Noah's Ark: Efficient Threshold-FHE Using Noise Flooding

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Abstract. We outline a secure and efficient methodology to do threshold distributed decryption for LWE based Fully Homomorphic Encryption schemes. The only "difficult" case being that of TFHE (due to the small parameters used in this scheme). We show that the standard technique of "noise flooding" can also be used with schemes with small parameters, by utilizing a switch to a scheme with slightly higher parameters and then utilizing the efficient bootstrapping operations which TFHE offers. Our protocol is proved secure via a simulation argument, making it's integration in bigger protocols easier to manage.

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1 Introduction

The problem of threshold decryption for Fully Homomorphic Encryption (FHE) schemes, called threshold-FHE from henceforth, is as old as FHE itself. The problem is for a set of n parties to have

a secret sharing of the underlying FHE secret key so that they can between them decrypt a given FHE ciphertext correctly, in the case where at most t of the parties are corrupt. Indeed, Gentry's original thesis [Gen09] mentioned threshold-FHE as a way of utilizing FHE to perform a very low round complexity semi-honest MPC protocol.

To understand the technical problem with threshold-FHE it is worth considering the "format" of a simple FHE scheme. To explain we utilize the format of BFV/TFHE [FV12, CGGI16, CGGI20] ciphertexts, but a similar discussion can be provided for other FHE schemes such as BGV [BGV12]. Consider encrypting an element $m \in \mathbb{Z}_p$, using a standard Learning-With-Errors (LWE) ciphertext of the form $(\mathbf{a}, b)$ with ciphertext modulus q , where $\mathbf{a} \in \mathbb{Z}_q^\ell$ and $b \in \mathbb{Z}_q$, using the equation

$$b = \mathbf{a} \cdot \mathbf{s} + e + \Delta \cdot m \pmod{q}$$

where $\Delta = \lfloor q/p \rfloor$, e is some "noise" term and $\mathbf{s} \in \mathbb{Z}_q^\ell$ is the secret key. Usually, in the FHE setting, $\mathbf{s}$ is chosen to be a vector small norm, for example $\mathbf{s} \in \{0,1\}^\ell$.

To enable threshold-FHE we first secret share the secret key $\mathbf{s}$ among n parties, a process which we shall denote by $[\mathbf{s}]^{\langle t,q \rangle}$ to signal a sharing modulo q with respect to a threshold $t < n$ linear secret sharing scheme. On input of the ciphertext $(\mathbf{a}, b)$ we can then produce trivially a secret sharing of the value $e + \Delta \cdot m$ by computing

$$[t]^{\langle t,q \rangle} = b - \mathbf{a} \cdot [\mathbf{s}]^{\langle t,q \rangle} = [e + \Delta \cdot m]^{\langle t,q \rangle}.$$

By opening the value of $[t]^{\langle t,q \rangle}$ all parties can then perform rounding to obtain m. However, this reveals the value of e, which combined with the ciphertext and the message, will reveal information about the secret key $\mathbf{s}$.

The way around this is to add some additional noise into the secret sharing before the opening. Thus the decrypting parties somehow generate an additional secret shared noise term $[E]^{\langle t,q \rangle}$, and the value which is opened is now

$$[t]^{\langle t,q \rangle} = b - \mathbf{a} \cdot [\mathbf{s}]^{\langle t,q \rangle} + [E]^{\langle t,q \rangle} = [e + E + \Delta \cdot m]^{\langle t,q \rangle}.$$

The key concern is then that E should introduce enough randomness to mask the e value after the shared value $[t]^{\langle t,q \rangle}$ is opened. If E is too small then too much information about e is revealed, if E is too big then the final rounding will not reveal the correct value of m. Diagrammatically we can consider this process as approximated by the diagram in Figure 1.

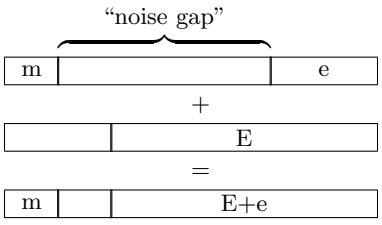

Fig. 1. Representation of the noise addition for threshold decryption

To mask, statistically, all information in e we would (naively) require E to be chosen uniformly from a range which is 2^{stat} larger than e. Thus if we can bound the ciphertext noise by $|e| < B$, then

we would require E to be chosen uniformly in the range $[-2^{\mathsf{stat}} \cdot B, \ldots, 2^{\mathsf{stat}} \cdot B]$. This process is often dubbed "noise flooding" in the literature. However, this would mean we require $\Delta > 2^{\mathsf{stat}} \cdot B$, which in turn means that the ciphertext modulus q needs to be "large".

We show that by adding an E term on, which is itself the addition of at least two uniform distributions in the range $[-2^{\mathsf{stat}} \cdot B, \ldots, 2^{\mathsf{stat}} \cdot B]$, we are able to obtain a statistical distance of actually $2^{-2 \cdot \mathsf{stat}}$. We can, hence, obtain enough security by selecting $\mathsf{stat} \approx 40$, and so reduce the need for very much larger q values.

In theory it may be possible to select E from a smaller range, and rely on some computational assumptions in order to argue about security. This approach is taken in two recent papers, [BS23] and [CSS+22], via the Renyi divergence. This methodology enables parameters to be chosen in which q is much smaller than the above analysis would require. As we discuss below this approach leads to additional problems in the larger protocols in which we embed our threshold decryption. Thus the use of Renyi divergence is not without problems in this situation.

Before proceeding we note that in many situations there is no problem with the increased size of q that the noise flooding approach requires. The key observation is that FHE enables the use of a bootstrapping operation. This operations purpose is to reduce the size of the noise e in the ciphertext $(\mathbf{a}, b)$ to be as small as possible. Thus if bootstrapping is performed, and the FHE scheme is such that the noise gap between e and m in Figure 1 is large enough, then the noise flooding methodology will work "out-of-the-box". Thus for BFV/BGV implementations which enable bootstrapping there is no problem to solve, as noise flooding is enough.

When using BFV/BGV in an SHE levelled mode then the problem also does not occur. In such schemes each level essentially adds an extra 14-24 bits (depending on the implementation) into the noise gap. Thus by simply increasing the number of levels by a small constant (say, two or three) one can obtain a noise gap which is enough to apply the flooding technique. Thus in such schemes our methodology in Section 5 can be applied, without any need for prior pre-processing.

Thus the only place where noise flooding is in practice a problem is when the FHE parameters are such that the noise gap is tiny, even after a bootstrapping operation is performed. This is exactly the situation in TFHE where one (usually) selects a relatively small q value (for example $q = 2^{64}$). This small q value, and associated small LWE dimension ℓ, requires the size of the noise even after bootstrapping to be around 2^{30} in order to ensure security. This means the noise gap is too small, but only by tens of bits. In this work we solve this problem for TFHE, by utilizing the fast bootstrapping enabled by TFHE. In some sense we protect the initial FHE ciphertext from the flooding operation, by placing the underlying message in a larger ciphertext (a kind of protective Noah's Ark).

1.1 Historical Discussion

At about the time of Gentry's thesis on FHE in 2009 [Gen09], the first threshold key generation and decryption for LWE based ciphertexts was given by Gendlin and Damgård [BD10]. Their methodology used replicated secret sharing to split the secret key, a method whose complexity scales with $\binom{n}{t}$. The simpler case of full-threshold, i.e. $t = n - 1$, decryption for LWE ciphertexts was combined with SHE and formed the basis of the SPDZ MPC protocol [DPSZ12]. This utilized the BGV encryption scheme, supporting circuits of multiplicative depth one, and used the noise flooding technique mentioned above.

The same techniques were then used in the context of FHE by Asharov et al [AJL+12] in the full threshold setting. A similar application and usage was given in [CLO+13], which considered

the threshold setting of $t < n/3$ via Shamir sharing. In our work we shall adopt the methodology of [CLO+13] for our main threshold decryption protocol.

A generic thresholdizer for arbitrary protocols was given by Boneh et al. in [BGG+18] using threshold-FHE. The construction of Boneh et al. utilizes a special form of secret sharing called $\{0,1\}$-LSSS, which is closely related to replicated sharing.

All of these prior works utilized noise flooding as a methodology. As remarked above this requires a super-polynomial gap between the bound on the noise term e and the ciphertext modulus q. Such super-polynomial blow-ups in other areas of cryptography based on LWE have recently been avoided by utilizing the Renyi divergence [BLR+18]. This, as an approach to threshold-FHE, was recently examined by [BS23] and [CSS+22].

The problem with using the Renyi divergence in the context is distributed decryption, is that the general technique of Renyi divergence is hard to apply to security problems which are inherently about distinguishing one distribution from another. In [CSS+22] and [BS23] a way around this was found by designing special security games for threshold-FHE usage, which enabled the use of the Renyi divergence. The problem is that these games need to cope with the homomorphic nature of the underlying encryption scheme, and thus cannot be adaptive. In the applications (such as to MPC) mentioned above we really require a threshold-FHE protocol which is indistinguishable, to an adversary, with a simulation interacting with an ideal functionality. The security games presented in [CSS+22] and [BS23] do not allow such a usage.

Thus we are led back to considering noise flooding. However, as detailed above, for FHE schemes such as BGV and BFV this is not a problem. The only issue comes with schemes such as TFHE, which utilize small parameters in order to achieve very fast bootstrapping operations. However, perhaps the very fast bootstrapping operation itself can be used to solve the problem?

1.2 Our Contribution

We present a simple method for threshold decryption for TFHE ciphertexts in the presence of $t < n/3$ actively (but statically) corrupted adversarial parties. Our methodology produces a threshold decryption functionality which is in the simulation paradigm, this makes it more amenable to being used as a black box in larger protocols than the game-based approaches based on Renyi divergence.

Our approach works for arbitrary prime power values of q, including the important case of $q = 2^{64}$. Adapting it to the case of non-prime power values of q is immediate via the Chinese-Remainder-Theorem. In doing so we utilize the (relatively standard) trick of applying Shamir secret sharing over Galois rings, thus we do not need to go via a replicated style secret sharing. In Shamir secret sharing the share sizes do not grow exponentially with the value of $\binom{n}{t}$.

When $\binom{n}{t}$ is "small" we apply a trick, which first appeared in [CLO+13], to enable threshold-FHE using a modified Pseudo-Random Secret Sharing (PRSS). In such a situation our protocol is a simple one round protocol, which is robust and works over asynchronous networks when $t < n/3$. When $t < n/2$ we note that we obtain a non-robust protocol, but one which has active-with-abort security. The proof of security in [CLO+13] has a number of minor bugs/missing details in it, and is overly complex, thus we also re-prove the main threshold-FHE result from this paper. It turns out that using a PRSS for small values of $\binom{n}{t}$, automatically means we are adding a sum of at least two uniform distributions in the flooding term; and thus we can apply to our improved statistical distance analysis in this case.

When $\binom{n}{t}$ is large we require slightly more work. In particular we divide our threshold-FHE protocol into two phases, an online and an offline phase. In the offline phase a "generic" MPC

protocol is used to generate random shares of bits, which are used to produce two uniformly random noise flooding terms of the correct size. Thus again, we are able to apply our improved statistical distance analysis in this case. In the online phase we consume these random shares of bits to perform the threshold-FHE operation. The online phase is again robust and works over asynchronous networks, when $t < n/3$; and is only active-with-abort secure when $t < n/2$. The security properties of the offline phase are inherited from the underlying MPC protocol used to generate the shares of random bits; if the underlying MPC protocol is robust over asynchronous networks then so is the offline phase of our threshold-FHE protocol; if it only provides active-with-abort security over synchronous networks then they are the properties of our offline phase.

Our methodology for threshold-FHE follows in three conceptually simple steps:

1. We take the input ciphertext with LWE parameters (ℓ, q) and then transform this into a ciphertext with LWE parameters with slightly larger parameters (L, Q) which encrypts the same message, where Q is a prime power with $q|Q$. The need to increase q to Q is to make sure we have the *possibility* of having a large enough noise gap to enable noise flooding.
2. We perform a bootstrap operation to obtain a ciphertext with smaller noise. The parameters for this operation are chosen to enable the noise gap to be sufficiently large.
3. We apply the traditional noise flooding operation, followed by a robust opening procedure on the secret shared value.

In practice the value q will be 2^{64}, and we will only need to boost the modulus to a value of $Q = 2^{128}$ in order to have a sufficient noise gap to perform threshold decryption. With such a value of Q it turns out that TFHE bootstrapping is still efficient, and thus the entire threshold decryption process is efficient.

In the special case when $\binom{n}{t}$ is small (say less than 100) we in addition obtain a one-round, threshold decryption protocol which is robustly secure when $t < n/3$, and which assumes only asynchronous, as opposed to synchronous, networks.

2 Preliminaries

2.1 Notation

Our basic input ciphertexts will come with ciphertext modulus q, and plaintext modulus p. For the underlying bootstrapping keys for TFHE we will utilize a cyclotomic ring of two-power degree N. The ring we define as

$$\mathcal{R} = \mathbb{Z}[X]/(X^N + 1),$$

with the reduction modulo the ciphertext (resp. plaintext) modulus q (resp. p) being given by

$$\mathcal{R}_q = (\mathbb{Z}/q\mathbb{Z})[X]/(X^N + 1) \quad (\text{ resp. } \mathcal{R}_p = (\mathbb{Z}/p\mathbb{Z})[X]/(X^N + 1) \ ).$$

We fix the global Δ as $\Delta = \lfloor q/p \rfloor$. This is the ratio between the ciphertext modulus q, and the *application* plaintext modulus p.

Elements in $\mathcal{R}$ (resp. $\mathcal{R}_q$, $\mathcal{R}_p$, etc) will be considered as vectors $\mathbf{A}$, $\mathbf{B}$, etc where we apply the component-wise addition operation. Multiplication, however, is performed with respect to the ring multiplication operation. Normal vectors, i.e. non-ring elements, will be written with lower case boldface, $\mathbf{a}$, $\mathbf{b}$, etc.

We let $\mathbf{a}[i]$ denote the i-th component of the vector $\mathbf{a}$, and $\mathbf{A}[i]$ denote the i-th coefficient of the ring element $\mathbf{A}$ when considered in the polynomial embedding. We assume the underlying ring is obvious from the context.

Multiplication of vectors $\mathbf{a} \cdot \mathbf{b}$ is assumed to be the normal dot-product, which results in a scalar value. We abuse notation by allowing $\mathbf{A} \leftarrow \mathbf{a}$ to denote a ring element is defined from a vector $\mathbf{a}$ of the same size. Thus, if $\mathbf{a} = (a_0, \ldots, a_{N-1})$ then we have

$$\mathbf{A} = a_0 + a_1 \cdot X + \cdots + a_{N-1} \cdot X^{N-1}.$$

2.2 Statistical Distance

Let $\Delta_{SD}(D_1, D_2)$ denote the standard statistical distance between two distributions D_1 and D_2 which are defined over a common domain X, i.e.

$$\Delta_{SD}(D_1, D_2) = \frac{1}{2} \sum_{x \in X} |D_1(x) - D_2(x)|.$$

Security for our threshold-FHE protocol when $\binom{n}{t}$ is small will rely on the following Lemma's, all of which are variants of the standard Smudging Lemma (see for example Lemma 2.1 of [AJW11])

Lemma 2.1 (Standard Smudging Lemma). *Let $e \in \mathbb{Z}$ and $B, m \in \mathbb{N}$ denote fixed integers, then we have*

$$\Delta_{SD}\left((e + U(-B, B))^m \ , \ U(-B, B)^m\right) \leq \frac{m \cdot |e|}{B},$$

where $U(-B, B)$ is the uniform distribution on the integer interval $(-B, \ldots, B]$.

From the data processing inequality, which says that the statistical distance between two distributions cannot increase by applying any (possibly randomized) function to them, one can immediately deduce

Lemma 2.2. *Let $e \in \mathbb{Z}$ and $B, m, v \in \mathbb{N}$ denote fixed integers, then we have*

$$\Delta_{SD}\left((e + \sum_{i=1}^{v} U(-B, B))^m \ , \ \sum_{i=1}^{v} U(-B, B)^m\right) \leq \frac{m \cdot |e|}{B},$$

where $U(-B, B)$ is the uniform distribution on the integer interval $(-B, \ldots, B]$.

However, a more accurate estimation, when $v \geq 2$, can be given by

Lemma 2.3. *Let $e \in \mathbb{Z}$ and $B, m, v \in \mathbb{N}$ denote fixed integers with $v \geq 2$, then we have,*

$$\Delta_{SD}\left((e + \sum_{i=1}^{v} U(-B, B))^m \ , \ \sum_{i=1}^{v} U(-B, B)^m\right) \leq \frac{m \cdot |e|}{B^2} + \sqrt{m \cdot \frac{|e|^2 \cdot \log B + 2}{2 \cdot (B^2 + B)}},$$

where $U(-B, B)$ is the uniform distribution on the integer interval $(-B, \ldots, B]$.

which follows, via the data processing inequality, from the following Lemma, whose proof is given in Appendix A,

Lemma 2.4. *Let $e \in \mathbb{Z}$ and $B, m \in \mathbb{N}$ denote fixed integers, and let $\mathcal{P} = U(-B, B) + U(-B, B)$. Then*

$$\Delta_{SD}(\mathcal{P}^m, (e + \mathcal{P})^m) \leq \frac{m \cdot |e|}{B^2} + \sqrt{m \cdot \frac{|e|^2 \cdot \log B + 2}{2 \cdot (B^2 + B)}}.$$

In our application we always utilize $v \geq 2$, in which case we apply Lemma 2.3. When we apply this for m distributed decryption queries we are actually sampling a different value of e per query. On each application, the specific e value used is the output noise term from a bootstrapping operation for a given input ciphertext. Thus the above distances are simplified, upper bounds in our application scenario of the actual statistical distances between the various distributions we analyze.

In our application we will set $B = 2^{\mathsf{stat}} \cdot |e|$, where $\mathsf{stat} = 40$, since Lemma 2.3 tells us that distinguishing the two distributions (for fixed e) requires around

$$\begin{aligned} \frac{B^2}{|e|^2 \cdot \log B} &= \frac{2^{2 \cdot \mathsf{stat}} \cdot |e|^2}{|e|^2 \cdot (\mathsf{stat} + \log |e|)} \\ &= \frac{2^{2 \cdot \mathsf{stat}}}{\mathsf{stat} + \log |e|} \approx 2^{2 \cdot \mathsf{stat}} \end{aligned}$$

samples.

2.3 Learning-with-Errors (LWE)

The (decision) LWE problem is to distinguish between samples drawn from the two distributions

$$\begin{aligned} D_1 &= \{ \ (\mathbf{a}, b) \ : \ \mathbf{a} \leftarrow \mathbb{Z}_q^\ell, \ b \leftarrow \mathbb{Z}_q \ \}, \\ D_2 &= \{ \ (\mathbf{a}, b) \ : \ \mathbf{a} \leftarrow \mathbb{Z}_q^\ell, \ e \leftarrow \mathcal{D}, \ b = \mathbf{a} \cdot \mathbf{s} + e \ \}, \end{aligned}$$

where $\mathbf{s} \in \mathbb{Z}_q^\ell$ is a fixed (secret) value, and $\mathcal{D}$ is the LWE-error distribution. In practice $\mathcal{D}$ is usually a discrete form of the Gaussian distribution with "small" standard deviation. For appropriate values of the parameters (q, ℓ) the problem is believed to be hard.

The Ring-LWE problem we define as trying to distinguish the two distributions

$$\begin{aligned} D_1 &= \{ \ (A, B) \ : \ A, B \leftarrow \mathcal{R}_q \ \}, \\ D_2 &= \{ \ (A, B) \ : \ A \leftarrow \mathcal{R}_q, \ E \leftarrow \mathcal{D}_\mathcal{R}, \ B = A \cdot S + E \ \}, \end{aligned}$$

where $S \in \mathcal{R}_q$ is a fixed (secret) value, and $\mathcal{D}_\mathcal{R}$ is the Ring-LWE-error distribution on elements of $\mathcal{R}$. We can think of the Ring-LWE problem as being a special version of the LWE problem in which N LWE samples of dimension N are obtained on every iteration.

To enable easier selection of such parameters which are "secure" we approximate the required standard deviation for the distribution $\mathcal{D}$ for given values of q and ℓ, and a given security level. In this work we select a security level of 128, namely distinguishing D_1 from D_2 should require a work effort of 2^{128}. We represent this function of the standard deviation for 128-bit security, as a function of q and ℓ, as the function $\Sigma_{\mathsf{LWE}}(q, \ell)$. This function can be approximated by fitting curves to the output of the LWE-estimator [APS15]. One should strictly speaking give a separate approximation for each value of q, but it turns out for the two values of q which are important to

us, namely $q = 2^{64}$ and $q = 2^{128}$, the same approximation can be used. For $\ell \geq 450$ we have the approximation[3]

$$\begin{aligned} \alpha &= -0.026599462343105 27, \\ \beta &= 2.98154318414599, \\ \Sigma_{\mathsf{LWE}}(q, \ell) &= \max(q \cdot 2^{\alpha \cdot \ell + \beta}, 4). \end{aligned}$$

We insert a minimum standard deviation of four into the approximation function to avoid problems when ℓ is very, very large.

2.4 TFHE

Our basic input TFHE ciphertext will be of the form $(\mathbf{a}, b)$ where $\mathbf{a} \in \mathbb{Z}_q^\ell$ and $b \in \mathbb{Z}_q$ such that

$$b = \mathbf{a} \cdot \mathbf{s} + e + \Delta \cdot m$$

for the message $m \in \mathbb{Z}_p$ and a noise value e. We assume the plaintext space $p = 2^{\varrho+1}$, where ϱ is the number of bits of plaintext and we add one bit to enable efficient non-negacylic operations. For our purposes we will not require specific details of the operations on TFHE ciphertexts, however we will require a detailed understanding of the associated noise growths in each operation. For the reader interested in the specific algorithm details we refer to [CLOT21] (also [CGGI20] and [CJP21]) as well as the details of how the following noise formulae are derived.

The operations we will perform (*modulus switch*, *keyswitch* and *bootstrap*) may require additional encryptions of the secret keys with respect to different LWE-style encryption schemes. Thus we have to also keep track of the different types of ciphertexts which each operation is performed on. For our purposes we can focus on just the basic LWE ciphertexts as above, plus a so-called "flattened-GLWE" ciphertext, or F-GLWE, which one can think of as a normal LWE ciphertext but with dimension $w \cdot N$, for the ring-LWE dimension N used in the GLWE ciphertexts and w an associated parameter. GLWE ciphertexts are a generalization of the RLWE ciphertexts introduced above.

For simplicity in this paper we present the noise formulae only for the case of q a power of two. This is the main application area of our work; small changes are needed for other prime power values of q. Note that with q and p both a power of two we have that p exactly divides q, which is what makes the noise formulae slightly easier to describe. We note that all the following operations are deterministic in nature; thus every party executing these operations will produce the same output; this assumes that the parties execute the same FFT algorithms internally and are working on identical hardware. This requirement of operating the same FFT algorithm on identical hardware can be relaxed, see Section 5.7 of [BBB+22].

Modulus Switch This operation takes an LWE ciphertext, with ciphertext modulus q, and switches it to an LWE ciphertext with modulus $2 \cdot N$. We present the high-level view of this operation in Figure 2. This algorithm is never *explicitly* called by our algorithms, however it is

[3] The coefficients a and b were estimated with the commit made on January 5, 2023: https://github.com/malb/lattice-estimator/tree/f9f4b3c69d5be6df2c16243e8b1faa80703f020c

the first stage of bootstrapping and thus we do need to take into account the noise added by this operation in our analysis. This algorithm will be correct (with probability pr_{MS}) if we have that

$$c_{MS} \cdot \sqrt{\sigma^2 + \sigma^2_{MS}} < \frac{\Delta}{2} \tag{1}$$

where

$$\mathsf{pr}_{MS} = 1 - \mathsf{erfc}\left(\frac{c_{MS}}{\sqrt{2}}\right) = \mathsf{erf}\left(\frac{c_{MS}}{\sqrt{2}}\right)$$

and

$$\sigma^2_{MS} = \frac{q^2}{48 \cdot N^2} - \frac{1}{12} + \ell \cdot \left(\frac{q^2}{96 \cdot N^2} + \frac{1}{48}\right).$$

We will take $c_{MS} \approx 7.2$ is our analysis, leading to an error probability of $\mathsf{erfc}(7.2/\sqrt{2}) = 2^{-40}$.

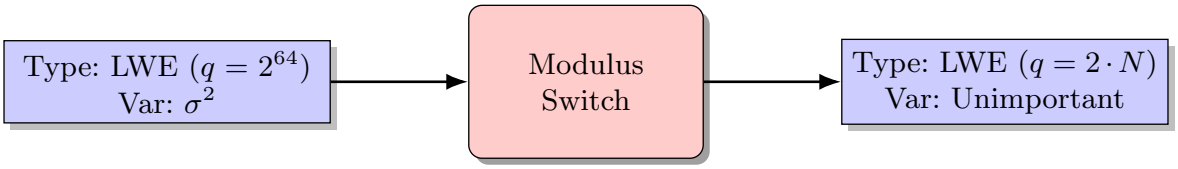

Fig. 2. Modulus Switch

Key Switch The KeySwitch operation takes a F-GLWE ciphertext and returns a normal LWE ciphertext. We require this operation as the bootstrapping operation below produces an F-GLWE ciphertext, and we need to translate it back to a standard LWE ciphertext for further processing by our algorithm. The high-level view of this algorithm is presented in Figure 3 where we have

$$\begin{aligned}
\sigma^2_{KS} &= w \cdot N \cdot \left(\frac{q^2}{12 \cdot \beta^{2 \cdot \nu_{ksk}}_{ksk}} - \frac{1}{12}\right) \cdot (\mathrm{Var}(s_i) + \mathbb{E}^2(s_i)) \\
&\quad + \frac{w \cdot N}{4} \cdot \mathrm{Var}(s_i) + w \cdot N \cdot \nu_{ksk} \cdot \sigma^2_{ksk} \cdot \left(\frac{\beta^2_{ksk} + 2}{12}\right) \\
&= \frac{w \cdot N}{2} \cdot \left(\frac{q^2}{12 \cdot \beta^{2 \cdot \nu_{ksk}}_{ksk}} - \frac{1}{12}\right) \\
&\quad + w \cdot N \cdot \left(\frac{1}{16} + \nu_{ksk} \cdot \sigma^2_{ksk} \cdot \left(\frac{\beta^2_{ksk} + 2}{12}\right)\right) \\
&= w \cdot N \cdot \left(\frac{q^2}{24 \cdot \beta^{2 \cdot \nu_{ksk}}_{ksk}} + \frac{1}{48} + \nu_{ksk} \cdot \sigma^2_{ksk} \cdot \left(\frac{\beta^2_{ksk} + 2}{12}\right)\right)
\end{aligned}$$

since for a binary secret key we have $Var[s_i] = 1/4$ and $\mathbb{E}[s_i] = 1/2$. The values ν_{ksk} and β_{ksk} are parameters associated with the key-switching keys, in particular how the decomposition gadget is formed. The value σ_{ksk} is the standard deviation used to generate the noise term in the key-switching keys. The latter is selected such that an LWE problem with dimension ℓ, modulus q and standard deviation for the noise term σ_{ksk} is hard to solve, i.e. $\sigma_{ksk} = \Sigma_{\mathsf{LWE}}(q, \ell)$.

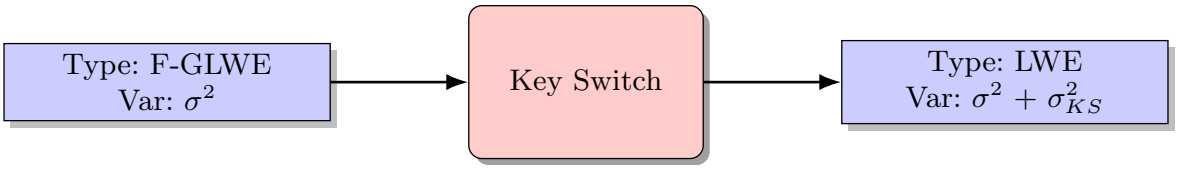

Fig. 3. Key Switch

Bootstrap Bootstrapping takes an LWE ciphertext and outputs a F-GLWE ciphertext but with (potentially) smaller noise, see Figure 4 for a high-level overview. The first thing a bootstrap operation performs is a modulus switch, therefore the input to the bootstrap operation (for it to be correct with a given probability) must satisfy equation (1). The specific details of how a bootstrap is performed is outside the scope of this paper, here we just describe its behavior. The noise output from bootstrap has variance σ^2_{BR} where

$$\sigma^2_{BR} = \ell \cdot \Bigg(\nu_{bk} \cdot (w+1) \cdot N \cdot \left(\frac{\beta_{bk}^2 + 2}{12}\right) \cdot \sigma^2_{bk} + \left(\frac{q^2 - \beta_{bk}^{2\cdot\nu_{bk}}}{24 \cdot \beta_{bk}^{2\cdot\nu_{bk}}}\right) \cdot \left(1 + \frac{w \cdot N}{2}\right) + \frac{w \cdot N}{32} + \frac{1}{16} \cdot \left(1 - \frac{w \cdot N}{2}\right)^2 \Bigg)$$

Again, the values ν_{bk} and β_{bk} are parameters associated with the decomposition gadget associated to the bootstrapping keys, and the value σ_{bk} is the standard deviation used to generate the noise term in the bootstrapping keys. The latter is selected such that an LWE problem with dimension $w \cdot N$, modulus q and standard deviation for the noise term σ_{bk} is hard to solve, $\sigma_{bk} = \Sigma_{\mathsf{LWE}}(q, w \cdot N)$.

Fig. 4. Bootstrap

Refresh Refresh is the key operation behind our method for threshold-FHE. It is the combination of bootstrap and keyswitch, see Figure 5. We shall refer to this operation by the notation $(\mathbf{a}, b) \leftarrow \mathsf{Refresh}((\mathbf{a}', b'), \mathfrak{pk})$, where $\mathfrak{pk}$ is the public key. As such the operation will be correct (with a given probability) only if the input noise satisfies equation (1).

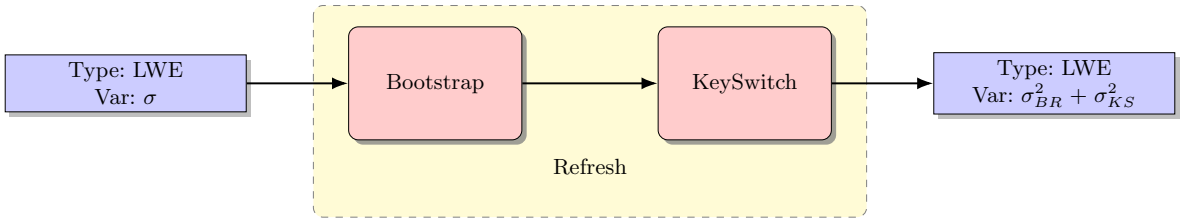

Fig. 5. The Refresh Algorithm

2.5 Secret Sharing

We want to utilize basic Shamir secret sharing over the ring $\mathbb{Z}_Q$ for the larger ciphertext modulus Q. For ease of exposition we will assume that Q is a prime power, with the most challenging case being $Q = 2^K$. To cope with Q being a power of two we need to use Shamir sharing over Galois rings.

Galois Ring Structures The use of Galois rings for Shamir sharing has a long history, going back to (at least) Serge Fehr's masters thesis [Feh93]. For more modern usage see [ACD+19, JSL22]. We first need to fix a Galois ring extension, and write $Q = \mathfrak{p}^K$, where in our case of interest $\mathfrak{p} = 2$. We then define

$$d = \lceil \log_{\mathfrak{p}}(n+1) \rceil$$

this means that the finite field $\mathbb{F}_{\mathfrak{p}^d}$ contains at least $n+1$ values. Fix a defining polynomial $F(Y)$, of degree d, for this finite field

$$\mathbb{K} = \mathbb{F}_{\mathfrak{p}^d} = \mathbb{F}_{\mathfrak{p}}[Y]/F(Y).$$

Elements in $\mathbb{F}_{\mathfrak{p}^d}$ will be represented by polynomials of degree less than d in a formal root θ of $F(Y)$, i.e. we write $\gamma = c_0 + c_1 \cdot \theta + \cdots + c_{d-1} \cdot \theta^{d-1} \in \mathbb{F}_{\mathfrak{p}^d}$ with $c_i \in \mathbb{F}_{\mathfrak{p}}$. We shall use *the same* polynomial to define the Galois ring extension

$$\mathcal{G} = \mathbb{Z}_Q[\theta] = \mathbb{Z}_Q[Y]/F(Y).$$

We assume that $\mathbb{F}_{\mathfrak{p}^d}$ is embedded into $\mathcal{G}$ in the obvious way, and so can freely talk about elements in $\mathbb{F}_{\mathfrak{p}^d}$ as if they are also in $\mathcal{G}$. Note, in the case where $Q = \mathfrak{p}$ (i.e. Q is a "large" prime) we have that $\mathcal{G} = \mathbb{F}_Q$.

We enumerate the non-zero elements in $\mathbb{F}_{\mathfrak{p}^d}$ as $\{\gamma_1, \ldots, \gamma_{\mathfrak{p}^d-1}\}$, and so for every player $\mathcal{P}_i$ we can refer to "their" element γ_i. Note, in the case where $Q = \mathfrak{p}$ we have $\gamma_i = i$ for $i \in [1, \ldots, n]$. Note that when thinking of the γ_i as elements of $\mathcal{G}$ we have that $\gamma_i - \gamma_j$ is invertible for every distinct pair (i, j). This allows us to define the following polynomials in $\mathcal{G}[X]$, for $i \in \{1, \ldots, n\}$. Note that

1. $\delta_i(\gamma_i) = 1$.
2. $\delta_i(\gamma_j) = 0$, if $i \neq j$.
3. $\deg \delta_i(X) = n - 1$.

More generally, $\delta_i(X)$ can be defined for any subset of at least $t+1$ players.

Shamir Sharing over $\mathbb{Z}_Q$ We now define a secret sharing scheme for elements $a \in \mathbb{Z}_Q$, given in Figure 6, which has threshold t out of n players. This means that the scheme perfectly hides a value if at most t parties combine their share, however if $t+1$ parties come together then the share value can be perfectly reconstructed (if no party deliberately introduces an error into their share value). We write $[a]^{\langle t,Q\rangle}$ to denote that a value $a \in \mathbb{Z}_Q$ is secret shared according to the sharing, and we write $[a]_i^{\langle t,Q\rangle} \in \mathcal{G}$ to denote player $\mathcal{P}_i$'s share. Note, that our sharing can also share elements in $\mathcal{G}$ and not just elements in $\mathbb{Z}_Q$, in which case upon opening such an element the opening procedure will abort.

The Secret Sharing Scheme $[x]^{\langle t,Q\rangle}$

$\mathsf{Share}(a)$: Given $a \in \mathbb{Z}_Q$ this produces a sharing, i.e. values $[a]_i^{\langle t,Q\rangle} \in \mathcal{G}$

1. Generate a polynomial $g_a(X) \in \mathcal{G}[X]$ of degree at most t such that $g_a(0) = a$.
2. Define $[a]_i^{\langle t,Q\rangle} = g_a(\gamma_i)$.

$\mathsf{Open}([a]_1^{\langle t,Q\rangle}, \ldots, [a]_n^{\langle t,Q\rangle})$:

1. Compute the polynomial
$$g_a(X) \leftarrow \sum_i [a]_i^{\langle t,Q\rangle} \cdot \delta_i(X).$$
2. If $\deg g_a(X) > t$ then abort.
3. If $g_a(0) \notin \mathbb{Z}_Q$ then abort.
4. Return $g_a(0)$.

Fig. 6. The Secret Sharing Scheme $[x]^{\langle t,Q\rangle}$.

Notice, that the opening algorithm, given in Figure 6, will abort if any of the n-parties send in a share value which is inconsistent. In addition it will abort if the shared value is not in $\mathbb{Z}_Q$, but in $\mathcal{G} \setminus \mathbb{Z}_Q$.

The secret sharing scheme is linear, namely given secret sharings $[a]^{\langle t,Q\rangle}$ and $[b]^{\langle t,Q\rangle}$ we can produce a secret sharing of the value $\alpha \cdot a + \beta \cdot b + \gamma$ for any values $\alpha, \beta, \gamma \in \mathbb{Z}_Q$ with no interaction. This is done by each party $\mathcal{P}_i$ computing

$$[\alpha \cdot a + \beta \cdot b + \gamma]_i^{\langle t,Q\rangle} \leftarrow \alpha \cdot [a]_i^{\langle t,Q\rangle} + \beta \cdot [b]_i^{\langle t,Q\rangle} + \gamma.$$

We shall write this as global operation in the notation

$$[\alpha \cdot a + \beta \cdot b + \gamma]^{\langle t,Q\rangle} \leftarrow \alpha \cdot [a]^{\langle t,Q\rangle} + \beta \cdot [b]^{\langle t,Q\rangle} + \gamma.$$

Error Correction Over Galois Rings In this section we explain how to do Reed-Solomon error correction over the Galois ring $\mathcal{G}$ when $t < n/3$. The methodology is taken from [ACD+19, Figure 1].

The standard Berlekamp–Welch or Gao algorithms for error correcting Reed-Solomon codes over $\mathbb{K}$ take as input $(x_1, \ldots, x_n)$ where $x_i \in \mathbb{K}$. We denote this by $\mathsf{RS\text{-}Decode}_{\mathbb{K}}(x_1, \ldots, x_n)$. It is assumed on input that $x_i = f(\gamma_i)$, for all except at most t values, and for a polynomial $f \in \mathbb{F}_{\mathfrak{p}^d}[X]$ of degree at most t. The "error" values x_i can either be incorrect values x_i or the $\perp$ symbol. One could think of $\perp$ as zero, but sometimes in decoding algorithms it is faster to keep data around

which we know to be a definite error. The output of $\mathsf{RS\text{-}Decode}_{\mathbb{K}}(x_1, \ldots, x_n)$ is the polynomial $f(X)$.

The Berlekamp–Welch and Gao algorithms can take an additional parameter r which specifies the maximum expected number of errors, with the algorithm returning $\perp$ if more than r errors are detected. In this context we write $\mathsf{RS\text{-}Decode}^{r}_{\mathbb{K}}(x_1, \ldots, x_n)$, with $r = \perp$ denoting the usual operation of no assumption on the errors.

In our Galois ring we have a similar decoding problem but now we have $x_i = f(\gamma_i)$, where f is a polynomial in $\mathcal{G}[X]$ of degree at most t, and the γ_i have been (trivially) lifted from $\mathbb{F}_{\mathfrak{p}^d}$ to $\mathcal{G}$. Note that every element $\alpha \in \mathcal{G}$ can be written as

$$\alpha = a_0 + a_1 \cdot \mathfrak{p} + \cdots + a_{K-1} \cdot \mathfrak{p}^{K-1}$$

where $a_i \in \mathbb{F}_{\mathfrak{p}^d}$. We will write the polynomial f in a similar manner as

$$f(X) = f_0(X) + f_1(X) \cdot \mathfrak{p} + \cdots + f_{K-1}(X) \cdot \mathfrak{p}^{K-1}$$

and we will recover the f_i values recursively using the standard algorithm $\mathsf{RS\text{-}Decode}_{\mathbb{K}}(x_1, \ldots, x_n)$ as a subroutine. This is explained in Figure 7, where $\pi : \mathcal{G} \longrightarrow \mathbb{K}$ denotes the reduction modulo p map. We adopt the convention that passing $\perp$ into π then the output is also $\perp$, and that any arithmetic operation on $\perp$ results in $\perp$.

Decoding in Galois Rings

$\mathsf{RS\text{-}Decode}^{r}_{\mathcal{G}}(x_1, \ldots, x_n)$.

1. $\mathbf{y} \leftarrow \mathbf{x}$.
2. For $i = 0, \ldots, K-1$ do
   (a) $\mathbf{z} \leftarrow \pi(\mathbf{y}/\mathfrak{p}^i)$.
   (b) $f_i(X) \leftarrow \mathsf{RS\text{-}Decode}^{e}_{\mathbb{K}}(\mathbf{z})$.
   (c) If $f_i(X) = \perp$ then return $\perp$. In this case either there are more than t errors, if $r = \perp$, or there are more than r errors, if $r \neq \perp$.
   (d) For $j = 1, \ldots, n$ set $t_j \leftarrow \sum_{l=0}^{i} f_l(\gamma_j) \cdot \mathfrak{p}^l \in \mathcal{G}$.
   (e) $\mathbf{y} \leftarrow \mathbf{x} - \mathbf{t}$.
   (f) If y_j is not divisible by $\mathfrak{p}^{i+1}$ then $y_j \leftarrow \perp$.
3. Output $\sum_{l=0}^{i} f_l(X) \cdot \mathfrak{p}^l \in \mathcal{G}[X]$.

Fig. 7. Reed-Solomon error decoding in Galois rings

Robust Opening We can now define an opening procedure called $\mathsf{RobustOpen}$ in Figure 8, which will robustly open the shared value, depending on the relationship between t and n, and the underlying network properties. We present the robust opening protocol for Shamir sharing of arbitrary degree d. When $d = t$ robust opening is only available when $t < n/3$. Note, that for asynchronous networks and $t = d < n/4$ we can execute less computational steps than for the case $t = d < n/3$ by simply waiting for more data to arrive. The method for asynchronous networks and $t = d < n/3$ is called "online error correction", and was first presented in [BCG93].

Assuming the input sharing is of an element in $\mathbb{Z}_Q$ then robust open protocol in Figure 8 will output the value in $\mathbb{Z}_Q$, even if adversarial parties introduce errors. This is despite the shares themselves, and the Lagrange interpolation coefficients, being defined by elements in $\mathcal{G}$.

RobustOpen

This protocol depends on the ratio between d, t, and n, and whether the underlying network is synchronous or asynchronous. It is run either by a player $\mathcal{P} = \mathcal{P}_i$ who already holds their share $[a]_i^{\langle d,Q\rangle}$ (we call this Case A), or by an external player $\mathcal{P}$ (which we call Case B).

$\mathsf{RobustOpen}(\mathcal{P}, [a]^{\langle d,Q\rangle})$:

1. Player $\mathcal{P}_i$ sends $[a]_i^{\langle d,Q\rangle}$ securely to player $\mathcal{P}$.
2. If $d + 3 \cdot t < n$ and the network is asynchronous, or $d + 2 \cdot t < n$ and the network is synchronous
   (a) $\mathcal{P}$ waits until they have received $d + 2 \cdot t$ (case A) or $d + 2 \cdot t + 1$ (case B) share values $\{[a]_j^{\langle d,Q\rangle}\}_j$.
   (b) Apply the Reed-Solomon decoding algorithm, $\mathsf{RS\text{-}Decode}_{\mathcal{G}}^{t}(\cdots)$, to the $d + 2 \cdot t + 1$ shares they hold to robustly compute $F(X)$,
   (c) Return $a = F(0)$.
3. If $d + 2 \cdot t < n$ and the network is asynchronous
   (a) For $r = 0, \ldots, t$ do
      i. $\mathcal{P}$ waits until $d + t + r$ (case A) / $d + t + r + 1$ (case B) shares have been received.
      ii. Apply the Reed–Solomon decoding algorithm, $\mathsf{RS\text{-}Decode}_{\mathcal{G}}^{r}(\cdots)$, on the $d + t + r + 1$ shares, assuming there are r errors in these shares.
      iii. If error correction outputs a degree d degree poly then, if there are at least $d+t+1$ shares (out of the $d + t + r + 1$ shares) which lie on the polynomial, then this is the correct polynomial so output the constant term and exit the loop. Note, this step requires just scanning the $d + t + r + 1$ shares and counting how may lie on the polynomial.

$\mathsf{RobustOpen}(\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}, [a]^{\langle d,Q\rangle})$: This is a short hand for all $i \in \{1, \ldots, n\}$ players executing $\mathsf{RobustOpen}(\mathcal{P}_i, [a]^{\langle d,Q\rangle})$ in parallel. Hence, all n players will obtain the value a.

Fig. 8. Robust Opening Protocol when $d + 2 \cdot t < n$

3 Switch-Up

The first step in our threshold decryption operation is to take an LWE ciphertext $(\mathbf{a}, b)$ defined with parameters (q, ℓ), with respect to a secret key $\mathbf{s} \in \{0,1\}^{\ell}$, and then switch it to a ciphertext $(\mathbf{a}', b')$ defined for parameters (Q, L) with $q|Q$, $L > \ell$, and for a secret key $\mathbf{s}' \in \{0,1\}^{L}$. Thus we increase both the ciphertext modulus and the LWE dimension. We perform this switch in two phases, the first is a modulus switch from q to Q, and the second is a switch of the dimension from ℓ to L (which is performed via a key switch). We denote the combined operation as $\mathfrak{ct}' \leftarrow \mathsf{SwitchUp}(\mathfrak{ct})$.

Modulus Switch Up Given a ciphertext $(\mathbf{a}, b = \mathbf{a} \cdot \mathbf{s} + e + \Delta \cdot m)$ modulo q, we can define a ciphertext encrypting the same message modulo Q (for $q|Q$) by setting

$$\mathbf{a}' = \frac{Q}{q} \cdot \mathbf{a} \pmod{Q} \text{ and } b' = \frac{Q}{q} \cdot b \pmod{Q}.$$

We then have that, for $\Delta' = Q/p$,

$$\begin{aligned} b' - \mathbf{a}' \cdot \mathbf{s} &= \frac{Q}{q} \cdot (b - \mathbf{a} \cdot \mathbf{s}) \pmod{Q} \\ &= \frac{Q}{q} \cdot e + \frac{Q}{q} \cdot \frac{q}{p} \cdot m \\ &= \frac{Q}{q} \cdot e + \Delta' \cdot m, \end{aligned}$$

$$= e' + \Delta' \cdot m.$$

Thus our noise has increased by a factor of Q/q. Note that if $e < \Delta/2$ then we have

$$e' = \frac{Q}{q} \cdot e < \frac{Q}{q} \cdot \Delta/2 = \Delta'/2.$$

Thus if the input ciphertext is decryptable modulo q, then the output ciphertext is also decryptable modulo Q. We express this diagramatically in Figure 9.

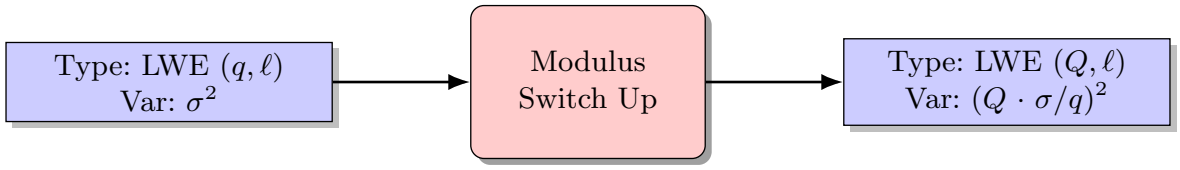

Fig. 9. Modulus Switch Up

Key Switch Up Our previous form of key switching took an F-GLWE ciphertext and produced an LWE ciphertext. Here we aim to take an LWE ciphertext with parameters (Q, ℓ) and produce a ciphertext with parameters (Q, L). The secret key switches from a value $\mathbf{s} \in \{0,1\}^{\ell}$ to one $\mathbf{s}' \in \{0,1\}^{L}$. This operation is virtually identical, in terms of how it works, to the algorithm we overviews in Section 2.4. However, it takes as input an LWE ciphertext as opposed to an F-GLWE ciphertext. This is described pictorially in Figure 10, with the associated noise formula being

$$\begin{aligned}
\sigma^2_{KS-Up} &= \ell \cdot \left(\frac{Q^2}{12 \cdot \beta^{2 \cdot \nu_{ksk-up}}_{ksk-up}} - \frac{1}{12} \right) \cdot (\mathrm{Var}(s_i) + \mathbb{E}^2(s_i)) \\
&\quad + \frac{\ell}{4} \cdot \mathrm{Var}(s_i) \\
&\quad + \ell \cdot \nu_{ksk-up} \cdot \sigma^2_{ksk-up} \cdot \left(\frac{\beta^2_{ksk-up} + 2}{12} \right) \\
&= \frac{\ell}{2} \cdot \left(\frac{Q^2}{12 \cdot \beta^{2 \cdot \nu_{ksk-up}}_{ksk-up}} - \frac{1}{12} \right) \\
&\quad + \ell \cdot \left(\frac{1}{16} + \nu_{ksk-up} \cdot \sigma^2_{ksk-up} \cdot \left(\frac{\beta^2_{ksk-up} + 2}{12} \right) \right) \\
&= \ell \cdot \left(\frac{Q^2}{24 \cdot \beta^{2 \cdot \nu_{ksk-up}}_{ksk-up}} + \frac{1}{48} + \right. \\
&\quad \left. + \nu_{ksk-up} \cdot \sigma^2_{ksk-up} \cdot \left(\frac{\beta^2_{ksk-up} + 2}{12} \right) \right)
\end{aligned}$$

since (again) for a binary secret key we have $Var[s_i] = 1/4$ and $\mathbb{E}[s_i] = 1/2$. Again ν_{ksk-up} and β_{ksk-up} are parameters associated with the associated decomposition gadget, and σ_{ksk-up} is chosen so that an LWE problem of dimension L and modulus Q is hard, i.e. $\sigma_{ksk-up} = \Sigma_{\mathsf{LWE}}(Q, L)$.

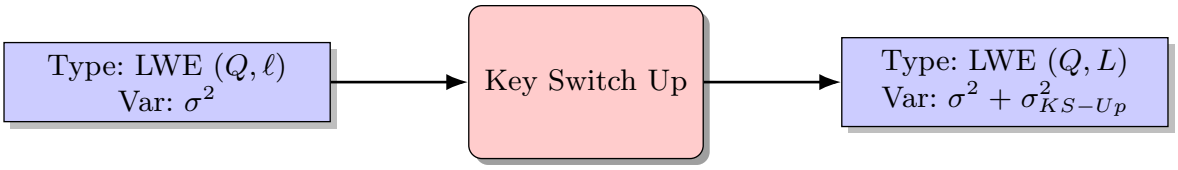

Fig. 10. Key Switch Up

4 Squashing the Noise

The reason for moving a ciphertext from parameter set (q, ℓ) to (Q, L) is to enable us to have a lot more room between the noise bound and the value of Δ'. In particular the noise-gap should be big enough to enable noise flooding for threshold decryption. But, after performing the switch up from the previous section we have a ciphertext $(\mathbf{a}', b')$ with relatively large noise. We thus, now, aim to squash the noise down to something for which there is a big gap between the maximum noise bound and the value $\Delta' = Q/p$. This gap has to be large enough to support threshold decryption via flooding. Thus we also need to select a bootstrapping key for the large parameter sets to enable this refresh operation.

If our input ciphertext with parameter set (q, ℓ) has noise variance σ^2, then, after the modulus switch and the key-switching operations, we obtain a ciphertext with parameter set (Q, L) with noise variance

$$\sigma'^2 = \left(\frac{Q}{q} \cdot \sigma\right)^2 + \sigma^2_{KS-Up}.$$

Our goal is now to define bootstrapping parameters which squash this noise σ' down to something we can then flood during threshold decryption.

The "bootstrapping" etc operations for the parameters (Q, L) are assumed to have associated standard deviations σ'_{MS}, σ'_{KS} and σ'_{BR}, which themselves are functions of L, Q, N', w', σ'_{ksk} etc as described earlier in the case of (q, l). Thus if we have

$$c_{MS} \cdot \sqrt{\sigma'^2 + {\sigma'_{MS}}^2} \leq \frac{\Delta'}{2}$$

then we can execute a refresh operation in order to produce a ciphertext with parameters (Q, L) with noise variance

$${\sigma'_{BR}}^2 + {\sigma'_{KS}}^2.$$

In the next section we will require the following equations to be satisfied, for some integer parameter pow. The parameter pow denote the extra factor of noise we will add during flooding, i.e. it is approximately $\log_2 |E/e|$. We make it slightly larger than stat (by an extra additive term of $\log_2 100$) in order to cope with a non-uniform value of E which will be used in our procedure when $\binom{n}{t}$ is small (see later for a further discussion of this case).

$$\begin{aligned} \mathsf{Bd} &= c_{Dec} \cdot \sqrt{{\sigma'_{BR}}^2 + {\sigma'_{KS}}^2}, \\ 2^{\mathsf{pow}} \cdot \mathsf{Bd} &\leq \frac{\Delta'}{2}, \\ \mathsf{pow} &\geq \mathsf{stat} + \log_2 100, \end{aligned}$$

where $c_{Dec} \approx 7.2$. Hence, combining these all together we have that

$$\mathsf{stat} + \log_2 100 \leq \mathsf{pow} \leq \log_2\left(\frac{\Delta'}{2}\right) - \log_2\left(c_{Dec} \cdot \sqrt{{\sigma'_{BR}}^2 + {\sigma'_{KS}}^2}\right).$$

In particular this means that we must have

$$\log_2\left(c_{Dec} \cdot \sqrt{{\sigma'_{BR}}^2 + {\sigma'_{KS}}^2}\right) \leq \log_2\left(\frac{\Delta'}{2}\right) - \mathsf{stat} - \log_2 100.$$

Given $\mathsf{stat} \approx 40$, we thus need to select parameters so that the noise after bootstrapping for these large parameters is at least stat bits smaller than the decryption correctness bound of $\Delta'/2$.

4.1 Searching For Parameters

For ϱ plaintext bits (with $\varrho = 1$ or $\varrho = 4$ being the two default values we use in our experiments) we select $p = 2^{\varrho+1}$, and so $\Delta' = Q/p$. We try to push the values of σ'_{KS} and σ'_{BR} to be around $2^d = \Delta'/2^{51}$. This is just a rule-of-thumb which seems to work well in practice.

Our first goal is to select L. The value of σ'_{KS} is certainly dependent on σ'_{ksk} so we first select σ'_{ksk} to be around 2^d. Thus we select L such that

$$2^{d-10} = \Sigma_{\mathsf{LWE}}(Q, L),$$

where $\Sigma_{\mathsf{LWE}}(Q, L)$ is the approximation of the curves from the LWE-estimator given earlier. i.e.

$$L = \left\lceil \frac{d - 10 - \log_2(Q) - \varrho}{a} \right\rceil.$$

The difference that looks like random value of ten here is to allow some gap to ensure σ_{KS-Up} can be made small enough (recall that σ^2_{ksk-up} is multiplied by a value dependent on ℓ, v_{ksk-up} and β_{ksk-up} to determine σ_{KS-Up}).

Our next goal is to select N'. The major affect N' has on our algorithms is via the σ'_{MS} value. The dominant term of ${\sigma'_{MS}}^2$ is given by

$$\frac{L \cdot Q^2}{96 \cdot N'^2}$$

and σ'_{MS} only affects our ability to execute the single bootstrap after we have pushed the ciphertext to parameter set (Q, L). Thus picking N' such that

$$\frac{L \cdot Q^2}{96 \cdot N'^2} > \frac{\Delta'^2}{2^8},$$

i.e.

$$N' \approx 2 \cdot \sqrt{L} \cdot \frac{Q}{\Delta'} > 2 \cdot p \cdot \sqrt{L}.$$

should be sufficient. Of course N' needs to be a power of two, and we also need to ensure security. Thus, in practice, one should probably pick the smallest power of two which is greater than L (since L provides enough security for the modulus Q already). So we select a power of two value of N' such that

$$N' > \max(L, 2 \cdot p \cdot \sqrt{L}).$$

We then select $w' = 1$, and set σ'_{bk} using our usual method of applying

$$\sigma'_{bk} = \Sigma_{\mathsf{LWE}}(Q, w' \cdot N').$$

We then search for the values β'_{ksk} and ν'_{ksk}. The value of ν'_{ksk} we try to minimize as much as possible, and β'_{ksk} we select to be a power of two. The goal is to select these two values so that

$$\sigma'_{KS} < 2^{d+1}.$$

A similar strategy can be applied to find β'_{bk} and ν'_{bk}, by finding values such that

$$\sigma'_{BR} < 2^{d+1}.$$

A summary of four potential parameter sets are given in Table 1. We give four sets of parameters; two for each plaintext size of $\varrho = 1$ and $\varrho = 4$, and for each plaintext size we give a variant with ℓ a non-power of two and ℓ a power of two. The former for use with the "traditional" methodology of giving out many encryptions of zero, and the latter for use with the more compact public key encryption methodology given in [Joy23]. We see that the standard deviation of the output noise after bootstrapping is around 2^{70}, which is gives us over 50 bits of noise gap for a ciphertext modulus of 2^{128}. Which is enough to fit in our flooding by a value of approximately $70 + 40 = 110$ bits. Note that for the input ciphertext, with parameters (q, ℓ), the noise gap is with overwhelming probability much smaller than 2^{50}, indeed it is less than 2^{10}.

Table 1. Parameters for switching up operations with the four sets of basic parameters.

	$\varrho = 1$	$\varrho = 4$	$\varrho = 1$	$\varrho = 4$
(q, ℓ)	$(2^{64}, 777)$	$(2^{64}, 870)$	$(2^{64}, 1024)$	$(2^{64}, 1024)$
(Q, L)	$(2^{128}, 2481)$	$(2^{128}, 2594)$	$(2^{128}, 2481)$	$(2^{128}, 2594)$
$\log_2 \sigma_{ksk-up}$	32	31	32	31
β_{ksk-up}	524288	1048576	524288	1048576
ν_{ksk-up}	3	3	3	3
$\log_2 \sigma_{KS-Up}$	74	71	74	71
pow	47	47	47	47
N'	4096	4096	4096	4096
w'	1	1	1	1
β'_{ksk}	524288	1048576	524288	1048576
ν'_{ksk}	3	3	3	3
β'_{bk}	2097152	4194304	2097152	4194304
ν'_{bk}	3	3	3	3
$\log_2 \sigma'_{ksk}$	22.0	22.0	22.0	22.0
$\log_2 \sigma'_{bk}$	22.0	22.0	22.0	22.0
$\log_2 \sigma'_{KS}$	74.7	71.7	74.7	71.7
$\log_2 \sigma'_{BR}$	73.8	70.9	73.8	70.9

5 Threshold Decryption Operation

After applying the methods from the previous sections we now have a ciphertext

$$(\mathbf{a}, b) = (\mathbf{a}, \mathbf{a} \cdot \mathbf{s}' + e + \Delta' \cdot m)$$

where $\mathbf{a} \in \mathbb{Z}_Q^L$, $\mathbf{s}' \in \{0,1\}^L$, the message m lies in $\mathbb{Z}_p$, and $\Delta' = Q/p$, and e is a noise term. The noise term is assumed to have variance ${\sigma'_{BR}}^2 + {\sigma'_{KS}}^2$, i.e. the LWE ciphertext instance is an output of the Refresh operation from the previous section. In what follows we shall assume $|e| \leq \mathsf{Bd}$, where we assume (with overwhelming probability) that

$$\mathsf{Bd} = c_{Dec} \cdot \sqrt{\sigma_{BR}^2 + \sigma_{KS}^2},$$

where $c_{Dec} \approx 7.2$. To fix ideas think of $Q = 2^{128}$ and the variance being of size roughly 2^{140}, and so $\mathsf{Bd} \approx 2^{70}$. Thus we have a noise gap of around 50 bits (assuming a plaintext space of at most 10 bits).

We assume the secret key $\mathbf{s}'$ has been secret shared with respect to our secret sharing scheme, i.e. we have a sharing $[\mathbf{s}']^{\langle t,Q\rangle}$. Formally we define the threshold decryption for the parameters (Q, L) via two ideal functionalities. The first $\mathcal{F}_{\mathsf{KeyGen}}$, in Figure 11, acts as a set-up assumption for our protocol, needed for the UC proof we provide. It generates a key pair, and secret shares the secret key among the players using the secret sharing scheme. One can realize this functionality using a generic MPC protocol, see Appendix D for an outline. Note, despite wanting active security we do not "complete" adversarially input shares into a complete sharing (as is often done in such situations), as the implementing actively protocol for $\mathcal{F}_{\mathsf{KeyGen}}$ does not actually need to do this.

$\mathcal{F}_{\mathsf{KeyGen}}$

$\mathsf{Init}()$:

1. Execute $(\mathfrak{pk}, \mathbf{s}') \leftarrow \mathsf{KeyGen}(1^\kappa)$ for the underlying TFHE encryption scheme with parameters (Q, L).
2. Generate a secret sharing $[\mathbf{s}']^{\langle t,Q\rangle}$ of the secret key.
3. Send $\mathfrak{pk}$ to all players (including the adversary), and send $[\mathbf{s}']_i^{\langle t,Q\rangle}$ to player $\mathcal{P}_i$ (including adversarially controlled players).

Fig. 11. The ideal functionality for distributed key generation

The key functionality we want to implement is $\mathcal{F}_{\mathsf{KeyGenDec}}$ given in Figure 12. Note, that this functionality always returns the correct result, irrespective of what the adversary does.

$\mathcal{F}_{\mathsf{KeyGenDec}}$

$\mathsf{Init}()$:

1. Execute $(\mathfrak{pk}, \mathbf{s}') \leftarrow \mathsf{KeyGen}(1^\kappa)$ for the underlying TFHE encryption scheme with parameters (Q, L).
2. Send $\mathfrak{pk}$ to all players, including the adversary and store the value $\mathbf{s}'$.

$\mathsf{DistDecrypt}(\mathfrak{ct}, \mathcal{U})$: For a ciphertext $\mathfrak{ct}$ with error e such that $|e| < \mathsf{Bd}$.

1. Compute $m \leftarrow \mathsf{Dec}(\mathfrak{ct}, \mathbf{s}')$.
2. If $\mathcal{U}$ is adversarially controlled then send $(\mathfrak{ct}, m)$ to the adversary.
3. Otherwise send m to player $\mathcal{U}$ and $\mathfrak{ct}$ to the adversary.

Fig. 12. The ideal functionality for distributed key generation and decryption

Our threshold decryption protocol comes in two flavours, one where $\binom{n}{t}$ is "small" and one where $\binom{n}{t}$ is "large"[4]. When $\binom{n}{t}$ is small our threshold decryption protocol requires only one round of interaction, whilst when $\binom{n}{t}$ is large the online phase of our threshold decryption still requires only one round, however there is a (slightly) complex, ciphertext independent, offline phase which needs to be completed first.

In both cases we assume $t < n/3$, as we wish to have a robust asynchronous threshold decryption protocol; at least in the online phase of our protocol. We also recall we require that the protocol's security should come via a simulation, as opposed to a game based argument. This is to enable composition of the threshold decryption protocol easily within other larger protocols.

5.1 Threshold Decryption for "Small" $\binom{n}{t}$

We start with the case of $\binom{n}{t}$ being small, where we can utilize a variant of the standard Pseudo-Random Secret Sharing (PRSS), originally introduced in [CDI05]. The problem is that the complexity of a PRSS depends on $\binom{n}{t}$, which can become exponentially big as n increases. Thus this method can only be used when $\binom{n}{t}$ is small. We use a slightly modified form of PRSS in that we do not output sharings of uniformly random values from $\mathbb{Z}_Q$, but from a different range. This form of PRSS was originally used in [CLO+13], for exactly the purpose of threshold-FHE.

The algorithms for a non-interactive PRSS are defined in Figure 13. The algorithm $\mathsf{PRSS.Init}()$ iterates over all sets A of size $n-t$. Thus the complexity of $\mathsf{PRSS}, \mathsf{Init}()$, i.e. the number of sets A we need to deal with, depends on $\binom{n}{t}$, which can become very large for large n and t.

The PRSS makes use of a PRF ψ of the form

$$\psi : \begin{cases} \{0,1\}^{\mathsf{sec}} \times S \longrightarrow & \mathbb{Z} \\ (\kappa, \mathsf{cnt}) \longmapsto & \psi(\kappa, \mathsf{cnt}) \end{cases}$$

where $\{0,1\}^{\mathsf{sec}}$ is the keyspace and S is a set of counters. The output of the function ψ is assumed to be bounded in absolute value by

$$\mathsf{Bd}_1 = \frac{(2^{\mathsf{pow}} - 1) \cdot \mathsf{Bd}}{\binom{n}{t}},$$

recall that pow is roughly speaking $\log_2 |E/e|$.

One can implement ψ using AES in an obvious counter mode, e.g. as $\log_2 \mathsf{Bd}_1 < 256$ one can set

$$\psi(\kappa, \mathsf{cnt}) = \left(\mathsf{AES}_\kappa(0 \| \mathsf{cnt}) + 2^{128} \cdot \mathsf{AES}_\kappa(1 \| \mathsf{cnt})\right) \pmod{\mathsf{Bd}_1},$$

where we treat the output block of the AES cipher as an integer in $[0, \ldots, 2^{128} - 1]$. Note, the output of ψ is only statistically uniform in the required range here if $\log_2 \mathsf{Bd}_1 < 256 - \mathsf{stat}$, which will be true in our usage. Since the output of ψ is bounded as above, we have that the value E is bound by $(2^{\mathsf{pow}} - 1) \cdot \mathsf{Bd}$, as the sum used in the PRSS has at most $\binom{n}{t}$ terms. Note, the shared value which is output by the PRSS invocation is the sharing of the value

$$E \leftarrow \sum_A \psi(r_A, \mathsf{cnt}).$$

[4] Think of the small/large regime being divided at a value such as 100

PRSS

PRSS.Init(): For every set $A \subseteq \{1, \ldots, n\}$ of size $n - t$:

1. $S \leftarrow \{\mathcal{P}_i\}_{i \in A}$.
2. Players $\mathcal{P}_i$ with $i \in A$ execute $r_A \leftarrow \mathsf{AgreeRandom}(S, \mathsf{sec})$, from Appendix B.
3. Define $f_A(X) \in \mathbb{Z}_q[X] = \mathbb{Z}_{2^k}[X]$ to be the polynomial of degree t such that $f_A(0) = 1$ and $f_A(\gamma_i) = 0$ for all $i \notin A$. Each party $\mathcal{P}_i$ only needs store $f_A(\gamma_i)$ though.
4. $\mathsf{cnt}_{\mathsf{PRSS}} \leftarrow 0$.

PRSS.Next():

1. Party $\mathcal{P}_i$ computes, where the sum is over every set A containing i,

$$[E]_i^{\langle t,Q \rangle} \leftarrow \sum_{A: i \in A} \psi(r_A, \mathsf{cnt}_{\mathsf{PRSS}}) \cdot f_A(\gamma_i).$$

2. $\mathsf{cnt}_{\mathsf{PRSS}} \leftarrow \mathsf{cnt}_{\mathsf{PRSS}} + 1$.
3. Return $[E]^{\langle t,Q \rangle}$.

Fig. 13. Pseudo-Random Secret Sharing PRSS

Given this PRSS we can define our threshold decryption protocol, which we give in Figure 14, where we assume a dedicated player $\mathcal{U}$ (possibly not one of the threshold decryption parties) will receive the final output. If all threshold decryption parties are to receive the output of the threshold decryption, or the output is to be public and not just to player $\mathcal{U}$, then the communication in step 3 does not need to be done securely.

Threshold Decryption - Protocol 1

Init():

1. The parties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ execute PRSS.Init().
2. The parties obtain $[\mathbf{s}']^{\langle t,Q \rangle}$ via a threshold key generation protocol, see Appendix D.

DistDecrypt($\mathfrak{ct}, [\mathbf{s}']^{\langle t,Q \rangle}, \mathcal{U}$): On input of $\mathfrak{ct} = (\mathbf{a}, b) \in \mathbb{Z}_Q^{L+1}$ this executes the following steps:

1. The parties $\mathcal{P}_i$ execute $[E]^{\langle t,Q \rangle} \leftarrow$ PRSS.Next().
2. The parties $\mathcal{P}_i$ compute $[v]^{\langle t,Q \rangle} \leftarrow b - \mathbf{a} \cdot [\mathbf{s}']^{\langle t,Q \rangle} + [E]^{\langle t,Q \rangle}$.
3. Party $\mathcal{P}_i$ sends the value $[v]_i^{\langle t,Q \rangle}$ *securely* to the player $\mathcal{U}$.
4. Player $\mathcal{U}$ applies algorithm RobustOpen from Figure 8 to robustly reconstruct the value $b - \mathbf{a} \cdot \mathbf{s}' + E$, and hence m.

Fig. 14. Threshold Decryption - Protocol 1

For correctness we require that

$$2^{\mathsf{pow}} \cdot \mathsf{Bd} \leq \frac{\Delta'}{2},$$

since then the PRSS addition will not effect the correctness of the final result as $E + \mathsf{Bd} \leq (2^{\mathsf{pow}} - 1) \cdot \mathsf{Bd} + \mathsf{Bd} = 2^{\mathsf{pow}} \cdot \mathsf{Bd} < \Delta'/2$.

On the other hand (see below) for security we require that

$$\mathsf{pow} \geq \mathsf{stat} + \log_2 \binom{n}{t},$$

where stat is the security parameter related to statistical distance. Thus this method is only applicable when we have a large Δ' in comparison to the noise bound Bd. This is why we needed to boost the ciphertext from one with parameters (q, ℓ) to one with parameters (Q, L) in the previous sections. Since we are assuming the small regime for $\binom{n}{t}$ is when $\binom{n}{t} \leq 100$, and we used the inequality

$$\mathsf{pow} \geq \mathsf{stat} + \log_2 100$$

in the previous section to derive the bounds on the noise after refreshing to ensure that

$$2^{\mathsf{pow}} \cdot \mathsf{Bd} \leq \frac{\Delta'}{2}$$

we are assured that the conditions of the following theorem are satisfied for our refresh parameters.

Simulator Threshold Decryption

On input of

1. A ciphertext $\mathfrak{ct} = (\mathbf{a}, b)$ and a public key $\mathfrak{pk}$.
2. The underlying message m encrypted by $\mathfrak{ct}$.
3. A set of adversarial parties I with $|I| \leq t$.
4. The share values $[\mathbf{s}']_i^{\langle t,Q\rangle}$ for $i \in I$.
5. The PRSS secret keys r_A for all sets A such that $A \cap I \neq \emptyset$.

this algorithm outputs the simulated shares $\{[v]_j^{\langle t,Q\rangle}\}_{j\notin I}$.

$\mathsf{Sim} - \mathsf{DistDecrypt}$:

1. The simulator computes, for $i \in I$,
$$[\hat{v}]_i^{\langle t,Q\rangle} = b - \mathbf{a} \cdot [\mathbf{s}']_i^{\langle t,Q\rangle} + \sum_{A:i\in A} \psi(r_A, \mathsf{cnt}_{\mathsf{PRSS}}) \cdot f_A(\gamma_i).$$
2. The simulator computes
$$E' = \sum_{A:A\cap I\neq\emptyset} \psi(r_A, \mathsf{cnt}_{\mathsf{PRSS}}) \cdot f_A(\gamma_i) + \sum_{B:B\cap I=\emptyset} r_B$$
where r_B is chosen uniformly at random so that $|r_B| \leq \mathsf{Bd}_1$.
3. The simulator computes $v = \Delta' \cdot m + E'$.
4. The simulator generates the decryption shares $\{[\hat{v}]_j^{\langle t,Q\rangle}\}_{j\notin I}$ via Lagrange interpolation (and possibly generating random shares if $|I| < t$) from v and the values $\{[\hat{v}]_i^{\langle t,Q\rangle}\}_{i\in I}$.
5. The simulator outputs $\{[v]_j^{\langle t,Q\rangle}\}_{j\notin I}$.

Fig. 15. Simulator for $\mathsf{DistDecrypt}(\mathfrak{ct}, [\mathbf{s}']^{\langle t,Q\rangle}, \mathcal{U})$

Theorem 5.1. *Assuming*

$$\mathsf{pow} \geq \mathsf{stat} + \log_2 \binom{n}{t},$$

in the $\mathcal{F}_{\mathsf{KeyGen}}$-hybrid model the protocol in Figure 14 implements $\mathcal{F}_{\mathsf{KeyGenDec}}$ with statistical security against any static active adversary corrupting I parties, with $|I| \leq t$.

Assuming

$$2^{\mathsf{pow}} \cdot \mathsf{Bd} \leq \frac{\Delta'}{2},$$

the protocol is correct.

Proof. Correctness follows on noticing that the values $\{[v]_j^{\langle t,Q\rangle}\}_{j\in I}$ produced by the simulator are the true decryption share values which the adversary should broadcast (even if he does not) if they acted honestly. The bounds on the noise described above then imply that the value v does encode the original message correctly. In which case the Lagrange interpolation will recover the shares for the honest players $\{[v]_j^{\langle t,Q\rangle}\}_{j\notin I}$ as required.

Security of the protocol follows by showing that the output of simulator in Figure 15 is statistically indistinguishable, with distance $2^{-\mathsf{stat}}$, from the output of an adversary controlling I parties, with $|I| \leq t$, in a real execution of the protocol.

The proof of this security claim follows essentially the argument in Section 7.5 of the full version of [CLO+13]; where we have to switch from a BGV style of looking at ciphertexts to one of BFV. However, the proof in [CLO+13] is overly complex and has a few minor bugs, which we correct here.

Let e denote the value of $b - \mathbf{a}\cdot\mathbf{s}' - \Delta'\cdot m \pmod{Q}$.

In a real execution of the protocol the shares output by the honest players are consistent and are enough to allow the honest parties to decrypt correctly, since $t < n/3$. The simulation has exactly the same properties.

The value E is the value output by the PRSS in the real execution of the protocol, and the value E' is the value simulated for the PRSS in the simulated protocol.

In the real protocol the adversary sees the value

$$\Delta'\cdot m + e + E$$

whereas in the simulated protocol he sees the value

$$\Delta'\cdot m + E'.$$

By the security of the PRSS the value of $e+E$ and $e+E'$ are indistinguishable. Thus we only need to show that $e+E'$ and E' are indistinguishable.

However, by Lemma 2.3 (applied with $v = \binom{n}{t} \geq n$, $B = \mathsf{Bd}_1$, and $|e| = \mathsf{Bd}$) we have that, when executing at most m distributed decryption operations,

$$\begin{aligned}
\Delta_{SD}\left((e+\sum_{i=1}^{v} U(-B,B))^m\ ,\ \sum_{i=1}^{v} U(-B,B)^m\right) &\leq \frac{m\cdot\mathsf{Bd}}{\mathsf{Bd}_1^2} + \sqrt{m\cdot\frac{\mathsf{Bd}^2\cdot\log\mathsf{Bd}_1+2}{2\cdot(\mathsf{Bd}_1^2+\mathsf{Bd}_1)}} \\
&\approx \frac{m\cdot v^2\cdot\mathsf{Bd}}{2^{2\cdot\mathsf{pow}}\cdot\mathsf{Bd}^2} + \sqrt{m\cdot\frac{v^2\cdot\mathsf{Bd}^2\cdot\log(2^{\mathsf{pow}}\cdot\mathsf{Bd})+2}{2^{2\cdot\mathsf{pow}+1}\cdot\mathsf{Bd}^2}} \\
&\approx \frac{m\cdot v^2}{2^{2\cdot\mathsf{pow}}\cdot\mathsf{Bd}} + \sqrt{m\cdot\frac{v^2\cdot\log(2^{\mathsf{pow}}\cdot\mathsf{Bd})+2}{2^{2\cdot\mathsf{pow}+1}}} \\
&\approx \frac{v}{2^{\mathsf{pow}}}\cdot\sqrt{m\cdot(\mathsf{pow}+\log\mathsf{Bd})} \\
&\leq c\cdot\sqrt{m}\cdot 2^{-\mathsf{stat}},
\end{aligned}$$

for some relatively 'small' constant c. Lemma 2.3 applies, since the number of uniform random variables $U(-B,B)$ added by the honest players is lower bounded by $n-t \geq 2$. □

5.2 Threshold Decryption for "Large" $\binom{n}{t}$

When $\binom{n}{t}$ is large we can no longer relay on a non-interactive PRSS. We can also not rely on "standard" interactive PRSS's, as our PRSS was used to create a small-ish element above and not a uniformly random one. Thus when $\binom{n}{t}$ is large we generate the masking value $[E]^{\langle t,Q\rangle}$ above, as a sum of two uniformly random values, using random bits provided by an "offline" phase. This offline phase is abstracted in the ideal functionality $\mathcal{F}_{\mathsf{Offline}}$ in Figure 16. We discuss how this can be implemented in Appendix C.

The Functionality $\mathcal{F}_{\mathsf{Offline}}$

We describe this functionality as a robust ideal functionality, the modifications to make a functionality which is only secure in an active-with-abort setting are easily made.

$\mathcal{F}_{\mathsf{Offline}}.\mathsf{Bits}(b)$:

1. The functionality samples uniformly random bits $b_i \in \{0,1\}$ for $i = 1,\ldots,b$.
2. The functionality creates random sharings $[b_i]^{\langle t,Q\rangle}$ of these bits.
3. The functionality distributes the shares $[b_i]_j^{\langle t,Q\rangle}$ to each player $\mathcal{P}_j$.

Fig. 16. The Offline Functionality $\mathcal{F}_{\mathsf{Offline}}$

In Protocol 1 the value E was a sum of $\binom{n}{t}$ uniform random variables in $[-2^{\mathsf{pow}-1}\cdot\mathsf{Bd},\ldots,2^{\mathsf{pow}-1}\cdot\mathsf{Bd})$, only two of which had to be truly random to ensure security. In Protocol 2 the value E is selected by adding two values obtained uniformly from the range $[-2^B,\ldots,2^B))$ where $B = \lceil\log_2 \mathsf{Bd}\rceil + \mathsf{pow}$. The full procedure is given in Figure 17. The correctness and security of the protocol follows from similar (but simpler) arguments to those presented above.

Threshold Decryption - Protocol 2

$\mathsf{Init}()$:

1. The parties obtain $[\mathbf{s}']^{\langle t,Q\rangle}$ via a threshold key generation protocol.

$\mathsf{DistDecrypt}(\mathfrak{ct}, [\mathbf{s}']^{\langle t,Q\rangle}, \mathcal{U})$: On input of $\mathfrak{ct} = (\mathbf{a}, b)$ this executes the following steps:

1. $([b_i]^{\langle t,Q\rangle})_{i=0}^{B} \leftarrow \mathcal{F}_{\mathsf{Offline}}.\mathsf{Bits}(B+1)$.
2. $([b'_i]^{\langle t,Q\rangle})_{i=0}^{B} \leftarrow \mathcal{F}_{\mathsf{Offline}}.\mathsf{Bits}(B+1)$.
3. The parties $\mathcal{Q}_i$ compute $[E]^{\langle t,Q\rangle} \leftarrow (-2^B + \sum_{i=0}^{B}[b_i]^{\langle t,Q\rangle}\cdot 2^i) + (-2^B + \sum_{i=0}^{B}[b'_i]^{\langle t,Q\rangle}\cdot 2^i)$.
4. The parties $\mathcal{Q}_i$ compute $[v]^{\langle t,Q\rangle} \leftarrow b - \mathbf{a}\cdot[\mathbf{s}']^{\langle t,Q\rangle} + [E]^{\langle t,Q\rangle}$.
5. Party $\mathcal{Q}_i$ sends the value $[v]_i^{\langle t,Q\rangle}$ *securely* to the player $\mathcal{U}$.
6. Player $\mathcal{U}$ applies algorithm $\mathsf{RobustOpen}$ from Figure 8 to robustly reconstruct the value $b - \mathbf{a}\cdot\mathbf{s} + E$, and hence m.

Fig. 17. Threshold Decryption - Protocol 2

6 Experiments

We can now present our threshold decryption procedure for TFHE ciphertexts, which we give in Figure 18. Recall, from the introduction, for BGV or BFV ciphertexts by selecting parameters suitably or by bootstrapping, one can proceed directly to step 3 in Figure 18.

Complete Threshold Decryption

$\mathsf{FullDistDecrypt}(\mathfrak{ct}, [\mathbf{s}']^{\langle t,Q\rangle}, \mathcal{U})$: On input of $\mathfrak{ct} = (\mathbf{a}, b) \in \mathbb{Z}_q^{\ell+1}$ this executes the following steps:

1. Execute $\mathfrak{ct}' \leftarrow \mathsf{SwitchUp}(\mathfrak{ct})$ to obtain $\mathfrak{ct}' \in \mathbb{Z}_Q^{L+1}$ encrypting the same value under the key $\mathbf{s}' \in \{0,1\}^L$.
2. Execute $\hat{ct} \leftarrow \mathsf{Refresh}(\mathfrak{ct}')$ to obtain a ciphertext with noise with variance ${\sigma'_{BR}}^2 + {\sigma'_{KS}}^2$.
3. Execute $m \leftarrow \mathsf{DistDecrypt}(\hat{\mathfrak{ct}}, [\mathbf{s}']^{\langle t,Q\rangle}, \mathcal{U})$ to obtain m.

Fig. 18. The complete threshold decryption protocol for TFHE ciphertexts

We note that lines 1 and 2 of Figure 18 do not require interaction, whereas 3 does. We thus first present experimental times for the evaluation of lines 1 (SwitchUp) and 2 (Refresh) for our four parameter sets. These we present in Table 2 of our Rust implementation. These results were obtained on an AWS `m6i.metal` instance with 128 Intel Xeon Gen 3 vCPUs and 512 GiB RAM, taking an average execution time over 100 runs of the relevant algorithms.

Table 2. Execution times (in milliseconds) for lines 1 (SwitchUp) and 2 (Refresh) of Figure 18

Parameters	SwitchUp	Refresh
$(2^{64}, 777) \to (2^{128}, 2481)$	2.2737	411.09
$(2^{64}, 870) \to (2^{128}, 2594)$	2.9697	573.57
$(2^{64}, 1024) \to (2^{128}, 2481)$	4.4843	541.97
$(2^{64}, 1024) \to (2^{128}, 2594)$	4.4990	670.53

Recall these timings are for a part of the computation which does not requite interaction, and which are amenable to acceleration by the FHE accelerators currently being developed. For example, the paper [BDV22] shows a three orders of magnitude acceleration using only FPGA acceleration (as opposed to ASIC acceleration).

To time line 3 of Figure 18 we need to consider various other factors; the number of parties n performing the distributed decryption, the threshold t, the type of network, the number of active corruptions. For each of our four parameter sets we utilized three different sets of (n, t) values; namely $(n, t) = (4, 1), (10, 3)$ and $(40, 13)$. For the first of these one can utilize the PRSS-based distributed decryption method, for the other two one needs to utilize the methodology requiring an offline phase. In our experiments we only timed the online phase for the latter two cases. In all cases we present the average run-time over 1000 iterations for a single honest party. Recall this party will terminate as soon as it has received enough shares to robustly reconstruct the underlying encrypted value.

We also investigated the effect of a LAN-like setting (1 Gbit/s with small ping times of ≈1 ms) versus a WAN-like setting (100 Mbit/s with high ping times of ≈100 ms), and whether we are optimistic or pessimistic in terms of the number of errors introduced by the adversary during the distributed decryption. If there are no errors then the online-error correction method of Figure 8 will execute faster than if there are maximal, i.e. t, adversarial errors. The asynchronous channels are implemented using gRPC with `tokio` and `tonic` Rust crates.

We measured our experiments on a single AWS `m6i.metal` instance as above. We ran the n protocol parties as individual docker containers and simulated the LAN/WAN connection between them. Our results are given in Table 3 of Appendix E.

In the most favorable situation, namely four parties where we can tolerate one dishonest party over a LAN, we obtain execution times for line 3 of Figure 18 of under 2 milliseconds. In the least favorable situation we investigated, namely 40 parties of which thirteen are malicious (and send invalid share values), and over a WAN, we are able to execute line 3 of Figure 18 in under 100 milliseconds on average.

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A Proof of Lemma 2.4

Proof. In the following, we show the case of $e = 1$. The general case follows from the triangle inequality and $|e|$ applications of the lemma for $e = 1$.

Let $b = 2 \cdot B + 1$ and note that $\mathcal{P}(x) = (b - |x|)/b^2$ for all $x \in [-2 \cdot B, 2 \cdot B]$. We begin by introducing two truncated distributions, $\mathcal{P}_\perp$ and $\mathcal{P}_\top$. Define $\mathcal{P}_\perp$ such that it is proportional to $\mathcal{P}$, except that $\mathcal{P}_\perp(-2 \cdot B) = 0$, i.e. we have $\mathcal{P}_\perp(x) = S \cdot (b - |x|)/b^2$ for all $x \in [-2 \cdot B + 1, 2 \cdot B]$, where $S = b^2/(b^2 - 1)$ is the normalization factor. Similarly, we define $\mathcal{P}_\top$ to be proportional to $1 + \mathcal{P}$ but truncated at the top such that we have $\mathcal{P}_\top(2 \cdot B + 1) = 0$. Note that the normalization factor of $\mathcal{P}_\top$ matches the one of $\mathcal{P}_\perp$ and that the two distributions have the same support. We will show the result by bounding $\Delta_{SD}(\mathcal{P}^m, \mathcal{P}_\perp^m)$, $\Delta_{SD}(\mathcal{P}_\perp^m, \mathcal{P}_\top^m)$, and $\Delta_{SD}(\mathcal{P}_\top^m, (1 + \mathcal{P})^m)$. The rest follows by the triangle inequality.

First note that $\Delta_{SD}(\mathcal{P}, \mathcal{P}_\perp) = \Delta_{SD}(\mathcal{P}_\top, 1 + \mathcal{P}) = 1/b^2$. Accordingly, we have $\Delta_{SD}(\mathcal{P}^m, \mathcal{P}_\perp^m) \leq m/b^2$ and $\Delta_{SD}(\mathcal{P}_\top^m, (1 + \mathcal{P})^m) \leq m/b^2$.

It remains to bound $\Delta_{SD}(\mathcal{P}_\perp^m, \mathcal{P}_\top^m)$. We first consider the KL-divergence between $\mathcal{P}_\perp$ and $\mathcal{P}_\top$:

$$\begin{aligned}
\Delta_{KL}(\mathcal{P}_\perp, \mathcal{P}_\top) &= -\sum_{x=-2\cdot B+1}^{2\cdot B} \mathcal{P}_\perp(x) \cdot \log \frac{\mathcal{P}_\top(x)}{\mathcal{P}_\perp(x)} \\
&= -S \cdot \sum_{x=-2\cdot B+1}^{2\cdot B} \mathcal{P}(x) \cdot \log \frac{\mathcal{P}(x-1)}{\mathcal{P}(x)} \\
&= -S \cdot \Big[\left(\mathcal{P}(0) \cdot \log \frac{\mathcal{P}(-1)}{\mathcal{P}(0)} \right) + \left(\mathcal{P}(2 \cdot B) \cdot \log \frac{\mathcal{P}(2 \cdot B - 1)}{\mathcal{P}(2 \cdot B)} \right) \\
&\quad + \sum_{x=-2\cdot B+1}^{-1} \mathcal{P}(x) \cdot \left(\log \frac{\mathcal{P}(x-1)}{\mathcal{P}(x)} + \log \frac{\mathcal{P}(x+1)}{\mathcal{P}(x)} \right) \Big]
\end{aligned}$$

Note that we have

$$\mathcal{P}(2 \cdot B) \cdot \log \frac{\mathcal{P}(2 \cdot B - 1)}{\mathcal{P}(2 \cdot B)} = \frac{\log 2}{b^2} \geq 0$$

so this term may be ignored (due to the negative sign of the expression). In the following, we make use of the fact that $\log(1 - 1/x) \geq -2/x$ for all $x \geq 2$. Then we have

$$\mathcal{P}(0) \cdot \log \frac{\mathcal{P}(-1)}{\mathcal{P}(0)} = \frac{1}{b} \cdot \log \frac{b-1}{b} = \frac{1}{b} \cdot \log(1 - 1/b) \geq -2/b^2 \ .$$

Similarly, we have for all $x \in [-2 \cdot B + 1, -1]$

$$\begin{aligned}
&\mathcal{P}(x) \cdot \left(\log \frac{\mathcal{P}(x-1)}{\mathcal{P}(x)} + \log \frac{\mathcal{P}(x+1)}{\mathcal{P}(x)} \right) \\
&= \mathcal{P}(x) \cdot \left(\log \left(\frac{\mathcal{P}(x-1)}{\mathcal{P}(x)} \frac{\mathcal{P}(x+1)}{\mathcal{P}(x)} \right) \right) \\
&= \frac{b+x}{b^2} \cdot \log \left(\frac{(b+x-1)(b+x+1)}{(b+x)^2} \right) \\
&= \frac{b+x}{b^2} \cdot \log \left(\frac{(b+x)^2 - 1}{(b+x)^2} \right) \\
&= \frac{b+x}{b^2} \cdot \log \left(1 - \frac{1}{(b+x)^2} \right) \\
&\geq -\frac{2}{b^2 \cdot (b+x)} \ .
\end{aligned}$$

Combined, we get

$$\begin{aligned}
\Delta_{KL}(\mathcal{P}_\perp, \mathcal{P}_\top) &\leq S \cdot \left[\frac{2}{b^2} + \sum_{x=-2\cdot B+1}^{-1} \frac{2}{b^2 \cdot (b+x)} \right] \\
&= \frac{2 \cdot S}{b^2} \cdot \left(1 + \sum_{x=2}^{2\cdot B} 1/x \right) \\
&= \frac{2 \cdot S}{b^2} \cdot \sum_{x=1}^{2\cdot B} 1/x \\
&\leq \frac{2 \cdot (\log(2 \cdot B) + 1)}{(b^2 - 1)} \\
&= \frac{\log B + 2}{2 \cdot (B^2 + B)}.
\end{aligned}$$

By the sub-additive property of Δ_{KL} we now have

$$\Delta_{KL}(\mathcal{P}_\perp^m, \mathcal{P}_\top^m) \leq m \cdot \frac{\log B + 2}{2 \cdot (B^2 + B)}$$

and by Pinsker's inequality

$$\Delta_{SD}(\mathcal{P}_\perp^m, \mathcal{P}_\top^m) \leq \sqrt{\Delta_{KL}(\mathcal{P}_\perp^m, \mathcal{P}_\top^m)} = \sqrt{m \cdot \frac{\log B + 2}{2 \cdot (B^2 + B)}}.$$

□

B Auxillary Protocols

B.1 Commitment Schemes

We will need a commitment scheme. The one we choose is secure in the random oracle model, and thus uses a hash function. The hash function $\mathcal{H} : \{0,1\}^* \longrightarrow \{0,1\}^{|\mathcal{H}|}$ is assumed to be one such as

SHA-256, or SHA-3. The output length $|\mathcal{H}|$ should be at least $(2 \cdot \mathsf{sec})$-bits in length. The scheme is defined in Figure 19. If we want to specify the randomness externally to the commitment then we write $\mathsf{Commit}(m; r)$.

Commitment Scheme

$\mathsf{Commit}(m)$: On input of a message $m \in \{0,1\}^*$ this proceeds as follows, to produce the commitment c and the opening information o.

1. Generate a random bit string $r \leftarrow \{0,1\}^{\mathsf{sec}}$.
2. Set $o \leftarrow m \| r$.
3. Compute the commitment $c \leftarrow \mathcal{H}(o)$.
4. Output (c, o).

$\mathsf{Open}(c, o)$: This opens the commitment.

1. Compute $c' \leftarrow \mathcal{H}(o)$.
2. If $c' \neq c$ then abort.
3. Write $m \| r \leftarrow o$, where r is sec-bits long.
4. Output m.

Fig. 19. Commitment Scheme

B.2 Agree on a Random Number

We require a protocol which enables a subset $S \subseteq \{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$ of our set of parties need to agree on a random value, often the key for a PRF, which is outside the control of all parties. This is done via the protocol $\mathsf{AgreeRandom}$ in Figure 20. Note, the protocol does not verify that the players all obtain the same random value. The idea being that if the same random value is not obtained then this should become apparent once the value is used in a PRF.

Protocol $\mathsf{AgreeRandom}(S, k)$

The input is a subset $S \subseteq \{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$ and a value $k \in \mathbb{N}$ which is the length of the output. Let I denote the index set of the players in S, i.e. $S = \{\mathcal{P}_i\}_{i \in I}$.

1. Each $\mathcal{P}_i \in S$ generates $s_i \leftarrow \{0,1\}^k$.
2. Each $\mathcal{P}_i \in S$ executes $(c_i, o_i) \leftarrow \mathsf{Commit}(s_i)$.
3. Each $\mathcal{P}_i \in S$ sends c_i to all other players in S.
4. When $\mathcal{P}_i$ has received c_j from all $j \in I \setminus \{i\}$ it sends o_i to all other players in S.
5. Each $\mathcal{P}_i \in S$ executes $s_j \leftarrow \mathsf{Open}(c_j, o_j)$ for all $j \in I \setminus \{i\}$.
6. Each $\mathcal{P}_i \in S$ outputs $s \leftarrow \bigoplus_{j \in I} s_i$.

Fig. 20. Protocol $\mathsf{AgreeRandom}$

C Offline Phase

Our methodology for large values of $\binom{n}{t}$ requires a method to generate shared random bits within an "offline" phase. We now elaborate on how this could be done. We first assume a generic MPC

protocol which works with the secret sharing scheme $[\cdot]^{\langle t,Q\rangle}$, which recall shares elements in the Galois ring $\mathcal{G}$. But we have to work with sharings modulo $Q' = 4 \cdot Q$ at one point, thus we assume our MPC protocol works natively modulo Q' in the ring $\mathcal{G}' = \mathbb{Z}_{Q'}[Y]/F(Y)$. Mapping from modulo Q' to modulo Q sharings at the end is trivial.

We present the ideal functionality $\mathcal{F}_{\mathsf{MPC}}$ in Figure 21, note we only require the ability to multiply elements and generate shares of random elements in this functionality. We describe this in terms of sharings in order to facilitate ease of understanding, however it can also be expressed in terms of the usual "handles" to variables maintained inside the functionality.

Ideal Functionality $\mathcal{F}_{\mathsf{MPC}}$

$\mathcal{F}_{\mathsf{MPC}}.\mathsf{RandomSharing}()$:
1. The ideal functionality generates a uniformly random $a \leftarrow \mathcal{G}'$.
2. The ideal functionality generates a random sharing $[a]^{\langle t,Q'\rangle}$.
3. The value $[a]_i^{\langle t,Q'\rangle}$ is given to player $\mathcal{P}_i$.

$\mathcal{F}_{\mathsf{MPC}}.\mathsf{Mult}([a]^{\langle t,Q'\rangle}, [b]^{\langle t,Q'\rangle})$:
1. The functionality forms the a random sharing $[c]^{\langle t,Q'\rangle}$ of the product $a \cdot b$.
2. The functionality passes $[c]_i^{\langle t,Q'\rangle}$ to player $\mathcal{P}_i$.

Fig. 21. The MPC Ideal Functionality $\mathcal{F}_{\mathsf{MPC}}$

To implement such a functionality there are a number of possibilities. To obtain a fully robust offline protocol over synchronous networks one could utilize the protocol from [ACD+19]. If one only requires an offline phase which is actively secure up to a possible abort then the potentially simpler protocols from [JSL22] may be preferred.

The standard method to generate shared random bits in an MPC protocol (for odd values of Q) is to use the $2:1$ nature of the squaring operation to produce random bits, see [DKL+13] for example. However, when Q is even we cannot use the above technique, since the squaring map is now a $4:1$ mapping. Instead we utilize a technique from [OSV20]. The method requires that we extend Q by an extra two bits, $Q' \geq 4 \cdot Q$ and work with values modulo Q'. What is nice, from an MPC perspective, about the even Q case is that we do not need to loop to produce a non-zero value.

1. $a \leftarrow \mathcal{G}$.
2. $v \leftarrow 1 + 2 \cdot a$.
3. $s \leftarrow v^2$.
4. $c \leftarrow \sqrt{s} \pmod{2 \cdot Q}$. We can think of s as an element modulo $4 \cdot Q$ here, when taking the square root. Note, since s is a square taking square roots is well defined. We fix a deterministic value of the root modulo $2 \cdot Q$. Since v was invertible, so is s, and hence so is c.
5. Write $c = 1 + 2 \cdot c'$ in the ring $\mathcal{G}$.
6. $b \leftarrow (1 + a + c')/c$.

This leads to the MPC-version of the bit generation protocol in Figure 22.

Example: We go through a small worked example to demonstrate this actually works using the simple ring $\mathcal{G} = \mathbb{Z}_Q$. We fix $k = 3$, i.e. $Q = 2^k = 8$ and $Q' = 2^{k+2} = 2^5 = 32$ for our baby example.

Bit Generation and Usage

Offline.GenBit():

1. Execute the following a "sufficient" number of times in parallel:
   (a) $[a]^{\langle t,Q'\rangle} \leftarrow \mathcal{F}_{\mathsf{MPC}}.\mathsf{RandomSharing}()$.
   (b) $[v]^{\langle t,Q'\rangle} \leftarrow 1 + 2 \cdot [a]^{\langle t,Q'\rangle}$.
   (c) $[s]^{\langle t,Q\rangle} \leftarrow \mathcal{F}_{\mathsf{MPC}}.\mathsf{Mult}([v]^{\langle t,Q'\rangle}, [v]^{\langle t,Q'\rangle})$.
   (d) $s \leftarrow \mathsf{RobustOpen}([s]^{\langle t,Q'\rangle})$.
   (e) $c \leftarrow \sqrt{s} \pmod{2 \cdot Q}$.
   (f) Write $c = 1 + 2 \cdot c'$.
   (g) $[b]^{\langle t,Q'\rangle} \leftarrow (1 + [a]^{\langle t,Q'\rangle} + c')/c$.
   (h) $[b]^{\langle t,Q\rangle} \leftarrow [b]^{\langle t,Q'\rangle} \pmod{Q}$.
   (i) $\mathcal{B} \leftarrow \mathcal{B} \cup \{[b]^{\langle t,Q\rangle}\}$.

Offline.Bits(v):

1. While $|\mathcal{B}| < v$ then execute Offline.GenBit().
2. Write $\mathcal{B} = \{[b_i]^{\langle t,Q\rangle}\}_{i=1}^{N}$.
3. $\mathcal{B} \leftarrow \mathcal{B} \setminus \{[b_i]^{\langle t,Q\rangle}\}_{i=1}^{v}$.
4. Return $\{[b_1]^{\langle t,Q\rangle}, \ldots, [b_v]^{\langle t,Q\rangle}\}$.

Fig. 22. Bit Generation and Bit Usage

1. $a \leftarrow \mathbb{Z}_{2^{3+2}} = \mathbb{Z}_{2^5}$. So take, for example $a = 20$.
2. $v \leftarrow 1 + 2 \cdot a \pmod{2^5}$. So in our example $v = 9$.
3. $s \leftarrow v \cdot v \pmod{2^5}$. So in our example $s = 17$.
4. $\mathsf{Open}(s)$
5. $c \leftarrow \sqrt{s} \pmod{2^3}$. Modulo 8 we have $s = 1$, thus the square roots modulo 8 of s are $1, 3, 5$, and 7. Suppose we take (for sake of argument) the one with the highest Hamming weight. So we have $c = 7$.
6. Since $c = 7$ we have $c' = 3$.
7. $b \leftarrow (1 + a + c')\ /\ c \pmod{2^5} = (1 + 20 + 3)\ /\ 7 \pmod{2^5} = 8$. Note that modulo 2^3 we have $b = 0 \in \{0, 1\}$.

D Threshold Key Generation

In this appendix we briefly discuss how a threshold key $[\mathbf{s}']^{\langle t,Q\rangle}$ can be generated.

The first technique would be to utilize the homomorphic properties of the underlying LWE encryption to "combine" different users individual keys into a single threshold key $[\mathbf{s}']^{\langle t,Q\rangle}$ with the correct properties. This approach typifies what is called *multi-key homomorphic encryption*. However, the approach tends to produce a threshold public key with different underlying noise distributions to that which would arise from a trusted third party generating the shared public key. This results in great inefficiencies in practice as (especially for TFHE) performance of homomorphic evaluation and bootstrapping is highly dependent on choosing exactly the correct parameters and noise distributions.

The second technique is to apply an MPC protocol to generate the underlying secret key data in an secret shared form. For LWE based ciphertexts this is relatively straight forward. One can utilize the MPC functionality from earlier, to obtain many sharings of random bits $[b]^{\langle t,Q\rangle}$. Given these almost all the operations needed to perform a threshold key generation are linear operations,

followed by openings of the shared values (when wishing to output the shared public values themselves). This approach has been used to generate threshold BGV style keys for the SPDZ MPC system [RST+22]. In our situation (generating TFHE keys) the application is even easier due to the ciphertext modulus being a power of a prime and not a product of multiple distinct primes.

E Timings for Distributed Decryption

The results for our various experiments for distributed decryption are given in Table 3.

Table 3. Execution times for line 3 (DistDecrypt) of Figure 18

	$(Q, L) = (2^{128}, 2481)$				$(Q, L) = (2^{128}, 2594)$			
(n, t)	Ping Time	Number Errors	DistDecrypt (ms)		(n, t)	Ping Time	Number Errors	DistDecrypt (ms)
$(4, 1)$	≈ 1 ms	0	1.43		$(4, 1)$	≈ 1 ms	0	1.52
$(4, 1)$	≈ 1 ms	1	1.64		$(4, 1)$	≈ 1 ms	1	1.66
$(4, 1)$	≈ 100 ms	0	50.78		$(4, 1)$	≈ 100 ms	0	50.97
$(4, 1)$	≈ 100 ms	1	53.68		$(4, 1)$	≈ 100 ms	1	53.49
$(10, 3)$	≈ 1 ms	0	2.40		$(10, 3)$	≈ 1 ms	0	2.41
$(10, 3)$	≈ 1 ms	3	3.31		$(10, 3)$	≈ 1 ms	3	3.42
$(10, 3)$	≈ 100 ms	0	53.85		$(10, 3)$	≈ 100 ms	0	52.51
$(10, 3)$	≈ 100 ms	3	57.90		$(10, 3)$	≈ 100 ms	3	56.12
$(40, 13)$	≈ 1 ms	0	18.57		$(40, 13)$	≈ 1 ms	0	18.69
$(40, 13)$	≈ 1 ms	13	41.58		$(40, 13)$	≈ 1 ms	13	41.79
$(40, 13)$	≈ 100 ms	0	68.38		$(40, 13)$	≈ 100 ms	0	67.51
$(40, 13)$	≈ 100 ms	13	91.59		$(40, 13)$	≈ 100 ms	13	91.27