

Vector Commitments With Short Proofs of Smallness

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Abstract. Vector commitment schemes are compressing commitments to vectors that make it possible to succinctly open a commitment for individual vector positions without revealing anything about other positions. We describe vector commitments enabling constant-size proofs that the committed vector is small (i.e., binary, ternary, or of small norm). As a special case, we obtain range proofs featuring the shortest proof length in the literature with only 3 group elements per proof. As another application, we obtain short pairing-based NIZK arguments for lattice-related statements. In particular, we obtain short proofs (comprised of 3 group elements) showing the validity of ring LWE ciphertexts and public keys. Our constructions are proven simulation-extractable in the algebraic group model and the random oracle model.

Keywords. Vector commitments, range proofs, ring LWE ciphertexts.

1 Introduction

Vector commitments (VCs) [84,26] allow a user to commit to a vector of messages $\mathbf{m} = (m_1, \dots, m_n) \in D^n$ over some domain D by generating a short commitment string. Later, the committer should be able to succinctly open individual entries of \mathbf{m} . Here, “succinctly” means that the partial opening information (called “proof”) should have constant size – no matter how large the committed vector is – and still convince the verifier that the opened coordinate is correct. As in standard commitments, a vector commitment scheme should satisfy two security properties: (i) The *binding* property, which ensures that no efficient adversary can open a commitment to two different values at the same position $i \in [n]$; (ii) The *hiding* property, which guarantees that revealing a subset of components does not reveal any information about messages at remaining positions.

Vector commitments found a number of applications in the context of zero-knowledge databases [84], verifiable data streaming [80], authenticated dictionaries [101], de-centralized storage [25], succinct arguments [11,81], cryptocurrencies [32,100] and blockchain transactions [11,63].

In this paper, we consider the problem of extending vector commitments with optimally short proofs that the committed vector \mathbf{m} has small entries. A straightforward solution is to generically use a general-purpose succinct non-interactive argument (SNARK) for all NP languages [91]. While the SNARKs of [67,56,69] would give constant-size proofs, they would require to represent the

statement as an arithmetic circuit. Then, the latter would have to compute the opening algorithm (including exponentiations in a group) of the commitment scheme, which would result in a complex circuit. In turn, this would require a large structured common reference string (CRS) and make the proof generation very expensive since, in pairing-based SNARKs with very short proofs [56,69,51], the CRS size grows linearly with the number of multiplication gates in the arithmetic circuit. Inevitably, the computational cost of the prover grows (at least) linearly with the circuit size as well. In this paper, we aim at proving smallness more efficiently than by generically using a SNARK for all NP statements.

1.1 Our Contributions

We revisit the vector commitment scheme of Libert and Yung [84] and propose a technique allowing to argue the smallness of committed vectors without changing the commitment algorithm nor its public parameters. Using a very small number of group elements (typically 2 or 3), we can prove that a committed vector is binary, ternary or that it has small infinity norm. By slightly increasing the proof length, we can also prove that a committed vector has small Euclidean norm or a small Hamming weight.

A key building block of our non-interactive arguments of smallness is a technique of generating a short proof that a committed \mathbf{m} has binary entries. The proof is made of only two group elements and is proven knowledge-sound in the combined algebraic group model (AGM) [49] and random oracle model. In addition, the scheme retains the useful properties of the original vector commitment [84]. In particular, its CRS size remains linear in the dimension n of committed vectors and it remains possible to succinctly open the commitment for individual vector positions. As in [83], it is also possible to prove that a committed (binary) $\mathbf{m} \in \mathbb{Z}_p$ satisfies a linear equation $\langle \mathbf{m}, \mathbf{t} \rangle = x$ for a public $\mathbf{t} \in \mathbb{Z}_p^n$ and a public $x \in \mathbb{Z}_p$. Finally, it retains the aggregation properties [83,63] that make it possible to generate a constant-size proof for a sub-vector opening.

As a first application of our arguments of binarity, we obtain a new construction of range proof featuring extremely short proofs. Regardless of the range magnitude, each proof consists of only 3 group elements, which matches the proof size of Groth’s SNARK [69] and improves upon the shortest known range proof due to Boneh *et al.* [14]. The construction extends to simultaneously prove possibly distinct ranges for the individual entries of a vector $\mathbf{x} = (x_1, \dots, x_m)$ without affecting the proof size. As a special case, it implies very short proofs that a committed $\mathbf{x} \in \mathbb{Z}^n$ has small infinity norm.

As a second main application, we provide short pairing-based non-interactive zero-knowledge (NIZK) arguments for many natural statements appearing in lattice-based cryptography. Specifically, we can argue knowledge of small-norm elements s_1, \dots, s_n of a cyclotomic ring $R = \mathbb{Z}[X]/(X^d + 1)$ that satisfy a linear relation $\sum_{i=1}^M \mathbf{a}_i \cdot s_i = \mathbf{t}$, for public vectors of ring elements $\mathbf{a}_1, \dots, \mathbf{a}_M, \mathbf{t} \in R_q^N$, where $R_q = R/(qR)$. Using only 3 group elements, we can prove the validity of a ring LWE (RLWE) ciphertext [88], an RLWE public key, or even FHE ciphertexts [19,45,33]. We can also prove that a committed vector is a solution to

an instance of the subset sum problem, which is useful for all the applications considered in [46]. For the specific task of proving the validity of a ciphertext in the Lyubashevsky-Peikert-Regev cryptosystem [88], we provide efficiency comparisons with Groth’s SNARK [69], which is the state-of-the art construction featuring the same proof size. We estimate that the size of the common reference string is reduced by a factor 2. While slower on the verifier’s side, our scheme decreases the number of exponentiations at the prover by a factor 4. The reason is that, on the prover and verifier sides, the number of exponentiations only depends on the length of the witness and not on the size of the arithmetic circuit describing the relation. Our construction thus provides a more balanced tradeoff than SNARKs between the complexities of the prover and the verifier. As such, it can be useful in cloud or blockchain applications where it is desirable to minimize the overhead of the client even at the cost of increasing the workload of the server. For example, in FHE-based private smart contracts [38,97] (which explicitly require ZK proofs of input awareness), a resource-constrained client has to prove the validity of its input FHE ciphertexts before sending them to a computationally powerful server performing homomorphic operations.

Our NIZK arguments of range membership and ciphertext validity can be proven simulation-extractable in the algebraic group model [49] and the random oracle model (recall that all such succinct arguments have to rely on an idealized model [59]). Simulation-extractability guarantees knowledge-soundness even when the adversary can observe proofs generated by honest parties. It thus provides non-malleability [42] guarantees against a malicious prover attempting to create a proof of its own by mauling honestly generated proofs. As pointed out in, e.g., [55,53], it is an important security property in all applications where succinct arguments are easily observable in the wild. For example, if a malleable range proof is used to demonstrate the validity of confidential transactions (as in the use case of [21]), it may fail to ensure transaction independence.

Luckily, we can prove simulation-extractability without increasing the proof length while even the random-oracle-optimized variants [18,6] of Groth’s SNARK have longer proofs. For the optimal proof length, existing SNARKs either provide a relaxed flavor of simulation-extractability [5] or they are computationally more demanding [70] than [69].

1.2 Technical Overview

In asymmetric pairings $e : \mathbb{G} \times \hat{\mathbb{G}} \rightarrow \mathbb{G}_T$, the scheme of [84] uses a CRS containing group elements $(g, \{g_i = g^{(\alpha^i)}\}_{i \in [2n] \setminus \{n+1\}})$ and $(\hat{g}, \{\hat{g}_i = \hat{g}^{(\alpha^i)}\}_{i=1}^n)$. The sender commits to $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_p^n$ by choosing $\gamma \xleftarrow{R} \mathbb{Z}_p$ and computing

$$C = g^\gamma \cdot \prod_{j=1}^n g_j^{m_j} = g^{\gamma + \sum_{j=1}^n m_j \cdot \alpha^j} .$$

To open a position $i \in [n]$ of \mathbf{m} , the committer reveals a proof

$$\pi_i = g_{n+1-i}^\gamma \cdot \prod_{j=1, j \neq i}^n g_{n+1-i+j}^{m_j} = \left(C/g^{m_i \cdot \alpha^i} \right)^{\alpha^{n+1-i}}$$

which is verified by checking that $e(C, \hat{g}_{n+1-i}) = e(\pi_i, \hat{g}) \cdot e(g_1, \hat{g}_n)^{m_i}$.

To aggregate multiple proofs, PointProofs [63] uses the observation [83] that the commitment of [84] allows proving that a committed $\mathbf{m} \in \mathbb{Z}_p^n$ satisfies an inner product relation $\langle \mathbf{m}, \mathbf{t} \rangle = x$ for public $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{Z}_p^n$ and $x \in \mathbb{Z}_p$. By raising the verification equation to the power $t_i \in \mathbb{Z}_p$ and taking the product over all $i \in [n]$, we obtain

$$e\left(C, \prod_{i=1}^n \hat{g}_{n+1-i}^{t_i}\right) = e\left(\prod_{i=1}^n \pi_i^{t_i}, \hat{g}\right) \cdot e(g_1, \hat{g}_n)^{\sum_{i=1}^n m_i \cdot t_i} . \quad (1)$$

PointProofs [63] aggregates proofs $\{\pi_i\}_{i \in S}$ for a sub-vector $S \subseteq [n]$ by deriving aggregation coefficients $\{t_i\}_{i \in S}$ from a random oracle and defining the aggregated proof as the product $\pi_S = \prod_{i \in S} \pi_i^{t_i}$. Verification then proceeds by testing the equality $e(C, \hat{g}_{n+1-i})^{\sum_{i \in S} t_i} = e(\pi_S, \hat{g}) \cdot e(g_1, \hat{g}_n)^{\sum_{i \in S} m_i \cdot t_i}$.

PROVING BINARITY. Let a commitment $\hat{C} = \hat{g}^\gamma \cdot \prod_{i=1}^n \hat{g}_i^{x_i}$ to $\mathbf{x} \in \{0, 1\}^n$.¹ Using its proof aggregation properties, we prove that, for each $i \in [n]$, we have $x_i \cdot (x_i - 1) = 0$. To this end, we use a similar batching technique to BulletProofs [21] and show that $\sum_{i=1}^n y_i \cdot x_i \cdot (x_i - 1) = 0$, where $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{Z}_p^n$ is a vector of random aggregation coefficients obtained by hashing $\mathbf{y} = H(\hat{C})$ using a random oracle $H : \{0, 1\}^* \rightarrow \mathbb{Z}_p^n$. As long as $\mathbf{y} \in \mathbb{Z}_p^n$ is chosen uniformly *after* $\{x_i\}_{i=1}^n$, the probability to have $\sum_{i=1}^n y_i \cdot x_i \cdot (x_i - 1) = 0$ is only $1/p$ if there exists $i \in [n]$ such that $x_i \notin \{0, 1\}$.

In order to prove the statement using a constant number of group elements, we first choose $\gamma_y \stackrel{R}{\leftarrow} \mathbb{Z}_p$ and generate an auxiliary commitment

$$C_y = g^{\gamma_y} \cdot \prod_{j=1}^n g_{n+1-j}^{y_j \cdot x_j}, \quad (2)$$

to the Hadamard product $\mathbf{y} \circ \mathbf{x} = (y_1 \cdot x_1, \dots, y_n \cdot x_n)$ (in the reversed order). Then, we proceed in two steps.

In a first step, our prover has to demonstrate that it really computed C_y as a commitment to $(y_n \cdot x_n, \dots, y_1 \cdot x_1)$. Since the commitment (2) satisfies

$$e(C_y, \hat{g}_i) = e\left(g_i^{\gamma_y} \cdot \prod_{j=1, j \neq i}^n g_{n+1-j+i}^{y_j \cdot x_j}, \hat{g}\right) \cdot e(g_1, \hat{g}_n)^{y_i \cdot x_i} \quad \forall i \in [n] \quad (3)$$

¹ For our applications, we will assume that the commitment is in $\hat{\mathbb{G}}$ rather than \mathbb{G} in order for the proof of knowledge-soundness to work out.

and the initial commitment $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^n \hat{g}_j^{x_j}$ satisfies

$$e(g_{n+1-i}, \hat{C}) = e(g_{n+1-i}^\gamma \cdot \prod_{j=1, j \neq i}^n g_{n+1-i+j}^{x_j}, \hat{g}) \cdot e(g_1, \hat{g}_n)^{x_i} \quad \forall i \in [n], \quad (4)$$

we can choose random exponents $\mathbf{t} = (t_1, \dots, t_n) \stackrel{R}{\leftarrow} \mathbb{Z}_p^n$ and use them to raise (4) to the power $t_i \cdot y_i$ and (3) to the power t_i , respectively. If we then take the products over all indices $i \in [n]$ and divide them, we find that

$$\pi_{eq} = \frac{\prod_{i=1}^n \left(g_{n+1-i}^\gamma \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-i+j}^{x_j} \right)^{t_i \cdot y_i}}{\prod_{i=1}^n \left(g_i^{\gamma y} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-j+i}^{y_j \cdot x_j} \right)^{t_i}}.$$

satisfies

$$\frac{e(\prod_{i=1}^n g_{n+1-i}^{t_i \cdot y_i}, \hat{C})}{e(C_y, \prod_{i=1}^n \hat{g}_i^{t_i})} = e(\pi_{eq}, \hat{g}), \quad (5)$$

The reason why π_{eq} is a convincing proof that the prover computed C_y as a commitment to $(y_n \cdot x_n, \dots, y_1 \cdot x_1)$ is the following. Suppose that C_y is a commitment $C_y = g^{\gamma y} \cdot \prod_{j=1}^n g_{n+1-j}^{z_{n+1-j}}$ to some (z_1, \dots, z_n) . Then, (3) becomes

$$e(C_y, \hat{g}_i) = e(\pi_{z,i}, \hat{g}) \cdot e(g_1, \hat{g}_n)^{z_{n+1-i}} \quad \forall i \in [n], \quad (6)$$

where $\pi_{z,i} = \prod_{j=1, j \neq i}^n g_{n+1-j+i}^{z_{n+1-j}}$ is the proof that a prover can compute to open the $(n+1-i)$ -th position of C_y . Now, if we raise (6) to the power t_i and divide it from (4) raised to the power $t_i \cdot y_i$, we obtain

$$\frac{e(\prod_{i=1}^n g_{n+1-i}^{t_i \cdot y_i}, \hat{C})}{e(C_y, \prod_{i=1}^n \hat{g}_i^{t_i})} = e\left(\prod_{i=1}^n (\pi_{x,i}^{y_i} / \pi_{z,i})^{t_i}, \hat{g}\right) \cdot e(g_1, \hat{g}_n)^{\sum_{i=1}^n t_i \cdot (y_i \cdot x_i - z_{n+1-i})},$$

where $\pi_{x,i} = \prod_{j \in [n] \setminus \{i\}} g_{n+1-i+j}^{x_j}$ is the computable proof that allows opening the i -th position of \hat{C} in (4). If \mathbf{t} is chosen uniformly after (z_1, \dots, z_n) , (y_1, \dots, y_n) and (x_1, \dots, x_n) , then the probability to have $\sum_{i=1}^n t_i \cdot (y_i \cdot x_i - z_{n+1-i}) = 0$ is $1/p$ if there exists $i \in [n]$ such that $z_{n+1-i} \neq y_i \cdot x_i$. In the construction, we derive $\mathbf{t} = (t_1, \dots, t_n) = H(\mathbf{y}, \hat{C}, C_y) \in \mathbb{Z}_p^n$ from a random oracle to make sure that \mathbf{t} is computed after \mathbf{y} , (x_1, \dots, x_n) and (z_1, \dots, z_n) .

The proof π_{eq} of the first step implies that $C_y \cdot \prod_{j=1}^n g_{n+1-j}^{-y_j}$ is a commitment to the vector $(y_n \cdot (x_n - 1), \dots, y_1 \cdot (x_1 - 1))$, where (x_1, \dots, x_n) is the vector committed in \hat{C} . In a second step, we prove that $(y_n \cdot (x_n - 1), \dots, y_1 \cdot (x_1 - 1))$ is orthogonal to (x_n, \dots, x_1) : i.e., $\sum_{i=1}^n y_i \cdot x_i \cdot (x_i - 1) = 0$. From (1), we notice that such a proof can be obtained as

$$\pi_y = C_y^\gamma \cdot \prod_{i=1}^n \left(g_i^{\gamma y} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-j+i}^{y_j \cdot (x_j - 1)} \right)^{x_i}$$

and satisfies

$$e(C_y, \prod_{j=1}^n g_{n+1-j}^{-y_j}, \hat{C}) = e(\pi_y, \hat{g}) \cdot e(g_1, \hat{g}_n)^{\sum_{i=1}^n y_i \cdot x_i \cdot (x_i - 1)} = e(\pi_y, \hat{g}) \quad (7)$$

In order to minimize the proof size, we will exploit the linearity of verification equations (5) and (7) to aggregate π_{eq} and π_y into a single group element $\pi = \pi_{eq}^{\delta_{eq}} \cdot \pi_y^{\delta_y}$ using random aggregation coefficients $(\delta_{eq}, \delta_y) \in \mathbb{Z}_p^2$.

Eventually, the proof $\pi = (C_y, \pi) \in \mathbb{G}^2$ that \hat{C} commits to a binary vector consists of the commitment C_y to $(y_n \cdot x_n, \dots, y_1 \cdot x_1)$ and $\pi \in \mathbb{G}$.

PROVING RANGE MEMBERSHIP. To obtain a constant-size range proof, we use the fact that the commitment scheme of [84] is also an inner product functional commitment. The prover has a Pedersen commitment [94] $\hat{V} = \hat{g}^r \cdot \hat{g}_1^x$ in the group $\hat{\mathbb{G}}$. In order to prove the statement $x \in [0, 2^\ell - 1]$, the prover considers the bit representation $(x_1, \dots, x_\ell) \in \{0, 1\}^\ell$ of x and computes a commitment $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^\ell g_j^{x_j}$, for a random $\gamma \in \mathbb{Z}_p$. Using the aggregation properties of the commitment, it will prove that the committed $\mathbf{x} = (x_1, \dots, x_\ell \mid \mathbf{0}^{n-\ell}) \in \mathbb{Z}_p^n$ satisfies: (i) $\sum_{i=1}^\ell x_i \cdot 2^{i-1} = x$; (ii) $x_i \in \{0, 1\}$ for each $i \in [n]$.

In order to prove (i), the prover can adapt (1) and generate a short proof $\prod_{i=1}^n \pi_i^{2^{i-1}} \in \mathbb{G}$ such that

$$e(\prod_{i=1}^\ell g_{n+1-i}^{2^{i-1}}, \hat{C}) = e(\prod_{i=1}^\ell \pi_i^{2^{i-1}}, \hat{g}) \cdot e(g_1, \hat{g}_n)^{\sum_{i=1}^\ell x_i \cdot 2^{i-1}} \quad (8)$$

and show that the exponent above $e(g_1, \hat{g}_n)$ in (8) is equal to the committed x in $\hat{V} = \hat{g}^r \cdot \hat{g}_1^x$. Since \hat{V} satisfies $e(g_n, \hat{V}) = e(g_1, \hat{g}_n)^x \cdot e(g_n^r, \hat{g})$, the prover can actually compute $\pi_x = \prod_{i=1}^\ell \pi_i^{2^{i-1}} / g_n^r$ such that

$$\frac{e(\prod_{i=1}^\ell g_{n+1-i}^{2^{i-1}}, \hat{C})}{e(g_n, \hat{V})} = e(\pi_x, \hat{g}) \quad (9)$$

Proving (ii) is addressed as explained earlier. Note that we do not need to prove that the $n - \ell$ last positions of \mathbf{x} are zeroes since the inner product in the right-hand-side member of (8) only involves the first ℓ positions of \mathbf{x} .

In order to minimize the proof size, we will exploit the linearity of verification equations (9), (5) and (7) to aggregate π_x , π_{eq} and π_y into a single group element. In order to ensure knowledge soundness in the algebraic group model, we also need to aggregate a proof element π_v showing that \hat{V} is a commitment to a vector of the form $(x, 0, \dots, 0)$. The entire proof $\pi = (\hat{C}, C_y, \pi) \in \hat{\mathbb{G}} \times \mathbb{G}^2$ eventually consists of the commitment \hat{C} to the bits of x , the auxiliary commitment C_y to $(y_n \cdot x_n, \dots, y_1 \cdot x_1)$ and the aggregated proof $\pi \in \mathbb{G}$.

BATCHING RANGE PROOFS. The above technique extends to prove multiple range membership statements at once about the entries of a committed vector.

For a commitment $\hat{V} = \hat{g}^r \cdot \prod_{k=1}^m \hat{g}_k^{x_k}$, the prover will convince the verifier that $x_k \in [0, 2^\ell - 1]$ for each $k \in [m]$ using only 3 group elements (we assume for now that the same range is proven for each x_k but distinct ranges can be handled).

To this end, we can use the same aggregation technique as BulletProofs [21, Section 4.3] and compute \hat{C} as a commitment to a vector of dimension $n = \bar{\ell} \cdot m$ (where $\bar{\ell}$ is an upper bound for ℓ) obtained by appending the binary expansions of all $\{x_k\}_{k=1}^m$. Then, we can use a single group element to prove that, for each $k \in [m]$, the k -th sub-vector $\mathbf{x}_k = (x_{k,1}, \dots, x_{k,\ell}, 0, \dots, 0)$ hidden by the commitment \hat{C} is a binary vector satisfying $x_k = \sum_{i=1}^{\ell} x_{k,i} \cdot 2^{i-1}$. Namely, for the k -th slot, the prover can compute $\pi_k \in \mathbb{G}$ such that

$$e\left(\prod_{i=1}^{\ell} g_{n+1-((k-1)\bar{\ell}+i)}^{2^{i-1}}, \hat{C}\right) = e(g_1, \hat{g}_n)^{x_k} \cdot e(\pi_k, \hat{g}) \quad (10)$$

Since \hat{V} is itself a vector commitment, the prover can compute $\pi_{v,k}$ such that

$$e(g_k, \hat{V}) = e(g_1, \hat{g}_n)^{x_k} \cdot e(\pi_{v,k}, \hat{g}) \quad (11)$$

Then, by dividing (11) from (10), raising the result to a random power $\xi_k \in \mathbb{Z}_p$ and taking the product over all indices $k \in [m]$, we find that the prover is able to compute a short $\pi = \prod_{k=1}^m (\pi_k / \pi_{v,k})^{\xi_k}$ such that

$$\frac{e\left(\prod_{k=1}^m \left(\prod_{i=1}^{\ell} g_{n+1-((k-1)\bar{\ell}+i)}^{2^{i-1}}\right)^{\xi_k}, \hat{C}\right)}{e\left(\prod_{k=1}^m g_k^{\xi_k}, \hat{V}\right)} = e(\pi, \hat{g}), \quad (12)$$

which argues that $x_k = \sum_{i=1}^{\ell} x_{k,i} \cdot 2^{i-1}$ for all $k \in [m]$. Indeed, otherwise, we have $\sum_{k=1}^m \xi_k \cdot (x_k - \sum_{i=1}^{\ell} x_{k,i} \cdot 2^{i-1}) = 0$ with negligible probability $1/p$ as long as (ξ_1, \dots, ξ_m) are chosen uniformly after the commitments \hat{V} and \hat{C} .

The remaining proof elements are computed exactly as in the single-slot setting, so that the final proof $\boldsymbol{\pi}$ still lives in $\hat{\mathbb{G}} \times \mathbb{G}^2$. This immediately provides a short proof that a committed vector has small infinity norm. By introducing a few more group elements in the proof, we can also prove small Euclidean norms.

PROVING RELATIONS IN LATTICES. Here, we build on an approach considered by del Pino, Lyubashevsky and Seiler [41] to prove lattice-related statements assuming the hardness of computing discrete logarithms. The difference that we replace the BulletProofs component [21] by our more compact proof that a committed vector is binary. We also exploit the fact that the underlying vector commitment [84] allows proving inner-product relations as in (1).

Let the polynomial rings $R = \mathbb{Z}[X]/(\Phi)$, for some cyclotomic polynomial Φ of degree d , and $R_q = R/(qR)$. As in [41], we aim at proving the existence of small-norm ring elements $\mathbf{s} = (s_1, \dots, s_M) \in R^M$ such that $\sum_{i=1}^M \mathbf{a}_i \cdot s_i = \mathbf{t} \bmod (q, \Phi)$, for public $\mathbf{t} \in R_q^N$ and $\mathbf{a}_1, \dots, \mathbf{a}_M \in R_q^N$. To this end, we proceed as in [41] and re-write the relation as the following equality over $\mathbb{Z}[X]/(\Phi)$

$$\sum_{i=1}^M \mathbf{a}_i \cdot s_i = \mathbf{t} + \mathbf{r} \cdot q \bmod (\Phi), \quad (13)$$

where $\mathbf{r} \in R^N$ is a vector of polynomials of degree $\leq d - 1$ and the components of $\{\mathbf{a}_i\}_{i=1}^M$ and \mathbf{t} are interpreted as polynomials with integer coefficients in $\{-\lfloor q/2 \rfloor, \dots, \lfloor q/2 \rfloor\}$. If $\|s_i\|_\infty \leq B_i$ for each $i \in [M]$, \mathbf{r} contains polynomials with coefficients of magnitude $\|\mathbf{r}\|_\infty \leq dM \cdot \max_{i \in [M]}(B_i)/2$.

If we denote by $\phi : R \rightarrow \mathbb{Z}^d$ the coefficient embedding that maps $s_i = \sum_{j=1}^d s_{i,j} \cdot X^{j-1}$ to its coefficient vector $\phi(s_i) = (s_{i,1}, \dots, s_{i,d}) \in \mathbb{Z}^d$, we can re-write (13) as a matrix-vector product over \mathbb{Z}

$$[\mathbf{A}_1 \mid \dots \mid \mathbf{A}_M \mid -q \cdot \mathbf{I}_{Nd}] \cdot \underbrace{[\phi(s_1) \mid \dots \mid \phi(s_M) \mid \phi(\mathbf{r})]^\top}_{\triangleq \mathbf{x}} = \phi(\mathbf{t}) \quad (14)$$

for structured matrices $\mathbf{A}_1, \dots, \mathbf{A}_M \in \mathbb{Z}_q^{Nd \times d}$ interpreted as integer matrices over $\{-\lfloor q/2 \rfloor, \dots, \lfloor q/2 \rfloor\}$. In order to prove (14), the prover can commit to the vector $\mathbf{x} \in \mathbb{Z}^{Md+Nd}$ using a vector commitment. Then, it can generate short proof that $\|\phi(s_i)\|_\infty \leq B_i$ for each $i \in [M]$ and $\|\phi(\mathbf{r})\|_\infty \leq dM \cdot \max_{i \in [M]}(B_i)/2$. Finally, it can prove that (14) holds over \mathbb{Z}_p , where p is the order of pairing-friendly groups. If $p > 2Mqd \max_i(B_i)$, this ensures that (14) also holds over the integers. In order to optimize the proof size, we commit to the binary decomposition of $(\phi(s_1), \dots, \phi(s_M), \phi(\mathbf{r}))$ and prove a relation like (14) where the blocks of the matrix are multiplied by a suitable ‘‘powers-of-2’’ gadget matrix [92].

In order to minimize the number of exponentiations, we apply the Schwartz-Zippel lemma in a different way than [41]: Instead of proving (13) by considering evaluations of degree- $2d$ polynomials,² we compress (14) by left-multiplying both members with a random vector $\boldsymbol{\theta} \in \mathbb{Z}_p^{Nd}$, which allows processing all the rows of (14) using a short proof for a single inner product relation.

Besides the shorter proof length, replacing BulletProofs by our arguments of binarity has the side effect of decreasing the number of exponentiations at the prover in the approach of del Pino *et al.* [41]: We need at most 340000 exponentiations to prove the validity of an LPR ciphertext [88] whereas [41] requires about 700000 exponentiations to prove an RLWE instance with smaller parameters.

Of course, neither of the two protocols preserves soundness against quantum adversaries. However, they still provide viable solutions in applications that only need to guarantee soundness at the moment of the protocol execution. In particular, they do not affect the post-quantum security of the encryption scheme as their zero-knowledge property does not rely on any assumption.

ACHIEVING SIMULATION-EXTRACTABILITY. In our security proofs, one of the main difficulties is to properly simulate proofs for adversarially-chosen statements while remaining able to extract a witness (or break some assumption) from a proof generated by the adversary. As noticed in, e.g. [53], the simulator cannot use the trapdoor $\alpha \in \mathbb{Z}_p$ of the CRS since it would be incompatible with a reduction from a q -type assumption in the AGM.

To address this problem, we build a trapdoor-less simulator [53] that can

² More precisely, [41] proceeds by proving a relation $\sum_{i=1}^M \mathbf{a}_i \cdot s_i - \mathbf{r}_1 \cdot q - \mathbf{r}_2 \cdot \Phi = \mathbf{t}$ over $\mathbb{Z}[X]$, where \mathbf{r}_1 and \mathbf{r}_2 contain polynomials of degree $2(d-1)$ and $d-2$, respectively.

simulate proofs for adversarially-chosen statements by programming the random oracles and without using α . To do this, we exploit the fact that our range proofs and our proof of valid RLWE encryption are obtained by aggregating various sub-proofs satisfying verifications of the form (5), (7) or (8). In each simulated proof $\pi = (\hat{C}, C_y, \pi)$, we compute \hat{C} and C_y as commitments to vectors which are programmed (as functions of previously chosen aggregation coefficients) in such a way that the uniquely determined proof π is computable without knowing the missing element $g^{(\alpha^{n+1})}$ of the CRS. At the same time, we can argue that, with overwhelming probability, the adversary can only come up with a statement and a proof π from which the reduction can extract either a witness or a representation of π that depends on $g^{(\alpha^{n+1})}$.

1.3 Related Work

Vector commitments with logarithmic-size proofs are known since the Merkle-tree-based construction [90]. In the last decade, a number of number-theoretic candidates have emerged and offered useful advantages such as additive homomorphism, very short proofs [84], stateless updatability [26], or sub-vector openings [81,11,100]. The first candidate with constant-size proofs was put forth by Libert and Yung [84] under a q -type assumption. Constructions based on the standard Diffie-Hellman assumption (in pairing-friendly groups) and the RSA assumption appeared in the work of Catalano and Fiore [26]. Lattice-based schemes were suggested by Peikert *et al.* [95]. While more versatile than their hash-based counterparts, algebraic VCs also seem to require more fancy mathematical tools. Indeed, Catalano *et al.* [27] recently proved negative results on the possibility of discrete-log-based vector commitments without pairings.

POLYNOMIAL AND FUNCTIONAL COMMITMENTS. As first introduced by Kate, Zaverucha and Goldberg [77], polynomial commitments allow one to commit to a polynomial and subsequently prove evaluations of this polynomial on specific inputs via a short proof (i.e., of length sub-linear in the degree of the committed polynomial). Succinct polynomial commitments were used in a number of SNARKs realizations (see, e.g., [89,51,22,13]). As shown in, e.g. [24, Section 3.1], polynomial commitments imply vector commitments.

Functional commitments (FC) for inner products [74,83] generalize both vector commitments and polynomial commitments by allowing the sender to commit to a vector \mathbf{m} and succinctly prove linear functions of the committed vector. The first flavor of inner product functional commitment was considered in the interactive setting [74] while non-interactive solutions with constant-size proofs are enabled by SNARKs. Libert, Ramanna and Yung [83] generalized the vector commitment of [84] into a non-interactive inner product FC in the standard model while preserving its short proof size. Constructions with short public parameters in hidden-order groups were put forth in [34,4]. Lai and Malavolta [81] proposed the notion of linear map commitments that allows a prover to reveal a linear map evaluation, instead of just an inner product. At the expense of losing the homomorphic property, Lipmaa and Pavlyk [86] provided an FC candidate

for sparse polynomials. Recently, lattice-based realizations were given for general bounded-depth circuits [40,102]. Boneh *et al.* [15] considered the dual notion of function-hiding FC schemes (where the committer commits to a function instead of a message) for arithmetic circuits, which also generalizes vector commitments and other primitives. More recently, Albrecht *et al.* described [1] a construction for constant-degree polynomials over the integers as a building block for lattice-based SNARKs. Back in 2015, Gorbunov *et al.* [64] implicitly described non-succinct functional commitments for circuits. Succinct FC candidates for circuits recently appeared in the work of Wee and Wu [102]. Independently, de Castro and Peikert [40] proposed a lattice-based function-hiding FC for circuits, but without fully succinct evaluation proofs.

Vector commitments with succinct proofs of smallness can be seen as a special case of functional commitments for Boolean predicates, where the smallness bound is hard-wired in the circuit. However, functional commitments for general circuits [40,102] seem ill-suited to our purposes since we aim at computationally efficient schemes with very short proofs. Indeed, the function-hiding FC scheme proposed by de Castro and Peikert [40] does not provide succinct openings (i.e., the opening size grows with the input length). While succinct, the construction of Wee and Wu [102] would not compete with ours in terms of proof length and CRS size (which is quadratic in the dimension of committed vectors in [102]). Moreover, in our application to NIZK arguments, the scheme of [102] would require the use of ad hoc knowledge assumptions in lattices for lack of a well-defined lattice analogue of the algebraic group model. Balbás *et al.* [7] suggested an alternative realization of FC for arithmetic circuits. However, its proof length grows at least linearly with the depth of the arithmetic circuit, which would translate into much longer proofs than ours.

In an earlier work, Catalano, Fiore and Tucker [28] proposed additively homomorphic FCs for constant-degree polynomials and monotone span programs. While their construction for polynomials and the Lipmaa-Pavlyk construction [86] are both amenable to proving smallness statements, they would be less efficient than our constructions, as discussed in Supplementary Material A. Moreover, their more complex CRS structure would make it harder to prove knowledge-soundness in our setting, where the evaluation-binding property considered in [86,28] would not suffice.

AGGREGATION AND SUB-VECTOR OPENINGS. On several occasions, we rely on sub-vector openings and proof aggregation in the vector commitment of [84].

The notion of sub-vector openings was independently introduced and realized by Lai and Malavolta [81] and by Boneh, Bünz and Fisch [11]. It allows a sender to generate a short proof π_S that opens a sub-vector \mathbf{m}_S of \mathbf{m} , for a subset $S \subseteq [n]$. Sub-vector openings are implied by the proof aggregation property considered in [11,101,100,25,63,98], which allows anyone (and not only the committer) to aggregate n individual proofs $\{\pi_i\}_{i \in S}$ for a committed sub-vector \mathbf{m}_S into a constant-size proof π_S . Boneh, Bünz and Fisch [11] and Tomescu *et al.* [100] realized same-commitment aggregation in hidden-order groups and under q -type assumptions in pairing-friendly groups, respectively. Campanelli *et*

al. [25] introduced incrementally aggregatable vector commitments, which allow different sub-vector openings to be merged into a shorter opening for the union of their sub-vectors. Moreover, aggregated proofs support further aggregation.

By leveraging the linearity properties of the vector commitment from [84], Gorbunov *et al.* [63] obtained the first VC scheme enabling cross-commitment aggregation, which is useful in blockchain applications. The same-commitment variant of their aggregation method is obtained by introducing a random oracle in the inner product functional commitment of [83]. Our technique of proving that a committed vector is a reversed Hadamard product of another committed vector \mathbf{x} and a public vector \mathbf{y} is inspired by the randomized aggregation technique of PointProofs [63]. The difference is that, while [63] uses proof aggregation to succinctly prove sub-vector openings, we use it to prove linear relations between related positions in distinct committed vectors.

By instantiating vector commitments from polynomial commitments, Boneh *et al.* [12,13] obtained an alternative VC system supporting cross-commitment aggregation. Hyperproofs *et al.* [98] is yet another VC scheme allowing cross-commitment aggregation with the additional feature that all proofs can be updated in sub-linear time when the vector changes.

OTHER PROOFS OF BINARITY. Prior works on pairing-based commitments [61,60] considered the problem of constructing constant-size proofs that a committed string is binary. However, these techniques apply to variants of Groth-Sahai commitments [71] that are *not* succinct vector commitments: i.e., either the commitment or partial openings (or both) do not have constant size. The first candidate [61] was designed for perfectly-binding commitments, where the commitment is longer than the committed message. The case of perfectly hiding (compressing) commitments was considered in [60, Section 4.2] but the underlying commitments do not natively support constant-size partial openings. As briefly alluded to in [60, Section 4.2.1], it is actually possible to build a succinct vector commitment to bitstrings on top of the perfectly hiding commitments from [60, Chapter 4]. However, the resulting construction has several limitations: (i) The CRS has quadratic size in the dimension of committed vectors (like the Diffie-Hellman-based vector commitment of [26]); (ii) It does not seem to support constant-size proofs that the committed $\mathbf{m} \in \mathbb{Z}_p^n$ satisfies inner product relations $\langle \mathbf{m}, \mathbf{t} \rangle = x$ for public $\mathbf{t} \in \mathbb{Z}_p^n$ and $x \in \mathbb{Z}_p$; (iii) Proofs are somewhat long and contain more than 20 group elements (according to Table 4.1 in [60]).

RANGE PROOFS. Range proofs were introduced by Brickell *et al.* [20] and investigated in a large body of work [30,23,17,85,68,29,37,62] since then.

A standard approach [20,23,68,62,21] consists in breaking integers into bits and committing to these bits using homomorphic commitments. When it comes to proving membership of a range $[0, 2^\ell - 1]$, the resulting proofs generally contain $O(\ell)$ group elements (and thus $O(\lambda \cdot \ell)$ bits, where λ is the security parameter) although somewhat shorter proofs [23,68,62] are achievable using pairings. Using a clever recursive folding technique, Bulletproofs [21] decreased the communication complexity to $O(\log \ell)$ group elements (i.e., $O(\lambda \cdot \log \ell)$ bits) in general

discrete-logarithm-hard groups without a bilinear map.

Another approach [17,85,65,37] relies on integer commitments in hidden-order groups, by decomposing positive integers as a sum of squares. The sum-of-squares method was transposed [36,35] to groups of (sufficiently large) public prime order. It was also adapted to class groups and lattices. For some parameters in the standard discrete logarithm setting, the constructions of [36,35] were shown to compare favorably with BulletProofs.

For some applications where the proof size is the primary concern (e.g., confidential transactions in the blockchain [21]), it may be desirable to have even shorter proofs than [21,36,35], even at the expense of losing the transparent setup property. Using polynomial commitments, Boneh *et al.* [14] suggested another range proof inspired by SNARK arithmetization techniques [51]. Their construction can be realized from a variety of polynomial commitments [77,22,82]. In instantiations from pairing-based polynomial commitments [77,83], it provides the smallest communication cost to date, with proofs as short as 3 group elements and 3 scalars. In our range proof construction, we further decrease the proof length to that of the shortest known SNARKs [69]. A detailed comparison with [14] is given in Section B.1.

DISCRETE-LOG-BASED PROOFS FOR LATTICE RELATIONS. The use of specialized pairing-based arguments to prove lattice relations was considered to prove the correct evaluation of FHE ciphertexts [48]. However, the modulus of the leveled FHE scheme had to match the group order of the pairing. This limitation does not appear in the del Pino *et al.* approach [41] nor in our construction. We note that the motivation of [48] was different since, in their setting, the prover was the server while the verifier was a computationally constrained client. Here, we consider use cases like [97] where the prover is the client (generating the proof on its browser using a single thread) and the verifier runs on a computationally powerful machine that can afford the use of multiple threads.

In applications to private FHE-based private smart contracts [97], the protocol of [41] was actually preferred to SNARKs in order to obtain faster prover. Our system can offer a similarly fast prover with the benefit of shorter proofs.

2 Background and Definitions

2.1 Hardness Assumptions

Let groups $(\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T)$ of prime order p with a bilinear map $e : \mathbb{G} \times \hat{\mathbb{G}} \rightarrow \mathbb{G}_T$.

We rely on the hardness of computing a discrete logarithm $\alpha \in \mathbb{Z}_p$ given $\{g^{\alpha^i}\}_{i \in [2n]}$ and $\{\hat{g}^{\alpha^i}\}_{i \in [n]}$. This assumption is similar to the n -discrete logarithm assumption considered in, e.g. [49], except that powers α^i are given in the exponents in both groups \mathbb{G} and $\hat{\mathbb{G}}$.

Definition 1 ([49]). Let $(\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T)$ be asymmetric bilinear groups of prime order p . For integers m, n , the (m, n) -**Discrete Logarithm** ((m, n) -DLOG) problem is, given $(g, g^\alpha, g^{\alpha^2}, \dots, g^{\alpha^m}, \hat{g}, \hat{g}^\alpha, \dots, \hat{g}^{\alpha^n})$, where $\alpha \stackrel{R}{\leftarrow} \mathbb{Z}_p$, $g \stackrel{R}{\leftarrow} \mathbb{G}$, $\hat{g} \stackrel{R}{\leftarrow} \hat{\mathbb{G}}$, to compute $\alpha \in \mathbb{Z}_p$.

2.2 Non-interactive Arguments

A NIZK argument for a language $L = \{(x, w) \in R\}$, where R is an NP relation, consists of algorithms $(\text{CRS-Gen}, \text{Prove}, \text{Verify})$ with the following specifications. On input of a security parameter $\lambda \in \mathbb{N}$ (and, optionally, language-dependent parameters), CRS-Gen generates a common reference string pp and a simulation trapdoor τ . Algorithm Prove takes as input the common reference string pp , a statement x and a witness w and outputs a proof π . Verify takes in pp , a statement x and a proof π and returns 0 or 1. Correctness requires that, for any $(x, w) \in R$, honestly generated proofs are always (or at least with overwhelming probability) accepted by the verifier.

From a security point of view, NIZK argument systems should satisfy two properties. The *zero-knowledge* property requires that proofs leak no information about the witness. This is formalized by asking that the trapdoor τ (hidden in pp) allows simulating proofs that are (statistically or computationally) indistinguishable from real proofs. The *knowledge-soundness* property requires that there exists an extractor that can compute a witness whenever the adversary generates a valid proof. The extractor has access to the adversary's internal state, including its random coins. In a NIZK argument for a relation R , these properties are defined as follows.

Completeness: For any $\lambda \in \mathbb{N}$, and any statement-witness pair $(x, w) \in R$, there is a negligible function $\text{negl} : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\Pr [\text{Verify}_{\text{pp}}(x, \pi) = 1 \mid (\text{pp}, \tau) \leftarrow \text{CRS-Gen}(1^\lambda), \pi \leftarrow \text{Prove}_{\text{pp}}(x, w)] = 1 - \text{negl}(\lambda).$$

Knowledge-soundness: For any PPT adversary \mathcal{A} , there is a PPT extractor $\mathcal{E}_{\mathcal{A}}$ that has access to \mathcal{A} 's internal state and random coins ρ such that

$$\Pr [\text{Verify}_{\text{pp}}(x, \pi) = 1 \wedge (x, w) \notin R \mid (\text{pp}, \tau) \leftarrow \text{CRS-Gen}(1^\lambda), (x, \pi) \leftarrow \mathcal{A}(\text{pp}; \rho), w \leftarrow \mathcal{E}_{\mathcal{A}}(\text{pp}, (x, \pi), \rho)] = \text{negl}(\lambda).$$

(Statistical) Zero-knowledge: There exists a PPT simulator Sim such that, for any $\lambda \in \mathbb{N}$ and any pair $(x, w) \in R$, the distributions $D_0 = \{\pi \leftarrow \text{Prove}_{\text{pp}}(x, w) : (\text{pp}, \tau) \leftarrow \text{CRS-Gen}(1^\lambda)\}$ and $D_1 = \{\pi \leftarrow \text{Sim}(\text{pp}, \tau, x) : (\text{pp}, \tau) \leftarrow \text{CRS-Gen}(1^\lambda)\}$ are statistically close.

For many applications, it is desirable to consider an adversary that can observe simulated proofs (for possibly false statements) and exploit some malleability of these proofs to generate a fake proof of its own. To prevent such attacks, the notion of simulation-extractability strengthens knowledge-soundness by giving the adversary access to a simulation oracle.

Simulation-Extractability: For any PPT adversary \mathcal{A} , there is a PPT extractor $\mathcal{E}_{\mathcal{A}}$ that has access to \mathcal{A} 's internal state/randomness ρ such that

$$\Pr [\text{Verify}_{\text{pp}}(x, \pi) = 1 \wedge (x, w) \notin R \wedge (x, \pi) \notin Q \mid (\text{pp}, \tau) \leftarrow \text{CRS-Gen}(1^\lambda), (x, \pi) \leftarrow \mathcal{A}^{\text{SimProve}}(\text{pp}; \rho), w \leftarrow \mathcal{E}_{\mathcal{A}}(\text{pp}, (x, \pi), \rho, Q)] = \text{negl}(\lambda),$$

where $\text{SimProve}(\text{pp}, \tau, \cdot)$ is an oracle that returns a simulated proof $\pi \leftarrow \text{Sim}(\text{pp}, \tau, x)$ for a given statement x and $Q = \{(x_i, \pi_i)\}_i$ denotes the set of queried statements and the simulated proofs returned by SimProve .

In the following sections, we extend the syntax with an algorithm Com that inputs a vector $\mathbf{x} \in D^n$ over a domain D and outputs a commitment C .

2.3 Algebraic Group Model

The algebraic group model (AGM) [49] is an idealized model, where all algorithms are assumed to be algebraic. Algebraic algorithms [16,93] generalize the notion of a generic algorithm [96] in that, whenever they compute a group element, they do it using generic operations, by taking linear combinations of available group elements so far. Hence, whenever they output a group element $X \in \mathbb{G}$, they also output a representation $\{\alpha_i\}_{i=1}^N$ of $X = \prod_{i=1}^N g_i^{\alpha_i}$ as a function of previously observed group elements $(g_1, \dots, g_N) \in \mathbb{G}^N$ in the same group.

In contrast with generic algorithms, algebraic algorithms can exploit the structure of the group and obtain more information than they would in the generic group model. Although its relation with the generic group model is unclear [78], the AGM provides a powerful framework to analyze the security of efficient protocols via reductions. In particular, it has been widely used in the context of SNARKs [49,89,51,13,55,53].

3 Proving That a Committed Vector is Binary

Our construction for binary strings goes as follows.

CRS-Gen($1^\lambda, 1^n$): On input of a security parameter λ and the maximal dimension $n \in \text{poly}(\lambda)$ of committed vectors, do the following:

1. Choose asymmetric bilinear groups $(\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T)$ of prime order $p > 2^{l(\lambda)}$, for some function $l : \mathbb{N} \rightarrow \mathbb{N}$, and $g \xleftarrow{R} \mathbb{G}$, $\hat{g} \xleftarrow{R} \hat{\mathbb{G}}$.
2. Pick $\alpha \xleftarrow{R} \mathbb{Z}_p$. Compute $g_1, \dots, g_n, g_{n+2}, \dots, g_{2n} \in \mathbb{G}$ and $\hat{g}_1, \dots, \hat{g}_n \in \hat{\mathbb{G}}$, where $g_i = g^{(\alpha^i)}$ for each $i \in [2n] \setminus \{n+1\}$ and $\hat{g}_i = \hat{g}^{(\alpha^i)}$ for each $i \in [n]$.
3. Choose hash functions $H, H_t : \{0, 1\}^* \rightarrow \mathbb{Z}_p^n$ and $H_{\text{agg}} : \{0, 1\}^* \rightarrow \mathbb{Z}_p^2$.

The public parameters are

$$\text{pp} = \left((\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T), g, \hat{g}, \{g_i\}_{i \in [2n] \setminus \{n+1\}}, \{\hat{g}_i\}_{i \in [n]}, \mathbf{H} \right)$$

where $\mathbf{H} = \{H, H_t, H_{\text{agg}}\}$ are hash functions.

Com_{pp}(\mathbf{x}) To commit to a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}_p^n$, choose a random $\gamma \xleftarrow{R} \mathbb{Z}_p$ and compute $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^n \hat{g}_j^{x_j}$. Return $\hat{C} \in \hat{\mathbb{G}}$ and the opening information $\text{aux} = \gamma \in \mathbb{Z}_p$.

Prove_{pp}($\hat{C}, (\mathbf{x}, \mathbf{aux})$): given a commitment \hat{C} and witnesses $(\mathbf{x}; \mathbf{aux})$ consisting of a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}_p^n$ and randomness $\mathbf{aux} = \gamma \in \mathbb{Z}_p$, return \perp if $(x_1, \dots, x_n) \notin \{0, 1\}^n$. Otherwise, do the following:

1. Compute $\mathbf{y} = (y_1, \dots, y_n) = H(\hat{C}) \in \mathbb{Z}_p^n$. Choose $\gamma_y \xleftarrow{R} \mathbb{Z}_p$ and compute

$$C_y = g^{\gamma_y} \cdot \prod_{j=1}^n g_{n+1-j}^{y_j \cdot x_j}$$

Then, compute $\mathbf{t} = (t_1, \dots, t_n) = H_t(\mathbf{y}, \hat{C}, C_y) \in \mathbb{Z}_p^n$.

2. Generate a proof

$$\pi_{eq} = \frac{\prod_{i=1}^n \left(g_{n+1-i}^\gamma \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-i+j}^{x_j} \right)^{t_i \cdot y_i}}{\prod_{i=1}^n \left(g_i^{\gamma_y} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-j+i}^{y_j \cdot x_j} \right)^{t_i}} \quad (15)$$

which satisfies

$$\frac{e\left(\prod_{i=1}^n g_{n+1-i}^{t_i \cdot y_i}, \hat{C}\right)}{e\left(C_y, \prod_{i=1}^n \hat{g}_i^{t_i}\right)} = e(\pi_{eq}, \hat{g}), \quad (16)$$

and argues that C_y commits to $(y_n \cdot x_n, \dots, y_1 \cdot x_1) \in \mathbb{Z}_p^n$.

3. Compute a proof

$$\pi_y = C_y^\gamma \cdot \prod_{i=1}^n \left(g_i^{\gamma_y} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-j+i}^{y_j \cdot (x_j - 1)} \right)^{x_i} \quad (17)$$

showing that $\sum_{i=1}^n y_i \cdot x_i \cdot (x_i - 1) = 0$ and satisfying

$$e\left(C_y \cdot \prod_{j=1}^n g_{n+1-j}^{-y_j}, \hat{C}\right) = e(\pi_y, \hat{g}) \quad (18)$$

4. Compute $(\delta_{eq}, \delta_y) = H_{\text{agg}}(\hat{C}, C_y) \in \mathbb{Z}_p^2$ and then $\pi = \pi_{eq}^{\delta_{eq}} \cdot \pi_y^{\delta_y}$.

Output the final proof $\boldsymbol{\pi} := (C_y, \pi) \in \mathbb{G}^2$.

Verify_{pp}($\hat{C}, \boldsymbol{\pi}$): Given $\hat{C} \in \hat{\mathbb{G}}$ and a purported proof $\boldsymbol{\pi} = (C_y, \pi) \in \mathbb{G}^2$,

1. Compute $\mathbf{y} = H(\hat{C}) \in \mathbb{Z}_p^n$, $(\delta_{eq}, \delta_y) = H_{\text{agg}}(\hat{C}, C_y) \in \mathbb{Z}_p^2$ and $\mathbf{t} = H_t(\mathbf{y}, \hat{C}, C_y) \in \mathbb{Z}_p^n$.
2. Return 1 if the following equations is satisfied and 0 otherwise:

$$\frac{e\left(C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{(\delta_{eq} \cdot t_i - \delta_y) \cdot y_i}, \hat{C}\right)}{e\left(C_y, \prod_{i=1}^n \hat{g}_i^{\delta_{eq} \cdot t_i}\right)} = e(\pi, \hat{g}). \quad (19)$$

Correctness follows from the observation that equation (19) is obtained by aggregating (16)-(18), for which a detailed proof of correctness can be found in Supplementary Material B.2.

In the algebraic group model, the construction can be proven zero-knowledge (for commitments chosen by an algebraic adversary) and knowledge-sound. The proof of knowledge-soundness can be inferred from the proof of Theorem 2 (in Section 4), of which it is a sub-case. In the upcoming sections, we will combine the system with other components in such a way that the combined arguments satisfy the stronger notion of simulation-extractability.

The construction easily extends to prove that a committed $\mathbf{x} \in \{-1, 0, 1\}^n$ is ternary, by showing that \mathbf{x} is the difference between two binary vectors. In Supplementary Material D, we give an optimized extension allowing to prove that a committed vector is ternary using only 3 group elements. In Supplementary Material F, we also explain how to prove the exact Hamming weight (or an upper bound thereof) of committed binary/ternary vectors using 4 group elements.

4 A Range Proof With Very Short Proofs

Using the non-interactive argument for binary vectors from Section 3, we can build range arguments made of a constant number of group elements.

In the description below, we assume ranges $[0, B]$ such that $B + 1$ is a power of 2 but the approach easily extends to general ranges. The standard approach to this problem is to consider the integer $\ell \in \mathbb{N}$ such that $2^{\ell-1} \leq B < 2^\ell$ and generate two range proofs showing that $x \in [0, 2^\ell - 1]$ and $x + (2^\ell - 1 - B) \in [0, 2^\ell - 1]$, where the second part is proven by leveraging the additive homomorphic property of the commitment. Instead of generating two independent range proofs, we can double the size of the CRS (by setting $n = 2\bar{\ell}$, where $\bar{\ell} \geq \ell$ is the maximal bitlength of the range) and avoid increasing the proof size. In Supplementary Material C.4, we provide more details on the treatment of general ranges.

4.1 Description

For public $\hat{g}, \hat{g}_1 \in \hat{\mathbb{G}}$, the range membership relation is formally defined as

$$\mathcal{R} = \left\{ (\mathbf{x}, \mathbf{w}) = ((\hat{C} = \hat{g}^r \cdot \hat{g}_1^x, \ell) \in \hat{\mathbb{G}} \times \mathbb{N}, (r, x) \in \mathbb{Z}_p \times [0, 2^\ell - 1]) \right\}$$

Since the commitment is perfectly-hiding, the proven relation is trivially satisfied because, for any group element \hat{V} , there exist a witness x in the stated range and a corresponding randomness $r \in \mathbb{Z}_p$ such that $\hat{V} = \hat{g}^r \cdot \hat{g}_1^x$. However, we prove that the scheme is an argument of knowledge of a valid (r, x) .

We assume that the initial Pedersen commitment $\hat{V} = \hat{g}^r \cdot \hat{g}_1^x$ to the witness $x \in [0, 2^\ell - 1]$ is computed using the group elements $(\hat{g}, \hat{g}_1) \in \hat{\mathbb{G}}^2$ contained in the CRS of the range argument.³ We note that a similar assumption is made (see, e.g., [99]) in the polynomial-commitment-based construction of [14], where the integer x is committed as a polynomial $f[X]$ such that $f(1) = x$.

³ We note that committing x to a different, pairing-free, group \mathbb{G} would not strengthen security in any way since an adversary that would be able to compute α from \mathbf{pp} would still break knowledge-soundness.

CRS-Gen($1^\lambda, 1^n$): On input of a security parameter λ and the maximal bitlength $n \in \text{poly}(\lambda)$ of ranges, do the following:

1. Choose asymmetric bilinear groups $(\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T)$ of prime order $p > 2^{l(\lambda)}$, for some polynomial function $l : \mathbb{N} \rightarrow \mathbb{N}$, and $g \xleftarrow{R} \mathbb{G}$, $\hat{g} \xleftarrow{R} \hat{\mathbb{G}}$.
2. Pick a random $\alpha \xleftarrow{R} \mathbb{Z}_p$ and compute $g_1, \dots, g_n, g_{n+2}, \dots, g_{2n} \in \mathbb{G}$ as well as $\hat{g}_1, \dots, \hat{g}_n \in \hat{\mathbb{G}}$, where $g_i = g^{(\alpha^i)}$ for each $i \in [2n] \setminus \{n+1\}$ and $\hat{g}_i = \hat{g}^{(\alpha^i)}$ for each $i \in [n]$.
3. Choose hash functions $H, H_t : \{0, 1\}^* \rightarrow \mathbb{Z}_p^n$, $H_s : \{0, 1\}^* \rightarrow \mathbb{Z}_p$ and $H_{\text{agg}} : \{0, 1\}^* \rightarrow \mathbb{Z}_p^4$ that will be modeled as random oracles.

The public parameters are defined to be

$$\text{pp} = \left((\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T), g, \hat{g}, \{g_i\}_{i \in [2n] \setminus \{n+1\}}, \{\hat{g}_i\}_{i \in [n]}, \mathbf{H} \right)$$

where $\mathbf{H} = \{H, H_s, H_t, H_{\text{agg}}\}$ are hash functions.

Com_{pp}(x) To commit to an integer $x \in \mathbb{Z}$, choose a random $r \xleftarrow{R} \mathbb{Z}_p$ and compute a Pedersen commitment $\hat{V} = \hat{g}^r \cdot \hat{g}_1^x \in \hat{\mathbb{G}}$. Return $\text{com} = \hat{V} \in \hat{\mathbb{G}}$ and the opening information $\text{aux} = r \in \mathbb{Z}_p$.

Prove_{pp}($\text{com}, (x, \text{aux})$): given $\text{com} = \hat{V}$ and witnesses $(x; \text{aux})$ consisting of an integer $x \in [0, 2^\ell - 1]$ with binary expansion $(x_1, \dots, x_\ell) \in \{0, 1\}^\ell$, where $\ell \leq n$, and $\text{aux} = r \in \mathbb{Z}_p$ such that $\hat{V} = \hat{g}^r \cdot \hat{g}_1^x$, do the following:

1. Set $(x_{\ell+1}, \dots, x_n) = \mathbf{0}^{n-\ell}$. Choose $\gamma \xleftarrow{R} \mathbb{Z}_p$ and compute

$$\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^{\ell} \hat{g}_j^{x_j}$$

together with a proof $\pi_x \in \mathbb{G}$ that \hat{C} commits to $(x_1, \dots, x_n) \in \mathbb{Z}_p^n$ such that $\sum_{i=1}^{\ell} x_i \cdot 2^{i-1} = x$. This proof π_x satisfies

$$\frac{e(\prod_{i=1}^{\ell} g_{n+1-i}^{2^{i-1}}, \hat{C})}{e(g_n, \hat{V})} = e(\pi_x, \hat{g}) \quad (20)$$

and is obtained as

$$\pi_x = g_n^{-r} \cdot \prod_{i=1}^{\ell} \left(g_{n+1-i}^\gamma \cdot \prod_{j \in [\ell] \setminus \{i\}} g_{n+1-i+j}^{x_j} \right)^{2^{i-1}}.$$

2. Compute $\mathbf{y} = (y_1, \dots, y_n) = H(\hat{V}, \hat{C}) \in \mathbb{Z}_p^n$. Pick $\gamma_y \xleftarrow{R} \mathbb{Z}_p$ and compute

$$C_y = g^{\gamma_y} \cdot \prod_{j=1}^{\ell} g_{n+1-j}^{y_j \cdot x_j}$$

Then, compute $\mathbf{t} = (t_1, \dots, t_n) = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^n$.

3. Prove that C_y commits to $(y_1 \cdot x_1, \dots, y_n \cdot x_n) \in \mathbb{Z}_p^n$ by computing a short $\pi_{eq} \in \mathbb{G}$ (as specified in (15)) satisfying

$$\frac{e(\prod_{i=1}^n g_{n+1-i}^{t_i \cdot y_i}, \hat{C})}{e(C_y, \prod_{i=1}^n \hat{g}_i^{t_i})} = e(\pi_{eq}, \hat{g}). \quad (21)$$

4. Prove that $\sum_{i=1}^n y_i \cdot x_i \cdot (x_i - 1) = 0$ by computing $\pi_y \in \mathbb{G}$ via (17), which satisfies

$$e(C_y \cdot \prod_{j=1}^n g_{n+1-j}^{-y_j}, \hat{C}) = e(\pi_y, \hat{g}) \quad (22)$$

5. Generate an aggregated proof that $\hat{V} = \hat{g}^r \cdot \hat{g}_1^x$ is a commitment to a vector that contains 0 in its last $n - 1$ coordinates. Namely, compute $\pi_v = \prod_{i=2}^n (g_{n+1-i}^r \cdot g_{n+2-i}^x)^{s_i} \in \mathbb{G}$ such that

$$e(\prod_{i=2}^n g_{n+1-i}^{s_i}, \hat{V}) = e(\pi_v, \hat{g}). \quad (23)$$

where $s_i = H_s(i, [2, n], \hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p$ for each $i \in [2, n]$.

6. Compute $(\delta_x, \delta_{eq}, \delta_y, \delta_v) = H_{\text{agg}}(\hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^4$ and an aggregated proof

$$\pi = \pi_x^{\delta_x} \cdot \pi_y^{\delta_y} \cdot \pi_{eq}^{\delta_{eq}} \cdot \pi_v^{\delta_v}.$$

Output the final range argument which consists of

$$\boldsymbol{\pi} := (\hat{C}, C_y, \pi). \quad (24)$$

Verify_{pp}(com, $\boldsymbol{\pi}$): Given a commitment $\text{com} = \hat{V} \in \hat{\mathbb{G}}$ and a purported proof $\boldsymbol{\pi}$, parse the latter as in (24).

1. Compute $\mathbf{y} = H(\hat{V}, \hat{C}) \in \mathbb{Z}_p^n$, $(\delta_x, \delta_{eq}, \delta_y, \delta_v) = H_{\text{agg}}(\hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^4$, $\mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^n$. Set $s_1 = 0$ and $s_i = H_s(i, [2, n], \hat{V}, \hat{C}, C_y)$ for all indices $i \in [2, n]$.
2. Return 1 if and only if

$$\frac{e(C_y^{\delta_y} \cdot \prod_{i=1}^{\ell} g_{n+1-i}^{\delta_{x,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) \cdot y_i}, \hat{C})}{e(g_n^{\delta_x} \cdot \prod_{i=2}^n g_{n+1-i}^{-\delta_v \cdot s_i}, \hat{V}) \cdot e(C_y, \prod_{i=1}^n \hat{g}_i^{\delta_{eq} \cdot t_i})} = e(\boldsymbol{\pi}, \hat{g}), \quad (25)$$

where $\delta_{x,i} = \delta_x$ if $i \in [\ell]$ and $\delta_{x,i} = 0$ if $i \in [\ell + 1, n]$.

CORRECTNESS. The verification equation (25) is obtained by raising equalities (20), (21), (22) and (23) to the powers δ_x , δ_{eq} , δ_y , and δ_v , respectively, and multiplying the results together. In Supplementary Material B.2, we provide detailed proofs of correctness for individual verification equations (20)-(23).

EFFICIENCY. The computational cost of the prover is dominated by $O(n)$ exponentiations in \mathbb{G} and two exponentiations in $\hat{\mathbb{G}}$. Indeed, computing \hat{C} at step 1 only requires one exponentiation and a subset product (which is cheaper than an exponentiation) in $\hat{\mathbb{G}}$. Step 2 requires $n + 1$ exponentiations in \mathbb{G} . Each of the proof elements $(\pi_x, \pi_{eq}, \pi_y, \pi_v)$ requires at most $2n$ exponentiations when terms are suitably re-arranged. The entire product π at step 6 is actually computable using roughly $2n$ exponentiations since $(\pi_x, \pi_{eq}, \pi_y, \pi_v)$ can be computed after the commitments (\hat{C}, C_y) and the aggregation coefficients. Overall, the prover’s overhead amounts to $3n$ exponentiations in \mathbb{G} , 2 exponentiations in $\hat{\mathbb{G}}$, and cheaper multiplications over \mathbb{Z}_p . Of course, the prover can be optimized using specialized multi-exponentiation algorithms. The verifier’s work is dominated by $2n + 1$ exponentiations in \mathbb{G} , n exponentiations in $\hat{\mathbb{G}}$ and 4 pairings.

In terms of proof length, π only requires one element of $\hat{\mathbb{G}}$, and 2 element of \mathbb{G} , which matches the optimal size of simulation-extractable pairing-based SNARKs [70]. Using the KSS18 family of pairing-friendly curves suggested by Kachisa *et al.* [76], each element of \mathbb{G} (resp. $\hat{\mathbb{G}}$) can have a 348-bit (resp. 1044-bit) representation at the 128-bit security level according to [44]. Assuming that elements of $\hat{\mathbb{G}}$ are three times as large as those of \mathbb{G} , the overall proof length does not exceed the equivalent of 5 elements of \mathbb{G} , which amounts to 1740 bits.

In Supplementary Material B.1, we give a detailed comparison among existing constant-size range proofs. As shown in Table 1, our scheme provides the shortest proof length and the smallest computational cost at the prover.

As shown in Supplementary Material C, the construction extends to prove multiple ranges at once for a committed vector of integers.

4.2 Security in the AGM & ROM

We first prove the zero-knowledge property in the random oracle model.

Theorem 1. *The construction provides statistical zero-knowledge in the ROM. (The proof is given in Supplementary Material B.3.)*

The simulator in the proof of Theorem 1 proceeds by programming the random oracles and also uses the trapdoor of the CRS. On the other hand, it works for any given $\hat{V} \in \hat{\mathbb{G}}$ without knowing an algebraic representation of \hat{V} . If we restrict \hat{V} to be chosen by an algebraic adversary, it is possible to build an algebraic simulator that does not rely on random oracles.

In the proof of Theorem 2, we assume that the construction is instantiated with either Type-2 or Type-3 pairings [52] (i.e., no isomorphism is efficiently computable from \mathbb{G} to $\hat{\mathbb{G}}$). Specifically, we rely on the fact that the Pedersen commitment \hat{V} lives in $\hat{\mathbb{G}}$ instead of \mathbb{G} . If it was in \mathbb{G} , the output of an algebraic adversary could depend on generators $\{g_i\}_{i=n+2}^{2n}$, which would hinder knowledge extraction. Since only $\{\hat{g}_i\}_{i \in [n]}$ are given in $\hat{\mathbb{G}}$, the algebraic group model in asymmetric bilinear groups (see, e.g., [10]) allows assuming that an adversarially-chosen commitment \hat{V} comes with a representation with respect to $(\hat{g}, \{\hat{g}_i\}_{i \in [n]})$.

We do not know how to extend the proof to support symmetric pairings but this is not really a restriction given that Type-1 pairings are much less efficient for the same security level.

Theorem 2. *Under the $(2n, n)$ -DLOG assumption, the scheme is simulation-extractable in the algebraic group model and in the random oracle model. (The proof is detailed in Supplementary Material B.4.)*

The intuition of the proof of Theorem 2 is as follows. The main difficulty is that we cannot immediately apply the simulator from the proof of Theorem 1 because it uses the secret exponent α as a simulation trapdoor. We thus use a trapdoor-less simulator [53] that can simulate proofs for adversarially-chosen statements by programming the random oracles and without using α .

We are to show that, if the adversary can output a proof for which the knowledge extractor fails, the reduction is able to compute $\alpha \in \mathbb{Z}_p$ from a $(2n, n)$ -discrete-logarithm instance $(g, \{g_i = g^{(\alpha^i)}\}_{i=1}^{2n})$, $(\hat{g}, \{\hat{g}_i = \hat{g}^{(\alpha^i)}\}_{i=1}^n)$. These elements are used to form the CRS, where $g_{n+1} = g^{(\alpha^{n+1})}$ is *not* included.

To simulate proofs for adversarially-chosen commitments \hat{V} , we use the observation that, in equations (20)-(23), the left-hand-side member can always be written $e(g, \hat{g})^{P_i(\alpha)}$, for some polynomial $P_i[X]$ which is computable by the reduction (either because it computes the paired commitments itself or using an algebraic representation of \hat{V} supplied by the adversary). In the aggregated verification equation (25), the left-hand-side member is thus of the form $e(g, \hat{g})^{\sum_i \delta_i \cdot P_i(\alpha)}$, where $\{\delta_i\}_i$ are random coefficients derived from a random oracle, and the simulator has to compute π such that $e(g, \hat{g})^{\sum_i \delta_i \cdot P_i(\alpha)} = e(\pi, \hat{g})$. In the simulated proof $\pi = (\hat{C}, C_y, \pi)$, it computes \hat{C} and C_y by programming the committed vectors as a function of all aggregation coefficients in such a way that the polynomial $\sum_i \delta_i \cdot P_i(\alpha)$ does not depend on the monomial X^{n+1} . Then, it patches the appropriate random oracles so as to make them hit the previously chosen aggregation coefficients. Since $\sum_i \delta_i \cdot P_i(\alpha)$ has no degree- $(n+1)$ term, the unique proof π satisfying $e(g, \hat{g})^{\sum_i \delta_i \cdot P_i(\alpha)} = e(\pi, \hat{g})$ is computable from $\{g_i\}_{i \in [2n] \setminus \{n+1\}}$. This way, the adversary never gets to see a group element whose algebraic representation depends on g_{n+1} .

Hence, when an algebraic adversary outputs a proof $\pi^* = (\hat{C}^*, C_y^*, \pi^*)$ of its own, it also outputs a representation of π^* that does not depend on g_{n+1} either. At the same time, if the knowledge extractor fails, the reduction can compute another representation of π^* that *does* depend on g_{n+1} w.h.p. Proving this fact requires some care as we have to distinguish cases where the adversary recycles \hat{C}^* or C_y^* from a simulated proof. For example, since $y = H(\hat{V}, \hat{C})$ does not depend on C_y , we have to prove that the adversary cannot output $(\hat{V}^*, \pi^* = (\hat{C}^*, C_y^*, \pi^*))$ where (\hat{V}^*, \hat{C}^*) appeared in a simulated proof. In this case, we can still rely on the fact that \mathbf{t} is defined after $(\hat{V}^*, \hat{C}^*, C_y^*)$ to obtain a representation of π^* that depends on g_{n+1} .

Then, the two distinct representations of π^* reveal a non-zero polynomial of which $\alpha \in \mathbb{Z}_p$ is a root.

5 Shorter Proofs for Ring LWE Ciphertexts

In this section, we show that the techniques of previous sections can be used to obtain very short proofs for natural statements that arise in lattice-based cryptography. For example, they can be used for all the applications described in [46]. It includes proving the validity of an LPR ciphertext [88], a BGV ciphertext [19], a ring GSW ciphertext [58], a TFHE ciphertext [33], a ring LWE public key or an NTRU public key [73]. They can also be used to prove that a committed vector contains a GPV signature [57].

We adapt the approach of del Pino *et al.* [41] with the difference that we replace the BulletProofs range proof by our more compact proofs of smallness. We also exploit the fact that the underlying vector commitment [84] allows proving inner-product relations as in [83].

Let the polynomial rings $R = \mathbb{Z}[X]/(X^d + 1)$ and $R_q = R/(qR)$, where d is a power of two. As in [41], we aim at proving the existence of a witness $\mathbf{s} = (s_1, \dots, s_M) \in R^M$ comprised of small-norm ring elements such that

$$\sum_{i=1}^M \mathbf{a}_i \cdot s_i = \mathbf{c} \pmod{(q, X^d + 1)} \quad (26)$$

for public $\mathbf{c}, \mathbf{a}_1, \dots, \mathbf{a}_M \in R_q^N$. The relation is defined as the set of pairs

$$(\mathbf{x}, \mathbf{w}) = \left((\mathbf{c}, \mathbf{a}_1, \dots, \mathbf{a}_M) \in (R_q^N)^{M+1}, (s_1, \dots, s_M) \in R^M \right)$$

satisfying (26). To prove this relation, we proceed as in [41] and re-write (26) as the following equality over $\mathbb{Z}[X]/(X^d + 1)$

$$\sum_{i=1}^M \mathbf{a}_i \cdot s_i = \mathbf{c} + \mathbf{r} \cdot q \pmod{(X^d + 1)}, \quad (27)$$

where $\mathbf{r} = (r_1, \dots, r_N)^\top \in R^N$ is a vector of polynomials of degree $\leq d - 1$ and the components of $\{\mathbf{a}_i\}_{i=1}^M$ and \mathbf{c} are interpreted as polynomials with coefficients in $\{-\lfloor q/2 \rfloor, \dots, \lfloor q/2 \rfloor\}$. If $\|s_i\|_\infty \leq B_i$ for each $i \in [M]$, \mathbf{r} contains polynomials with coefficients of magnitude smaller than $\|\mathbf{r}\|_\infty \leq B_r \triangleq dM \cdot \max_{i \in [M]} (B_i)/2$.

Let us parse $\mathbf{a}_i = (a_{i,1}, \dots, a_{i,N})^\top \in R_q^N$. Let the coefficient embedding $\phi : R \rightarrow \mathbb{Z}^d$ that maps s_i to its coefficient vector $\phi(s_i) \in \mathbb{Z}^d$. Let $\mathbf{rot}(a_{i,j}) \in \mathbb{Z}^{d \times d}$ the anti-circulant matrix such that $\phi(a_{i,j} \cdot s_i \pmod{(X^d + 1)}) = \mathbf{rot}(a_{i,j}) \cdot \phi(s_i) \in \mathbb{Z}^d$. If we re-write (27) as a matrix-vector product over \mathbb{Z} , we obtain the relation

$$[\mathbf{A}_1 \mid \dots \mid \mathbf{A}_M] \cdot [\phi(s_1) \mid \dots \mid \phi(s_M)]^\top = \sum_{i=1}^M \mathbf{A}_i \cdot \phi(s_i) = \phi(\mathbf{c}) + \phi(\mathbf{r}) \cdot q$$

where $\mathbf{A}_i = [\mathbf{rot}(a_{i,1})^\top \mid \dots \mid \mathbf{rot}(a_{i,N})^\top]^\top \in \mathbb{Z}_q^{Nd \times d}$ for all $i \in [M]$, $\phi(\mathbf{c}) \in \mathbb{Z}_q^{Nd}$, and $\phi(\mathbf{r}) \in \mathbb{Z}^{Nd}$. Equivalently, this can be written

$$[\mathbf{A}_1 \mid \dots \mid \mathbf{A}_M \mid -q \cdot \mathbf{I}_{Nd}] \cdot \underbrace{[\phi(s_1) \mid \dots \mid \phi(s_M) \mid \phi(\mathbf{r})]}_{\triangleq \tilde{\mathbf{w}}}^\top = \phi(\mathbf{c}) \quad (28)$$

In order to prove (28), a natural idea is to have the prover commit to the vector $\tilde{\mathbf{w}} \in \mathbb{Z}^{Md+Nd}$ using a vector commitment over $\hat{\mathbb{G}}$. Then, using the batched range proof of Section C.1, it can generate short range proof that $\|\phi(s_i)\|_\infty \leq B_i$ for each $i \in [M]$ and $\|\phi(\mathbf{r})\|_\infty \leq dM \cdot \max_{i \in [M]}(B_i)/2$. Using the approach of [41], it can then prove that (28) holds over \mathbb{Z}_p ,⁴ where p is the order of $\hat{\mathbb{G}}$. If $p > 2Mqd \max_i(B_i)$, this ensures that (28) also holds over the integers. Instead of using the batched range proof of Section C.1, we can make the proof shorter (and spare one commitment in $\hat{\mathbb{G}}$) by directly committing to the bits of $\tilde{\mathbf{w}}$.

For any integer $z \in \mathbb{Z}$, we define $\mathbf{g}_z = (1, 2, 4, \dots, 2^{z-2}, -2^{z-1})^\top \in \mathbb{Z}^{1 \times z}$ and $\mathbf{G}_z = \mathbf{I}_d \otimes \mathbf{g}_z^\top \in \mathbb{Z}^{d \times dz}$. We also define $\mathbf{G}_z^{-1}(\mathbf{v})$ as the decomposition function that inputs an integer vector $\mathbf{v} \in [-2^{z-1}, 2^{z-1}-1]^d$ and outputs a decomposition $\mathbf{G}_z^{-1}(\mathbf{v}) \in \{0, 1\}^{dz}$ such that $\mathbf{G}_z \cdot \mathbf{G}_z^{-1}(\mathbf{v}) = \mathbf{v}$. Then, for each $i \in [M]$, we define

$$\tilde{\mathbf{A}}_i \triangleq [\mathbf{G}_{1+\log B_i}^\top \cdot \text{rot}(a_{i,1})^\top \mid \dots \mid \mathbf{G}_{1+\log B_i}^\top \cdot \text{rot}(a_{i,N})^\top]^\top \in \mathbb{Z}_q^{Nd \times d(1+\log B_i)}$$

and we prove that

$$\underbrace{[\tilde{\mathbf{A}}_1 \quad \dots \quad \tilde{\mathbf{A}}_M \mid -q \cdot (\mathbf{I}_N \otimes \mathbf{G}_{1+\log B_r})]}_{\triangleq \tilde{\mathbf{A}}} \cdot \underbrace{\begin{bmatrix} \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_M \\ \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_N \end{bmatrix}}_{\triangleq \tilde{\mathbf{w}}} = \phi(\mathbf{c}), \quad (29)$$

where we set $\mathbf{s}_i = \mathbf{G}_{1+\log B_i}^{-1}(\phi(s_i)) \in \{0, 1\}^{d \cdot (1+\log B_i)}$ for each $i \in [M]$, and $\mathbf{r}_i = \mathbf{G}_{1+\log B_r}^{-1}(\phi(r_i)) \in \{0, 1\}^{d \cdot (1+\log B_r)}$ for each $i \in [N]$.

The prover will thus commit to the bit-decomposition $\tilde{\mathbf{w}} \in \{0, 1\}^D$ of the witness, where $D = d \cdot (\sum_{i=1}^M (1 + \log B_i) + N(1 + \log B_r))$. In order to prove that relation (29) holds modulo p (and thus also over \mathbb{Z} since both members have infinity norm smaller than $p/2$), the prover will use a random $\boldsymbol{\theta} \in \mathbb{Z}_p^{Nd}$ (derived from a random oracle) and prove that the committed $\tilde{\mathbf{w}} \in \{0, 1\}^D$ satisfies $\boldsymbol{\theta}^\top \cdot \tilde{\mathbf{A}} \cdot \tilde{\mathbf{w}} = \boldsymbol{\theta}^\top \cdot \phi(\mathbf{c}) \pmod p$. If $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{w}} \neq \phi(\mathbf{c}) \pmod p$, then we have $\boldsymbol{\theta}^\top \cdot (\tilde{\mathbf{A}} \cdot \tilde{\mathbf{w}} - \phi(\mathbf{c})) = 0 \pmod p$ with probability $1/p$. Proving $\boldsymbol{\theta}^\top \cdot \tilde{\mathbf{A}} \cdot \tilde{\mathbf{w}} = \boldsymbol{\theta}^\top \cdot \phi(\mathbf{c})$ is doable using one element of \mathbb{G} as explained in the introduction.⁵

⁴ Here, $x \pmod p$ is defined as the value $y \in (-p/2, p/2)$ such that $y \equiv x \pmod p$

⁵ We actually prove (29) using the linear map commitment of Lai and Malavolta [81, Appendix D.2]. While their scheme is only proven weakly function-binding (as defined in [81]) in the random oracle model, it can be proven strongly function-binding in the AGM+ROM and it still allows us to prove simulation-extractability in the AGM+ROM.

5.1 Description

In the description below, the CRS does not depend on a specific public key, but we allow it to depend on upper bounds on the RLWE dimension d , the modulus q of (26) and the infinity norms $\{B_i\}_{i=1}^M$. The reason is that they impact the dimension n of committed vectors and/or the order of the pairing-friendly group. Therefore the CRS-Gen algorithm inputs an upper bound \bar{d} for the dimension, an upper bound \bar{q} for the modulus and maximal values \bar{B}_i for the infinity norm bounds B_i to be proven. The prover is allowed to choose a different dimension $d \leq \bar{d}$, a different noise bound $B_i \leq \bar{B}_i$, and a different modulus $q \leq \bar{q}$ in each proof. For simplicity, we assume that each norm bound B_i is a power of two.

CRS-Gen($1^\lambda, 1^{\bar{d}}, 1^{\bar{q}}, 1^{\bar{M}}, 1^{\bar{N}}, \{1^{\bar{B}_i}\}_{i=1}^{\bar{N}}$): Given a security parameter λ , a maximal dimension $\bar{d} \in \text{poly}(\lambda)$, integers $\bar{N}, \bar{M} \in \text{poly}(\lambda)$, $\bar{q} \in \text{poly}(\lambda)$, $\bar{B}_i \in \text{poly}(\lambda)$, set $B_r \triangleq \bar{d}\bar{M} \cdot \max_{i \in [M]}(\bar{B}_i)/2$ and do the following.

1. Generate asymmetric pairing-friendly groups $(\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T)$ of prime order $p > \max(2^{l(\lambda)}, 2\bar{M}\bar{q}\bar{d}\max_i(\bar{B}_i))$, for some polynomial $l : \mathbb{N} \rightarrow \mathbb{N}$. Let $n > \bar{d} \cdot (\sum_{i=1}^{\bar{N}}(1 + \log \bar{B}_i) + \bar{N}(1 + \log B_r))$.
2. Pick a random $\alpha \xleftarrow{R} \mathbb{Z}_p$ and compute $g_1, \dots, g_n, g_{n+2}, \dots, g_{2n} \in \mathbb{G}$ as well as $\hat{g}_1, \dots, \hat{g}_n \in \hat{\mathbb{G}}$, where $g_i = g^{(\alpha^i)}$ for each $i \in [2n] \setminus \{n+1\}$ and $\hat{g}_i = \hat{g}^{(\alpha^i)}$ for each $i \in [n]$.
3. Choose hash functions $H, H_t : \{0, 1\}^* \rightarrow \mathbb{Z}_p^n$, $H_{\text{agg}} : \{0, 1\}^* \rightarrow \mathbb{Z}_p^3$ and $H_{\text{map}} : \{0, 1\}^* \rightarrow \mathbb{Z}_p^{\bar{N}\bar{d}+1}$ that will be modeled as random oracles.

Output the CRS

$$\text{pp} = \left((\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T), g, \hat{g}, \{g_i\}_{i \in [2n] \setminus \{n+1\}}, \{\hat{g}_i\}_{i \in [n]}, \mathbf{H} \right)$$

where $\mathbf{H} = \{H, H_t, H_{\text{agg}}, H_{\text{map}}\}$ are hash functions.

Prove_{pp}(\mathbf{x}, \mathbf{w}): Given a statement $\mathbf{x} = (q, d, M, N, \{B_i\}_{i=1}^N, \{\mathbf{a}_i\}_{i=1}^M, \mathbf{c})$ consisting of dimensions $d \leq \bar{d}$, $M \leq \bar{M}$, $N \leq \bar{N}$, a modulus $q \leq \bar{q}$, vectors of ring elements $\{\mathbf{a}_i \in R_q^N\}_{i=1}^M$, $\mathbf{c} \in R_q^N$, and norm bounds $B_i \leq \bar{B}_i$, as well as a witness $\mathbf{w} = (s_1, \dots, s_M) \in R^M$ such that (26) holds with $\|s_i\|_\infty \leq B_i$ for each $i \in [M]$, do the following.

1. Compute polynomials $(r_1, \dots, r_N) \in R^N$ such that $\|r_i\|_\infty \leq B_r$ for each $i \in [N]$ and satisfying (27). Encode (s_1, \dots, s_M) and (r_1, \dots, r_N) as

$$\tilde{\mathbf{w}} = [\mathbf{s}_1^\top \mid \dots \mid \mathbf{s}_M^\top \mid \mathbf{r}_1^\top \mid \dots \mid \mathbf{r}_N^\top]^\top \in \{0, 1\}^D,$$

using bit decompositions $\mathbf{s}_i = \mathbf{G}_{1+\log B}^{-1}(\phi(s_i)) \in \{0, 1\}^{d(1+\log B_i)}$ for each $i \in [M]$ and $\mathbf{r}_i = \mathbf{G}_{\log d}^{-1}(\phi(r_i)) \in \{0, 1\}^{d(1+\log B_r)}$ for each $i \in [N]$, where $D = d \cdot (\sum_{i=1}^N(1 + \log B_i) + N(1 + \log B_r))$.

2. Commit to $\mathbf{w} \triangleq (\tilde{\mathbf{w}} \mid \mathbf{0}^{n-D}) = (w_1, \dots, w_D, 0, \dots, 0) \in \{0, 1\}^n$ and prove that \mathbf{w} is binary. Namely,

- a. Choose $\gamma \xleftarrow{R} \mathbb{Z}_p$ and compute $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^D \hat{g}_j^{w_j}$.
- b. Compute $\mathbf{y} = (y_1, \dots, y_n) = H(\mathbf{x}, \hat{C}) \in \mathbb{Z}_p^n$. Next, choose $\gamma_y \xleftarrow{R} \mathbb{Z}_p$ and compute

$$C_y = g^{\gamma_y} \cdot \prod_{j=1}^D g_{n+1-j}^{y_j \cdot w_j}$$

Then, compute $\mathbf{t} = (t_1, \dots, t_n) = H_t(\mathbf{y}, \mathbf{x}, \hat{C}, C_y) \in \mathbb{Z}_p^n$.

- c. Using (15), compute $\pi_{eq} \in \mathbb{G}$ such that

$$\frac{e(\prod_{i=1}^n g_{n+1-i}^{t_i \cdot y_i}, \hat{C})}{e(C_y, \prod_{i=1}^n \hat{g}_i^{t_i})} = e(\pi_{eq}, \hat{g}). \quad (30)$$

which shows that C_y commits to the (reversed) product $\mathbf{y} \circ \mathbf{w} \in \mathbb{Z}_p^n$.

- d. Compute $\pi_y = C_y^\gamma \cdot \prod_{i=1}^n \left(g_i^{\gamma_y} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-j+i}^{y_j \cdot (w_j - 1)} \right)^{w_i}$ such that

$$e(C_y \cdot \prod_{j=1}^n g_{n+1-j}^{-y_j}, \hat{C}) = e(\pi_y, \hat{g}) \quad (31)$$

which shows that $\sum_{i=1}^n y_i \cdot w_i \cdot (w_i - 1) = 0$.

3. Compute $\tilde{\boldsymbol{\theta}} = H_{\text{map}}(\mathbf{x}, \hat{C}, C_y) \in \mathbb{Z}_p^{\tilde{N}d+1}$ and define $\tilde{\mathbf{A}} \in \mathbb{Z}^{Nd \times D}$ and $\phi(\mathbf{c}) \in \mathbb{Z}^{Nd}$ as in (29). Let $\boldsymbol{\theta} \in \mathbb{Z}_p^{Nd+1}$ the first $Nd+1$ entries of $\tilde{\boldsymbol{\theta}}$.
4. Parse $\boldsymbol{\theta}$ as $\boldsymbol{\theta} = (\boldsymbol{\theta}_0^\top \mid \delta_\theta)^\top$, with $\boldsymbol{\theta}_0 \in \mathbb{Z}_p^{Nd}$. Let $t_\theta = \boldsymbol{\theta}_0^\top \cdot \phi(\mathbf{c}) \bmod p$ and $\mathbf{a}_\theta^\top = \boldsymbol{\theta}_0^\top \cdot \tilde{\mathbf{A}} \bmod p$. Generate a proof $\pi_\theta \in \mathbb{G}$ satisfying

$$e\left(\prod_{k=1}^D g_{n+1-k}^{\mathbf{a}_\theta[k]}, \hat{C}\right) \cdot e(g_1, \hat{g}_n)^{-t_\theta} = e(\pi_\theta, \hat{g}) \quad (32)$$

by computing $\pi_\theta = \prod_{k=1}^D (g_{n+1-k}^\gamma \cdot \prod_{j \in [D] \setminus [k]} g_{n+1-k+j}^{w_j})^{\mathbf{a}_\theta[k]}$.

5. Compute $(\delta_{eq}, \delta_y, \delta_\theta) = H_{\text{agg}}(\mathbf{x}, \hat{C}, C_y) \in \mathbb{Z}_p^3$ and an aggregated proof

$$\pi = \pi_y^{\delta_y} \cdot \pi_{eq}^{\delta_{eq}} \cdot \pi_\theta^{\delta_\theta}.$$

Output the final proof $\boldsymbol{\pi} = (\hat{C}, C_y, \pi)$.

Verify_{pp}($\mathbf{x}, \boldsymbol{\pi}$): Given a statement $\mathbf{x} = (q, d, M, N, \{B_i\}_{i=1}^N, \{\mathbf{a}_i\}_{i=1}^M, \mathbf{c})$ and a candidate $\boldsymbol{\pi}$, return 0 if $\boldsymbol{\pi}$ does not parse properly. Otherwise,

1. Compute $(\delta_{eq}, \delta_y, \delta_\theta) = H_{\text{agg}}(\mathbf{x}, \hat{C}, C_y) \in \mathbb{Z}_p^3$, $\mathbf{y} = H(\mathbf{x}, \hat{C}) \in \mathbb{Z}_p^n$, $\mathbf{c} = H_t(\mathbf{y}, \mathbf{x}, \hat{C}, C_y) \in \mathbb{Z}_p^n$.

2. Compute $\bar{\theta} = H_{\text{map}}(\mathbf{x}, \hat{C}, C_y) \in \mathbb{Z}_p^{\bar{N}\bar{d}+1}$ and let $\theta = (\theta_0^\top \mid \delta_\theta)^\top \in \mathbb{Z}_p^{Nd+1}$ the first $Nd + 1$ coordinates of θ . Compute $t_\theta = \theta_0^\top \cdot \phi(\mathbf{c}) \in \mathbb{Z}_p$ and $\tilde{\mathbf{a}}_\theta^\top = \theta_0^\top \cdot \tilde{\mathbf{A}} \in \mathbb{Z}_p^D$. Define $\mathbf{a}_\theta^\top = (\tilde{\mathbf{a}}_\theta^\top \mid \mathbf{0}^{n-D}) \in \mathbb{Z}_p^n$.
3. Return 1 if the following equality holds and 0 otherwise:

$$e(\pi, \hat{g}) = e\left(C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{(\delta_{eq} \cdot t_i - \delta_y) \cdot y_i + \delta_\theta \cdot \mathbf{a}_\theta[i]}, \hat{C}\right) \cdot e\left(C_y, \prod_{i=1}^D \hat{g}_i^{\delta_{eq} \cdot t_i}\right)^{-1} \cdot e(g_1, \hat{g}_n)^{-t_\theta \cdot \delta_\theta}. \quad (33)$$

CORRECTNESS. Equation (33) is obtained by aggregating (30), (31), and (32) using randomness $(\delta_{eq}, \delta_y, \delta_\theta)$. The correctness of (30)-(31) can be shown as in Section 4 while (32) is a special case of the verification equation of the inner product functional commitment of [83] (recalled in the introduction, cf. (1)).

EFFICIENCY. We note that a 256-bit p is more than enough to satisfy the constraint $p > 2M \cdot qd \max_i(B_i)$ since d is typically 1024 or 2048, $q \approx 2^{64}$, and M is a small constant (concrete numbers are given in Supplementary Material G.3).

The CRS is comprised of $2n$ elements of \mathbb{G} and n elements of $\hat{\mathbb{G}}$. As in PointProofs [63], the verifier does not need $\{g_i\}_{i=n+2}^{2n}$, which are only used by the prover. The proof only consists of one element of $\hat{\mathbb{G}}$ and two elements of \mathbb{G} . Compared to the most efficient simulation-extractable variant [6] of Groth’s SNARK [69], our proofs are shorter by one element of $\hat{\mathbb{G}}$. This matches the optimal proof size of the simulation-extractable SNARK of Groth and Maller [70], which is significantly more expensive than [69] in terms of prover time and CRS size (see, e.g., [6] for detailed comparisons among them).

In terms of computation, $\pi = \pi_y^{\delta_y} \cdot \pi_{eq}^{\delta_{eq}} \cdot \pi_\theta^{\delta_\theta}$ can be computed using $2n$ exponentiations. At first, computing the corresponding exponents seems to require $O(n^2)$ multiplications over \mathbb{Z}_p , which can be quite expensive for a very large n . Fortunately, these exponents can be obtained via two products of degree- n polynomials,⁶ using only $O(n \cdot \log n)$ \mathbb{Z}_p -multiplications for a suitable prime p . At step 4, the prover computes a product $\mathbf{a}_\theta^\top = \theta_0^\top \cdot \tilde{\mathbf{A}} \bmod p$, which takes time $O(Nd \cdot D)$ in general. When it comes to proving many natural statements in structured lattices, the matrix $\tilde{\mathbf{A}}$ has a special structure allowing to compute $\theta_0^\top \cdot \tilde{\mathbf{A}}$ using only $O(d \cdot \log d)$ multiplications in \mathbb{Z}_p , as explained in Supplementary Material G.4. The prover’s cost is thus dominated by $3n$ exponentiations in \mathbb{G} and a product of $D = d \cdot (\sum_{i=1}^N (1 + \log B_i) + N(1 + \log B_r))$ elements in $\hat{\mathbb{G}}$. The verifier computes 3 pairings and n exponentiations in each source group.

The scheme is not fully succinct since the number of exponentiations at the verifier grows with the length of the witness. On the prover side, however, it enables significant savings compared to R1CS-based SNARKs as the number of

⁶ These polynomial products are implicitly computed in the exponent by the pairings in the right-hand-side member of (33). One of these two products is much faster to compute as it involves a polynomial of which almost all coefficients are binary.

exponentiations only grows with the size of the witness, rather than the size of the arithmetic circuit that computes the encryption function. Indeed, the number of ring operations in the encryption algorithm does not affect the number of exponentiations in the argument system.

In Supplementary Material G.3, we provide concrete proof/CRS sizes together with an estimation of the prover’s complexity when it comes to proving the validity of a ciphertext in the LPR cryptosystem [88]. For such a statement, we provide a detailed comparison with SNARKs [69] providing similarly short proofs. Our construction is shown advantageous in applications (e.g., [97]) that seek to decrease the prover’s computational effort, even at the cost of increasing the verifier’s. We also provide a comparison with [41].

5.2 Security

We first describe a simple zero-knowledge simulator.

Theorem 3. *The above non-interactive argument is perfectly zero-knowledge.* (The proof is given in Supplementary Material G.1.)

We note that the zero-knowledge simulator of Theorem 3 is not trapdoor-less [53] as it relies on the trapdoor of the CRS to simulate proofs. On the other hand, it works in the standard model, without relying on random oracles. In the proof of Theorem 4, we describe a trapdoor-less simulator that does not use the trapdoor of the CRS, but rather proceeds by programming the random oracles.

Theorem 4. *If the $(2n, n)$ -DLOG assumption holds, the above non-interactive argument provides simulation-extractability in the algebraic group model and in the random oracle model.* (The proof is given in Supplementary Material G.2.)

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Supplementary Material

A On Proving Smallness via Functional Commitments for Constant-Degree Polynomials

Catalano, Fiore and Tucker [28] recently built additively homomorphic FCs for constant-degree multivariate polynomials and monotone span programs. The former could be used to build short proofs of binarity by showing that the degree-2 polynomial $f(x_1, \dots, x_n) = \sum_{i=1}^n y_i \cdot x_i \cdot (x_i - 1)$ evaluates to 0 for random coefficients $\{y_i\}_{i=1}^n$. Nevertheless, their construction for degree- d polynomials has a CRS size $O(n^d)$. If we were to use it as is to prove that a committed vector is binary, we would end up with a quadratic-size CRS (instead of linear in our construction) and longer commitments containing two group elements. Moreover, the shape of the CRS would make it harder to prove knowledge-soundness in the algebraic group model (note that their notion of evaluation-binding⁷ would not suffice for our purposes). The reason is that their CRS contains elements of the form $(g^{\alpha^j}, g^{\beta \cdot \alpha^j})_{j \in [n^2]}$, for some secret $\beta, \alpha \in \mathbb{Z}_p$, while some components of honestly generated commitments are of the form $g^{\sum_{j=1}^n x_j \cdot (\alpha^j)}$ and only depend on $\{g^{\alpha^j}\}_{j \in [n]}$. Hence, it is not clear how the AGM would enable knowledge extraction from an adversarially-generated commitment/proof since the commitment can depend on all generators contained in the CRS, including $\{g^{\beta \cdot \alpha^j}\}_{j \in [n^2]}$.

To avoid these difficulties and decrease the CRS size to $O(n)$ group elements, it is tempting to exploit the sparsity of the polynomial $\sum_i y_i \cdot x_i \cdot (x_i - 1)$. Then, in the closest adaptation of the technique from [28, Section 4] that we can think of, either the commitment or the opening is longer than ours by at least one group element: The prover would include a commitment $\tilde{C} \in \mathbb{G}$ to the product $\mathbf{x} \circ \mathbf{x} = (x_1^2, \dots, x_n^2)$ in the opening before proving that $\mathbf{x} \circ \mathbf{x} - \mathbf{x}$ satisfies an inner product relation $\langle \mathbf{x} \circ \mathbf{x} - \mathbf{x}, \mathbf{y} \rangle = 0$ and that \tilde{C} is consistent with the initial commitment $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^n g_j^{x_j}$ to \mathbf{x} , which is part of the statement. To do this, the prover would have to include at least one additional group element (typically, an auxiliary commitment C to a reversed version of $\mathbf{y} \circ \mathbf{x}$ in \mathbb{G} if the initial commitment \hat{C} lives in $\hat{\mathbb{G}}$) either in the commitment or in the opening. Then, it would have to prove that C , and \tilde{C} and \hat{C} are consistent with one another by computing a pairing $e(C, \hat{C})$ and a pairing of \tilde{C} with some public encoding of \mathbf{y} . Hence, the auxiliary commitment C would have to be part of either the initial commitment or the opening, thus increasing the global communication overhead (besides the main commitment \hat{C}) to 3 group elements (C, \tilde{C}, π) if π is an aggregated proof showing the consistency of all commitments. In our applications to range proofs and short proofs for ring LWE ciphertexts, this would increase the proof length by at least one group element. Our approach avoids this overhead

⁷ In short, evaluation-binding means that no PPT adversary can prove distinct evaluations for a given function of the committed vector.

since, instead of including a commitment to (x_1^2, \dots, x_n^2) in the proof, we include a commitment to the reversed Hadamard product $(y_n \cdot x_n, \dots, y_1 \cdot x_1)$ so that we only need two group elements to argue that $\sum_i y_i \cdot x_i \cdot (x_i - 1) = 0$. This allows us to reach the smallest proof length of SNARKs [69] in our proofs of smallness and valid ring LWE encryption.

We also note that the technique of [28, Section 4.1] could be used to prove that a committed vector has infinity norm $\leq B$ by showing that the polynomial $P_y(x_1, \dots, x_n) = \sum_{i=1}^n y_i \cdot \prod_{j \in [-B, B]} (x_i - j)$ evaluates to 0 for a random $\mathbf{y} \in \mathbb{Z}_p^n$. However, the commitment size would grow with B (since it grows with the degree of the polynomial) while the proof length would grow with $\log B$. In contrast, both sizes are constant in our construction of Section C.1.

In an earlier work [86], Lipmaa and Pavlyk used the arithmetization of SNARKs [56] to construct succinct FC for sparse polynomials, where the number monomials is small w.r.t. the number n of variables. Their construction could be used as well to prove that a committed vector (x_1, \dots, x_n) satisfies $\sum_{i=1}^n y_i \cdot x_i \cdot (x_i - 1) = 0$, for a random $\mathbf{y} \in \mathbb{Z}_p$ derived from a random oracle. While their openings only consist of one group element, their scheme is more complex and using it in our setting would be significantly less efficient than our construction from Section 3 in other metrics. First, their commitments are larger and contain element of both sources groups \mathbb{G} and $\hat{\mathbb{G}}$ (concretely, 2 elements of \mathbb{G} and one element of $\hat{\mathbb{G}}$). In our applications of sections 4, C and 5, this would lengthen the proofs by at least one element of \mathbb{G} . Also, their CRS is more complex and contains $2\nu + \mu$ elements of \mathbb{G} and ν elements of $\hat{\mathbb{G}}$, where ν is the number of multiplication gates in the arithmetic circuit that computes the polynomial (which would be $\nu = 2n$ in our setting) and μ is the number of wires (here, we would have $\mu \geq 2n$). Their prover is more expensive as well and computes more than $\nu + \mu + \mu_\alpha + 2\mu_\beta$ exponentiations in \mathbb{G} , where μ_α and μ_β denote the lengths of private and public inputs (in our setting, this would amount to at least $7n$ exponentiations in \mathbb{G}). Moreover, their verification algorithm computes a product of 5 pairings (instead of 3 in Section 3) and $\mu_\beta = n$ exponentiations in both source groups.

Finally, the complex structure of their CRS would make it harder to prove knowledge-soundness in our context as it contains multiple monomials $\alpha^i y^j$ in the exponent (with $j > 1$), while “valid” commitments have components that only depend on monomials $\alpha^i y$, which have degree one in y . In the AGM, this would complicate the task of the knowledge extractor since maliciously generated commitments come with a representation that possibly depends on all group elements contained in the CRS.

B Deferred Material for the Range Proof of Section 4

B.1 Comparisons

Our construction of Section 4.1 assumes that the witness x is committed using a Pedersen commitment in the pairing-friendly group specified by the CRS of the

range proof. The range proof of Boneh *et al.* [14] makes a similar assumption as it requires x to be committed as a constant polynomial using the CRS of a polynomial commitment scheme.

The BFGW range proofs [14] have the shortest length to date and they also feature constant verification time (whereas our verifier computes $O(n)$ exponentiations, where n is the maximal bitlength of the range, as in BulletProofs). When they are instantiated with KZG commitments [77] and the cross-commitment evaluation techniques of [13, Section 4.1], their proofs consist of 2 commitments to polynomials (each of which takes an element of \mathbb{G}), 3 elements of \mathbb{Z}_p representing evaluations of committed polynomials, and a batched evaluation proof comprised of a group element and at least one scalar.⁸ If their construction is instantiated with the polynomial commitment of [83, Section 4.1]⁹ and the batched evaluation protocol of [13, Section 4.1], the communication cost decreases to 2 elements of \mathbb{G} (which commit to polynomials), 3 scalars (for polynomial evaluations) and a single element of \mathbb{G} for the batched evaluation proof. In the latter case, the range proof of [14] only requires 3 elements of \mathbb{G} and 3 elements of \mathbb{Z}_p . On the downside, combining [14,83] induces $2n$ exponentiations in $\hat{\mathbb{G}}$ at the verifier (instead of $O(1)$ using KZG commitments) and increases the prover’s overhead to $7n$ exponentiations in \mathbb{G} .

Not only does our construction ensure simulation-extractability in the AGM, it also features the shortest proof length and the smallest number of exponentiations at the prover (which is reduced by at least 40%, as shown in Table 1). In addition, the underlying vector commitment can be used to prove other properties about the committed short vector (e.g., upper bounds on its Hamming weight or its Euclidean norm) besides its infinity norm.

In terms of space, the above construction also improves upon BulletProofs [21], which requires the prover to send $2\lceil \log \ell \rceil + 4$ group elements and 5 elements of \mathbb{Z}_p . If we compare our construction with SNARKs, we obtain the same proof size as optimally short candidates [69,70] with the advantage that our CRS size is much shorter: It only depends on the maximal bitlength n of a range rather than the size of a circuit representation of the statement. Also, our prover only needs to compute $O(n)$ exponentiations instead of a number of exponentiations growing with the size of an arithmetic circuit that computes a commitment opening (which would be very large as the circuit would have to compute modular exponentiations).

In Table 1, we compare our constant-size range proofs with existing pairing-based solutions (including SNARK-based ones) featuring similarly short proofs. Several instantiations of [12] are considered for different polynomial commit-

⁸ In randomized versions of the KZG commitment (described in [77, Section 3.3], [13, Appendix B.2] and [103]), each evaluation proof consists of an element of \mathbb{G} and at least one scalar or an additional element of \mathbb{G} .

⁹ In order to prove the knowledge soundness of the range proof of [14] when the polynomial commitment of [83] is used, it is necessary to rely on the latter’s knowledge soundness in the AGM (as defined in [13, Appendix C.1.3]) but we believe this property holds under the $(2n, n)$ -DLOG assumption.

ment schemes that are known to provide constant-size evaluation proofs. Among schemes that do not generically rely on SNARKs, we only consider those where the CRS size is at most logarithmic in the range magnitude (i.e., linear in n). For example, Table 1 does not include range proofs based on lookup arguments [50, Section 5] which do not meet the latter criterion when instantiated with KZG commitments.

Table 1. Efficiency comparisons between constant-size range proofs

Schemes	Proof size	CRS size [◊]	Prover cost [†]	Verifier cost [*]
BFGW [14] + KZG [77, Section 3.3]	$3 \times \hat{\mathbb{G}} $ $+ 4 \times \mathbb{Z}_p $	$(4n + 2) \times \mathbb{G} $ $+ 4 \times \hat{\mathbb{G}} $	$5n \text{ exp}_{\mathbb{G}}$	$3P + 4 \text{ exp}_{\hat{\mathbb{G}}}$ $+ 1 \text{ exp}_{\mathbb{G}}$
BFGW + Zhang <i>et al.</i> [103]	$4 \times \hat{\mathbb{G}} $ $+ 3 \times \mathbb{Z}_p $	$(2n + 1) \times \mathbb{G} $ $+ 3 \times \hat{\mathbb{G}} $	$5n \text{ exp}_{\mathbb{G}}$	$3P + 4 \text{ exp}_{\hat{\mathbb{G}}}$ $+ 1 \text{ exp}_{\mathbb{G}}$
BFGW + LRY [83]	$3 \times \mathbb{G} $ $+ 3 \times \mathbb{Z}_p $	$4n \times \mathbb{G} $ $+ 2n \times \hat{\mathbb{G}} $	$7n \text{ exp}_{\mathbb{G}}$	$3P + 2n \text{ exp}_{\hat{\mathbb{G}}}$ $+ 2 \text{ exp}_{\mathbb{G}}$
Groth16 [69]	$1 \times \hat{\mathbb{G}} $ $+ 2 \times \mathbb{G}$	$3 \cdot \mathcal{C} \times \mathbb{G} ^{\Delta}$ $+ \mathcal{C} \times \hat{\mathbb{G}} $	$4 \cdot \mathcal{C} \text{ exp}_{\mathbb{G}}$ $ \mathcal{C} \text{ exp}_{\hat{\mathbb{G}}}$	$3P + O(1) \text{ exp}_{\mathbb{G}}$
New construction (Section 4)	$1 \times \hat{\mathbb{G}} $ $+ 2 \times \mathbb{G}$	$2n \times \mathbb{G} $ $+ n \times \hat{\mathbb{G}} $	$3n \text{ exp}_{\mathbb{G}}$ $+ 1 \text{ exp}_{\hat{\mathbb{G}}} + n \text{ mult}_{\hat{\mathbb{G}}}$	$4P + 2n \text{ exp}_{\mathbb{G}}$ $+ n \text{ exp}_{\hat{\mathbb{G}}}$

[◊] $\text{exp}_{\mathbb{G}}$ and $\text{exp}_{\hat{\mathbb{G}}}$ denote exponentiations in \mathbb{G} and $\hat{\mathbb{G}}$ while $\text{mult}_{\hat{\mathbb{G}}}$ denotes a multiplication in $\hat{\mathbb{G}}$.

[†] n denotes the bitlength of the range.

^{*} P stands for a pairing computation.

^Δ $|\mathcal{C}|$ denotes the number of multiplication gates in the arithmetic circuit computing a commitment opening.

B.2 Proof of Correctness

The first verification equation (20) is satisfied because

$$\begin{aligned}
e\left(\prod_{i=1}^n g_{n+1-i}^{2^{i-1}}, \hat{C}\right) &= \prod_{i=1}^n e(g_{n+1-i}, \hat{C})^{2^{i-1}} \\
&= \prod_{i=1}^n e(g_{n+1-i}, \hat{g}^{\gamma} \cdot \prod_{j=1}^n \hat{g}_j^{x_j})^{2^{i-1}} = \prod_{i=1}^n e(g^{\gamma} \cdot \prod_{j=1}^n g_j^{x_j}, \hat{g}_{n+1-i})^{2^{i-1}}, \\
&= \prod_{i=1}^n e(g_{n+1-i}^{\gamma} \cdot \prod_{j=1}^n g_{n+1+j-i}^{x_j}, \hat{g})^{2^{i-1}} \\
&= e(g_1, \hat{g}_n)^{\sum_{i=1}^n x_i \cdot 2^{i-1}} \cdot e\left(\prod_{i=1}^n (g_{n+1-i}^{\gamma} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-i+j}^{x_j})^{2^{i-1}}, \hat{g}\right)
\end{aligned}$$

and $e(g_n, \hat{V}) = e(g_1, \hat{g}_n)^x \cdot e(g_n^r, \hat{g})$, so that dividing out the two equations yields $e(\prod_{i=1}^n g_{n+1-i}^{2^{i-1}}, \hat{C}) / e(g_n, \hat{V}) = e(\pi_x, \hat{g})$ when $x = \sum_{i=1}^n x_i \cdot 2^{i-1}$.

Similarly, the second verification equation (21) follows by dividing the following two equalities:

$$\begin{aligned}
e\left(\prod_{i=1}^n g_{n+1-i}^{t_i \cdot y_i}, \hat{C}\right) &= \prod_{i=1}^n e(g_{n+1-i}, \hat{C})^{t_i \cdot y_i} \\
&= \prod_{i=1}^n \left(e(g_1, \hat{g}_n)^{x_i} \cdot e(g, \hat{g}_{n+1-i}^\gamma \cdot \prod_{j \in [n] \setminus \{i\}} \hat{g}_{n+1+j-i}^{x_j}) \right)^{t_i \cdot y_i} \\
&= \prod_{i=1}^n \left(e(g_1, \hat{g}_n)^{x_i} \cdot e(g_{n+1-i}^\gamma \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1+j-i}^{x_j}, \hat{g}) \right)^{t_i \cdot y_i} \\
&= e(g_1, \hat{g}_n)^{\sum_{i=1}^n y_i \cdot t_i \cdot x_i} \\
&\quad \cdot e\left(\prod_{i=1}^n \left(g_{n+1-i}^\gamma \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-i+j}^{x_j} \right)^{t_i \cdot y_i}, \hat{g}\right)
\end{aligned}$$

and

$$\begin{aligned}
e\left(C_y, \prod_{i=1}^n \hat{g}_i^{t_i}\right) &= \prod_{i=1}^n e(C_y, \hat{g}_i)^{t_i} = \prod_{i=1}^n e(g^{\gamma_y} \cdot \prod_{j=1}^n g_{n+1-j}^{y_j \cdot x_j}, \hat{g}_i)^{t_i} \\
&= \prod_{i=1}^n e\left(g_i^{\gamma_y} \cdot \prod_{j=1}^n g_{n+1-j+i}^{y_j \cdot x_j}, \hat{g}\right)^{t_i} \\
&= e(g_1, \hat{g}_n)^{\sum_{i=1}^n y_i \cdot t_i \cdot x_i} \cdot e\left(\prod_{i=1}^n \left(g_i^{\gamma_y} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-j+i}^{y_j \cdot x_j} \right)^{t_i}, \hat{g}\right).
\end{aligned}$$

As for equation (22), we have

$$\begin{aligned}
e\left(C_y \cdot \prod_{j=1}^n g_{n+1-j}^{-y_j}, \hat{C}\right) &= e\left(g^{\gamma_y} \cdot \prod_{j=1}^n g_{n+1-j}^{y_j \cdot (x_j-1)}, \hat{g}^\gamma \cdot \prod_{i=1}^n \hat{g}_i^{x_i}\right) \\
&= e(C_y, \hat{g})^\gamma \cdot \prod_{i=1}^n e\left(\left(g_i^{\gamma_y} \cdot \prod_{j=1}^n g_{n+1-j+i}^{y_j \cdot (x_j-1)}\right)^{x_i}, \hat{g}\right) \\
&= e(g_1, \hat{g}_n)^{\sum_{i=1}^n y_i \cdot x_i \cdot (x_i-1)} \cdot e(C_y^\gamma, \hat{g}) \\
&\quad \cdot \prod_{i=1}^n e\left(\left(g_i^{\gamma_y} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-j+i}^{y_j \cdot (x_j-1)}\right)^{x_i}, \hat{g}\right) \\
&= e\left(C_y^\gamma \cdot \prod_{i=1}^n \left(g_i^{\gamma_y} \cdot \prod_{j \in [n] \setminus \{i\}} g_{n+1-j+i}^{y_j \cdot (x_j-1)} \right)^{x_i}, \hat{g}\right),
\end{aligned}$$

where the last equality holds because $x_i(x_i - 1) = 0$ for each $i \in [n]$.

Equation (23) is satisfied by $\pi_v = \prod_{i=2}^n (g_{n+1-i}^r \cdot g_{n+2-i}^x)^{s_i} \in \mathbb{G}$ since

$$\begin{aligned} e\left(\prod_{i=2}^n g_{n+1-i}^{s_i}, \hat{V}\right) &= \prod_{i=2}^n e(g_{n+1-i}, \hat{g}^r \cdot \hat{g}_1^x)^{s_i} \\ &= \prod_{i=2}^n e(g^r \cdot g_1^x, \hat{g}_{n+1-i})^{s_i} = \prod_{i=2}^n e(g_{n+1-i}^r \cdot g_{n+2-i}^x, \hat{g})^{s_i} = e(\pi_v, \hat{g}). \end{aligned}$$

B.3 Proof of Theorem 1

Proof. We describe a simulator that perfectly simulates proofs using a trapdoor $\text{tk} = \alpha \in \mathbb{Z}_p$ and by programming the random oracles. Given a commitment $\hat{V} \in \hat{\mathbb{G}}$, the simulator computes $C_y = g^{\theta_y} \in \mathbb{G}$ for a randomly chosen $\theta_y \xleftarrow{R} \mathbb{Z}_p$. Next, it obtains $s_i = H_s(i, [2, n], \hat{V}, \hat{C}, C_y)$ for each index $i \in [2, n]$. It then uniformly chooses $\mathbf{y} = (y_1, \dots, y_n) \xleftarrow{R} \mathbb{Z}_p^n$, $\mathbf{t} = (t_1, \dots, t_n) \xleftarrow{R} \mathbb{Z}_p^n$, $\boldsymbol{\delta} = (\delta_x, \delta_{eq}, \delta_y, \delta_v) \xleftarrow{R} \mathbb{Z}_p^4$, sets $s_0 = 0$, and computes

$$\lambda = \frac{\delta_x \cdot (\alpha^n) - \sum_{i=2}^n \delta_v \cdot s_i \cdot (\alpha^{n+1-i})}{\theta_y \cdot \delta_y + \sum_{i=1}^n (\delta_{x,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) \cdot y_i + \delta_0 \cdot u_i) \cdot (\alpha^{n+1-i})},$$

where $\delta_{x,i} = \delta_x$ if $i \in [\ell]$ and $\delta_{x,i} = 0$ for all $i \in [\ell + 1, n]$.

Note that the denominator is uniformly distributed over \mathbb{Z}_p and non-zero with probability $1 - 1/p$. Then, it chooses $\gamma \xleftarrow{R} \mathbb{Z}_p$ and computes a commitment $\hat{C} = \hat{V}^\lambda \cdot \hat{g}^\gamma$. It aborts if $\mathbf{y} = H(\hat{V}, \hat{C})$, or $H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$ or $H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$ was already defined. If the simulator does not fail, it computes

$$\pi = \left(C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{\delta_{x,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) y_i} \right)^\gamma \cdot \left(\prod_{i=1}^n \hat{g}_i^{\delta_{eq} \cdot t_i} \right)^{-\theta_y}. \quad (34)$$

Then, it programs the random oracles to have $H_{\text{agg}}(\hat{V}, \hat{C}, C_y) = (\delta_x, \delta_{eq}, \delta_y, \delta_v)$, $\mathbf{y} = H(\hat{V}, \hat{C})$, $\mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$ for each $i \in [n]$. This provides a valid proof $\boldsymbol{\pi} = (\hat{C}, C_y, \pi)$, where \hat{C} and C_y are uniformly distributed over \mathbb{G} and $\hat{\mathbb{G}}$, respectively. Moreover, π satisfies the verification equation (25) since we have

$$g_n^{\delta_x} \cdot \prod_{i=2}^n g_{n+1-i}^{-\delta_v \cdot s_i} = \left(C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{\delta_{x,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) y_i} \right)^\lambda,$$

which implies

$$\begin{aligned}
& \frac{e(C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{\delta_{x,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) \cdot y_i}, \hat{C})}{e(g_n^{\delta_x} \cdot \prod_{i=2}^n g_{n+1-i}^{-\delta_v \cdot s_i}, \hat{V}) \cdot e(C_y, \prod_{i=1}^n \hat{g}_i^{\delta_{eq} \cdot t_i})} \\
&= \frac{e(C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{\delta_{x,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) \cdot y_i}, \hat{V}^\lambda \cdot \hat{g}^\gamma)}{e(g_n^{\delta_x} \cdot \prod_{i=2}^n g_{n+1-i}^{-\delta_v \cdot s_i}, \hat{V}) \cdot e(C_y, \prod_{i=1}^n \hat{g}_i^{\delta_{eq} \cdot t_i})} \\
&= \frac{e(C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{\delta_{x,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) \cdot y_i}, \hat{V}^\lambda)}{e(g_n^{\delta_x} \cdot \prod_{i=2}^n g_{n+1-i}^{-\delta_v \cdot s_i}, \hat{V})} \\
&\quad \cdot \frac{e((C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{\delta_{x,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) \cdot y_i})^\gamma, \hat{g})}{e((\prod_{i=1}^n \hat{g}_i^{\delta_{eq} \cdot t_i})^{\theta_y}, \hat{g})} = e(\pi, \hat{g})
\end{aligned}$$

□

B.4 Proof of Theorem 2

Proof. In the AGM+ROM model, we show that, unless the $(2n, n)$ -DLOG assumption is false, there exists an extractor that can extract a witness from any adversarially-generated proof $\pi = (\hat{C}, C_y, \pi)$ and statement $(\hat{V}, [0, 2^\ell - 1])$. Specifically, we give an algorithm \mathcal{B} that can either extract a witness (x, r) with $x \in [0, 2^\ell - 1]$ or solve an $(2n, n)$ -DLOG instance by computing $\alpha \in \mathbb{Z}_p$ from $\{(g, g_1, \dots, g_{2n}), (\hat{g}_1, \dots, \hat{g}_n)\}$, where $g_i = g^{(\alpha^i)}$ and $\hat{g}_i = \hat{g}^{(\alpha^i)}$ for all i .

The given problem instance $\{(g, g_1, \dots, g_{2n}), (\hat{g}_1, \dots, \hat{g}_n)\}$ is used to define the CRS pp. Note that $g_{n+1} = g^{(\alpha^{n+1})}$ is *not* included in pp although it is part of \mathcal{B} 's input.¹⁰ Our reduction/extractor \mathcal{B} then interacts with \mathcal{A} as follows.

Queries: When \mathcal{A} makes a random oracle query, \mathcal{B} returns the previously defined value if it exists. Otherwise, it returns a random element in the appropriate range. When \mathcal{A} queries a hash value $H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$, \mathcal{B} makes the corresponding hash queries $\mathbf{y} = H(\hat{V}, \hat{C})$, $\mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$, $\{s_i = H_s(i, \hat{V}, [2, n])\}_{i=2}^n$ for itself before returning a tuple $(\delta_x, \delta_{eq}, \delta_y, \delta_v)$. Since we are in the algebraic group model, at the first hash query involving a group element in \mathbb{G} , $\hat{\mathbb{G}}$, or \mathbb{G}_T , \mathcal{A} provides a representation of this group element as a linear combination of all the group elements that \mathcal{A} previously observed in the same group.

At any time, \mathcal{A} can choose a commitment $\text{com} = \hat{V}$ and ask for a simulated proof that \hat{V} is a commitment to some integer in $[0, 2^\ell - 1]$ for some $\ell \leq n$ of its choice. Since \mathcal{A} is an algebraic adversary, it must provide a representation of \hat{V} with respect to the generators $\{\hat{g}_i\}_{i \in [0, n]}$ and the commitments \hat{C} contained in earlier proofs generated by the simulator. However, the simulator used by

¹⁰ In fact, $g_{n+1} = g^{(\alpha^{n+1})}$ will not be used by \mathcal{B} at all, so that the reduction also works under the weaker assumption where g_{n+1} is not given. For simplicity, we consider the $(2n, n)$ -DLOG assumption, which is more widely used.

\mathcal{B} is itself algebraic and always simulates proofs by computing commitments \hat{C} as linear combinations of $\{\hat{g}_i\}_{i \in [0, n]}$ for coefficients of its choice. Hence, for any \hat{V} chosen by the adversary, \mathcal{B} can always compute a representation $\{v_i\}_{i=0}^n$ such that $\hat{V} = \hat{g}^{v_0} \cdot \prod_{i=1}^n \hat{g}_i^{v_i}$. We assume w.l.o.g. that either $v_1 \notin [0, 2^\ell - 1]$ or $(v_2, \dots, v_n) \neq \mathbf{0}$ since, otherwise, \mathcal{B} can generate a real proof using (v_1, v_0) as witnesses. Then, \mathcal{B} proceeds as follows to simulate a proof without using g_{n+1} :

1. Choose random vectors $\boldsymbol{\delta} = (\delta_x, \delta_{eq}, \delta_y, \delta_v) \xleftarrow{R} \mathbb{Z}_p^4$, $\mathbf{y} = (y_1, \dots, y_n) \xleftarrow{R} \mathbb{Z}_p^n$, $\mathbf{t} = (t_1, \dots, t_n) \xleftarrow{R} \mathbb{Z}_p^n$.
2. Let $f_{n+1} = \sum_{i=2}^n v_i \cdot s_i$ for randomly chosen $s_2, \dots, s_n \xleftarrow{R} \mathbb{Z}_p$. Define

$$\begin{aligned} a_1 &= v_1 - \frac{\delta_v \cdot f_{n+1}}{\delta_x} \\ a_i &= 0 & \forall i \in [2, n] \\ z_n &= y_1 \end{aligned}$$

Note that $a_1 \notin \{0, 1\}$ w.h.p. if $v_1 \notin [0, 2^\ell - 1]$ or $(v_2, \dots, v_n) \neq \mathbf{0}$. Then, find an arbitrary $(z_1, \dots, z_{n-1}) \in \mathbb{Z}_p^{n-1}$ such that

$$\sum_{i=2}^n t_i \cdot z_{n+1-i} = t_1 \cdot (a_1 \cdot y_1 - y_1).$$

3. Choose random $a_0, z_0 \xleftarrow{R} \mathbb{Z}_p$ and compute simulated commitments

$$\hat{C} = \hat{g}^{a_0} \cdot \prod_{i=1}^n \hat{g}_i^{a_i} = \hat{g}^{a_0} \cdot \hat{g}^{a_1}, \quad C_y = g^{z_0} \cdot \prod_{i=1}^n g_i^{z_i}.$$

4. If one of the random oracle values $H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$, $H(\hat{V}, \hat{C})$, $H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$ or $\{H_s(i, [2, n], \hat{V}, \hat{C}, C_y)\}_{i=2}^n$ was already defined, then abort and report failure. Otherwise, set $\mathbf{y} = H(\hat{V}, \hat{C})$, $\mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$, $\boldsymbol{\delta} = H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$ and $s_i = H_s(i, [2, n], \hat{V}, \hat{C}, C_y)$ for each $i \in [2, n]$.
5. Define the polynomials

$$\begin{aligned} Q_x[X] &= \left(\sum_{i=0}^n a_i \cdot X^i \right) \cdot \left(\sum_{i=1}^{\ell} 2^{i-1} \cdot X^{n+1-i} \right) - \left(\sum_{i=0}^n v_i \cdot X^{i+n} \right) = \sum_{i=0}^{n+\ell} q_i \cdot X^i, \\ Q_y[X] &= \left(\sum_{i=0}^n z_i \cdot X^i - \sum_{i=1}^n y_i \cdot X^{n+1-i} \right) \cdot \left(\sum_{i=0}^n a_i \cdot X^i \right) \\ &= \left(z_0 + \sum_{i=1}^n (z_{n+1-i} - y_i) \cdot X^{n+1-i} \right) \cdot \left(\sum_{i=0}^n a_i \cdot X^i \right) = \sum_{i=0}^{2n} \sigma_i \cdot X^i \end{aligned}$$

$$\begin{aligned}
Q_{eq}[X] &= \left(\sum_{i=0}^n a_i \cdot X^i \right) \cdot \left(\sum_{i=1}^n t_i \cdot y_i \cdot X^{n+1-i} \right) \\
&\quad - \left(\sum_{i=0}^n z_i \cdot X^i \right) \cdot \left(\sum_{i=1}^n t_i \cdot X^i \right) = \sum_{j=0}^{2n} e_j \cdot X^j, \\
Q_v[X] &= \left(\sum_{i=0}^n v_i \cdot X^i \right) \cdot \left(\sum_{i=2}^n s_i \cdot X^{n+1-i} \right) = \sum_{j=0}^{2n} f_j \cdot X^j.
\end{aligned}$$

Their degree- $(n+1)$ coefficients are $f_{n+1} = \sum_{i=2}^n v_i \cdot s_i$ and

$$q_{n+1} = -v_1 + \sum_{i=1}^{\ell} a_i \cdot 2^{i-1} = -v_1 + a_1 = -\frac{\delta_v \cdot f_{n+1}}{\delta_x},$$

$$\sigma_{n+1} = \sum_{i=1}^n a_i \cdot (z_{n+1-i} - y_i) = a_1 \cdot (z_n - y_1) = 0$$

$$e_{n+1} = \sum_{i=1}^n t_i \cdot (a_i \cdot y_i - z_{n+1-i}) = t_1 \cdot (a_1 \cdot y_1 - y_1) - \sum_{i=2}^n t_i \cdot z_{n+1-i} = 0$$

due to the definition of committed $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$. Observe that

$$\delta_x \cdot q_{n+1} + \delta_{eq} \cdot e_{n+1} + \delta_y \cdot \sigma_{n+1} + \delta_v \cdot f_{n+1} = 0 \quad (35)$$

6. Define the polynomial

$$\begin{aligned}
Q_{agg}[X] &= \delta_x \cdot Q_x[X] + \delta_{eq} \cdot Q_{eq}[X] + \delta_y \cdot Q_y[X] + \delta_v \cdot Q_v[X] \\
&= \sum_{i=0}^{2n} \eta_i \cdot X^i
\end{aligned}$$

for which $\eta_{n+1} = 0$ by construction. Compute

$$\pi = \prod_{i=1, i \neq n+1}^{2n} g_i^{\eta_i} \quad (36)$$

using $(g, \{g_i\}_{i \in [2n] \setminus \{n+1\}})$ and return the simulated proof $\boldsymbol{\pi} = (\hat{C}, C_y, \pi)$.

Note that the simulated π from (36) satisfies the verification equation

$$\frac{e(C_y^{\delta_y} \cdot \prod_{i=1}^{\ell} g_{n+1-i}^{\delta_x \cdot 2^{i-1} + (\delta_{eq} \cdot t_i - \delta_y) \cdot y_i} \cdot \prod_{i=\ell+1}^n g_{n+1-i}^{(\delta_{eq} \cdot t_i - \delta_y) \cdot y_i}, \hat{C})}{e(g_n^{\delta_x} \cdot \prod_{i=2}^n g_{n+1-i}^{-\delta_v \cdot s_i}, \hat{V}) \cdot e(C_y, \prod_{i=1}^n \hat{g}_i^{\delta_{eq} \cdot t_i})} = e(\pi, \hat{g}). \quad (37)$$

Moreover, $\boldsymbol{\pi}$ has the same distribution as a proof generated by the zero-knowledge simulator in the proof of Theorem 1. Indeed, $\boldsymbol{\pi}$ is uniquely determined by the commitments (\hat{C}, \hat{V}, C_y) and the \mathbb{Z}_p -elements $\mathbf{y}, \mathbf{t}, \{s_i\}_{i=2}^n$, and $\boldsymbol{\delta}$ in (37). Also,

while the committed vectors $\mathbf{a}, \mathbf{z} \in \mathbb{Z}_p^n$ are programmed in a special way, they are perfectly hidden by the randomness a_0 and z_0 in \hat{C} and C_y .

Consequently, the simulation is perfect, unless a collision occurs when random oracles are programmed in one of the simulation queries. If Q_S (reps. Q_H) denotes the number of queries made by \mathcal{A} to the simulator (resp. to random oracles), this happens with probability at most $(Q_S + Q_H) \cdot Q_H/p$.

Output: When the adversary \mathcal{A} halts, it outputs a statement $(\hat{V}, [0, 2^{\ell-1}])$, for some $\ell \in [1, n]$, together with a verifying proof $\boldsymbol{\pi} = (\hat{C}, C_y, \boldsymbol{\pi})$.

Since we are in the AGM, any malicious prover that comes up with a commitment $\text{com} = \hat{V}$ and a proof $\boldsymbol{\pi} = (\hat{C}, C_y, \boldsymbol{\pi})$ must also provide a representation of each group element w.r.t. the group elements $(g, \hat{g}, \{g_i\}_{i \in [2n] \setminus \{n+1\}}, \{\hat{g}_i\}_{i \in [n]})$ and all other group elements that have been observed so far.¹¹ In particular, \mathcal{A} must provide a representation of C_y w.r.t to $(g, \{g_i\}_{i \in [2n] \setminus \{n+1\}})$ and the group elements $\{C_y^{(i)}, \pi^{(i)}\}_{i \in [Q_S]}$ contained in responses $\{\pi^{(i)}\}_{i \in [Q_S]}$ to simulation queries. For the same reason, \mathcal{A} must provide a representation of \hat{C} w.r.t $(\hat{g}, \{\hat{g}_i\}_{i \in [n]})$ and the commitments $\{\hat{C}^{(i)}\}_{i \in [Q_S]}$ contained in simulated proofs $\{\pi^{(i)}\}_{i \in [Q_S]}$. However, for each $i \in [Q_S]$, \mathcal{B} knows a representation of $\hat{C}^{(i)}$ w.r.t. $(\hat{g}, \{\hat{g}_i\}_{i \in [n]})$ and a representation of C_y w.r.t. $(g, \{g_i\}_{i=1}^n)$. It also knows a representation of each simulated $\pi^{(i)}$ w.r.t $(g, \{g_i\}_{i \in [2n] \setminus \{n+1\}})$. From \mathcal{A} 's output and the random coins of the simulation, \mathcal{B} is able to compute scalars $\{(\theta_i, z_i) \in \mathbb{Z}_p^2\}_{i \in [0, 2n] \setminus \{n+1\}}, \{(a_i, v_i) \in \mathbb{Z}_p^2\}_{i \in [0, n]}$ such that

$$\hat{C} = \prod_{i=0}^n \hat{g}_i^{a_i}, \quad C_y = \prod_{i=0, i \neq n+1}^{2n} g_i^{z_i}, \quad \hat{V} = \prod_{i=0}^n \hat{g}_i^{v_i}, \quad \boldsymbol{\pi} = \prod_{i=0, i \neq n+1}^{2n} g_i^{\theta_i},$$

where we define $g_0 = g$ and $\hat{g}_0 = \hat{g}$ for convenience.

If the representation $(v_0, v_1, \dots, v_n) \in \mathbb{Z}_p^2$ of \hat{V} is such that $v_1 \in [0, 2^\ell - 1]$ and $v_i = 0$ for all $i \in [2, n]$, then \mathcal{B} is done as it can simply output $(v_1, v_0) \in \mathbb{Z}_p^2$ as a valid opening of the Pedersen commitment \hat{V} to an integer v_1 in the proper range. We now assume that either $v_1 \notin [0, 2^\ell - 1]$ or $(v_2, \dots, v_n) \neq \mathbf{0}^{n-1}$.

Solving $(2n, n)$ -DLOG: By hypothesis, \mathcal{A} 's statement ($\text{com} = \hat{V}, [0, 2^{\ell-1}]$) and proof $\boldsymbol{\pi} = (\hat{C}, C_y, \boldsymbol{\pi})$ satisfy the verification equation (37), where $\mathbf{y} = H(\hat{V}, \hat{C})$, $\mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$, $s_0 = 0$, $s_i = H_s(i, [2, n], \hat{V}, \hat{C}, C_y)$ for each $i \in [2, n]$, and $(\delta_x, \delta_{eq}, \delta_y, \delta_v) = H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$.

We first note that a non-trivial valid $\boldsymbol{\pi}$ cannot recycle (\hat{V}, \hat{C}, C_y) obtained from the simulation oracle (namely, we must have $(\hat{V}, \hat{C}, C_y) \neq (\hat{V}^{(i)}, \hat{C}^{(i)}, C_y)$ for all $i \in [Q_S]$) since the left-hand-side member of (37) is uniquely determined by $(\hat{V}^{(i)}, \hat{C}^{(i)}, C_y^{(i)})$ and it in turn determines a unique valid $\pi^{(i)}$. Consequently, the hash values $H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$, $H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$ and $\{H_s(i, [2, n], \hat{V}, \hat{C}, C_y)\}_{i=2}^n$

¹¹ These representations are actually supplied by \mathcal{A} at the first random oracle query (or the first simulation query in the case of V) involving the corresponding group elements.

were *not* programmed by the simulator in a simulation query.

We also note that the left-hand-side member of (37) is obtained by raising the left-hand-side members of equations (20)-(23) to the powers $(\delta_x, \delta_{eq}, \delta_y, \delta_v)$ and multiplying the results together. Hence, it can be written $e(g, \hat{g})^{P_{\text{agg}}(\alpha)}$, where $P_{\text{agg}}[X]$ is the polynomial

$$P_{\text{agg}}[X] = \delta_x \cdot P_x[X] + \delta_y \cdot P_y[X] + \delta_{eq} \cdot P_{eq}[X] + \delta_v \cdot P_v[X]$$

obtained as a linear combination of the polynomials

$$\begin{aligned} P_x[X] &= \left(\sum_{i=0}^n a_i \cdot X^i \right) \cdot \left(\sum_{i=1}^{\ell} 2^{i-1} \cdot X^{n+1-i} \right) - \left(\sum_{i=0}^n v_i \cdot X^{n+i} \right) = \sum_{i=0}^{n+\ell} \omega_i \cdot X^i, \\ P_y[X] &= \left(\sum_{i=0, i \neq n+1}^{2n} z_i \cdot X^i - \sum_{i=1}^n y_i \cdot X^{n+1-i} \right) \cdot \left(\sum_{i=0}^n a_i \cdot X^i \right) \\ &= \left(z_0 + \sum_{i=1}^n (z_{n+1-i} - y_i) \cdot X^{n+1-i} + \sum_{i=n+2}^{2n} z_i \cdot X^i \right) \cdot \left(\sum_{i=0}^n a_i \cdot X^i \right) \\ &= \sum_{i=0}^{3n} \gamma_i \cdot X^i \\ P_{eq}[X] &= \left(\sum_{i=0}^n a_i \cdot X^i \right) \cdot \left(\sum_{i=1}^n t_i \cdot y_i \cdot X^{n+1-i} \right) \\ &\quad - \left(\sum_{i=0, i \neq n+1}^{2n} z_i \cdot X^i \right) \cdot \left(\sum_{i=1}^n t_i \cdot X^i \right) = \sum_{j=0}^{3n} \beta_j \cdot X^j, \\ P_v[X] &= \left(\sum_{i=0}^n v_i \cdot X^i \right) \cdot \left(\sum_{i=2}^n s_i \cdot X^{n+1-i} \right) = \sum_{j=0}^{2n} \mu_j \cdot X^j \end{aligned}$$

for which the left-hand-side members of (20)-(23) can be written $e(g, \hat{g})^{P_x(\alpha)}$, $e(g, \hat{g})^{P_{eq}(\alpha)}$, $e(g, \hat{g})^{P_y(\alpha)}$, and $e(g, \hat{g})^{P_v(\alpha)}$, respectively.

Letting $P_{\text{agg}}[X] = \sum_{i=0}^{3n} \nu_i \cdot X^i$, the coefficient ν_{n+1} of the degree- $(n+1)$ term can be written

$$\begin{aligned} \nu_{n+1} &= \underbrace{\delta_x \cdot \left(\sum_{i=1}^{\ell} a_i \cdot 2^{i-1} - v_1 \right)}_{\triangleq \omega_{n+1}} + \underbrace{\delta_y \cdot \sum_{i=1}^n (z_{n+1-i} - y_i) \cdot a_i}_{\triangleq \gamma_{n+1}} \\ &\quad + \underbrace{\delta_{eq} \cdot \sum_{i=1}^n t_i \cdot (a_i \cdot y_i - z_{n+1-i})}_{\triangleq \beta_{n+1}} + \underbrace{\delta_v \cdot \sum_{i=2}^n v_i \cdot s_i}_{\triangleq \mu_{n+1}}, \end{aligned}$$

where $(\omega_{n+1}, \gamma_{n+1}, \beta_{n+1}, \mu_{n+1})$ are the coefficients of the degree- $(n+1)$ terms of $(P_x[X], P_y[X], P_{eq}[X], P_v[X])$, respectively. We argue that, if $v_1 \notin [0, 2^\ell - 1]$ or

$(v_2, \dots, v_n) \neq \mathbf{0}^{n-1}$, we cannot have $\nu_{n+1} = 0$, except with negligible probability. This follows from the following two arguments:

- The probability that $\boldsymbol{\rho} \triangleq (\omega_{n+1}, \gamma_{n+1}, \beta_{n+1}, \mu_{n+1}) = \mathbf{0}$ is negligible if $v_1 \notin [0, 2^\ell - 1]$ or $(v_2, \dots, v_n) \neq \mathbf{0}^{n-1}$. Indeed, when $(v_2, \dots, v_n) \neq \mathbf{0}^{n-1}$, we have $\mu_{n+1} = 0$, with probability $1/p$ over the random choice of $\{s_i\}_{i=2}^n$ since $\{s_i = H_s(i, [2, n], \hat{V}, \hat{C}, C_y)\}_{i=2}^n$ are derived uniformly *after* the choice of $\{v_i\}_{i=2}^n$. Likewise, when $z_{n+1-i} \neq a_i \cdot y_i$ for some $i \in [n]$, we have $\beta_{n+1} = 0$ with probability $1/p$ since $\mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$ is derived *after* the choice of \mathbf{y} , $\{a_i\}_{i=0}^n$ and $\{z_i\}_{i \in [0, 2n] \setminus \{n+1\}}$. Then, if $z_{n+1-i} = a_i \cdot y_i$ for all $i \in [n]$, we have $\gamma_{n+1} = \sum_{i=1}^n y_i \cdot (a_i - 1) \cdot a_i$, which cancels with probability $1/p$ if there exists $i \in [n]$ such that $a_i \notin \{0, 1\}$. To see this, we distinguish two cases:
 - a. If $\mathbf{y} = H(\hat{V}, \hat{C})$ was defined when answering a simulation query, we can only have $\gamma_{n+1} = 0$ with probability $1/p$ since the simulator programmed (a_1, \dots, a_n) to have $\gamma_{n+1} = \sum_{i=1}^n y_i \cdot a_i \cdot (a_i - 1) = y_1 \cdot a_1 \cdot (a_1 - 1)$ with $y_1 \in_R \mathbb{Z}_p$ and $a_1 \notin \{0, 1\}$. This covers the case of \mathcal{A} attempting to recycle $(\hat{V}, \hat{C}) = (\hat{V}^{(i)}, \hat{C}^{(i)})$ from a simulated proof $\boldsymbol{\pi}^{(i)} = (\hat{C}^{(i)}, C_y^{(i)}, \pi^{(i)})$, with a modified $C_y \neq C_y^{(i)}$.
 - b. If $H(\hat{V}, \hat{C})$ was not programmed by the simulator, then the hash value $\mathbf{y} = H(\hat{V}, \hat{C})$ was defined after \mathcal{B} obtained the algebraic representation $\{a_i\}_{i=0}^n$ of \hat{C} . Over the choice of \mathbf{y} , we have $\sum_{i=1}^n y_i \cdot (a_i - 1) \cdot a_i = 0$ with probability $1/p$.

If none of the above unlikely events occurs and $\omega_{n+1} = 0$, then we have $v_1 = \sum_{i=1}^\ell a_i \cdot 2^{i-1}$ and $a_i \in \{0, 1\}$ for all $i \in [\ell]$, which contradicts the hypothesis that $v_1 \notin [0, 2^\ell - 1]$.

- If $\boldsymbol{\rho} \neq \mathbf{0}$, then we have $\nu_{n+1} \neq 0$ with probability $1 - 1/p$ since $\boldsymbol{\delta} \triangleq (\delta_x, \delta_{eq}, \delta_y, \delta_v) = H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$ is derived from a random oracle *after* the choice of $\{(a_i, v_i)\}_{i=0}^n$, and $\{z_i\}_{i \in [0, 2n] \setminus \{n+1\}}$, which determine the coordinates of $\boldsymbol{\rho}$. Hence, a random independent $\boldsymbol{\delta} \in \mathbb{Z}_p^4$ can only satisfy $\langle \boldsymbol{\delta}, \boldsymbol{\rho} \rangle = 0$ with probability $1/p$.

If $\nu_{n+1} \neq 0$, \mathcal{B} can compute $\alpha \in \mathbb{Z}_p$ by observing that the aggregated verification equation (37) implies

$$\pi = g_{n+1}^{\nu_{n+1}} \cdot \prod_{i \in [0, 3n] \setminus \{n+1\}} g_i^{\nu_i}, \quad (38)$$

where $g_{2n+1} = g^{(\alpha^{2n+1})}, \dots, g_{3n} = g^{(\alpha^{3n})}$ are not available to \mathcal{B} . However, \mathcal{B} knows $\{\nu_i\}_{i=0}^{3n}$. Since $\nu_{n+1} \neq 0$, we are guaranteed that the representation (38) of π is different from its representation $\pi = \prod_{i=0, i \neq n+1}^{2n} g_i^{\theta_i}$ revealed by \mathcal{A} as part of its output. This means that $\alpha \in \mathbb{Z}_p$ is a root of the non-zero polynomial

$$R[X] = \sum_{i \in [0, 2n] \setminus \{n+1\}} (\nu_i - \theta_i) \cdot X^i + \nu_{n+1} \cdot X^{n+1} + \sum_{i=2n+1}^{3n} \nu_i,$$

which allows computing $\alpha \in \mathbb{Z}_p$ by factoring $R[X]$. \square

C Short Proofs that a Committed Vector is Small

We now show that the range proof of Section 4 can be batched in order to simultaneously prove possibly distinct ranges for the different slots of a multi-base Pedersen commitment. In particular, we can prove that a vector commitment commits to a vector of small infinity norm.

C.1 Description

Given a commitment $\hat{V} = \hat{g}^r \cdot \prod_{k=1}^m \hat{g}_k^{x_k}$, the prover will convince the verifier that $x_k \in [0, 2^{\ell_k} - 1]$ for each $k \in [m]$ using only 3 group elements. The construction proceeds as follows.

CRS-Gen($1^\lambda, 1^m, \{1^{n_k}\}_{k=1}^m$): On input of a security parameter λ , a number of slots $m \in \text{poly}(\lambda)$, and the maximal bitlength $n_k \in \text{poly}(\lambda)$ of ranges for each slot $k \in [m]$, set $n = \sum_{k=1}^m n_k$ and do the following:

1. Choose asymmetric bilinear groups $(\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T)$ of prime order $p > 2^{\ell_p}$, where $\ell_p = \max(l(\lambda), n_1, \dots, n_m)$ for some polynomial $l : \mathbb{N} \rightarrow \mathbb{N}$, and generators $g \stackrel{R}{\leftarrow} \mathbb{G}$, $\hat{g} \stackrel{R}{\leftarrow} \hat{\mathbb{G}}$.
2. Pick a random $\alpha \stackrel{R}{\leftarrow} \mathbb{Z}_p$ and compute $g_1, \dots, g_n, g_{n+2}, \dots, g_{2n} \in \mathbb{G}$ as well as $\hat{g}_1, \dots, \hat{g}_n \in \hat{\mathbb{G}}$, where $g_i = g^{(\alpha^i)}$ for each $i \in [2n] \setminus \{n+1\}$ and $\hat{g}_i = \hat{g}^{(\alpha^i)}$ for each $i \in [n]$.
3. Choose hash functions $H, H_t : \{0, 1\}^* \rightarrow \mathbb{Z}_p^n$, $H_s : \{0, 1\}^* \rightarrow \mathbb{Z}_p$, $H_{\text{agg}} : \{0, 1\}^* \rightarrow \mathbb{Z}_p^4$ and $H_\xi : \{0, 1\}^* \rightarrow \mathbb{Z}_p^m$ modeled as random oracles.

The public parameters are defined to be

$$\text{pp} = \left((\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T), n, g, \hat{g}, \{g_i\}_{i \in [2n] \setminus \{n+1\}}, \{\hat{g}_i\}_{i \in [n]}, \mathbf{H} \right)$$

where $\mathbf{H} = \{H, H_s, H_t, H_{\text{agg}}, H_\xi\}$ are hash functions.

Com_{pp}(x) To commit to a vector of integers $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$, choose a random $r \stackrel{R}{\leftarrow} \mathbb{Z}_p$ and compute $\hat{V} = \hat{g}^r \cdot \prod_{k=1}^m \hat{g}_k^{x_k} \in \hat{\mathbb{G}}$. Return the commitment $\text{com} = \hat{V} \in \hat{\mathbb{G}}$ and the opening information $\text{aux} = r \in \mathbb{Z}_p$.

Prove_{pp}($(\text{com}, \{1^{\ell_k}\}_{k=1}^m), (\mathbf{x}, \text{aux})$): given a commitment $\text{com} = \hat{V}$ and witnesses $(\mathbf{x}; \text{aux})$ consisting of an integer vector $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ such that $x_k \in [0, 2^{\ell_k} - 1]$ for each $k \in [m]$, where $\ell_k \leq n_k$, and $\text{aux} = r \in \mathbb{Z}_p$ is the randomness such that $\hat{V} = \hat{g}^r \cdot \prod_{k=1}^m \hat{g}_k^{x_k}$, do the following:

1. For each $k \in [m-1]$, set $j_k = n_1 + \dots + n_{k-1}$ with $j_1 = 0$. For each $k \in [m]$, let the binary expansion $(x_{k,1}, \dots, x_{k,\ell_k}) \in \{0, 1\}^{\ell_k}$ of x_k . Set $x_{k,i} = 0$ for each $i \in [\ell_k + 1, n_k]$ and define $\bar{\mathbf{x}}_k = (x_{k,1}, \dots, x_{k,n_k}) \in \{0, 1\}^{n_k}$.

2. Choose $\gamma \xleftarrow{R} \mathbb{Z}_p$ and compute

$$\hat{C} = \hat{g}^\gamma \cdot \prod_{k=1}^m \prod_{j=1}^{\ell_k} \hat{g}_{j_k+j}^{x_{k,j}}.$$

Compute $\mathbf{y} = (y_1, \dots, y_n) = H(\hat{V}, \hat{C}) \in \mathbb{Z}_p^n$. Then, choose $\gamma_y \xleftarrow{R} \mathbb{Z}_p$ and compute

$$C_y = g^{\gamma_y} \cdot \prod_{k=1}^m \prod_{j=1}^{\ell_k} g_{n+1-(j_k+j)}^{y_{j_k+j} \cdot x_{k,j}}$$

3. Compute $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) = H_\xi(\hat{V}, \hat{C}, C_y)$ and generate a proof π_x that \hat{C} commits to $(\bar{\mathbf{x}}_1 \mid \dots \mid \bar{\mathbf{x}}_m) \in \mathbb{Z}_p^n$ such that $\sum_{i=1}^{\ell_k} x_{k,i} \cdot 2^{i-1} = x_k$ for each $k \in [m]$. This proof $\pi_x \in \mathbb{G}$ satisfies

$$\frac{e\left(\prod_{k=1}^m \left(\prod_{i=1}^{\ell_k} g_{n+1-(j_k+i)}^{2^{i-1}}\right)^{\xi_k}, \hat{C}\right)}{e\left(\prod_{k=1}^m g_{n+1-k}^{\xi_k}, \hat{V}\right)} = e(\pi_x, \hat{g}) \quad (39)$$

and is obtained as per formula (46) in Supplementary Material C.2.

4. Compute $\mathbf{t} = (t_1, \dots, t_n) = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^n$. Generate a proof π_{eq} (as per formula (49) in Supplementary Material C.2) satisfying

$$\frac{e\left(\prod_{k=1}^m \prod_{i=1}^{n_k} g_{n+1-(j_k+i)}^{t_{j_k+i} \cdot y_{j_k+i}}, \hat{C}\right)}{e\left(C_y, \prod_{k=1}^m \prod_{i=1}^{n_k} \hat{g}_{j_k+i}^{t_{j_k+i}}\right)} = e(\pi_{eq}, \hat{g}), \quad (40)$$

which shows that C_y is consistent with \mathbf{y} and \hat{C} .

5. Prove that $\sum_{k=1}^m \sum_{i=1}^{n_k} y_{j_k+i} \cdot x_{k,i} \cdot (x_{k,i} - 1) = 0$ by computing $\pi_y \in \mathbb{G}$ (as specified by (50) in Supplementary Material C.2) satisfying

$$e\left(C_y \cdot \prod_{k=1}^m \prod_{i=1}^{n_k} g_{n+1-(j_k+i)}^{-y_{j_k+i}}, \hat{C}\right) = e(\pi_y, \hat{g}). \quad (41)$$

6. Generate an aggregated proof that $\hat{V} = \hat{g}^r \cdot \prod_{k=1}^m \hat{g}_k^{x_k}$ is a commitment to a vector that contains 0 in its last $n - m$ coordinates. Namely, compute $\pi_v = \prod_{i=m+1}^n (g_{n+1-i}^r \cdot \prod_{k=1}^m g_{n+2-i+k}^{x_k})^{s_i} \in \mathbb{G}$ such that

$$e\left(\prod_{i=m+1}^n g_{n+1-i}^{s_i}, \hat{V}\right) = e(\pi_v, \hat{g}). \quad (42)$$

where $s_i = H_s(i, [m+1, n], \hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p$ for each $i \in [m+1, n]$.

7. Compute $(\delta_x, \delta_{eq}, \delta_y, \delta_v) = H_{\text{agg}}(\hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^4$ and compute an aggregated proof

$$\pi = \pi_x^{\delta_x} \cdot \pi_y^{\delta_y} \cdot \pi_{eq}^{\delta_{eq}} \cdot \pi_v^{\delta_v}.$$

Output the final range argument which consists of

$$\boldsymbol{\pi} := (\hat{C}, C_y, \pi). \quad (43)$$

Verify_{pp}(com, $\boldsymbol{\pi}$): Given a commitment com = $\hat{V} \in \hat{\mathbb{G}}$ and a candidate proof $\boldsymbol{\pi}$, parse the latter as in (43).

1. Compute $(\delta_x, \delta_{eq}, \delta_y, \delta_v) = H_{\text{agg}}(\hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^4$, $\mathbf{y} = H(\hat{V}, \hat{C}) \in \mathbb{Z}_p^n$, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) = H_\xi(\hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^m$, $\mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y) \in \mathbb{Z}_p^n$, and $s_i = H_s(i, [m+1, n], \hat{V}, \hat{C}, C)$ for all indices $i \in [m+1, n]$. Define the vector $(s_1, \dots, s_n) = (\mathbf{0}^m, s_{m+1}, \dots, s_n)$.
2. For each $k \in [m]$, define $\delta_{x,k,i} = \delta_x$ if $i \in [\ell_k]$ and $\delta_{x,k,i} = 0$ if $i \in [\ell_k + 1, n_k]$. Return 1 if

$$\begin{aligned} & e\left(C_y^{\delta_y} \prod_{k=1}^m \prod_{i=1}^{n_k} g_{n+1-(j_k+i)}^{\xi_k \cdot \delta_{x,k,i} \cdot 2^{i-1} + (\delta_{eq} \cdot t_{j_k+i} - \delta_y) \cdot y_{j_k+i}}, \hat{C}\right) \\ & \cdot e\left(\prod_{k=1}^m g_{n+1-k}^{\xi_k} \prod_{i=m+1}^n g_{n+1-i}^{-\delta_v \cdot s_i}, \hat{V}\right)^{-1} \cdot e\left(C_y, \prod_{k=1}^m \prod_{i=1}^{n_k} \hat{g}_{j_k+i}^{\delta_{eq} \cdot t_{j_k+i}}\right)^{-1} = e(\pi, \hat{g}) \end{aligned} \quad (44)$$

and 0 otherwise.

The correctness of the scheme can be proven in the same way as in the base scheme of Section 4 and details are given in Supplementary Material C.2.

EFFICIENCY. The proof size remains the same as in Section 4. If m denotes the number of simultaneously performed range proofs, the computational cost of the prover is now dominated by $3n$ exponentiations in \mathbb{G} and $m+1$ exponentiations in $\hat{\mathbb{G}}$ since each subset product $\prod_{j=1}^{n_k} \hat{g}_{j_k+j}^{x_k \cdot j}$ is cheaper to compute than an exponentiation in $\hat{\mathbb{G}}$ (recall that $n_k < \log p$).

The verifier's workload amounts to $2n+1$ exponentiations in \mathbb{G} , n exponentiations in $\hat{\mathbb{G}}$ and 4 pairings.

EXTENSION TO PROVE SMALL EUCLIDEAN NORMS. As explained in Supplementary Material E, the construction can be used to prove that $\hat{V} = \hat{g}^r \cdot \prod_{k=1}^m \hat{g}_k^{x_k}$ commits to a vector of small Euclidean norm. In short, this can be achieved by: (i) Generating an auxiliary commitment to $\bar{\mathbf{x}} = (x_1^2, \dots, x_m^2)$ and proving that it was done correctly; (ii) Proving that $\langle \bar{\mathbf{x}}, (1, 1, \dots, 1) \rangle \bmod p$ is sufficiently small using its binary decomposition; (iii) Proving that $\|\mathbf{x}\|_\infty^2 \leq p/m$, so that $\sum_{i=1}^m x_i^2 \bmod p$ is also $\sum_{i=1}^m x_i^2$ over \mathbb{Z} . To make sure that step (iii) does not affect the zero-knowledge property, we restrict the scheme to prove Euclidean norm bounds smaller than $\sqrt{p/m}$. Without optimizations, the proof length only increases to 9 group elements.

C.2 Correctness of Aggregated Range Proofs

The verification equation (44) is obtained by raising equalities (39), (40), (41) and (42) to the powers δ_x , δ_{eq} , δ_y and δ_v , and multiplying them together. We now consider the generation of proofs for individual verification equations (39)-(42).

The prover can compute $\pi_x \in \mathbb{G}$ satisfying equation (39) because, for each $k \in [m]$, we have

$$\begin{aligned}
e\left(\prod_{i=1}^{\ell_k} g_{n+1-(j_k+i)}^{2^{i-1}}, \hat{C}\right) &= \prod_{i=1}^{\ell_k} e(g_{n+1-(j_k+i)}, \hat{C})^{2^{i-1}} \\
&= \prod_{i=1}^{\ell_k} e\left(g_{n+1-(j_k+i)}, \hat{g}^\gamma \cdot \prod_{\kappa=1}^m \prod_{j=1}^{\ell_\kappa} \hat{g}_{j_\kappa+j}^{x_{\kappa,j}}\right)^{2^{i-1}} \\
&= \prod_{i=1}^{\ell_k} e\left(g^\gamma \cdot \prod_{\kappa=1}^m \prod_{j=1}^{\ell_\kappa} g_{j_\kappa+j}^{x_{\kappa,j}}, \hat{g}_{n+1-(j_k+i)}\right)^{2^{i-1}}, \\
&= \prod_{i=1}^{\ell_k} e\left(g_{n+1-(j_k+i)}^\gamma \cdot \prod_{\kappa=1}^m \prod_{j=1}^{\ell_\kappa} g_{n+1+(j_\kappa+j)-(j_k+i)}^{x_{\kappa,j}}, \hat{g}\right)^{2^{i-1}} \\
&= e(g_1, \hat{g}_n)^{\sum_{i=1}^{\ell_k} x_{k,i} \cdot 2^{i-1}} \\
&\quad \cdot e\left(\prod_{i=1}^{\ell_k} (g_{n+1-(j_k+i)}^\gamma \cdot \prod_{\substack{j \in [\ell_\kappa] \\ (\kappa,j) \neq (k,i)}} g_{n+1-(j_k+i)+(j_\kappa+j)}^{x_{\kappa,j}})\right)^{2^{i-1}}, \hat{g}
\end{aligned}$$

and thus

$$\begin{aligned}
\prod_{k=1}^m e\left(\prod_{i=1}^{\ell_k} g_{n+1-(j_k+i)}^{2^{i-1}}, \hat{C}\right)^{\xi_k} &= e(g_1, \hat{g}_n)^{\sum_{k=1}^m \xi_k \cdot (\sum_{i=1}^{\ell_k} x_{k,i} \cdot 2^{i-1})} \\
&\quad \cdot e\left(\prod_{k=1}^m \left(\prod_{i=1}^{\ell_k} (g_{n+1-(j_k+i)}^\gamma \cdot \prod_{\substack{j \in [\ell_\kappa] \\ (\kappa,j) \neq (k,i)}} g_{n+1-(j_k+i)+(j_\kappa+j)}^{x_{\kappa,j}})\right)^{2^{i-1}}, \hat{g}\right).
\end{aligned} \tag{45}$$

We also have

$$e\left(\prod_{k=1}^m g_{n+1-k}^{\xi_k}, \hat{V}\right) = e(g_1, \hat{g}_n)^{\xi_k \cdot x_k} \cdot e\left(\prod_{k=1}^m (g_{n+1-k}^r \cdot \prod_{i=1, i \neq k}^m g_{n+1+i-k}^{x_i})^{\xi_k}, \hat{g}\right),$$

so that dividing the latter from (45) yields (39) when $x_k = \sum_{i=1}^n x_{k,i} \cdot 2^{i-1}$ for each $k \in [m]$ and

$$\pi_x = \frac{\prod_{k=1}^m \left(\prod_{i=1}^{\ell_k} (g_{n+1-(j_k+i)}^\gamma \cdot \prod_{\kappa=1}^m \prod_{j \in [\ell_\kappa], (\kappa,j) \neq (k,i)} g_{n+1-(j_k+i)+(j_\kappa+j)}^{x_{\kappa,j}})\right)^{2^{i-1}} \xi_k}{\prod_{k=1}^m (g_{n+1-k}^r \cdot \prod_{i=1, i \neq k}^m g_{n+1+i-k}^{x_i})^{\xi_k}}. \tag{46}$$

The prover can also compute π_{eq} satisfying the second verification equation (40) by observing that, for each $k \in [m]$, we have

$$\begin{aligned}
e\left(\prod_{i=1}^{n_k} g_{n+1-(j_k+i)}^{t_{j_k+i} \cdot y_{j_k+i}}, \hat{C}\right) &= \prod_{i=1}^{n_k} e(g_{n+1-(j_k+i)}, \hat{C})^{t_{j_k+i} \cdot y_{j_k+i}} \tag{47} \\
&= \prod_{i=1}^{n_k} e(g_{n+1-(j_k+i)}, \hat{g}^\gamma \prod_{\kappa=1}^m \prod_{j=1}^{n_\kappa} \hat{g}_{j_\kappa+j}^{x_{\kappa,j}})^{t_{j_k+i} \cdot y_{j_k+i}} \\
&= \prod_{i=1}^{n_k} \left(e(g_1, \hat{g}_n)^{x_{k,i}} \cdot e(g, \hat{g}_{n+1-(j_k+i)}^\gamma) \cdot \prod_{\kappa=1}^m \prod_{\substack{j \in [n_\kappa] \\ (\kappa,j) \neq (k,i)}} \hat{g}_{n+1+(j_\kappa+j)-(j_k+i)}^{x_{\kappa,j}} \right)^{t_{j_k+i} \cdot y_{j_k+i}} \\
&= \prod_{i=1}^{n_k} \left(e(g_1, \hat{g}_n)^{x_{k,i}} \cdot e(g_{n+1-(j_k+i)}^\gamma) \cdot \prod_{\kappa=1}^m \prod_{\substack{j \in [n_\kappa] \\ (\kappa,j) \neq (k,i)}} g_{n+1+(j_\kappa+j)-(j_k+i)}^{x_{\kappa,j}} \right)^{t_{j_k+i} \cdot y_{j_k+i}} \\
&= e(g_1, \hat{g}_n)^{\sum_{i=1}^{\ell_k} y_{j_k+i} \cdot t_{j_k+i} \cdot x_{k,i}} \\
&\quad \cdot e\left(\prod_{i=1}^{n_k} \left(g_{n+1-(j_k+i)}^\gamma \cdot \prod_{\kappa=1}^m \prod_{\substack{j \in [n_\kappa] \\ (\kappa,j) \neq (k,i)}} g_{n+1+(j_\kappa+j)-(j_k+i)}^{x_{\kappa,j}} \right)^{t_{j_k+i} \cdot y_{j_k+i}}, \hat{g}\right).
\end{aligned}$$

We also have

$$\begin{aligned}
e(C_y, \prod_{k=1}^m \prod_{i=1}^{n_k} \hat{g}_{j_k+i}^{t_{j_k+i}}) &= \prod_{k=1}^m \prod_{i=1}^{n_k} e(C_y, \hat{g}_{j_k+i})^{t_{j_k+i}} \tag{48} \\
&= \prod_{k=1}^m \prod_{i=1}^{n_k} e(g^{\gamma_y} \cdot \prod_{\kappa=1}^m \prod_{j=1}^{n_\kappa} g_{n+1-(j_\kappa+j)}^{y_{j_\kappa+j} \cdot x_{\kappa,j}} \cdot \hat{g}_{j_k+i})^{t_{j_k+i}} \\
&= \prod_{k=1}^m \prod_{i=1}^{n_k} e(g_{j_k+i}^{\gamma_y} \cdot \prod_{\kappa=1}^m \prod_{j=1}^{n_\kappa} g_{n+1-(j_\kappa+j)+(j_k+i)}^{y_{j_\kappa+j} \cdot x_{\kappa,j}} \cdot \hat{g})^{t_{j_k+i}} \\
&= e(g_1, \hat{g}_n)^{\sum_{k=1}^m \sum_{i=1}^{\ell_k} y_{j_k+i} \cdot t_{j_k+i} \cdot x_{k,i}} \\
&\quad \cdot e\left(\prod_{k=1}^m \prod_{i=1}^{n_k} \left(g_{j_k+i}^{\gamma_y} \cdot \prod_{\kappa=1}^m \prod_{\substack{j \in [n_\kappa] \\ (\kappa,j) \neq (k,i)}} g_{n+1-(j_\kappa+j)+(j_k+i)}^{y_{j_\kappa+j} \cdot x_{\kappa,j}} \right)^{t_{j_k+i}}, \hat{g}\right).
\end{aligned}$$

By taking the product of (47) for all $k \in [m]$ and dividing (48) out of the result, we obtain (40) when π_{eq} is computed as

$$\pi_{eq} = \frac{\prod_{k=1}^m \prod_{i=1}^{n_k} \left(g_{n+1-(j_k+i)}^\gamma \cdot \prod_{\kappa=1}^m \prod_{j \in [n_\kappa], (\kappa,j) \neq (k,i)} g_{n+1+(j_\kappa+j)-(j_k+i)}^{x_{\kappa,j}} \right)^{t_{j_k+i} \cdot y_{j_k+i}}}{\prod_{k=1}^m \prod_{i=1}^{n_k} \left(g_{j_k+i}^{\gamma_y} \cdot \prod_{\kappa=1}^m \prod_{j \in [n_\kappa], (\kappa,j) \neq (k,i)} g_{n+1-(j_\kappa+j)+(j_k+i)}^{y_{j_\kappa+j} \cdot x_{\kappa,j}} \right)^{t_{j_k+i}}} \tag{49}$$

As for equation (41), the prover can compute

$$\pi_y = C_y^\gamma \cdot \prod_{\kappa=1}^m \prod_{j=1}^{n_\kappa} (g_{j_\kappa+j}^{\gamma_y} \cdot \prod_{k=1}^m \prod_{\substack{i \in [n_k] \\ (k,i) \neq (\kappa,j)}} g_{n+1-(j_k+i)+(j_\kappa+j)}^{y_{j_k+i} \cdot (x_{k,i}-1)})^{x_{\kappa,j}}, \quad (50)$$

which satisfies (41) because we have

$$\begin{aligned} e(C_y \cdot \prod_{k=1}^m \prod_{i=1}^{n_k} g_{n+1-(j_k+i)}^{-y_{j_k+i}}, \hat{C}) &= e\left(g^{\gamma_y} \cdot \prod_{k=1}^m \prod_{i=1}^{n_k} g_{n+1-(j_k+i)}^{y_{j_k+i} \cdot (x_{k,i}-1)}, \hat{g}^\gamma \cdot \prod_{\kappa=1}^m \prod_{j=1}^{n_\kappa} \hat{g}_{j_\kappa+j}^{x_{\kappa,j}}\right) \\ &= e(C_y, \hat{g})^\gamma \cdot \prod_{\kappa=1}^m \prod_{j=1}^{n_\kappa} e\left((g_{j_\kappa+j}^{\gamma_y} \cdot \prod_{k=1}^m \prod_{i=1}^{n_k} g_{n+1-(j_k+i)+(j_\kappa+j)}^{y_{j_k+i} \cdot (x_{k,i}-1)})^{x_{\kappa,j}}, \hat{g}\right) \\ &= e(g_1, \hat{g}_n)^{\sum_{k=1}^m \sum_{i=1}^{n_k} y_{j_k+i} \cdot x_{k,i} \cdot (x_{k,i}-1)} \cdot e(C_y^\gamma, \hat{g}) \\ &\quad \cdot \prod_{\kappa=1}^m \prod_{j=1}^{n_\kappa} e\left((g_{j_\kappa+j}^{\gamma_y} \cdot \prod_{k=1}^m \prod_{\substack{i \in [n_k] \\ (k,i) \neq (\kappa,j)}} g_{n+1-(j_k+i)+(j_\kappa+j)}^{y_{j_k+i} \cdot (x_{k,i}-1)})^{x_{\kappa,j}}, \hat{g}\right) \\ &= e\left(C_y^\gamma \cdot \prod_{\kappa=1}^m \prod_{j=1}^{n_\kappa} (g_{j_\kappa+j}^{\gamma_y} \cdot \prod_{k=1}^m \prod_{\substack{i \in [n_k] \\ (k,i) \neq (\kappa,j)}} g_{n+1-(j_k+i)+(j_\kappa+j)}^{y_{j_k+i} \cdot (x_{k,i}-1)})^{x_{\kappa,j}}, \hat{g}\right), \end{aligned}$$

where the last equality holds since $x_{k,i} \cdot (x_{k,i}-1) = 0$ for all indices $k \in [m]$ and $i \in [n_k]$.

Equation (42) is satisfied by $\pi_v = \prod_{i=m+1}^n (g_{n+1-i}^r \cdot \prod_{k=1}^m g_{n+2-i+k}^{x_k})^{s_i}$ since

$$\begin{aligned} e\left(\prod_{i=m+1}^n g_{n+1-i}^{s_i}, \hat{V}\right) &= \prod_{i=m+1}^n e\left(g_{n+1-i}, \hat{g}^r \cdot \prod_{k=1}^m \hat{g}_k^{x_k}\right)^{s_i} \\ &= \prod_{i=m+1}^n e\left(g^r \cdot \prod_{k=1}^m g_k^{x_k}, \hat{g}_{n+1-i}\right)^{s_i} \\ &= \prod_{i=m+1}^n e\left(g_{n+1-i}^r \cdot \prod_{k=1}^m g_{n+1-i+k}^{x_k}, \hat{g}\right)^{s_i} = e(\pi_v, \hat{g}). \end{aligned}$$

C.3 Security

Theorem 5. *The construction provides zero-knowledge in the ROM.*

Proof. The proof is identical to that of Theorem 1 and omitted. \square

Theorem 6. *Under the $(2n, n)$ -DLOG assumption, the scheme is simulation-extractable in the algebraic group model and in the random oracle model.*

Proof. The proof is similar to that of Theorem 2 and we only detail the changes in the interaction between the reduction/extractor \mathcal{B} and the adversary \mathcal{A} .

Queries: At any time, \mathcal{A} can choose a commitment $\text{com} = \hat{V}$ and ask for a simulated proof that \hat{V} commits to an integer vector $(x_1, \dots, x_m) \in \mathbb{Z}^m$ such that $x_k \in [0, 2^{\ell_k} - 1]$ for some integers $\{\ell_k\}_{k \in [m]}$ of its choice such that $\ell_k \leq n_k$. Since \mathcal{A} is algebraic, it must provide a representation of \hat{V} with respect to the generators $\{\hat{g}_i\}_{i \in [0, n]}$ and the commitments \hat{C} contained in previous simulated proofs. Since the simulator \mathcal{B} is itself algebraic, for any \hat{V} chosen by \mathcal{A} , \mathcal{B} can always find a representation $\{v_i\}_{i=0}^n$ such that $\hat{V} = \hat{g}^{v_0} \cdot \prod_{i=1}^n \hat{g}_i^{v_i}$. We assume w.l.o.g. that either: (i) There exists $k \in [m]$ such that $v_k \notin [0, 2^{\ell_k} - 1]$; or (ii) $(v_{m+1}, \dots, v_n) \neq \mathbf{0}$. Otherwise, \mathcal{B} can faithfully generate a proof using (v_0, v_1, \dots, v_m) as witnesses. Then, \mathcal{B} proceeds as follows to simulate a proof without using g_{n+1} :

1. Choose random vectors $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \xleftarrow{R} \mathbb{Z}_p^m$, $\boldsymbol{\delta} = (\delta_x, \delta_{eq}, \delta_y, \delta_v) \xleftarrow{R} \mathbb{Z}_p^4$, $\mathbf{y} = (y_1, \dots, y_n) \xleftarrow{R} \mathbb{Z}_p^n$, $\mathbf{t} = (t_1, \dots, t_n) \xleftarrow{R} \mathbb{Z}_p^n$.
2. Let $f_{n+1} = \sum_{i=m+1}^n v_i \cdot s_i$, for randomly chosen $s_i \xleftarrow{R} \mathbb{Z}_p$ for all indices $i \in [m+1, n]$. Let an arbitrary $k \in [m]$ such that

$$a_{j_k+1} \triangleq v_k + \frac{1}{\xi_k} \cdot \left(-\frac{\delta_v \cdot f_{n+1}}{\delta_x} + \sum_{\kappa \in [m] \setminus \{k\}} \xi_\kappa v_\kappa \right) \notin \{0, 1\}$$

Such a $k \in [m]$ must exist w.h.p. since we assumed that $(v_{m+1}, \dots, v_n) \neq \mathbf{0}$ or there exists $k \in [m]$ such that $v_k \notin [0, 2^{\ell_k} - 1]$. Then, set

$$\begin{aligned} a_i &= 0 & \forall i \in [n] \setminus \{j_k + 1\} \\ z_{n-j_k} &= y_{j_k+1} \end{aligned}$$

Then, find arbitrary scalars $\{z_j\}_{j \in [n] \setminus \{n-j_k\}}$ such that

$$\begin{aligned} \sum_{i=2}^{n_k} t_{j_k+i} \cdot z_{n+1-(j_k+i)} + \sum_{\substack{\kappa=1 \\ \kappa \neq k}}^m \sum_{i=1}^{n_\kappa} t_{j_\kappa+i} \cdot z_{n+1-(j_\kappa+i)} \\ = t_{j_k+1} \cdot (a_{j_k+1} \cdot y_{j_k+1} - y_{j_k+1}). \end{aligned}$$

3. Choose random $a_0, z_0 \xleftarrow{R} \mathbb{Z}_p$ and compute simulated commitments

$$\hat{C} = \hat{g}^{a_0} \cdot \prod_{i=1}^n \hat{g}_i^{a_i} = \hat{g}^{a_0} \cdot \hat{g}_{j_k+1}^{a_{j_k+1}}, \quad C_y = g^{z_0} \cdot \prod_{i=1}^n g_i^{z_i}.$$

4. If one of the hashes $H_\xi(\hat{V}, \hat{C}, C_y)$, $H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$, $H(\hat{V}, \hat{C})$, $H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$ or $\{s_i = H_s(i, [m+1, n], \hat{V}, \hat{C}, C_y)\}_{i=m+1}^n$ was already defined, abort. Otherwise, set $\boldsymbol{\delta} = H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$,

$$\boldsymbol{\xi} = H_\xi(\hat{V}, \hat{C}, C_y), \quad \mathbf{y} = H(\hat{V}, \hat{C}), \quad \mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y),$$

and $s_i = H_s(i, [m+1, n], \hat{V}, \hat{C}, C_y)$ for each $i \in [m+1, n]$.

5. Define the polynomials

$$\begin{aligned}
Q_x[X] &= \left(\sum_{i=0}^n a_i \cdot X^i \right) \cdot \left(\sum_{k=1}^m \sum_{i=1}^{\ell_k} 2^{i-1} \cdot \xi_k \cdot X^{n+1-(j_k+i)} \right) \\
&\quad - \left(\sum_{i=0}^n v_i \cdot X^i \right) \cdot \left(\sum_{k=1}^m \xi_k \cdot X^{n+1-k} \right) = \sum_{i=0}^{2n} q_i \cdot X^i, \\
Q_y[X] &= \left(\sum_{i=0}^n z_i \cdot X^i - \sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} y_{j_\kappa+i} \cdot X^{n+1-(j_\kappa+i)} \right) \cdot \left(\sum_{i=0}^n a_i \cdot X^i \right) \\
&= \left(z_0 + \sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} (z_{n+1-(j_\kappa+i)} - y_{j_\kappa+i}) \cdot X^{n+1-(j_\kappa+i)} \right) \cdot \left(\sum_{i=0}^n a_i \cdot X^i \right) \\
&= \sum_{i=0}^{2n} \sigma_i \cdot X^i \\
Q_{e_q}[X] &= \left(\sum_{i=0}^n a_i \cdot X^i \right) \cdot \left(\sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} t_{j_\kappa+i} \cdot y_{j_\kappa+i} \cdot X^{n+1-(j_\kappa+i)} \right) \\
&\quad - \left(z_0 + \sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} z_{n+1-(j_\kappa+i)} \cdot X^{n+1-(j_\kappa+i)} \right) \cdot \left(\sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} t_{j_\kappa+i} \cdot X^{j_\kappa+i} \right) \\
&= \sum_{j=0}^{2n} e_j \cdot X^j, \\
Q_v[X] &= \left(\sum_{i=0}^n v_i \cdot X^i \right) \cdot \left(\sum_{i=m+1}^n s_i \cdot X^{n+1-i} \right) = \sum_{j=0}^{2n} f_j \cdot X^j.
\end{aligned}$$

Their degree- $(n+1)$ coefficients are $f_{n+1} = \sum_{i=m+1}^n v_i \cdot s_i$ and

$$\begin{aligned}
q_{n+1} &= \sum_{\kappa=1}^m \xi_\kappa \cdot \left(-v_\kappa + \sum_{i=1}^{\ell_\kappa} a_{j_\kappa+i} \cdot 2^{i-1} \right) \\
&= \xi_k \cdot a_{j_k+1} - \sum_{\kappa \in [m]} \xi_\kappa \cdot v_\kappa = -\frac{\delta_v \cdot f_{n+1}}{\delta_x}, \\
\sigma_{n+1} &= \sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} a_{j_\kappa+i} \cdot (z_{n+1-(j_\kappa+i)} - y_{j_\kappa+i}) = a_{j_k+1} \cdot (z_{n-j_k} - y_{j_k+1}) = 0 \\
e_{n+1} &= \sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} t_{j_\kappa+i} \cdot (a_{j_\kappa+i} \cdot y_{j_\kappa+i} - z_{n+1-(j_\kappa+i)}) \\
&= t_{j_k+1} \cdot (a_{j_k+1} \cdot y_{j_k+1} - y_{j_k+1}) - \sum_{i=2}^{n_k} t_{j_k+i} \cdot z_{n+1-(j_k+i)} \\
&\quad - \sum_{\substack{\kappa=1 \\ \kappa \neq k}}^m \sum_{i=1}^{n_\kappa} t_{j_\kappa+i} \cdot z_{n+1-(j_\kappa+i)} = 0
\end{aligned}$$

due to the definition of committed $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$. Observe that

$$\delta_x \cdot q_{n+1} + \delta_{eq} \cdot e_{n+1} + \delta_y \cdot \sigma_{n+1} + \delta_v \cdot f_{n+1} = 0 \quad (51)$$

6. Define the polynomial

$$Q_{\text{agg}}[X] = \delta_x \cdot Q_x[X] + \delta_{eq} \cdot Q_{eq}[X] + \delta_y \cdot Q_y[X] + \delta_v \cdot Q_v[X] = \sum_{i=0}^{2n} \eta_i \cdot X^i$$

for which $\eta_{n+1} = 0$ by construction. Compute

$$\pi = \prod_{i=1, i \neq n+1}^{2n} g_i^{\eta_i} \quad (52)$$

using $(g, \{g_i\}_{i \in [2n] \setminus \{n+1\}})$ and return the simulated proof $\boldsymbol{\pi} = (\hat{C}, C_y, \pi)$.

We remark that the simulated π from (52) satisfies the verification equation (44) by construction. Moreover, the proof $\boldsymbol{\pi}$ has the same distribution as a proof generated by the zero-knowledge simulator. Indeed, π is uniquely determined by the commitments (\hat{C}, \hat{V}, C_y) and the aggregation coefficients $\boldsymbol{\xi}, \mathbf{y}, \mathbf{t}, \{s_i\}_{i=m+1}^n$ and $\boldsymbol{\delta}$ in (44). Also, the committed vectors $\mathbf{a}, \mathbf{z} \in \mathbb{Z}_p^n$ are perfectly hidden by the randomness a_0 and z_0 in \hat{C} and C_y , respectively.

Therefore the simulation is perfect, unless a collision occurs when random oracles are programmed in the simulation queries. If Q_S (reps. Q_H) is the number of queries made by \mathcal{A} to the simulator (resp. to random oracles), this happens with probability $\leq (Q_S + Q_H) \cdot Q_H/p$.

Output: When \mathcal{A} terminates, it outputs a statement $(\hat{V}, \{1^{\ell_k}\}_{k=1}^m)$, for some integers $\ell_1 \in [n_1], \dots, \ell_m \in [n_m]$, together with a valid proof $\boldsymbol{\pi} = (\hat{C}, C_y, \pi)$.

Since we are in the AGM, \mathcal{A} must provide a representation of C_y w.r.t to $(g, \{g_i\}_{i \in [2n] \setminus \{n+1\}})$ and the group elements $\{C_y^{(i)}, \pi^{(i)}\}_{i \in [Q_S]}$ contained in responses $\{\pi^{(i)}\}_{i \in [Q_S]}$ to simulation queries. Likewise, it must provide a representation of \hat{C} w.r.t $(\hat{g}, \{\hat{g}_i\}_{i \in [n]})$ and the commitments $\{\hat{C}^{(i)}\}_{i \in [Q_S]}$ contained in simulated proofs $\{\pi^{(i)}\}_{i \in [Q_S]}$. Also, for each $i \in [Q_S]$, \mathcal{B} knows a representation of $\hat{C}^{(i)}$ w.r.t. $(\hat{g}, \{\hat{g}_i\}_{i \in [n]})$ and a representation of C_y w.r.t. $(g, \{g_i\}_{i=1}^n)$. It also knows a representation of each simulated $\pi^{(i)}$ w.r.t $(g, \{g_i\}_{i \in [2n] \setminus \{n+1\}})$. From \mathcal{A} 's output and the randomness of the simulation, \mathcal{B} can infer scalars $\{(\theta_i, z_i) \in \mathbb{Z}_p^2\}_{i \in [0, 2n] \setminus \{n+1\}}, \{(a_i, v_i) \in \mathbb{Z}_p^2\}_{i \in [0, n]}$ such that

$$\hat{C} = \prod_{i=0}^n \hat{g}_i^{a_i}, \quad C_y = \prod_{i=0, i \neq n+1}^{2n} g_i^{z_i}, \quad \hat{V} = \prod_{i=0}^n \hat{g}_i^{v_i}, \quad \pi = \prod_{i=0, i \neq n+1}^{2n} g_i^{\theta_i},$$

where we define $g_0 = g$ and $\hat{g}_0 = \hat{g}$ for convenience.

If the representation $(v_0, v_1, \dots, v_n) \in \mathbb{Z}_p^2$ of \hat{V} is such that $v_k \in [0, 2^{\ell_k} - 1]$

for all $k \in [m]$ and $v_i = 0$ for all $i \in [m+1, n]$, then \mathcal{B} can simply output $(v_0, v_1, \dots, v_m) \in \mathbb{Z}_p^{m+1}$ as a valid witness. We henceforth assume that either $(v_{m+1}, \dots, v_n) \neq \mathbf{0}^{n-m}$ or there exists $k \in [m]$ such that $v_k \notin [0, 2^{\ell_k} - 1]$.

Solving $(2n, n)$ -DLOG: We first note that a non-trivial valid proof π cannot recycle (\hat{V}, \hat{C}, C_y) from an output of the simulation oracle (namely, we must have $(\hat{V}, \hat{C}, C_y) \neq (\hat{V}^{(i)}, \hat{C}^{(i)}, C_y)$ for all $i \in [Q_S]$) since the left-hand-side member of (44) is uniquely determined by $(\hat{V}^{(i)}, \hat{C}^{(i)}, C_y^{(i)})$ and it in turn determines a unique valid $\pi^{(i)} \in \mathbb{G}$. As a consequence, $H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$, $H_\xi(\hat{V}, \hat{C}, C_y)$, $H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$ and $\{H_s(i, [m+1, n], \hat{V}, \hat{C}, C_y)\}_{i=m+1}^n$ are not part of the random oracle values that have been programmed by the simulator.

Since the left-hand-side member of (44) is obtained by raising the right-hand-side members of (39)-(42) to the powers $(\delta_x, \delta_{eq}, \delta_y, \delta_v)$ and multiplying the results, it can be written $e(g, \hat{g})^{P_{\text{agg}}(\alpha)}$, where $P_{\text{agg}}[X]$ is the polynomial

$$P_{\text{agg}}[X] = \delta_x \cdot P_x[X] + \delta_y \cdot P_y[X] + \delta_{eq} \cdot P_{eq}[X] + \delta_v \cdot P_v[X]$$

obtained as a linear combination of the polynomials

$$\begin{aligned} P_x[X] &= \left(\sum_{i=0}^n a_i \cdot X^i \right) \cdot \left(\sum_{k=1}^m \sum_{i=1}^{\ell_k} 2^{i-1} \cdot \xi_k \cdot X^{n+1-(j_k+i)} \right) \\ &\quad - \left(\sum_{i=0}^n v_i \cdot X^i \right) \cdot \left(\sum_{k=1}^m \xi_k \cdot X^{n+1-k} \right) = \sum_{i=1}^{2n} \omega_i \cdot X^i, \end{aligned}$$

$$\begin{aligned} P_y[X] &= \left(\sum_{i=0, i \neq n+1}^{2n} z_i \cdot X^i - \sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} y_{j_\kappa+i} \cdot X^{n+1-(j_\kappa+i)} \right) \cdot \left(\sum_{i=0}^n a_i \cdot X^i \right) \\ &= \left(z_0 + \sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} (z_{n+1-(j_\kappa+i)} - y_{j_\kappa+i}) \cdot X^{n+1-(j_\kappa+i)} + \sum_{i=n+2}^{2n} z_i \cdot X^i \right) \\ &\quad \cdot \left(\sum_{i=0}^n a_i \cdot X^i \right) = \sum_{i=0}^{3n} \gamma_i \cdot X^i \end{aligned}$$

$$\begin{aligned} P_{eq}[X] &= \left(\sum_{i=0}^n a_i \cdot X^i \right) \cdot \left(\sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} t_{j_\kappa+i} \cdot y_{j_\kappa+i} \cdot X^{n+1-(j_\kappa+i)} \right) \\ &\quad - \left(\sum_{i=0, i \neq n+1}^{2n} z_i \cdot X^i \right) \cdot \left(\sum_{\kappa=1}^m \sum_{i=1}^{n_\kappa} t_{j_\kappa+i} \cdot X^{j_\kappa+i} \right) = \sum_{j=0}^{3n} \beta_j \cdot X^j, \end{aligned}$$

$$P_v[X] = \left(\sum_{i=0}^n v_i \cdot X^i \right) \cdot \left(\sum_{i=m+1}^n s_i \cdot X^{n+1-i} \right) = \sum_{j=0}^{2n} \mu_j \cdot X^j$$

for which the left-hand-side members of (39)-(42) can be written $e(g, \hat{g})^{P_x(\alpha)}$, $e(g, \hat{g})^{P_{eq}(\alpha)}$, $e(g, \hat{g})^{P_y(\alpha)}$ and $e(g, \hat{g})^{P_v(\alpha)}$, respectively.

If we write $P_{\text{agg}}[X] = \sum_{i=0}^{3n} \nu_i \cdot X^i$, the coefficient ν_{n+1} of its degree- $(n+1)$ term can be written

$$\begin{aligned} \nu_{n+1} = & \underbrace{\delta_x \cdot \sum_{\kappa=1}^m \xi_{\kappa} \cdot \left(\sum_{i=1}^{\ell_{\kappa}} a_{j_{\kappa}+i} \cdot 2^{i-1} - v_{\kappa} \right)}_{\triangleq \omega_{n+1}} + \underbrace{\delta_y \cdot \sum_{\kappa=1}^m \sum_{i=1}^{n_{\kappa}} (z_{n+1-(j_{\kappa}+i)} - y_{j_{\kappa}+i}) \cdot a_{j_{\kappa}+i}}_{\triangleq \gamma_{n+1}} \\ & + \underbrace{\delta_{eq} \cdot \sum_{\kappa=1}^m \sum_{i=1}^{n_{\kappa}} t_{j_{\kappa}+i} \cdot (a_{j_{\kappa}+i} \cdot y_{j_{\kappa}+i} - z_{n+1-(j_{\kappa}+i)})}_{\triangleq \beta_{n+1}} + \underbrace{\delta_v \cdot \sum_{i=m+1}^n v_i \cdot s_i}_{\triangleq \mu_{n+1}}, \end{aligned}$$

where $(\omega_{n+1}, \gamma_{n+1}, \beta_{n+1}, \mu_{n+1})$ are the coefficients of the degree- $(n+1)$ terms of $(P_x[X], P_y[X], P_{eq}[X], P_v[X])$.

We now argue that, if there exists $k \in [m]$ such that $v_k \notin [0, 2^{\ell_k} - 1]$ or if $(v_{m+1}, \dots, v_n) \neq \mathbf{0}^{n-m}$, then we can only have $\nu_{n+1} = 0$ with negligible probability. This follows from the following arguments:

- The probability to have $\boldsymbol{\rho} \triangleq (\omega_{n+1}, \gamma_{n+1}, \beta_{n+1}, \mu_{n+1}, \zeta_{n+1}) = \mathbf{0}$ is negligible. Indeed, when $(v_{m+1}, \dots, v_n) \neq \mathbf{0}^{n-m}$, we have $\mu_{n+1} = 0$, with probability $1/p$ over the choice of $\{s_i = H_s(i, [m+1, n], \hat{V}, \hat{C}, C_y)\}_{i=m+1}^n$. When $z_{n+1-(j_{\kappa}+i)} \neq a_{j_{\kappa}+i} \cdot y_{j_{\kappa}+i}$ for some $\kappa \in [m]$ and $i \in [n_{\kappa}]$, we have $\beta_{n+1} = 0$ with probability $1/p$ since $\mathbf{t} = H_t(\mathbf{y}, \hat{V}, \hat{C}, C_y)$ is derived *after* the choice of \mathbf{y} , $\{a_i\}_{i=0}^n$ and $\{z_i\}_{i \in [0, 2n] \setminus \{n+1\}}$. Then, if $z_{n+1-(j_{\kappa}+i)} = a_{j_{\kappa}+i} \cdot y_{j_{\kappa}+i}$ for all $\kappa \in [m]$, $i \in [n_{\kappa}]$, we have $\gamma_{n+1} = \sum_{\kappa=1}^m \sum_{i=1}^{n_{\kappa}} y_{j_{\kappa}+i} \cdot (a_{j_{\kappa}+i} - 1) \cdot a_{j_{\kappa}+i}$, which cancels with probability $1/p$ if there exists $\kappa \in [m]$ and $i \in [n_{\kappa}]$ such that $a_{j_{\kappa}+i} \notin \{0, 1\}$. This can be seen by distinguishing two cases:

a. If $\mathbf{y} = H(\hat{V}, \hat{C})$ was programmed when answering a simulation query, we can only have $\gamma_{n+1} = 0$ with probability $1/p$ since the simulator programmed (a_1, \dots, a_n) to have

$$\gamma_{n+1} = a_{j_k+1} \cdot (z_{n-j_k} - y_{j_k+1}) = y_{j_k+1} \cdot a_{j_k+1} \cdot (a_{j_k+1} - 1)$$

for some index $j_k \in [n]$ such that $a_{j_k+1} \notin \{0, 1\}$ and $y_{j_k+1} \in_R \mathbb{Z}_p$. This captures the case of \mathcal{A} attempting to re-use $(\hat{V}, \hat{C}) = (\hat{V}^{(i)}, \hat{C}^{(i)})$ contained in an output $\boldsymbol{\pi}^{(i)} = (\hat{C}^{(i)}, C_y^{(i)}, \boldsymbol{\pi}^{(i)})$ of the simulator, with a different $C_y \neq C_y^{(i)}$.

b. If $H(\hat{V}, \hat{C})$ was not programmed by the simulator, then $\mathbf{y} = H(\hat{V}, \hat{C})$ was defined after \mathcal{B} obtained the algebraic representation $\{a_i\}_{i=0}^n$ of \hat{C} . Over the choice of \mathbf{y} , we have $\sum_{\kappa=1}^m \sum_{i=1}^{n_{\kappa}} y_{j_{\kappa}+i} \cdot (a_{j_{\kappa}+i} - 1) \cdot a_{j_{\kappa}+i} = 0$ with probability $1/p$.

If there exists $k \in [m]$ such that $v_k \neq \sum_{i=1}^{\ell_k} a_{j_k+i} \cdot 2^{i-1}$, the probability to have $\omega_{n+1} = 0$ is only $1/p$ since $\boldsymbol{\xi} = H_{\xi}(\hat{V}, \hat{C}, C_y)$ are chosen uniformly after $\{v_k\}_{k=1}^m$ and $\{a_i\}_{i=0}^n$. If none of the above events occurs, then we have

$v_k = \sum_{i=1}^{\ell_k} a_{j_k+i} \cdot 2^{i-1}$ for each $k \in [m]$ and $a_{j_k+i} \in \{0, 1\}$ for each $i \in [\ell_k]$. This contradicts the hypothesis that $v_k \notin [0, 2^{\ell_k} - 1]$ for some $k \in [m]$.

- If $\rho \neq \mathbf{0}$, then we have $\nu_{n+1} \neq 0$ with probability $1 - 1/p$ since the aggregation coefficients $\delta \triangleq (\delta_x, \delta_{eq}, \delta_y, \delta_v) = H_{\text{agg}}(\hat{V}, \hat{C}, C_y)$ are derived *after* the choice of $\{(a_i, v_i)\}_{i=0}^n$, and $\{z_i\}_{i \in [0, 2n] \setminus \{n+1\}}$, which determine the coordinates of ρ . Hence, a random independent $\delta \in \mathbb{Z}_p^5$ can only satisfy $\langle \delta, \rho \rangle = 0$ with probability $1/p$.

If $\nu_{n+1} \neq 0$, then \mathcal{B} can compute $\alpha \in \mathbb{Z}_p$ by factoring a non-zero polynomial as in the proof of Theorem 2. \square

C.4 Range Proofs for Non-Power-of-Two Ranges

In order to prove membership of a range $[0, B]$ where $B + 1$ is not a power of 2, a common approach is to use an additively homomorphic commitment and consider the integer $\ell \in \mathbb{N}$ such that $2^{\ell-1} < B < 2^\ell$. Then, we generate two range proofs showing that $x \in [0, 2^\ell - 1]$ and $x + (2^\ell - 1 - B) \in [0, 2^\ell - 1]$.

To do this without increasing the proof size, we can commit to vectors of dimension $n = 2\bar{\ell}$ (where $\bar{\ell}$ is an upper bound on ℓ) containing a concatenation $(\mathbf{x} \mid \mathbf{x}')$ of the binary decompositions $\mathbf{x} = (x_1, \dots, x_\ell, 0, \dots, 0)$ and $\mathbf{x}' = (x'_1, \dots, x'_\ell, 0, \dots, 0)$ of x and $x + (2^\ell - 1 - B)$, respectively. Then, the prover can compute $\pi_x, \pi'_x \in \mathbb{G}$

$$\frac{e(\prod_{i=1}^{\bar{\ell}} g_{n+1-i}^{2^{i-1}}, \hat{C})}{e(g_n, \hat{V})} = e(\pi_x, \hat{g}) \quad \frac{e(\prod_{i=\bar{\ell}+1}^n g_{n+1-i}^{2^{i-\bar{\ell}-1}}, \hat{C})}{e(g_n, \hat{V} \cdot \hat{g}^{2^\ell-1-B})} = e(\pi'_x, \hat{g}),$$

where $\hat{V} = \hat{g}^r \cdot \hat{g}_1^x$. In the above equalities, π_x and π'_x show that $x = \sum_{i=1}^{\bar{\ell}} x_i \cdot 2^{i-1}$ and $x + (2^\ell - 1 - B) = \sum_{i=1}^{\bar{\ell}} x'_i \cdot 2^{i-1}$. The rest of the proof follows the basic construction of Section 4. The two proofs π_x and π'_x can be aggregated (using additional randomization components derived from a random oracle) with other proof components to obtain a proof of the same form as in Section 4.

D Proving that a Committed Vector is Ternary

We now describe arguments showing that a commitment $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^{\ell} \hat{g}_j^{x_j}$ commits to a ternary vector $\mathbf{x} = (x_1, \dots, x_\ell) \in \{-1, 0, 1\}^\ell$.

A natural approach is to write \mathbf{x} as the difference $\mathbf{x} = \mathbf{x}_0 - \mathbf{x}_1$ between two binary vectors $\mathbf{x}_0, \mathbf{x}_1 \in \{0, 1\}^\ell$. The prover can generate two vector commitments \hat{C}_0, \hat{C}_1 to \mathbf{x}_0 and \mathbf{x}_1 , respectively, before proving that: (i) \hat{C}_0, \hat{C}_1 are both commitments to bitstrings; (ii) $\hat{C}_0 \cdot \hat{C}_1^{-1}$ and \hat{C} commit to the same vector. Even if we aggregate proofs, each proof π still requires 2 elements of $\hat{\mathbb{G}}$ and 3 elements of \mathbb{G} since \hat{C}_0 and \hat{C}_1 both require an auxiliary commitment in \mathbb{G} .

At the expense of doubling the CRS size, we can reduce the size of proofs to

one element of $\hat{\mathbb{G}}$ and 2 elements of \mathbb{G} . To do this, we can generate a commitment to a binary vector $\bar{\mathbf{x}} = (\mathbf{x}_0 \mid \mathbf{x}_1) \in \{0, 1\}^n$, where $n = 2\ell$. We can then prove that: (i) $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n)$ is binary; (ii) For each $i \in [\ell]$, $\bar{x}_i - \bar{x}_{i+\ell} = x_i$. In order to ensure knowledge-soundness (and even simulation-extractability), we also need a proof component π_0 showing that \hat{C} commits to a vector containing zeroes in its last ℓ positions. We can prove (i) as in previous constructions. As for (ii), we can use the properties of the underlying commitment. Namely, if $\hat{C}_x = \hat{g}^{\gamma x} \cdot \prod_{j=1}^n \hat{g}_j^{\bar{x}_j}$ is a commitment to $\bar{\mathbf{x}} = (\mathbf{x}_1 \mid \mathbf{x}_2)$, the prover can compute $\bar{\pi}_{\theta, i, 0}, \bar{\pi}_{\theta, i, 1} \in \mathbb{G}$ such that

$$e(g_{n+1-i}, \hat{C}_x) = e(g_1, \hat{g}_n)^{\bar{x}_i} \cdot e(\bar{\pi}_{\theta, i, 0}, \hat{g}) \quad (53)$$

and

$$e(g_{n+1-i-\ell}, \hat{C}_x) = e(g_1, \hat{g}_n)^{\bar{x}_{i+\ell}} \cdot e(\bar{\pi}_{\theta, i, 1}, \hat{g}) \quad (54)$$

for each $i \in [\ell]$. In the initial commitment $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^\ell \hat{g}_j^{x_j}$ to the ternary $\mathbf{x} = (x_1, \dots, x_\ell)$, we also have

$$e(g_{n+1-i}, \hat{C}) = e(g_1, \hat{g}_n)^{x_i} \cdot e(\tilde{\pi}_{\theta, i}, \hat{g}) \quad \forall i \in [\ell] \quad (55)$$

for some $\tilde{\pi}_{\theta, i} \in \mathbb{G}$ computable by the prover. By combining (53)-(55), we have

$$\frac{e(g_{n+1-i} \cdot g_{n+1-i-\ell}^{-1}, \hat{C}_x)}{e(g_{n+1-i}, \hat{C})} = e(\underbrace{\bar{\pi}_{\theta, i, 0} / (\bar{\pi}_{\theta, i, 1} \cdot \tilde{\pi}_{\theta, i})}_{\triangleq \pi_{\theta, i}}, \hat{g}), \quad (56)$$

where $\pi_{\theta, i}$ is computable by the prover and argues that $x_i = \bar{x}_i - \bar{x}_{i+\ell}$. We can then aggregate proofs for all positions $i \in [\ell]$ at once by raising (56) to a random power $\theta_i \in \mathbb{Z}_p$ and taking the product over all indices $i \in [\ell]$. Specifically, after having computed \hat{C}_x , the prover computes $(\theta_1, \dots, \theta_\ell) = H_\theta(\hat{C}, \hat{C}_x) \in \mathbb{Z}_p^\ell$, which are used as aggregation coefficients to compute a proof $\pi_\theta \in \mathbb{G}$ such that

$$\frac{e(\prod_{i=1}^\ell g_{n+1-i}^{\theta_i} \cdot g_{n+1-i-\ell}^{-\theta_i}, \hat{C}_x)}{e(\prod_{i=1}^\ell g_{n+1-i}^{\theta_i}, \hat{C})} = e(\pi_\theta, \hat{g}), \quad (57)$$

which shows that $\mathbf{x} = \mathbf{x}_0 - \mathbf{x}_1$.

In the final proof $\boldsymbol{\pi} = (\hat{C}_x, C_y, \pi) \in \hat{\mathbb{G}} \times \mathbb{G}^2$, the last component π is then obtained by aggregating π_θ with the proof that $\bar{\mathbf{x}} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ is binary.

E Proving Small Euclidean Norms

In this section, we extend the construction of Section C to prove that a commitment $\hat{V} = \hat{g}^r \cdot \prod_{k=1}^m \hat{g}_k^{x_k}$ is a commitment to some $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ such that $\|\mathbf{x}\| \leq B$. In order to preserve the zero-knowledge property, we need to choose the group order p so that $\sqrt{p/m} > B$ for any proven norm bound B .

When the CRS is generated, we thus assume a maximal value \bar{B} for the norm bounds to be proven and choose p so that $B_\infty \triangleq \sqrt{p/m} > \bar{B}$. For simplicity, we assume that $B^2 + 1$ is a power of two but this restriction can be lifted using the observations in Section C.4. In the setup phase, we also need parameters allowing commitments to vectors of dimension $n = \max(m \cdot (1 + \lceil \log B_\infty \rceil), \lceil \log(\bar{B}^2 + 1) \rceil)$ in order to use the scheme of Section C.1.

The prover has a commitment $\hat{V} = \hat{g}^{\hat{r}} \cdot \prod_{k=1}^m \hat{g}_k^{x_k} \in \hat{\mathbb{G}}$ to an integer vector $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ and wishes to convince the verifier that $\|\mathbf{x}\| \leq B$ without revealing anything else. To this end, it chooses $r \xleftarrow{R} \mathbb{Z}_p$ and computes $V = g^r \cdot \prod_{k=1}^m g_{n+1-k}^{x_k}$, which commits to $\bar{\mathbf{x}} = (\mathbf{0}^{n-m} \parallel (x_m, \dots, x_1))$ in the first source group \mathbb{G} . Then, the prover generates a proof showing that $V \in \mathbb{G}$ commits to the same vector as \hat{V} , but in the reversed order. This is done by computing $\theta = (\theta_1, \dots, \theta_m) = H_\theta(\hat{V}, V) \in \mathbb{Z}_p^m$ using a random oracle H_θ and computing a proof

$$\pi_\theta = \frac{\prod_{j=1}^m \left(g_{n+1-j}^{\hat{r}} \cdot \prod_{k \in [m] \setminus \{j\}} g_{n+1+k-j}^{x_k} \right)^{\theta_j}}{\prod_{j=1}^m \left(g_j^r \cdot \prod_{k \in [m] \setminus \{j\}} g_{n+1-k+j}^{x_k} \right)^{\theta_j}},$$

satisfying

$$\frac{e\left(\prod_{j=1}^m g_{n+1-j}^{\theta_j}, \hat{V}\right)}{e\left(V, \prod_{j=1}^m g_j^{\theta_j}\right)} = e(\pi_\theta, \hat{g}). \quad (58)$$

By itself, (58) only argues that the first m entries of $\mathbf{x} \in \mathbb{Z}_p^n$ coincide with the last $n - m$ entries of $\bar{\mathbf{x}}$ in the reversed order. To ensure knowledge soundness, we also need to prove that the last $n - m$ positions of \mathbf{x} are zeroes, but this will be addressed at a later step.

Next, assuming that \hat{V} and V were indeed computed by the prover as commitments to $((x_1, \dots, x_m) \parallel \mathbf{0}^{n-m})$ and $((*, \dots, *) \parallel (x_m, \dots, x_1))$, respectively, we observe that the pairing $e(V, \hat{V})$ computes a product of polynomials in the exponent, where the coefficient of α^{n+1} is $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$. This allows the prover to compute $\pi_B = \prod_{k=1}^m (g_k^{r_v} \cdot \prod_{\kappa=1, \kappa \neq k}^m g_{n+1-\kappa+k}^{x_\kappa})^{x_k}$ such that

$$e(V, \hat{V}) = e(g_1, \hat{g}_n)^{\langle \mathbf{x}, \mathbf{x} \rangle} \cdot e(\pi_B, \hat{g}). \quad (59)$$

However, π_B is not disclosed. Instead, the prover computes the ℓ_B -bit representation of $\|\mathbf{x}\|^2 = \sum_{k=1}^m x_k^2$ (where $\ell_B = \log(B^2 + 1)$) and commits to the vector $\mathbf{w} = (w_1, \dots, w_{\ell_B}, 0, \dots, 0) \in \{0, 1\}^n$ by choosing $\gamma \xleftarrow{R} \mathbb{Z}_p$, computing

$$\hat{C}_w = \hat{g}^\gamma \cdot \prod_{j=1}^{\ell_B} \hat{g}_j^{w_j} \quad (60)$$

and proving that the committed \mathbf{w} is a binary vector. This is done by generating a proof $\pi_w = (C_{y,w}, \pi_w) \in \mathbb{G}^2$ as in Section 3. Now, we observe that the prover can compute $\pi'_B = \prod_{i=1}^{\ell_B} \left(g_{n+1-i}^\gamma \cdot \prod_{j=1, j \neq i}^{\ell_B} g_{n+1-i+j}^{w_j} \right)^{2^{i-1}}$ such that

$$e\left(\prod_{i=1}^{\ell_B} g_{n+1-i}^{2^{i-1}}, \hat{C}_w\right) = e(g_1, \hat{g}_n)^{\langle \mathbf{x}, \mathbf{x} \rangle} \cdot e(\pi'_B, \hat{g}). \quad (61)$$

By dividing (61) from (59), we see that the prover is able to compute a short $\bar{\pi}_B = \pi_B/\pi'_B \in \mathbb{G}$ such that

$$\frac{e(V, \hat{V})}{e(\prod_{i=1}^{\ell_B} g_{n+1-i}^{2^{i-1}}, \hat{C}_w)} = e(\bar{\pi}_B, \hat{g}). \quad (62)$$

Together with the proof π_θ satisfying (58) and $\pi_w, \bar{\pi}_B$ shows that \hat{V} commits to a vector $((x_1, \dots, x_m) \mid \mathbf{0}^{n-m}) \in \mathbb{Z}_p^n$ (we still assume that its last $n-m$ positions are proven to be 0) such that $\sum_{k=1}^m x_k^2 \bmod p$ is at most $2^{\ell_B} - 1$.

However, we also need to prove that $\sum_{k=1}^m x_k^2 \bmod p$ is actually $\sum_{k=1}^m x_k^2$ over \mathbb{Z} , in which case we have $\|\mathbf{x}\| \leq B$. To this end, the prover uses the construction of Section C.1 to prove that $\|\mathbf{x}\|_\infty \leq \sqrt{p/m}$, which ensures that $\sum_{k=1}^m x_k^2$ does not wrap around modulo p . We note that this additional proof component does not affect the zero-knowledge property because the proven statement $\|\mathbf{x}\| \leq B$ already implies $\|\mathbf{x}\|_\infty \leq \sqrt{p/m}$ (recall that we assumed $B \leq \sqrt{p/m}$ and we always have $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|$). The prover thus generates a proof $\pi_\infty = (\hat{C}, C_y, \pi_\infty) \in \hat{\mathbb{G}} \times \mathbb{G}^2$ that $\|\mathbf{x}\|_\infty \leq B_\infty$ by proving that $0 \leq x_k + \sqrt{p/m} < 2\sqrt{p/m}$ for each $k \in [m]$ using the construction of Section C, which also demonstrates that \hat{V} commits to a vector containing zeroes in its last $n-m$ entries.

The entire proof $\pi = (V, \pi_\infty, \hat{C}_w, \pi_w, \pi_\theta, \bar{\pi}_B)$ is eventually comprised of $\pi_\infty = (\hat{C}, C_y, \pi_\infty) \in \hat{\mathbb{G}} \times \mathbb{G}^2$, the commitment \hat{C}_w and its proof of binarity $\pi_w = (C_{y,w}, \pi_w) \in \mathbb{G}^2$, the commitment V and the proof π_θ satisfying (58), and the proof $\bar{\pi}_B$ satisfying (62).

From a security standpoint, the knowledge-soundness property follows from that of underlying proof components. Simulation-extractability is also preserved as long as these components are bound together in a non-malleable way. To do this, one option is to use a short one-time signature (such as the one from [66, Section 5.4]) whose verification key is included in all random oracle inputs. However, more efficient solutions are possible by suitably combining the various sub-proofs together and including previously computed commitments in each random oracle input.

In terms of efficiency, it is also possible to exploit the linearity of verification equations and compress $(\pi_\infty, \pi_\theta, \pi_w, \bar{\pi}_B) \in \mathbb{G}^4$ into a single group element $\pi = \pi_\infty^{\delta_\infty} \cdot \pi_\theta^{\delta_\theta} \cdot \pi_w^{\delta_w} \cdot \bar{\pi}_B^{\delta_B}$ using aggregation coefficients $(\delta_\infty, \delta_\theta, \delta_w, \delta_B)$ derived from a random oracle. This shrinks the proof to 2 elements of $\hat{\mathbb{G}}$ and 4 elements of \mathbb{G} while verification boils down to a product of 8 pairings.

F Proving Small Hamming Weights

In this section, we show that our argument of Section 3 can be extended to prove that committed vectors have small Hamming weights.

Compact proofs of small Hamming weights were previously considered by Damgård *et al.* [39] in the context of perfectly binding commitments. To our

knowledge, no efficient solution to this problem has been reported in the case of perfectly hiding commitments if we aim at constant-size proofs. The only solution we are aware of is to rely on SNARKs for general NP relations via an expensive Karp reduction.

F.1 Proving Exact Hamming Weights for Binary Vectors

For a commitment $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^n \hat{g}_j^{x_j}$ to a binary $\mathbf{x} \in \{0, 1\}^n$, we can also prove that \mathbf{x} has a fixed Hamming weight. This is useful in the context of FHE, where secret keys are sometimes chosen with a special structure for efficiency reasons. To prove that a committed \mathbf{x} is a binary vector of Hamming weight k , we can prove that: (i) \mathbf{x} is binary; (ii) Its inner product with the all-one vector $(1, 1, \dots, 1)$ is exactly k . Our technique from Section 3 allows handling (i). In order to prove (ii), the prover can generate a short $\pi_k \in \mathbb{G}$ such that

$$e\left(\prod_{i=1}^n g_{n+1-i}, \hat{C}\right) = e(g_1, \hat{g}_n)^k \cdot e(\pi_k, \hat{g}), \quad (63)$$

which is possible as in (1). Again, we can aggregate π_k with other proof components to obtain a proof comprised of one element of $\hat{\mathbb{G}}$ and two elements of \mathbb{G} .

If we combine the above idea with the range proof construction, it is also possible to prove that the Hamming weight $HW(\mathbf{x})$ of the committed \mathbf{x} is such that $HW(\mathbf{x}) \leq B$, for some bound B , without revealing the exact weight. In Appendix F.3, we provide a more efficient way to prove the inequality $HW(\mathbf{x}) \leq B$ for arbitrary (i.e., not necessarily binary) vectors.

F.2 Proving Exact Hamming Weights for Ternary Vectors

If we want to prove the *exact* Hamming weight a committed $\mathbf{x} \in \{-1, 0, 1\}^\ell$, we can first generate a commitment to $\bar{\mathbf{x}} = \mathbf{x} \circ \mathbf{x}$, which is the Hadamard product of \mathbf{x} with itself. We can then prove that: (i) $\bar{\mathbf{x}}$ is indeed the product $\mathbf{x} \circ \mathbf{x} = (x_1^2, \dots, x_n^2)$; (ii) $\bar{\mathbf{x}}$ is binary and has Hamming weight B .

In order to prove (i), we need to compute $C_{sq} = g^{\gamma_{sq}} \cdot \prod_{j=1}^n g_j^{x_j^2}$ as an auxiliary commitment to $\bar{\mathbf{x}} = \mathbf{x} \circ \mathbf{x}$, for a random $\gamma_{sq} \xleftarrow{R} \mathbb{Z}_p$, and prove that C_{sq} commits to $\mathbf{x} \circ \mathbf{x}$. This can be done by adapting the technique of Section 3. The prover generates yet another commitment $C_y = g^{\gamma_y} \cdot \prod_{i=1}^n g_{n+1-i}^{y_i \cdot x_i}$ and proves that it commits to $(y_n \cdot x_n, \dots, y_1 \cdot x_1)$ by proceeding exactly as in Section 3. Specifically, by computing $(y_1, \dots, y_n) = H(\hat{C}, C_{sq})$ and adapting (7), the prover can generate a short $\pi_y \in \mathbb{G}$ such that

$$e(C_y, \hat{C}) = e(\pi_y, \hat{g}) \cdot e(g_1, \hat{g}_n)^{\sum_{i=1}^n y_i \cdot x_i^2} \quad (64)$$

Since the prover can also compute a short $\pi'_y \in \mathbb{G}$ such that C_{sq} satisfies

$$e\left(C_{sq}, \prod_{i=1}^n g_{n+1-i}^{y_i}\right) = e(\pi'_y, \hat{g}) \cdot e(g_1, \hat{g}_n)^{\sum_{i=1}^n y_i \cdot x_i^2}, \quad (65)$$

it can compute $\pi_{sq} = \pi_y/\pi'_y \in \mathbb{G}$ such that

$$\frac{e(C_y, \hat{C})}{e(C_{sq}, \prod_{i=1}^n g_{n+1-i}^{y_i})} = e(\pi_{sq}, \hat{g}), \quad (66)$$

which follows by dividing (65) from (64). From $\hat{C} \in \hat{\mathbb{G}}$, $C_y \in \mathbb{G}$, $C_{sq} \in \mathbb{G}$, π_{sq} and π_{eq} (which shows that C_y commits to $(y_n \cdot x_n, \dots, y_1 \cdot x_1)$), the verifier is convinced that C_{sq} is a commitment to $\mathbf{x} \circ \mathbf{x}$.

We note that proving that $\mathbf{x} \circ \mathbf{x}$ is binary provides an alternative method of proving that \mathbf{x} is ternary, but it is less efficient than the argument of Section D.

F.3 Proving Bounded Hamming Weights for Arbitrary Vectors

We now consider the problem of proving small Hamming weights for an arbitrary vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}_q^n$ committed as $\hat{C} = \hat{g}^\gamma \cdot \prod_{j=1}^n \hat{g}_j^{x_j}$. Using the additive homomorphic property of the commitment scheme, this also allows proving that two committed vectors are close in terms of Hamming distance.

In order to prove that \mathbf{x} has at most B non-zero positions, we first generate a commitment C_w to a random vector $\mathbf{w} = (w_1, \dots, w_n) \in \{0, 1\}^n$ of Hamming weight $HW(\mathbf{w}) = B$ for which $w_i = 1$ for all $i \in [n]$ such that $x_i \neq 0$. We can then prove that: (i) \mathbf{w} is binary and has Hamming weight B ; (ii) For each $i \in [n]$, $w_i = 1$ whenever $x_i \neq 0$, which implies $HW(\mathbf{x}) \leq HW(\mathbf{w})$.

We can prove (i) as explained in Section F.1. In order to prove (ii), we will prove that $\sum_{i=1}^n y_i \cdot (1 - w_i) \cdot x_i = 0$ for a random vector $\mathbf{y} = (y_1, \dots, y_n)$, which ensures that $\forall i \in [n] : (x_i \neq 0) \Rightarrow (w_i = 1)$ with probability $1 - 1/p$. Indeed, if there exists $i \in [n]$ such that $x_i \neq 0$ and $w_i = 0$, we have $\sum_{i=1}^n y_i \cdot (1 - w_i) \cdot x_i = 0$ with probability $1/p$ since \mathbf{y} is computed after \mathbf{w} and \mathbf{x} .

In more details, the prover computes a commitment $\hat{C}_w = \hat{g}^{\gamma_w} \cdot \prod_{j=1}^n \hat{g}_j^{w_j}$ to $\mathbf{w} \in \{0, 1\}^n$, for some random $\gamma_w \xleftarrow{R} \mathbb{Z}_p$, and proves that \hat{C}_w is a commitment to a binary vector using a short proof $\pi_w = (C_{y,w}, \pi_w) \in \mathbb{G}^2$. Then, the prover generates another commitment $C_y = g^{\gamma_y} \cdot \prod_{i=1}^n g_{n+1-i}^{y_i \cdot w_i}$ and proves that it commits to $(y_n \cdot w_n, \dots, y_1 \cdot w_1)$, where $(y_1, \dots, y_n) = H(\hat{C}, \hat{C}_w)$, by proceeding exactly as in Section 3. Next, the prover can generate a short $\pi_y \in \mathbb{G}$ such that

$$e\left(\prod_{i=1}^n g_{n+1-i}^{y_i} \cdot C_y^{-1}, \hat{C}\right) = e(\pi_y, \hat{g}) \cdot e(g_1, \hat{g}_n)^{\sum_{i=1}^n y_i \cdot (1 - w_i) \cdot x_i} = e(\pi_y, \hat{g}), \quad (67)$$

which is possible since $\prod_{i=1}^n g_{n+1-i}^{y_i} \cdot C_y^{-1} = g^{\gamma_y} \cdot \prod_{i=1}^n g_{n+1-i}^{y_i \cdot (1 - w_i)}$, so that the sum $\sum_{i=1}^n y_i \cdot (1 - w_i) \cdot x_i$ is the coefficient of α^{n+1} when we see the left-hand-side member of (67) as a product of polynomials in the exponent.

The final proof then consists of $\hat{C}_w \in \hat{\mathbb{G}}$, $C_{y,w}, C_y \in \mathbb{G}$, and a short $\pi \in \mathbb{G}$ obtained by aggregating the various proof components (including π_w, π_y , the proof π_B that \hat{C}_w satisfies (63), and the proof that C_y is correctly formed).

G Deferred Material for the Argument of Section 5

G.1 Proof of Theorem 3

Proof. To simulate a proof for a statement $\mathbf{x} = (q, d, M, N, \{B_i\}_{i=1}^N, \{\mathbf{a}_i\}_{i=1}^M, \mathbf{c})$, we can use the trapdoor $\alpha \in \mathbb{Z}_p$ of the CRS as follows. First, the simulator samples $\gamma, r, \gamma_y \xleftarrow{R} \mathbb{Z}_p$ and computes $\hat{C} = \hat{g}^\gamma$ and $C_y = g^{\gamma_y}$ as commitments to the all-zeroes vector. Next, it computes the polynomial

$$\begin{aligned} P[X] = & \gamma \cdot \gamma_y \cdot \delta_y + \gamma \cdot \sum_{i=1}^n \left((\delta_{eq} \cdot t_i - \delta_y) \cdot y_i + \delta_\theta \cdot \mathbf{a}_\theta[i] \right) \cdot X^{n+1-i} \\ & - \gamma_y \cdot \delta_{eq} \cdot \sum_{i=1}^n t_i \cdot X^i - t_\theta \cdot \delta_\theta \cdot X^{n+1} \end{aligned}$$

for which the right-hand-side member of (33) can be written $e(g, \hat{g})^{P(\alpha)}$. Using the secret exponent $\alpha \in \mathbb{Z}_p$, the simulator can then simulate a proof by computing $\pi = g^{P(\alpha)}$. It is easy to see that the resulting tuple $\boldsymbol{\pi} = (\hat{C}, C_y, \pi)$ is distributed as a real proof since the commitments \hat{C} and C_y are uniformly distributed in their group and $\pi \in \mathbb{G}$ is uniquely determined by (\hat{C}, C_y) and the aggregation coefficients. \square

G.2 Proof of Theorem 4

Proof. In the AGM+ROM model, we show that, under the $(2n, n)$ -DLOG assumption, there is an extractor that can extract a witness from any adversarially-generated proof $\boldsymbol{\pi}$ and statement $\mathbf{x} = (q, d, M, N, \{B_i\}_{i=1}^N, \{\mathbf{a}_i\}_{i=1}^M, \mathbf{c})$. Concretely, we show an algorithm \mathcal{B} that either extracts a witness or solves an $(2n, n)$ -DLOG instance by computing $\alpha \in \mathbb{Z}_p$ from $\{(g, g_1, \dots, g_{2n}), (\hat{g}_1, \dots, \hat{g}_n)\}$, where $g_i = g^{(\alpha^i)}$ and $\hat{g}_i = \hat{g}^{(\alpha^i)}$ for all i .

The problem instance $\{(g, g_1, \dots, g_{2n}), (\hat{g}_1, \dots, \hat{g}_n)\}$ is used to define pp . Note that $g_{n+1} = g^{(\alpha^{n+1})}$ is not included in pp although it is part of \mathcal{B} 's input. Our reduction/extractor \mathcal{B} interacts with \mathcal{A} as follows.

Queries: At any time, \mathcal{A} can provide $\mathbf{x} = (q, d, M, N, \{B_i\}_{i=1}^N, \{\mathbf{a}_i\}_{i=1}^M, \mathbf{c})$ and ask for a simulated proof that $\mathbf{c} \in R_q^N$ is a valid ciphertext for the public key $(\mathbf{a}_1, \dots, \mathbf{a}_M)$. To generate such a proof, the reduction \mathcal{B} defines the public-key-dependent matrix $\tilde{\mathbf{A}} \in \mathbb{Z}^{Nd \times D}$ and the ciphertext-dependent vector $\phi(\mathbf{c}) \in \mathbb{Z}^{Nd}$ as in (29). It chooses $\boldsymbol{\theta}_0 \xleftarrow{R} \mathbb{Z}_p^{Nd}$ and $\delta_\theta \xleftarrow{R} \mathbb{Z}_p$ and computes $\tilde{\mathbf{a}}_\theta^\top = \boldsymbol{\theta}_0^\top \cdot \tilde{\mathbf{A}} \bmod p$ and $t_\theta = \boldsymbol{\theta}_0^\top \cdot \phi(\mathbf{c}) \bmod p$. We note that the first component $\tilde{\mathbf{a}}_\theta[1] \in \mathbb{Z}_p$ of $\tilde{\mathbf{a}}_\theta \in \mathbb{Z}_p^D$ is non-zero with overwhelming probability (as we may assume that the first column $\tilde{\mathbf{A}}[1]$ of $\tilde{\mathbf{A}} \in \mathbb{Z}^{Nd \times D}$ is non-zero). If $\boldsymbol{\theta}_0^\top \cdot \tilde{\mathbf{A}}[1] = \boldsymbol{\theta}_0^\top \cdot \phi(\mathbf{c})$, \mathcal{B} can generate a real proof using the witness $\tilde{\mathbf{w}} = (1, 0, \dots, 0)$. We thus assume $\boldsymbol{\theta}_0^\top \cdot \tilde{\mathbf{A}}[1] \neq \boldsymbol{\theta}_0^\top \cdot \phi(\mathbf{c})$, so that \mathcal{B} can compute a non-binary $\mathbf{w} = (w_1 \mid \mathbf{0}^{n-1}) \in \mathbb{Z}_p^n$ satisfying the equation

$$\mathbf{a}_\theta^\top \cdot \mathbf{w} = t_\theta \bmod p,$$

where $\mathbf{a}_\theta^\top = (\tilde{\mathbf{a}}_\theta^\top \mid \mathbf{0}^{n-D})$. It commits to \mathbf{w} by computing $\hat{C} = \hat{g}^\gamma \cdot \hat{g}_1^{w_1}$ for a random $\gamma \xleftarrow{R} \mathbb{Z}_p$. Next, \mathcal{B} simulates other proof elements as follows.

1. Choose random vectors $\boldsymbol{\delta} = (\delta_{eq}, \delta_y, \delta_\theta) \xleftarrow{R} \mathbb{Z}_p^3$, $\mathbf{y} = (y_1, \dots, y_n) \xleftarrow{R} \mathbb{Z}_p^n$, $\mathbf{t} = (t_1, \dots, t_n) \xleftarrow{R} \mathbb{Z}_p^n$.
2. Set $z_n = y_1$ and find an arbitrary $(z_1, \dots, z_{n-1}) \in \mathbb{Z}_p^{n-1}$ such that

$$\sum_{i=2}^n t_i \cdot z_{n+1-i} = t_1 \cdot y_1 \cdot (w_1 - 1).$$

3. Choose a random $z_0 \xleftarrow{R} \mathbb{Z}_p$ and compute a simulated commitment

$$C_y = g^{z_0} \cdot \prod_{i=1}^n g_i^{z_i}.$$

4. If one of the random oracle values $H_{\text{agg}}(\mathbf{x}, \hat{C}, C_y)$, $H(\mathbf{x}, \hat{C})$, $H_{\text{map}}(\mathbf{x}, \hat{C}, C_y)$ or $H_t(\mathbf{y}, \mathbf{x}, \hat{C}, C_y)$ was already defined, then abort and report failure. Otherwise, set $\mathbf{y} = H(\mathbf{x}, \hat{C})$, $\mathbf{t} = H_t(\mathbf{y}, \mathbf{x}, \hat{C}, C_y)$, $\boldsymbol{\delta} = H_{\text{agg}}(\mathbf{x}, \hat{C}, C_y)$ and $\boldsymbol{\theta} = H_{\text{map}}(\mathbf{x}, \hat{C}, C_y) \in \mathbb{Z}_p^{Nd+1}$ for a random vector $\boldsymbol{\theta} \in \mathbb{Z}_p^{Nd+1}$ whose first $Nd+1$ components are $(\boldsymbol{\theta}_0 \mid \delta_\theta)$.
5. Define the polynomials

$$\begin{aligned} Q_y[X] &= \left(\sum_{i=0}^n z_i \cdot X^i - \sum_{i=1}^n y_i \cdot X^{n+1-i} \right) \cdot (\gamma + w_1 \cdot X) \\ &= \left(z_0 + \sum_{i=1}^n (z_{n+1-i} - y_i) \cdot X^{n+1-i} \right) \cdot (\gamma + w_1 \cdot X) = \sum_{i=0}^{n+1} \sigma_i \cdot X^i \\ Q_{eq}[X] &= (\gamma + w_1 \cdot X) \cdot \left(\sum_{i=1}^n t_i \cdot y_i \cdot X^{n+1-i} \right) \\ &\quad - \left(\sum_{i=0}^n z_i \cdot X^i \right) \cdot \left(\sum_{i=1}^n t_i \cdot X^i \right) = \sum_{j=0}^{2n} e_j \cdot X^j, \\ Q_\theta[X] &= \left(\sum_{k=1}^D \mathbf{a}_\theta[k] \cdot X^{n+1-k} \right) \cdot (\gamma + w_1 \cdot X) - t_\theta \cdot X^{n+1} = \sum_{i=0}^{n+1} \zeta_i \cdot X^i \end{aligned}$$

Their degree- $(n+1)$ coefficients are

$$\begin{aligned} \sigma_{n+1} &= w_1 \cdot (z_n - y_1) = 0 \\ e_{n+1} &= w_1 t_1 y_1 - \sum_{i=1}^n t_i \cdot z_{n+1-i} = t_1 \cdot y_1 \cdot (w_1 - 1) - \sum_{i=2}^n t_i \cdot z_{n+1-i} = 0 \\ \zeta_{n+1} &= \mathbf{a}_\theta[1] \cdot w_1 - t_\theta = 0 \end{aligned}$$

due to the definition of committed $\mathbf{w} = (w_1, \dots, w_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$.

6. Define the polynomial

$$Q_{\text{agg}}[X] = \delta_{eq} \cdot Q_{eq}[X] + \delta_y \cdot Q_y[X] + \delta_\theta \cdot Q_\theta[X] = \sum_{i=0}^{2n} \eta_i \cdot X^i$$

for which $\eta_{n+1} = 0$ by construction. Compute

$$\pi = \prod_{i=1, i \neq n+1}^{2n} g_i^{\eta_i} \quad (68)$$

using $(g, \{g_i\}_{i \in [2n] \setminus \{n+1\}})$ and return the simulated proof $\pi = (\hat{C}, C_y, \pi)$.

The proof π has the same distribution as an output of the simulator in the proof of Theorem 3. Indeed, π is uniquely determined by \mathbf{x} , (\hat{C}, C_y) and the \mathbb{Z}_p -elements \mathbf{y} , \mathbf{t} , and $\boldsymbol{\delta}$ in the right-hand-side member (33). Moreover, while the committed $\mathbf{w}, \mathbf{z} \in \mathbb{Z}_p^n$ are programmed in a special way, they are completely independent of \mathcal{A} 's view due to the randomness γ and z_0 in (\hat{C}, C_y) .

Consequently, the simulation is perfect, unless one of the random oracles has to be programmed on an input where it was previously defined. If Q_S (reps. Q_H) is the number of queries made by \mathcal{A} to the simulator (resp. to random oracles), this happens with probability $\leq (Q_S + Q_H) \cdot Q_H/p$.

Output: When \mathcal{A} halts, it outputs $\mathbf{x} = (q, d, M, N, \{B_i\}_{i=1}^N, \{\mathbf{a}_i\}_{i=1}^M, \mathbf{c})$ and a valid proof $\pi = (\hat{C}, C_y, \pi)$. Let $\tilde{\mathbf{A}} \in \mathbb{Z}^{N \times D}$ the matrix obtained by encoding $\{\mathbf{a}_i \in R_q^N\}_{i=1}^M$ in (29).

Since we are in the AGM, \mathcal{A} must provide representations of \hat{C} w.r.t to the set of all $\hat{\mathbb{G}}$ -elements that it could observe during the game. It also has to provide representations of C_y and π w.r.t to all \mathbb{G} -elements that it was allowed to see. Since the simulator used by \mathcal{B} is algebraic, it also knows a representation of each simulated $C_y^{(i)}$ and $\pi^{(i)}$ w.r.t $(g, \{g_i\}_{i \in [2n] \setminus \{n+1\}})$. It also knows a representation of each simulated $\hat{C}^{(i)}$ w.r.t $(\hat{g}, \{\hat{g}_i\}_{i \in [n]})$. From \mathcal{A} 's output and the randomness of the simulation, \mathcal{B} can therefore compute scalars $\{(\psi_i, z_i) \in \mathbb{Z}_p^2\}_{i \in [0, 2n] \setminus \{n+1\}}$ and $\{w_i \in \mathbb{Z}_p\}_{i \in [0, n]}$ such that

$$\hat{C} = \prod_{i=0}^n \hat{g}_i^{w_i}, \quad C_y = \prod_{i=0, i \neq n+1}^{2n} g_i^{z_i}, \quad \pi = \prod_{i=0, i \neq n+1}^{2n} g_i^{\psi_i},$$

where $g_0 = g$ and $\hat{g}_0 = \hat{g}$.

If the representation $\mathbf{w} = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_p^n$ of the commitment \hat{C} satisfies the conditions

- (i) $w_k \in \{0, 1\}$ for all $k \in [1, D]$;
- (ii) $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{w}} = \phi(\mathbf{c}) \bmod p$, where $\tilde{\mathbf{w}} = (w_1, \dots, w_D) \in \mathbb{Z}_p^D$;

then \mathcal{B} can use the bits $(w_0, w_1, \dots, w_D) \in \{0, 1\}^D$ to reconstruct witnesses $s_1, \dots, s_M \in \mathbb{Z}[X]/(X^d + 1)$ such that $\|s_i\|_\infty \leq B_i$ for all $i \in [M]$ and (26)

holds, meaning that $\{s_i\}_{i=1}^M$ are valid outputs for the knowledge extractor. We now assume that at least one of the conditions (i)-(ii) does not hold.

Solving $(2n, n)$ -DLOG: We remark that a non-trivial valid proof π cannot recycle $(\mathbf{x}, \hat{C}, C_y)$ from a simulated proof. That is, for all $i \in [Q_S]$, we must have $(\mathbf{x}, \hat{C}, C_y) \neq (\mathbf{x}^{(i)}, \hat{C}^{(i)}, C_y^{(i)})$ since the right-hand-side member of (33) is uniquely determined by $(\mathbf{x}^{(i)}, \hat{C}^{(i)}, C_y^{(i)})$ and it in turn determines a unique valid proof element $\pi^{(i)} \in \mathbb{G}$ in the left-hand-side member. This implies that $(\delta_y, \delta_{eq}, \delta_\theta) = H_{\text{agg}}(\mathbf{x}, \hat{C}, C_y)$ is not one of the hashes programmed by the simulator and neither are $\mathbf{t} = H_t(\mathbf{y}, \mathbf{x}, \hat{C}, C_y)$, and $\bar{\theta} = H_{\text{map}}(\mathbf{x}, \hat{C}, C_y)$.

Let the vector $\mathbf{a}_\theta^\top = (\tilde{\mathbf{a}}_\theta^\top \mid \mathbf{0}^{n-D}) \in \mathbb{Z}_p^n$ defined in the Verify algorithm. From the algebraic representations of \mathcal{A} 's commitments and proof π , \mathcal{B} can compute

$$P_\theta[X] = \left(\sum_{k=1}^D \mathbf{a}_\theta[k] \cdot X^{n+1-k} \right) \cdot \left(\sum_{i=0}^n w_i \cdot X^i \right) - t_\theta \cdot X^{n+1} = \sum_{i=0}^{n+D} \omega_i \cdot X^i$$

as well as the polynomials

$$\begin{aligned} P_y[X] &= \left(\sum_{i=0, i \neq n+1}^{2n} z_i \cdot X^i - \sum_{i=1}^n y_i \cdot X^{n+1-i} \right) \cdot \left(\sum_{i=0}^n w_i \cdot X^i \right) \\ &= \left(z_0 + \sum_{i=1}^n (z_{n+1-i} - y_i) \cdot X^{n+1-i} + \sum_{i=n+2}^{2n} z_i \cdot X^i \right) \cdot \left(\sum_{i=0}^n w_i \cdot X^i \right) \\ &= \sum_{i=0}^{3n} \gamma_i \cdot X^i \\ P_{eq}[X] &= \left(\sum_{i=0}^n w_i \cdot X^i \right) \cdot \left(\sum_{i=1}^n t_i \cdot y_i \cdot X^{n+1-i} \right) \\ &\quad - \left(\sum_{i=0, i \neq n+1}^{2n} z_i \cdot X^i \right) \cdot \left(\sum_{i=1}^n t_i \cdot X^i \right) = \sum_{j=0}^{3n} \beta_j \cdot X^j, \end{aligned}$$

for which the left-hand-side members of (30)-(32) can be written $e(g, \hat{g})^{P_{eq}(\alpha)}$, $e(g, \hat{g})^{P_y(\alpha)}$, and $e(g, \hat{g})^{P_\theta(\alpha)}$, respectively.

The right-hand-side member of (33) can be written $e(g, \hat{g})^{P_{\text{agg}}(\alpha)}$, where $P_{\text{agg}}[X]$ is the polynomial

$$P_{\text{agg}}[X] = \delta_y \cdot P_y[X] + \delta_{eq} \cdot P_{eq}[X] + \delta_\theta \cdot P_\theta[X] = \sum_{i=0}^{3n} \nu_i \cdot X^i$$

In $P_{\text{agg}}[X]$, the coefficient ν_{n+1} of the degree- $(n+1)$ term can be written

$$\nu_{n+1} = \underbrace{\delta_y \cdot \sum_{i=1}^n (z_{n+1-i} - y_i) \cdot w_i}_{\triangleq \gamma_{n+1}} + \underbrace{\delta_{eq} \cdot \sum_{i=1}^n t_i \cdot (w_i \cdot y_i - z_{n+1-i})}_{\triangleq \beta_{n+1}} + \underbrace{\delta_\theta \cdot \left(\sum_{k=1}^D \mathbf{a}[k] \cdot w_k - t_\theta \right)}_{\triangleq \omega_{n+1}},$$

where $\boldsymbol{\rho} \triangleq (\gamma_{n+1}, \beta_{n+1}, \omega_{n+1})$ is the vector containing the coefficients of the degree- $(n+1)$ terms of $(P_y[X], P_{eq}[X], P_\theta[X])$.

We now argue that, if one of the conditions (i)-(ii) does not hold, the probability to have $\nu_{n+1} = 0$ is negligible. This follows from the following observations:

- The probability to have $\boldsymbol{\rho} = \mathbf{0}$ is negligible. First, if $z_{n+1-i} \neq w_i \cdot y_i$ for some $i \in [n]$, we have $\beta_{n+1} = 0$ with probability $1/p$ since $\mathbf{t} = H_t(\mathbf{y}, \mathbf{x}, \hat{C}, C_y)$ is derived *after* the choice of \mathbf{y} , $\{w_i\}_{i=0}^n$ and $\{z_i\}_{i \in [0, 2n] \setminus \{n+1\}}$. Now, if we assume that $z_{n+1-i} = w_i \cdot y_i$ for all $i \in [n]$, then we have

$$\gamma_{n+1} = \sum_{i=1}^n y_i \cdot (w_i - 1) \cdot w_i,$$

which vanishes with probability $1/p$ if there exists $i \in [n]$ such that $w_i \notin \{0, 1\}$. This can be seen by distinguishing two cases:

- a. If $\mathbf{y} = H(\mathbf{x}, \hat{C})$ was programmed when answering a simulation query, we can only have $\gamma_{n+1} = 0$ with probability $1/p$ since the simulator programmed (w_1, \dots, w_n) so has to have

$$\gamma_{n+1} = \sum_{i=1}^n y_i \cdot w_i \cdot (w_i - 1) = y_1 \cdot w_1 \cdot (w_1 - 1)$$

where $w_1 \notin \{0, 1\}$ and $y_1 \in_R \mathbb{Z}_p$. This captures the case of an adversary attempting to re-use the components $(\mathbf{x}, \hat{C}) = (\mathbf{x}^{(i)}, \hat{C}^{(i)})$ of a simulated proof $\boldsymbol{\pi}^{(i)} = (\hat{C}^{(i)}, C_y^{(i)}, \pi^{(i)})$ with a modified $C_y \neq C_y^{(i)}$.

- b. If $H(\mathbf{x}, \hat{C})$ was not programmed by the simulator, then $\mathbf{y} = H(\mathbf{x}, \hat{C})$ was defined after \mathcal{B} obtained the scalars $\{w_i\}_{i=0}^n$ underlying \hat{C} . We then have the equality $\sum_{i=1}^n y_i \cdot (w_i - 1) \cdot w_i = 0$ with probability $1/p$ over the random choice of $\{y_i\}_{i=1}^n$.

If none of the previous events occurs, we have $w_i \in \{0, 1\}$ for all $i \in [D]$. Then, we are left with bounding the probability that $\omega_{n+1} = 0$ when condition (ii) does not hold. In this case, we have $\boldsymbol{\theta}_0^\top \cdot (\tilde{\mathbf{A}} \cdot \tilde{\mathbf{w}} - \phi(\mathbf{c})) = 0 \pmod p$ with probability $1/p$ since $\tilde{\boldsymbol{\theta}} = H_{\text{map}}(\mathbf{x}, \hat{C}, C_y)$ is defined after \mathbf{A} and $\phi(\mathbf{c})$.

where $\bar{\mathbf{A}}$ is interpreted as a $2d \times 6d$ matrix with coefficients in $\{-\lfloor q/2 \rfloor, \dots, \lfloor q/2 \rfloor\}$.

The prover commits to the bits of $\bar{\mathbf{w}}$ and proves that

$$\underbrace{\begin{bmatrix} \text{rot}(a) & & \mathbf{G}_{1+\log B} & & -q \cdot \mathbf{G}_{\log d} \\ \text{rot}(b) & \Delta \cdot \mathbf{I}_d & & \mathbf{G}_{1+\log B} & -q \cdot \mathbf{G}_{\log d} \end{bmatrix}}_{\cong \bar{\mathbf{A}} \in \mathbb{Z}^{2d \times D}} \cdot \underbrace{\begin{bmatrix} \phi(r) \\ \phi(m) \\ e_1 \\ e_2 \\ r_1 \\ r_2 \end{bmatrix}}_{\cong \bar{\mathbf{w}}} = \underbrace{\begin{bmatrix} \phi(c_1) \\ \phi(c_2) \end{bmatrix}}_{\cong \phi(\mathbf{c})}, \quad (71)$$

where $e_1 = \mathbf{G}_{1+\log B}^{-1}(\phi(e_1))$, $e_2 = \mathbf{G}_{1+\log B}^{-1}(\phi(e_2))$, $r_1 = \mathbf{G}_{\log d}^{-1}(\phi(r_1))$ and $r_2 = \mathbf{G}_{\log d}^{-1}(\phi(r_2))$. We note that there is no need to decompose $m, r \in R/(2R)$ since they are natively binary. The prover will thus commit to the decomposition of the witness $\bar{\mathbf{w}} \in \{0, 1\}^D$, where $D = 2d(2 + \log B + \log d)$.

It is also interesting to consider proofs of validity in an encryption scheme proposed by Joye [75], which was designed to be used as a component of the TFHE [33] homomorphic encryption scheme. The scheme of [75] can be seen as a variant of the LPR cryptosystem where the second ciphertext component computes an inner product over \mathbb{Z}_q instead of a multiplication over R_q . For a plaintext $m \in \mathbb{Z}_p$ and a noise $e_2 \in \mathbb{Z}$, ciphertexts are of the form

$$(c_1, c_2) = (a \cdot r + e_1, \langle \phi(\bar{b}), \phi(r) \rangle + e_2 + \Delta \cdot m) \in R_q \times \mathbb{Z}_q$$

where $\Delta = \lfloor q/t \rfloor$ (for a plaintext modulus t) and $\phi(\bar{b}) = (b_{n-1}, \dots, b_0) \in \mathbb{Z}_q^n$ contains the coefficients of the polynomial $b(x) = b_0 + b_1X + \dots + b_{n-1}X^{n-1} \in R_q$ in reversed order. Note that $c_2 \in \mathbb{Z}_q$ can be seen as the extraction of the last slot from the second component $b \cdot r + \Delta \cdot m + \text{noise}$ of an LPR ciphertext since the degree- $(n-1)$ coefficient of the polynomial product $b \cdot r \in R_q$ is $\langle \phi(\bar{b}), \phi(r) \rangle$. In this case, relation (70) simplifies as

$$\underbrace{\begin{bmatrix} \text{rot}(a) & & \mathbf{I}_d & & -q \cdot \mathbf{I}_d \\ \phi(\bar{b}) & \Delta & & 1 & -q \end{bmatrix}}_{\cong \bar{\mathbf{A}}} \cdot \underbrace{\begin{bmatrix} \phi(r) \\ m \\ \phi(e_1) \\ e_2 \\ \phi(r_1) \\ r_2 \end{bmatrix}}_{\cong \bar{\mathbf{w}}} = \underbrace{\begin{bmatrix} \phi(c_1) \\ c_2 \end{bmatrix}}_{\cong \phi(\mathbf{c})}. \quad (72)$$

while (71) becomes

$$\underbrace{\begin{bmatrix} \text{rot}(a) & & \mathbf{G}_{1+\log B} & & -q \cdot \mathbf{G}_{\log d} \\ \phi(\bar{b}) & \Delta \cdot \mathbf{g}_{\log t} & & \mathbf{g}_{1+\log B} & -q \cdot \mathbf{g}_{\log d} \end{bmatrix}}_{\triangleq \tilde{\mathbf{A}} \in \mathbb{Z}^{(d+1) \times D}} \cdot \underbrace{\begin{bmatrix} \phi(r) \\ m \\ e_1 \\ e_2 \\ r_1 \\ r_2 \end{bmatrix}}_{\triangleq \tilde{\mathbf{w}}} = \underbrace{\begin{bmatrix} \phi(c_1) \\ c_2 \end{bmatrix}}_{\triangleq \phi(\mathbf{c})}, \quad (73)$$

with $\mathbf{e}_2 = \mathbf{g}_{1+\log B}^{-1}(e_2) \in \{0, 1\}^{1+\log B}$ and $\mathbf{r}_2 = \mathbf{g}_{\log d}^{-1}(r_2) \in \{0, 1\}^{\log d}$. The prover then commits to the decomposition of the witness $\tilde{\mathbf{w}} \in \{0, 1\}^D$, where $D = d + \log t + (d + 1)(1 + \log B + \log d)$, where t is the plaintext modulus.

G.4 Efficiency Estimations

Application to the LPR cryptosystem. In an instantiation of LPR for $\lambda = 128$, a common choice of parameters is $d = 1024$, $q \approx 2^{64}$, with binary uniform $r \in R/(2R)$ while e_1, e_2 are sampled from a discrete Gaussian distribution with standard deviation $\alpha q \approx 2^{39}$. In this case, a given noise vector $e_i \sim D_{\mathbb{Z}^d, \alpha q}$ has infinity norm $\|e_i\|_\infty \leq B = \alpha q \sqrt{\lambda} < 2^{43}$ with overwhelming probability by [87, Lemma 4.4]. The computational complexity and the CRS size are then determined by $n = D = 112640$.

In order to obtain the coefficients allowing to compute π from the generators $\{g_i\}_{i \in [2n] \setminus \{n+1\}}$, the prover has to evaluate two products of degree- n polynomials,¹³ which can be done using $O(n \cdot \log n)$ multiplications. The prover also has to compute the product $\mathbf{a}_\theta^\top = \boldsymbol{\theta}_0^\top \cdot \tilde{\mathbf{A}} \bmod p$. In (71), each block of $\tilde{\mathbf{A}}$ has a special structure allowing to compute the matrix-vector product using $O(d \cdot \log d)$ multiplications over \mathbb{Z}_p . Indeed, computing $\boldsymbol{\theta}_0^\top \cdot [\text{rot}(a)^\top \mid \text{rot}(b)^\top]^\top$ takes $O(d \cdot \log d)$ multiplications while computing $\boldsymbol{\theta}_0^\top \cdot (\mathbf{I}_2 \otimes \mathbf{G}_z)$ can be done using $2dz$ additions over \mathbb{Z}_p since \mathbf{G}_z is of the form $\mathbf{I}_d \otimes (1, 2, 4, \dots, 2^{z-2}, -2^{z-1})$.

Eventually, the prover's cost is dominated by 337920 exponentiations in \mathbb{G} and $D + 1$ multiplications, which are used to compute \hat{C} . If we assume that exponentiations in $\hat{\mathbb{G}}$ are three times as expensive as in \mathbb{G} , the overall workload of the prover is roughly equivalent to 339150 exponentiations in \mathbb{G} .

Given the relatively large value of $n \approx 2^{17}$, we need to increase the group order p by about 20 bits in order to obtain a sufficient security margin against Cheon's algorithm [31]. If we use a 275-bit group order, elements of \mathbb{G} (resp.

¹³ The first product is cheaper since one of the factors is of the form $\gamma + \sum_{i=1}^D w_i \cdot X^i$ for binary $w_i \in \{0, 1\}$

$\hat{\mathbb{G}}$) can have a 374-bit (resp. 1122-bit) representation using KSS18 curves. With these parameters, the CRS size amounts to 25712 KB and proofs fit in 1870 bits.

Application to Joye’s scheme. We now consider an instantiation of the scheme in [75] for parameters of interest where $d = 1024$, $q \approx 2^{64}$ and when the noise has magnitude $B \leq 2^{42}$. We assume that the plaintext modulus t has 8 bits due to compatibility constraints with the bootstrapping of TFHE. In order to encrypt 256-bit messages, we consider a packed version of the scheme allowing to encrypt $k = 32$ slots of 8-bit messages each and where all slots $\{e_{2,i} = \langle \phi(\bar{b}_i), \phi(r) \rangle + e_{2,i} + \Delta \cdot m_i\}_{i=1}^k$ share the same secret $r \in R/(2R)$ but use independent noise terms $e_{2,i}$ such that $\|e_{2,i}\|_\infty \leq B$.¹⁴ To prove a packed version of relation (73), the prover commits to a vector $\hat{\boldsymbol{w}}$ of dimension

$$n > D = (d + k \log t) + (d + k)(1 + \log B + \log d) = 57248.$$

Using KSS18 or BLS24 curves, this requires a CRS of 13068 KB or 14340 KB, respectively. The prover computes at most 171744 exponentiations in \mathbb{G} (143000 on average if the noise is sampled from a uniform distribution over $[-B, B]$) besides 57248 multiplications in $\hat{\mathbb{G}}$.

The prover also computes 2 multiplications of polynomials with degree 57248 over a 275-bit field \mathbb{Z}_p and a large matrix-vector product over \mathbb{Z}_p (which can be fast since the matrix is structured). For a suitable prime p such that $p - 1$ is divisible by the smallest power of 2 above n , all \mathbb{Z}_p -operations can be optimized using the FFT.

Verification requires 57248 exponentiations in each source group of the pairing.¹⁵ Assuming multiple threads at the verifier, we can speed up its computation by splitting exponentiations in smaller batches to be processed in parallel. Also, we can reduce the cost of $\hat{\mathbb{G}}$ -exponentiations by observing that the exponents $\boldsymbol{t} = (t_1, \dots, t_n)$ do not need to be uniformly distributed over \mathbb{Z}_p since they are only used to perform a batch verification in the proof of Theorem 4 (i.e., to guarantee that $\beta_{n+1} \neq 0$ w.h.p. in the expression of ν_{n+1}). By [47, Theorem 3.2], we can choose each t_i uniformly in a 128-bit interval (instead of a 275-bit one) and change the verification equation (33) into

$$e(\pi, \hat{g}) = e\left(C_y^{\delta_y} \cdot \prod_{i=1}^n g_{n+1-i}^{(\delta_{eq} \cdot t_i - \delta_y) \cdot y_i + \delta_\theta \cdot \alpha_\theta[i]}, \hat{C}\right) \\ \cdot e\left(C_y^{\delta_{eq}}, \prod_{i=1}^D \hat{g}_i^{t_i}\right)^{-1} \cdot e(g_1, \hat{g}_n)^{-t_\theta \cdot \delta_\theta},$$

¹⁴ In this case, the matrix $\tilde{\mathbf{A}}$ in (73) is modified to have $d + k$ rows, where the last k rows encode public keys components $\{\phi(b_i)\}_{i=1}^k$ in the lower block.

¹⁵ We note that, in applications to private smart contracts [97], this is acceptable since transaction validators can proceed in parallel, regardless of the number of validators. Moreover, a transaction is often considered valid when $2N/3$ out of N validators have verified the proof.

which makes it faster to compute $\prod_{i=1}^D \hat{g}_i^{t_i}$ for 128-bit exponents $\{t_i\}_i$.

We note that, in both schemes [88,75], the NIZK argument of ciphertext validity does not impose any constraint on the modulus q of the encryption scheme (so that any NTT-friendly ring can be used). We only need to choose a sufficiently large group order p to make sure that no implicit modular reduction occurs when we want to prove relations (70) and (72) over the integers.

Possible choices of elliptic curve. In order to optimize the prover’s cost, one may prefer using pairing-friendly curves enabling faster exponentiations in the first source group \mathbb{G} . One option is to choose \mathbb{G} as a subgroup of a curve $E(\mathbb{F}_r)$, for which the base prime field \mathbb{F}_r is as small as possible. In this case, the BLS24 curves [9] are good candidates as they offer the fastest exponentiations in \mathbb{G} (but slower exponentiations in $\hat{\mathbb{G}}$). In order to obtain a sufficient security margin against Cheon’s attack, we can choose $p = |\mathbb{G}| > 2^{275}$, in which case r has $1.25 \cdot 275 = 342$ bits. By keeping r small, we also have a short representation for group elements in \mathbb{G} while elements of $\hat{\mathbb{G}}$ are typically 4 times as long as those of \mathbb{G} (when they live in the twisted curve $E'(\mathbb{F}_{r^4})$). This yields a proof size of 2052 bits and a CRS size of 14340 KB.

As suggested in [2, Table 1], BLS24 curves $E(\mathbb{F}_r) \triangleq \{(x, y) \in \mathbb{F}_r^2 : y^2 = x^3 + b\}$ can be generated via the following parameterization

$$r = \frac{u^{10} - 2u^9 + u^8 - u^6 + 2u^5 - u^4 + u^2 + u + 1}{3}$$

$$p = u^8 - u^4 + 1$$

where u can be tuned until r and p are both prime and p satisfies the conditions $2^{16} | p - 1$ and $p \sim 2^{275}$. A concrete example is given by $E(\mathbb{F}_r) \triangleq \{(x, y) \in \mathbb{F}_r^2 : y^2 = x^3 + 4\}$ with¹⁶

$$u = -2^{34} - 2^{32} - 2^{17} + 2^{14} + 2^{12} = -21474947072$$

$$r = 0x31ac4ff9bfee67ff4d8255eff7c02ac913dc3c878932925def1a1036c086ae44b1456815ff5549daa1aab$$

$$p = 0x5f5f117f282a670f9e6589ab38900f75a509f7e50f735d99fe437e40f000000000001$$

and

$$t = (r + 1) - \#E(F_r) = -0x50001afff$$

(which yields the co-factor $\#E(F_r)/|\mathbb{G}|$). Note that

$$p - 1 = 2^{48} \cdot 3 \cdot 5 \cdot 23 \cdot 269 \cdot 281 \cdot 569 \cdot 677 \cdot 1709$$

$$\cdot 5242907^4 \cdot 37741559 \cdot 328237261028613077$$

¹⁶ In the literature on pairing-friendly curves, p often denotes the characteristic of the base field of the curve while r denotes the group order. Here, it is the other way around.

is divisible by 2^{48} , which allows high-dimension FFTs over \mathbb{Z}_p^* .

At the higher 192-bit security level, BLS24 curves with $|p| > 384$ and $|r| \approx 477$ were suggested in [3, Section 7].¹⁷

One disadvantage of BLS24 curves is their slower arithmetic in $\hat{\mathbb{G}}$. In order to obtain a more balanced tradeoff between the costs of multi-exponentiations in \mathbb{G} and $\hat{\mathbb{G}}$, one may prefer using BLS12 curves. Using BLS12-379 curves, a multi-exponentiation in $\hat{\mathbb{G}}$ with 57248 elements is computable in less than a second according to the timings given in [43, Figure 4.2] for a 256-bit group order. In order to obtain 128 bits of security and taking Cheon’s attack, one may use the BLS12-446 curve suggested in [72],¹⁸ which is obtained from the parameterization

$$\begin{aligned} p &= u^4 - u^2 + 1 \\ r &= (u^6 - 2 \cdot u^5 + 2 \cdot u^3 - u + 1)/3 + u \end{aligned}$$

with the seed

$$u = -(2^{74} + 2^{73} + 2^{63} + 2^{57} + 2^{50} + 2^{17} + 1)$$

and yields a group order $p \approx 2^{299}$ (such that 2^{16} divides $p - 1$) whereas elements of \mathbb{G} (resp. $\hat{\mathbb{G}}$) fit within 446 (resp. 892) bits.

G.5 Comparisons

For the above choice of parameters in the LPR cryptosystem, we commit to vectors of dimension $n = 112640$, which translates into a prover computing 337920 exponentiations in \mathbb{G} and 8196 exponentiations in $\hat{\mathbb{G}}$. In general, exponentiations in $\hat{\mathbb{G}}$ are at least 3 times as expensive as in \mathbb{G} using KSS18 curves (see, e.g., [8, Table 12]). In our setting, the prover computes the equivalent of ≈ 339150 exponentiations in \mathbb{G} . In the example given in [41, Section 5.3] for a smaller modulus q , del Pino *et al.* need about 724986 exponentiations at the prover (and 200667 at the verifier). In general, their construction [41, Section 5.2] incurs up to $10n + 6 \log n$ exponentiations at the prover (and $2n + 4 \log n$ exponentiations at the verifier) in order to generate a proof for a vector of dimension n . Here, we only need $3n$ exponentiations in \mathbb{G} (and n multiplications in $\hat{\mathbb{G}}$) at the prover and the equivalent of $4n$ \mathbb{G} -exponentiations at the verifier. Although we need a slightly larger group order than theirs (i.e., 275 bits vs 256), we expect our prover to be faster and our verification algorithm to be slower. On the other hand, we lose the transparent setup property of BulletProofs and we need to rely on the algebraic group model.

If we want to prove the same statement using Groth’s SNARK [69] in order to obtain a similar proof size, we have to express the statement in the language of Quadratic Arithmetic Programs (QAPs) [56] and obtain a CRS size growing

¹⁷ See <https://neuromancer.sk/std/bls/BLS24-477> for a concrete curve.

¹⁸ See also <https://neuromancer.sk/std/bls/BLS12-446>.

with the number of arithmetic constraints. Then, we run into two issues that increase the size of the arithmetic circuit. First, proving the smallness of noise terms $e_1 = (e_{1,1}, \dots, e_{1,d})^\top, e_2 = (e_{2,1}, \dots, e_{2,d})^\top \in \mathbb{Z}[X]/(X^d + 1)$ requires to break their components into bits. Then, for each pair $(b, j) \in \{0, 1\} \times [d]$, proving that $\{e_{b,j,\tau}\}_{\tau=1}^{1+\log B}$ are all bits requires $d \cdot (1 + \log B)$ constraints of the form $e_{b,j,\tau} \cdot (e_{b,j,\tau} - 1) = 0 \pmod p$, each of which contributes to the number of multiplication gates. In our example, this would amount to $2d \cdot (1 + \log B) = 90112$ arithmetic constraints. We also need $2d$ constraints to prove that r and m are binary. Then, we need $O(d \cdot \log d)$ additional constraints to compute the products $a \cdot r$ and $b \cdot r$ over $\mathbb{Z}_p[X]/(X^d + 1)$ and map e_1, e_2 to the FFT domain. We note that the prover can pre-compute $(\text{FFT}(a), \text{FFT}(b))$ and $(\text{FFT}(c_1), \text{FFT}(c_2))$ instead of leaving it to the circuit,¹⁹ but the circuit still needs $O(d \cdot \log d)$ multiplication gates to compute $\text{FFT}(r)$, $\text{FFT}(e_1)$ and $\text{FFT}(e_2)$.

The second issue is that constant-size SNARKs like [69,56] are designed to handle arithmetic circuits over a large prime field \mathbb{F}_p (where $p > 2^{275}$ is the order of the pairing-friendly group), whereas we need to prove a statement over a ring R_q where $q \approx 2^{64}$. As observed in [54], using finite field arithmetic to emulate arithmetic over rings induces some overhead. For example, additions in R_q may no longer be for free since adding two $\log q$ -bit integers over \mathbb{F}_p may result in a $(1 + \log q)$ -bit sum to be reduced in R_q . In [79], short-integer arithmetic is emulated over \mathbb{F}_p by reducing intermediate computation values modulo q on carefully chosen occasions. In order to prove that a modular reduction $x \bmod q$ is performed correctly (when q is not a power of 2), the prover is required to provide wires $x \div q$ and $x \bmod q$, allowing the circuit to check that $x = q \cdot (x \div q) + (x \bmod q)$ and $(x \bmod q) < q$. In turn, the latter comparison requires access to the bits of $x \bmod q$, which introduces $\log q$ arithmetic constraints. Using a greedy approach that only performs one reduction modulo q per component of $(c_1, c_2) = (a \cdot r + e_1, b \cdot r + e_2 + \Delta \cdot m)$, the remainder checking technique of [79] would require $2d \cdot \log q = 131072$ constraints, thus leading to an arithmetic circuit with more than 250000 multiplication gates. To improve this, we can instead interpret the components of $(c_1, c_2) \in R_q^2$ as remainders of the long division and prove the smallness of its quotients, which is also what our construction is doing. Since each quotient has magnitude $\approx d/2$, this decreases the number of constraints from $2d \cdot \log q$ to $2d \cdot \log d$ when it comes to proving correct reductions modulo q . Overall, we estimate that the entire process would still cost $n_m = 2d(1 + \log B) + 4d + 5d \cdot \log d = 145408$ arithmetic constraints to prove the global statement.

While the number $n_m = 145408$ of multiplication gates might seem only slightly larger than our vector dimension $n = 112640$, it has a significant impact. In the SNARK, the prover has to compute $n_m = 145408$ exponentiations in \mathbb{G} (with possibly large exponents over \mathbb{Z}_p) besides $3n_m + (n_w - \ell_s) \approx 806913$ \mathbb{G} -

¹⁹ This allows computing $\text{FFT}(a \cdot r)$ and $\text{FFT}(b \cdot r)$ using $2d$ multiplications.

exponentiations, where $n_w \approx 374785$ is the number of wires²⁰ and $\ell_s = 4d = 4096$ is the number of field elements describing the statement. In comparison, we only need 337920 exponentiations in \mathbb{G} and only 410 exponentiations in $\hat{\mathbb{G}}$. If we count each exponentiation over $\hat{\mathbb{G}}$ as 3 exponentiations in \mathbb{G} , the SNARK of [69] computes about 1243137 equivalent \mathbb{G} -exponentiations. On the other hand, our verification algorithm is more demanding and computes n exponentiations in both groups $\hat{\mathbb{G}}$ and \mathbb{G} when the SNARK only needs ℓ_s exponentiations in \mathbb{G} .

As far as the CRS size goes, the SNARK approach would cost $n_m \approx 145408$ elements of $\hat{\mathbb{G}}$ and $2n_m + (n_w - \ell_s) \approx 661505$ elements of \mathbb{G} . Using KSS18 curves with a 275-bit group order, it would take about 50116 KB. On the other hand, the verifier only needs to store a small part of the CRS in Groth's SNARK.

²⁰ The number of wires is $n_w = n_m + n_{in} + n_{out}$, where $n_{in} = 4d + 2d + 2d(\log B + 1) + 2d \log d$ is the number of input wires and $n_{out} = 1 + 2d + 2d(\log B + 1) + 2d \log d$ is the number of output wires (where we count one output wire per bit-proving constraint).