}}) = 0.$$

This is only possible with $(1 - (-1)^n) = 0$ and when $n \equiv 0 \pmod 2$. Conversely, suppose that $n \equiv 0 \pmod 2$ then $\frac{n}{2} \in \mathbb{N}$ and we can write

$$2^n - 1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1)$$

and we see that

$$\eta(2^n - 1) = 0.$$

□

Integers of the form $2^n - 1$ with high degrees of carry serve as an obstruction to achieving the inequality

$$\iota(2^n - 1) \leq n - 1 + \iota(n)$$

[$\iota(n)$ is the shortest addition chain producing n] using our current method. At best avoiding them can yield progress on the conjecture using the current method but only for a specialized set of integers of the form $2^n - 1$ with low degrees of carry. It turns out that the nature of the exponents in large part characterizes integers with high degree (resp. low degree) carries. Encountering integers of the form $2^n - 1$ with exponents giving rise to high degree carries can be controlled in a way to minimize the corresponding length of the addition chain. At the moment we prove that we can obtain a chain of small length for numbers $2^n - 1$ with exponents giving rise to low degree carries.

3. Improved inequality using the notion of carries

In this section, we prove an explicit upper bound for the length of the shortest addition chain producing numbers of the form $2^n - 1$. We begin with the following important but fundamental result.

Theorem 3.1. *If $2^n - 1$ has carries of degree at most*

$$\kappa(2^n - 1) = \frac{1}{2(1 + c)}\lfloor\frac{\log n}{\log 2}\rfloor - 1$$

for $c > 0$ fixed, then the inequality

$$\iota(2^n - 1) \leq n - 1 + (1 + \frac{1}{1 + c})\lfloor\frac{\log n}{\log 2}\rfloor$$

holds for all $n \in \mathbb{N}$ with $n \geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

Proof. For a fixed $c > 0$ let $2^n - 1$ has at most

$$\frac{1}{2(1 + c)}\lfloor\frac{\log n}{\log 2}\rfloor - 1$$

degrees of carries. Next decompose the number $2^n - 1$ and obtain the decomposition

$$2^n - 1 = (2^{\lfloor\frac{n}{2}\rfloor} - 1)(2^{\lfloor\frac{n}{2}\rfloor} + 1) + \eta(2^n - 1)$$

where

$$\eta(2^n - 1) := \frac{(1 - (-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}})$$

is the level one carry of 2^n-1. It is easy to see that we can recover the general factorization of 2^n-1 from this identity according to the parity of the exponent n. In particular, if $n\equiv 0 \pmod 2$, then we have

$$2^n-1=(2^{\frac{n}{2}}-1)(2^{\frac{n}{2}}+1)$$

and

$$2^n-1=(2^{\frac{n-1}{2}}-1)(2^{\frac{n-1}{2}}+1)+2^{n-1}$$

if $n\equiv 1 \pmod 2$. By combining both cases, we obtain the inequality

$$\iota(2^n-1)\leq \iota((2^{\lfloor \frac{n}{2}\rfloor}-1)(2^{\lfloor \frac{n}{2}\rfloor}+1))+\eta(2^n-1).$$

Applying Lemma 1.7, we obtain further the inequality

$$\iota(2^n-1)\leq \iota(2^{\lfloor \frac{n}{2}\rfloor}-1)+\iota(2^{\lfloor \frac{n}{2}\rfloor}+1)+\eta(2^n-1) \tag{3.1}$$

Again let us set $\lfloor \frac{n}{2}\rfloor=k$ in (3.12), then we obtain the general decomposition

$$2^k-1=(2^{\lfloor \frac{k}{2}\rfloor}-1)(2^{\lfloor \frac{k}{2}\rfloor}+1)+\eta(2^k-1)$$

where

$$\eta(2^k-1)=\frac{(1-(-1)^k)}{2}(2^{k-(1-(-1)^k)\frac{1}{2}})$$

is the carry of 2^k-1. It is easy to see that we can recover the general factorization of 2^k-1 from this identity according to the parity of the exponent k. In particular, if $k\equiv 0 \pmod 2$, then we have

$$2^k-1=(2^{\frac{k}{2}}-1)(2^{\frac{k}{2}}+1)$$

and

$$2^k-1=(2^{\frac{k-1}{2}}-1)(2^{\frac{k-1}{2}}+1)+2^{k-1}$$

if $k\equiv 1 \pmod 2$. By combining both cases, we obtain the inequality

$$\iota(2^k-1)\leq \iota((2^{\lfloor \frac{k}{2}\rfloor}-1)(2^{\lfloor \frac{k}{2}\rfloor}+1))+\eta(2^k-1).$$

Applying Lemma 1.7, we obtain further the inequality

$$\begin{aligned}\iota(2^k-1)&\leq \iota(2^{\lfloor \frac{k}{2}\rfloor}-1)+\iota(2^{\lfloor \frac{k}{2}\rfloor}+1)+\eta(2^k-1)\\&=\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}-1)+\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}+1)+\eta(2^{\lfloor \frac{n}{2}\rfloor}-1)\end{aligned} \tag{3.2}$$

so that by inserting (3.13) into (3.12), we obtain the inequality

$$\begin{aligned}\iota(2^n-1)&\leq \iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}-1)+\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}+1)+\eta(2^{\lfloor \frac{n}{2}\rfloor}-1)\\&+\iota(2^{\lfloor \frac{n}{2}\rfloor}+1)+\eta(2^n-1).\end{aligned} \tag{3.3}$$

Next we iterate the factorization up to frequency s to obtain

$$\begin{aligned}\iota(2^n-1)&\leq \iota(2^{\lfloor \frac{n}{2}\rfloor}+1)+\eta(2^n-1)+\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}-1)+\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}+1)+\eta(2^{\lfloor \frac{n}{2}\rfloor}-1)\\&+\cdots+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}+1)+\eta(2^{\lfloor \frac{n}{2^{s-1}}\rfloor}-1)\end{aligned} \tag{3.4}$$

where $0\leq \xi(n,s)<1$ for an integer $2\leq s:=s(n)$ fixed to be chosen later. For instance,

$$\xi(n,1)=(1-(-1)^n)\frac{1}{4}<1$$

and

$$\xi(n,2)=(1-(-1)^n)\frac{1}{8}+(1-(-1)^k)\frac{1}{4}<1$$

with

$$k := \lfloor \frac{n}{2} \rfloor$$

and so on. Indeed the function $\xi(n,s)$ for values of $s \geq 3$ can be read from exponents of the terms arising from the iteration process. It follows from (3.15) the inequality

$$\iota(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + s + 2\sum_{j=1}^{s} \sum_{\substack{\eta(2^m-1)\neq 0 \\ m=\lfloor \frac{n}{2^{j-1}} \rfloor}} 1 - \theta(n,s) + \iota(2^{\frac{n}{2^s} - \xi(n,s)} - 1)$$

$$= n(1 - \frac{1}{2^s}) + s + 2\sum_{j=1}^{s} \sum_{\substack{\eta(2^m-1)\neq 0 \\ m=\lfloor \frac{n}{2^{j-1}} \rfloor}} 1 - \theta(n,s) + \iota(2^{\frac{n}{2^s} - \xi(n,s)} - 1) \tag{3.5}$$

where the term

$$\sum_{j=1}^{s} \sum_{\substack{\kappa(2^m-1)\neq 0 \\ m=\lfloor \frac{n}{2^{j-1}} \rfloor}} 1$$

counts the number of all non-zero carry of $2^n - 1$ up to level s and $0 \leq \theta(n,s) := \sum_{j=1}^{s} \xi(n,j)$ and $2 \leq s := s(n)$ fixed, an integer to be chosen later. It is worth noting that

$$\theta(n,s) := \sum_{j=1}^{s} \xi(n,j) = 0$$

if $n = 2^r$ for some $r \in \mathbb{N}$, since $\xi(n,j) = 0$ for each $1 \leq j \leq s$ for all n which are powers of 2. It is also important to note that the $2s$ term is obtained by noting that there are at most s terms with odd exponents under the iteration process and each term with odd exponent contributes 2, and the other s term comes from summing 1 with frequency s finding the total length of the short addition chains producing numbers of the form $2^v + 1$. Now, we set $k = \frac{n}{2^s} - \xi(n,s)$ and construct the addition chain producing 2^k as $1, 2, 2^2, \ldots, 2^{k-1}, 2^k$ with corresponding sequence of partition

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3 \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}, 2^k = 2^{k-1} + 2^{k-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq k+1$, where a_i and r_i denotes the determiner and the regulator of the i^{th} generator of the chain. Let us consider only the complete sub-addition chain

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}.$$

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor}, 2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-1}{2^2} \rfloor}, \ldots, 2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-1}{2^2} \rfloor} + \cdots + 2^1.$$

Since $\xi(n,s) = 0$ if $n = 2^r$ and $0 \leq \xi(n,s) < 1$ if $n \neq 2^r$, we note that the adjoined sequence contributes at most

$$\lfloor \frac{\log k}{\log 2} \rfloor = \lfloor \frac{\log(\frac{n}{2^s} - \xi(n,s))}{\log 2} \rfloor = \lfloor \frac{\log n - s\log 2}{\log 2} \rfloor = \lfloor \frac{\log n}{\log 2} \rfloor - s$$

terms to the original complete sub-addition chain, where the upper bound follows by virtue of Lemma 1.6. Since the inequality holds

$$2^{k-1}+2^{\lfloor \frac{k-1}{2}\rfloor}+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1<\sum_{i=1}^{k-1}2^i$$
$$=2^k-2$$

we insert terms into the sum

$$2^{k-1}+2^{\lfloor \frac{k-1}{2}\rfloor}+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1 \tag{3.6}$$

so that we have

$$\sum_{i=1}^{k-1}2^i=2^k-2.$$

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert $2^{k-2}+2^{k-3}+\cdots+2^{\lfloor \frac{k-1}{2}\rfloor+1}$ into (3.17) and this comes at the cost of adjoining

$$k-2-\lfloor \frac{k-1}{2}\rfloor$$

terms to the term in (3.17). The last term of the adjoined sequence is given by

$$2^{k-1}+(2^{k-2}+2^{k-3}+\cdots+2^{\lfloor \frac{k-1}{2}\rfloor+1})+2^{\lfloor \frac{k-1}{2}\rfloor}+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1. \tag{3.7}$$

Again we have to insert $2^{\lfloor \frac{k-1}{2}\rfloor-1}+\cdots+2^{\lfloor \frac{k-1}{2^2}\rfloor+1}$ into (3.18) and this comes at the cost of adjoining

$$\lfloor \frac{k-1}{2}\rfloor-\lfloor \frac{k-1}{2^2}\rfloor-1$$

terms to the term in (3.18). The last term of the adjoined sequence is given by

$$2^{k-1}+(2^{k-2}+2^{k-3}+\cdots+2^{\lfloor \frac{k-1}{2}\rfloor+1})+2^{\lfloor \frac{k-1}{2}\rfloor}+(2^{\lfloor \frac{k-1}{2}\rfloor-1}+\cdots+2^{\lfloor \frac{k-1}{2^2}\rfloor+1})+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1. \tag{3.8}$$

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (3.19) and this comes at the cost of adjoining

$$\lfloor \frac{k-1}{2^j}\rfloor-\lfloor \frac{k-1}{2^{j+1}}\rfloor-1$$

terms to the term in (3.19) for $j\leq \lfloor \frac{\log n}{\log 2}\rfloor-s$, since we are filling in at most $\lfloor \frac{\log k}{\log 2}\rfloor$ blocks with $k=\frac{n}{2^s}-\xi(n,s)$. It follows that the contribution of these new terms is at most

$$k-1-\left\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2}\rfloor}}\right\rfloor-\lfloor \frac{\log k}{\log 2}\rfloor \tag{3.9}$$

obtained by adding the numbers in the chain

$$k-1-\lfloor \frac{k-1}{2}\rfloor-1$$

$$\lfloor \frac{k-1}{2}\rfloor-\lfloor \frac{k-1}{2^2}\rfloor-1$$

....
....

....
....

$$\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2} \rfloor}} \rfloor - \lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2} \rfloor + 1}} \rfloor - 1.$$

By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain producing $2^k - 1$ with $k = \frac{n}{2^s} - \xi(n,s)$ is

$$\delta(2^k - 1) \leq k + k - 1 - \left\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2} \rfloor + 1}} \right\rfloor - \lfloor \frac{\log k}{\log 2} \rfloor + \lfloor \frac{\log n}{\log 2} \rfloor - s$$

$$\leq \frac{n}{2^{s-1}} - 1 - \left\lfloor \frac{\frac{n}{2^s} - \xi(n,s) - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor + 1 - s}} \right\rfloor - \lfloor \frac{\log n}{\log 2} \rfloor + s + \lfloor \frac{\log n}{\log 2} \rfloor - s$$

$$= \frac{n}{2^{s-1}} - 1 - \left\lfloor \frac{\frac{n}{2^s} - \xi(n,s) - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor + 1 - s}} \right\rfloor . \tag{3.10}$$

By plugging the inequality (3.21) into the inequalities in (3.16) and noting that $\iota(\cdot) \leq \delta(\cdot)$, we obtain the inequality

$$\iota(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + s + 2\sum_{j=1}^{s} \sum_{\substack{\eta(2^m - 1) \neq 0 \\ m = \lfloor \frac{n}{2^{j-1}} \rfloor}} 1 - \theta(n,s) + \iota(2^{\frac{n}{2^s} - \xi(n,s)} - 1)$$

(3.11)

$$= n(1 - \frac{1}{2^s}) + \frac{n}{2^{s-1}} - 1 + s + 2\sum_{j=1}^{s} \sum_{\substack{\eta(2^m - 1) \neq 0 \\ m = \lfloor \frac{n}{2^{j-1}} \rfloor}} 1 - \theta(n,s) - \left\lfloor \frac{\frac{n}{2^s} - \xi(n,s) - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor + 1 - s}} \right\rfloor$$

$$= n - 1 + \frac{n}{2^s} + s + 2\sum_{j=1}^{s} \sum_{\substack{\eta(2^m - 1) \neq 0 \\ m = \lfloor \frac{n}{2^{j-1}} \rfloor}} 1 - \theta(n,s) - \left\lfloor \frac{\frac{n}{2^s} - \xi(n,s) - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor + 1 - s}} \right\rfloor$$

where we note that

$$\sum_{j=1}^{s} \sum_{\substack{\eta(2^m - 1) \neq 0 \\ m = \lfloor \frac{n}{2^{j-1}} \rfloor}} 1$$

counts the number of non-zero carries up to the s level for the number $2^n - 1$. By taking $2 \leq s := s(n)$ such that $s = \lfloor \frac{\log n}{\log 2} \rfloor$ which is the maximum frequency of the iteration, then

$$\left\lfloor \frac{\frac{n}{2^s} - \xi(n,s) - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor + 1 - s}} \right\rfloor = 0$$

and we obtained that

$$\sum_{j=1}^{s} \sum_{\substack{\eta(2^m - 1) \neq 0 \\ m = \lfloor \frac{n}{2^{j-1}} \rfloor}} 1 \leq \frac{1}{2(1+c)} \lfloor \frac{\log n}{\log 2} \rfloor - 1$$

and the inequality

$$\iota(2^n-1)\leq n-1-\theta(n,\lfloor\frac{\log n}{\log 2}\rfloor)+\lfloor\frac{\log n}{\log 2}\rfloor+2+\frac{1}{(1+c)}\lfloor\frac{\log n}{\log 2}\rfloor-2$$

for $\theta(n,\lfloor\frac{\log n}{\log 2}\rfloor):=\sum\limits_{j=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}\xi(n,j)>0$ with $n\geq 4$ and the claimed inequality follows as a consequence. □

Now we show that numbers of the form 2^n-1 with low degree carries almost satisfy the Scholz conjecture.

Theorem 3.2. *If 2^n-1 has carries of degree at most*

$$\kappa(2^n-1):=(\frac{1}{1+\log n})\lfloor\frac{\log n}{\log 2}\rfloor-1$$

then the inequality

$$\iota(2^n-1)\leq n-1+(1+\frac{2}{1+\log n})\lfloor\frac{\log n}{\log 2}\rfloor$$

holds for all $n\in\mathbb{N}$ with $n\geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

Proof. let 2^n-1 has at most

$$\frac{1}{(1+\log n)}\lfloor\frac{\log n}{\log 2}\rfloor-1$$

degrees of carries. Next decompose the number 2^n-1 and obtain the decomposition

$$2^n-1=(2^{\lfloor\frac{n}{2}\rfloor}-1)(2^{\lfloor\frac{n}{2}\rfloor}+1)+\eta(2^n-1)$$

where

$$\eta(2^n-1):=\frac{(1-(-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}})$$

is the level one carry of 2^n-1. It is easy to see that we can recover the general factorization of 2^n-1 from this identity according to the parity of the exponent n. In particular, if $n\equiv 0 \pmod 2$, then we have

$$2^n-1=(2^{\frac{n}{2}}-1)(2^{\frac{n}{2}}+1)$$

and

$$2^n-1=(2^{\frac{n-1}{2}}-1)(2^{\frac{n-1}{2}}+1)+2^{n-1}$$

if $n\equiv 1 \pmod 2$. By combining both cases, we obtain the inequality

$$\iota(2^n-1)\leq\iota((2^{\lfloor\frac{n}{2}\rfloor}-1)(2^{\lfloor\frac{n}{2}\rfloor}+1))+\eta(2^n-1).$$

Applying Lemma 1.7, we obtain further the inequality

$$\iota(2^n-1)\leq\iota(2^{\lfloor\frac{n}{2}\rfloor}-1)+\iota(2^{\lfloor\frac{n}{2}\rfloor}+1)+\eta(2^n-1)\tag{3.12}$$

Again let us set $\lfloor\frac{n}{2}\rfloor=k$ in (3.12), then we obtain the general decomposition

$$2^k-1=(2^{\lfloor\frac{k}{2}\rfloor}-1)(2^{\lfloor\frac{k}{2}\rfloor}+1)+\eta(2^k-1)$$

where

$$\eta(2^k-1)=\frac{(1-(-1)^k)}{2}(2^{k-(1-(-1)^k)\frac{1}{2}})$$

is the carry of 2^k-1. It is easy to see that we can recover the general factorization of 2^k-1 from this identity according to the parity of the exponent k. In particular, if $k \equiv 0 \pmod 2$, then we have

$$2^k - 1 = (2^{\frac{k}{2}} - 1)(2^{\frac{k}{2}} + 1)$$

and

$$2^k - 1 = (2^{\frac{k-1}{2}} - 1)(2^{\frac{k-1}{2}} + 1) + 2^{k-1}$$

if $k \equiv 1 \pmod 2$. By combining both cases, we obtain the inequality

$$\iota(2^k - 1) \leq \iota((2^{\lfloor \frac{k}{2} \rfloor} - 1)(2^{\lfloor \frac{k}{2} \rfloor} + 1)) + \eta(2^k - 1).$$

Applying Lemma 1.7, we obtain further the inequality

$$\begin{aligned}\iota(2^k - 1) &\leq \iota(2^{\lfloor \frac{k}{2} \rfloor} - 1) + \iota(2^{\lfloor \frac{k}{2} \rfloor} + 1) + \eta(2^k - 1)\\ &= \iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2} \rfloor\rfloor} - 1) + \iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2} \rfloor\rfloor} + 1) + \eta(2^{\lfloor \frac{n}{2} \rfloor} - 1) \qquad (3.13)\end{aligned}$$

so that by inserting (3.13) into (3.12), we obtain the inequality

$$\begin{aligned}\iota(2^n - 1) &\leq \iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2} \rfloor\rfloor} - 1) + \iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2} \rfloor\rfloor} + 1) + \eta(2^{\lfloor \frac{n}{2} \rfloor} - 1)\\ &\quad + \iota(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \eta(2^n - 1). \qquad (3.14)\end{aligned}$$

Next we iterate the factorization up to frequency s to obtain

$$\begin{aligned}\iota(2^n - 1) &\leq \iota(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \eta(2^n - 1) + \iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2} \rfloor\rfloor} - 1) + \iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2} \rfloor\rfloor} + 1) + \eta(2^{\lfloor \frac{n}{2} \rfloor} - 1)\\ &\quad + \cdots + \iota(2^{\frac{n}{2^s} - \xi(n,s)} - 1) + \iota(2^{\frac{n}{2^s} - \xi(n,s)} + 1) + \eta(2^{\lfloor \frac{n}{2^{s-1}} \rfloor} - 1) \qquad (3.15)\end{aligned}$$

where $0 \leq \xi(n,s) < 1$ for an integer $2 \leq s := s(n)$ fixed to be chosen later. For instance,

$$\xi(n,1) = (1 - (-1)^n)\frac{1}{4} < 1$$

and

$$\xi(n,2) = (1 - (-1)^n)\frac{1}{8} + (1 - (-1)^k)\frac{1}{4} < 1$$

with

$$k := \lfloor \frac{n}{2} \rfloor$$

and so on. Indeed the function $\xi(n,s)$ for values of $s \geq 3$ can be read from exponents of the terms arising from the iteration process. It follows from (3.15) the inequality

$$\begin{aligned}\iota(2^n - 1) &\leq \sum_{v=1}^{s} \frac{n}{2^v} + s + 2\sum_{j=1}^{s} \sum_{\substack{\eta(2^m - 1) \neq 0\\ m = \lfloor \frac{n}{2^{j-1}} \rfloor}} 1 - \theta(n,s) + \iota(2^{\frac{n}{2^s} - \xi(n,s)} - 1)\\ &= n(1 - \frac{1}{2^s}) + s + 2\sum_{j=1}^{s} \sum_{\substack{\eta(2^m - 1) \neq 0\\ m = \lfloor \frac{n}{2^{j-1}} \rfloor}} 1 - \theta(n,s) + \iota(2^{\frac{n}{2^s} - \xi(n,s)} - 1) \qquad (3.16)\end{aligned}$$

where the term

$$\sum_{j=1}^{s} \sum_{\substack{\kappa(2^m - 1) \neq 0\\ m = \lfloor \frac{n}{2^{j-1}} \rfloor}} 1$$

counts the number of all non-zero carry of 2^n-1 up to level s and $0\leq \theta(n,s):=\sum\limits_{j=1}^{s}\xi(n,j)$ and $2\leq s:=s(n)$ fixed, an integer to be chosen later. It is worth noting that

$$\theta(n,s):=\sum_{j=1}^{s}\xi(n,j)=0$$

if $n=2^r$ for some $r\in \mathbb{N}$, since $\xi(n,j)=0$ for each $1\leq j\leq s$ for all n which are powers of 2. It is also important to note that the $2s$ term is obtained by noting that there are at most s terms with odd exponents under the iteration process and each term with odd exponent contributes 2, and the other s term comes from summing 1 with frequency s finding the total length of the short addition chains producing numbers of the form 2^v+1. Now, we set $k=\frac{n}{2^s}-\xi(n,s)$ and construct the addition chain producing 2^k as $1,2,2^2,\ldots,2^{k-1},2^k$ with corresponding sequence of partition

$$2=1+1,2+2=2^2,2^2+2^2=2^3\ldots,2^{k-1}=2^{k-2}+2^{k-2},2^k=2^{k-1}+2^{k-1}$$

with $a_i=2^{i-2}=r_i$ for $2\leq i\leq k+1$, where a_i and r_i denotes the determiner and the regulator of the i^{th} generator of the chain. Let us consider only the complete sub-addition chain

$$2=1+1,2+2=2^2,2^2+2^2=2^3,\ldots,2^{k-1}=2^{k-2}+2^{k-2}.$$

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor},2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor},\ldots,2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor}+\cdots+2^1.$$

Since $\xi(n,s)=0$ if $n=2^r$ and $0\leq \xi(n,s)<1$ if $n\neq 2^r$, we note that the adjoined sequence contributes at most

$$\lfloor\frac{\log k}{\log 2}\rfloor=\lfloor\frac{\log(\frac{n}{2^s}-\xi(n,s))}{\log 2}\rfloor=\lfloor\frac{\log n-s\log 2}{\log 2}\rfloor=\lfloor\frac{\log n}{\log 2}\rfloor-s$$

terms to the original complete sub-addition chain, where the upper bound follows by virtue of Lemma 1.6. Since the inequality holds

$$\begin{aligned}2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor}+\cdots+2^1&<\sum_{i=1}^{k-1}2^i\\&=2^k-2\end{aligned}$$

we insert terms into the sum

$$2^{k-1}+2^{\lfloor\frac{k-1}{2}\rfloor}+2^{\lfloor\frac{k-1}{2^2}\rfloor}+\cdots+2^1 \tag{3.17}$$

so that we have

$$\sum_{i=1}^{k-1}2^i=2^k-2.$$

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert $2^{k-2}+2^{k-3}+\cdots+2^{\lfloor\frac{k-1}{2}\rfloor+1}$ into (3.17) and this comes at the cost of adjoining

$$k-2-\lfloor\frac{k-1}{2}\rfloor$$

terms to the term in (3.17). The last term of the adjoined sequence is given by

$$2^{k-1} + (2^{k-2} + 2^{k-3} + \cdots + 2^{\lfloor \frac{k-1}{2} \rfloor + 1}) + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-1}{2^2} \rfloor} + \cdots + 2^1. \tag{3.18}$$

Again we have to insert $2^{\lfloor \frac{k-1}{2} \rfloor - 1} + \cdots + 2^{\lfloor \frac{k-1}{2^2} \rfloor + 1}$ into (3.18) and this comes at the cost of adjoining

$$\lfloor \frac{k-1}{2} \rfloor - \lfloor \frac{k-1}{2^2} \rfloor - 1$$

terms to the term in (3.18). The last term of the adjoined sequence is given by

$$2^{k-1} + (2^{k-2} + 2^{k-3} + \cdots + 2^{\lfloor \frac{k-1}{2} \rfloor + 1}) + 2^{\lfloor \frac{k-1}{2} \rfloor} + (2^{\lfloor \frac{k-1}{2} \rfloor - 1} + \cdots + 2^{\lfloor \frac{k-1}{2^2} \rfloor + 1}) + 2^{\lfloor \frac{k-1}{2^2} \rfloor} + \cdots + 2^1. \tag{3.19}$$

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (3.19) and this comes at the cost of adjoining

$$\lfloor \frac{k-1}{2^j} \rfloor - \lfloor \frac{k-1}{2^{j+1}} \rfloor - 1$$

terms to the term in (3.19) for $j \leq \lfloor \frac{\log n}{\log 2} \rfloor - s$, since we are filling in at most $\lfloor \frac{\log k}{\log 2} \rfloor$ blocks with $k = \frac{n}{2^s} - \xi(n,s)$. It follows that the contribution of these new terms is at most

$$k - 1 - \left\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2} \rfloor}} \right\rfloor - \lfloor \frac{\log k}{\log 2} \rfloor \tag{3.20}$$

obtained by adding the numbers in the chain

$$k - 1 - \lfloor \frac{k-1}{2} \rfloor - 1$$

$$\lfloor \frac{k-1}{2} \rfloor - \lfloor \frac{k-1}{2^2} \rfloor - 1$$

$$\vdots$$

$$\vdots$$

$$\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2} \rfloor}} \rfloor - \lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2} \rfloor + 1}} \rfloor - 1.$$

By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain producing $2^k - 1$ with $k = \frac{n}{2^s} - \xi(n,s)$ is

$$\begin{aligned}
\delta(2^k - 1) &\leq k + k - 1 - \left\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2} \rfloor + 1}} \right\rfloor - \lfloor \frac{\log k}{\log 2} \rfloor + \lfloor \frac{\log n}{\log 2} \rfloor - s \\
&\leq \frac{n}{2^{s-1}} - 1 - \left\lfloor \frac{\frac{n}{2^s} - \xi(n,s) - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor + 1 - s}} \right\rfloor - \lfloor \frac{\log n}{\log 2} \rfloor + s + \lfloor \frac{\log n}{\log 2} \rfloor - s \\
&= \frac{n}{2^{s-1}} - 1 - \left\lfloor \frac{\frac{n}{2^s} - \xi(n,s) - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor + 1 - s}} \right\rfloor . \qquad (3.21)
\end{aligned}$$

By plugging the inequality (3.21) into the inequalities in (3.16) and noting that $\iota(\cdot) \leq \delta(\cdot)$, we obtain the inequality

$$\iota(2^n-1) \leq \sum_{v=1}^{s}\frac{n}{2^v}+s+2\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1-\theta(n,s)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1)$$

(3.22)

$$=n(1-\frac{1}{2^s})+\frac{n}{2^{s-1}}-1+s+2\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1-\theta(n,s)-\left\lfloor\frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor\frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor$$

$$=n-1+\frac{n}{2^s}+s+2\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1-\theta(n,s)-\left\lfloor\frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor\frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor$$

where we note that

$$\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1$$

counts the number of non-zero carries up to the s level for the number 2^n-1. By taking $2 \leq s := s(n)$ such that $s=\lfloor\frac{\log n}{\log 2}\rfloor$ which is the maximum frequency of the iteration, then

$$\left\lfloor\frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor\frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor=0$$

and we obtained that

$$\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1 \leq \frac{1}{2(1+c)}\lfloor\frac{\log n}{\log 2}\rfloor-1$$

and the inequality

$$\iota(2^n-1)\leq n-1-\theta(n,\lfloor\frac{\log n}{\log 2}\rfloor)+\lfloor\frac{\log n}{\log 2}\rfloor+2+\frac{2}{(1+\log n)}\lfloor\frac{\log n}{\log 2}\rfloor-2$$

for $\theta(n,\lfloor\frac{\log n}{\log 2}\rfloor) := \sum_{j=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}\xi(n,j)>0$ with $n\geq 4$ and the claimed inequality follows as a consequence. □

The proofs presented in Theorem 3.1 and 3.2 serves as model for obtaining improved upper bound for the shortest length of addition chains producing numbers of the form 2^n-1. Indeed, without using the notion of carries one can obtain the weaker upper bound which holds for all exponents $n \geq 4$.

Theorem 3.3.

$$\iota(2^n-1)\leq n+1-\sum_{j=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}\xi(n,j)+3\lfloor\frac{\log n}{\log 2}\rfloor$$

for all $n \in \mathbb{N}$ with $n \geq 4$ for $0 \leq \xi(n,j) < 1$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

It follows similarly from the proofs the following result which holds for numbers of the form $2^n - 1$ with low degree of carries

$$\frac{1}{(1+\log\log n)}\lfloor\frac{\log n}{\log 2}\rfloor - 1$$

Theorem 3.4. *If $2^n - 1$ has carries of degree at most*

$$\kappa(2^n-1) := (\frac{1}{1+\log\log n})\lfloor\frac{\log n}{\log 2}\rfloor - 1$$

then the inequality

$$\iota(2^n-1) \leq n - 1 + (1+\frac{2}{1+\log\log n})\lfloor\frac{\log n}{\log 2}\rfloor$$

holds for all $n \in \mathbb{N}$ with $n \geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

We obtain the more general theorem

Theorem 3.5. *If $2^n - 1$ has carries of degree at most*

$$\kappa(2^n-1) := (\frac{1}{1+f(n)})\lfloor\frac{\log n}{\log 2}\rfloor - 1$$

where $f(n) = o(\log n)$ with $f(n) \longrightarrow \infty$ as $n \longrightarrow \infty$, then the inequality

$$\iota(2^n-1) \leq n - 1 + (1+\frac{2}{1+f(n)})\lfloor\frac{\log n}{\log 2}\rfloor$$

holds for all $n \in \mathbb{N}$ with $n \geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

The following chain of results we have obtained illustrates that to make progress on the Scholz conjecture, it suffices to study possible way of controlling numbers of the form $2^n - 1$ with high carries. In other words, the degree of carries of numbers of the form $2^n - 1$ determines the quality of the upper bound for its corresponding length of the shortest addition using the current method.
[1].

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