

Proofs for Deep Thought: Accumulation for large memories and deterministic computations

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Abstract. We construct two new accumulation schemes. The first one is for checking that ℓ read and write operations were performed correctly from a memory of size T . Unlike all prior work, the prover time is entirely independent of T and only depends on ℓ . The second one is for deterministic computations. It does not require committing to the intermediate wires of the computation but only the input and output. This is achieved by building an accumulation scheme for a modified version of the famous GKR protocol. We show that these schemes are highly compatible and that the accumulation for GKR can further reduce the cost of the memory-checking scheme. Using the BCLMS (Crypto 21) compiler, these protocols yield an efficient incrementally verifiable computation (IVC) scheme that is particularly useful for machine computations with large memories and deterministic steps.

Keywords: Proof system · Accumulation Scheme · Incrementally Verifiable Computation.

1 Introduction

Imagine an untrusted prover P^{vm} that is performing an unbounded machine computation. P^{vm} wants to produce a convincing certificate that it ran the computation correctly. The certificate’s size and the complexity of checking its correctness should ideally be independent of the length of the computation. P^{vm} should also be able to stop the machine at any point and output its state along with a certificate, such that anyone can check the computation up to that point and even continue it. This is the problem statement of *incrementally verifiable computation (IVC)* [Val08a]¹.

IVC enables P^{vm} to produce a proof π^{IVC} that convinces a verifier V^{IVC} that some (possibly non-deterministic) machine computation was run correctly. IVC and its generalization to graphs, PCD [CT10], have many applications, which range from outsourcing computation to untrusted servers [BCTV14b], over distributed proving [CTV15], to verifiable delay functions [BBBF18]. IVC has also

¹ The literary application of IVC is the machine Deep Thought from the Hitchhikers Guide to the Galaxy. It computes the answer to the ultimate question of the universe and life over several thousand years. Given the non-sensical answer (42), it would have been helpful to be able to efficiently verify the correctness of the computation.

been deployed in practice, mostly in the context of succinct blockchain clients [KB20; BMRS20; CCDW20] and zk-Rollups [Tea22] which use IVC to show that an entire block of transactions is valid. Recently, a line of work has shown that IVC can be constructed more efficiently and from weaker assumptions using so-called accumulation or folding schemes [BGH19; BCMS20; BCLMS21; BDFG21; KST22; BC23; EG23].

Despite this progress, significant barriers for IVC remain: most machine computations require a memory. Proving the correctness of the computation requires proving that all the memory accesses were performed correctly. This can be handled by a separate prover P^{mem} , which performs the so-called *memory-checking*. Memory checking allows P^{mem} to convince a verifier V^{mem} that all reads and write instructions were executed correctly. Given this, P^{vm} can focus on proving that a computation was performed correctly *given* a list of reads and writes. The most efficient prior approaches² to memory checking require P^{vm} to perform work linear in the memory size T . This is true even if ℓ , the number of reads and writes in each computation step, is significantly smaller than T . Protostar recently showed that for static read-only memory, there can be a prover who only performs one group scalar multiplication per read instruction; however, their approach does not support write instructions [BC23]. This naturally brings us to our first research question:

Research Question 1: $o(T)$ Memory-checking Can we perform efficient memory-checking for large memories, such that the prover’s work is independent of the memory size T ?

Another limitation of IVC is that even the most efficient constructions built from accumulation schemes require cryptographically committing to a witness per intermediate wire. Practically, this results in exactly one multi-scalar multiplication (MSM), which has the size of the circuit, even if the inputs (private and public) to the circuit are significantly smaller than the circuit. Another way to phrase this limitation is that the prover’s witness is different from the computation’s witness even for deterministic computations. Imagine an arithmetic circuit, consisting of addition and multiplication gates, that checks $x^{16} = y$ for some input x, y . The circuit’s wires consist of 5 values (y, x, x^2, x^4 and x^8) even though the input is only 2 values. One way to resolve this is to have more expressive circuits with fewer but higher degree gates. Unfortunately, this approach quickly hits a limit as the degree grows exponentially when combining multiple gates into one, e.g. you’d need one degree 16 gate to prove that $x^{16} = y$. An alternative approach relies on the fundamental work of GKR [GKR08]. GKR enables proving that a deterministic depth d computation was executed correctly while only requiring the verifier to have access to the input and output of the computation and not any of the intermediate values. Moreover, GKR only requires committing to and sending values which are linear in the depth of the circuit and

² Using a Merkle Tree (as in [BFRSBW13]) combined with a SNARK the prover can perform memory checking by showing the correctness of Merkle Tree updates. This requires proving the correctness of $\Theta(\ell \cdot \log T)$ hashes.

the degree of the circuit gates, but independent of the width and thus the total size of the circuit. This is helpful for both the verifier’s computation and, even more importantly, the prover’s computation. Subsequent work [CMT12; XZ-ZPS19; CBBZ23], most notably CMT, showed how to further reduce the prover cost such that proving a computation is barely more expensive than running it. Unfortunately, GKR in its plain version, due to its many rounds of interaction, is not well-suited for constructing accumulation schemes. Nevertheless, GKR’s existence motivates our second research question.

Research Question 2: Breaking the witness barrier Can we build an efficient accumulation-based IVC scheme that does not require committing to each intermediate circuit wire?

In this paper we answer both research questions positively and show a connection between them.

$O(\ell)$ memory-checking. Firstly, we built a memory-checking scheme that only requires the prover to do $O(\ell)$ computation where ℓ is the number of reads and writes done in each computation step, independent of the size of the memory T . This significantly improves on prior work, which either required \mathcal{V}^{mem} to do a linear-scan of the entire memory [BCGTV13] or relied on Merkle trees which require $\log T$ hashes per memory read or write. These methods are especially expensive in the context of IVC, where the prover proves recursively that \mathcal{V}^{mem} would have accepted the memory-checking proof. At its core, we build an accumulation-based memory-checking scheme, which builds on a prior lookup argument first introduced in [Hab22]. Protostar already notes that this lookup argument is well suited for accumulation and enables checking that a set of witness values exists in a static table of values [BC23]. We extend the argument to support reading and writing to an entirely dynamic table. A key challenge we overcame is proving only $O(\ell)$ table values were changed in a table of size $T \gg \ell$ while keeping the prover cost independent of T . The lookup argument is practically efficient with the prover only having to commit to 10ℓ field values for ℓ read and write operations. Using the Protostar compiler this accumulation scheme directly yields IVC with essentially the same computational overhead. One limitation of the scheme is that the prover needs to commit to 6ℓ large field elements, i.e. of λ bit size, even if the memory entries itself are small. Our second contribution for deterministic computation resolves this overhead.

IVC for deterministic computations. We build an accumulation scheme for the GKR protocol which retains GKR’s efficiency properties, in particular the ability to prove deterministic layered computations without committing to the intermediate values. We build an accumulation scheme for GKR by utilizing the ProtoStar compiler, which takes any interactive special-sound public coin protocol with an algebraic verifier and turns it into an accumulation scheme. The accumulation scheme, in turn, can be used to construct IVC. The specific accumulation scheme we build is based on a variant of GKR that uses a bivariate sumcheck protocol instead of relying on multi-linear sumcheck, since bivariate

sumcheck has the advantage of only requiring 3 prover messages. In the context of the ProtoStar compiler, this reduced number of rounds leads to a significantly reduced recursive overhead in IVC. While a bivariate sumcheck requires committing to messages of size $O(\sqrt{n})$, where n is the width of the circuit, this is generally not a bottleneck as the prover already needs to commit to the input of the circuit, which is of size $\Theta(n)$.

Improving memory checking with GKR. We use the GKR protocol to improve our memory-checking protocol. In the memory-checking protocol, the prover needs to send 6 vectors consisting of inverses, with each inverse as large as the field. These inverses are used to compute sums of fractions that have the form $\sum_{i=1}^{\ell} \frac{1}{r+t_i}$ where r is a constant, and each t_i is a small table entry. We rely on ideas from Lasso, and logUp [STW23; PH23] to compute this sum using formal fractions and a $\log \ell$ depth GKR protocol. The protocol never requires formally computing and committing to the inverses. Thus, if we read/write ℓ c -bit values from memory, the number of group operations goes from $O(\lambda\ell)$ to $O(c\ell)$, i.e., the actual size of the data that is read/written. We briefly describe the resulting efficiency of our protocol in Table 1.

We also introduce several optimizations for our GKR-powered memory checking, that help to further reduce the number of GKR rounds. GKR has many applications beyond improving our memory-checking protocol. In fact, it can reduce the commitment cost of *any* deterministic computation to just the inputs and outputs. We show the utility of this by briefly describing a GKR protocol for computing group scalar multiplications, the dominant cost inside the recursive circuit.

Table 1. Efficiency Table for our Memory-Checking Protocol. See Table 3 for an explanation of the columns and symbols.

	P _{ACC} Time	V _{ACC} Time
Plain	$(4\ell, \mathbb{T})\text{-MSM} + (6\ell, \mathbb{F})\text{-MSM}$	$3\mathbb{G}$
Using GKR	$(4\ell, \mathbb{T})\text{-MSM}$	$O(\log T)\mathbb{G}$

1.1 Related Work

IVC and Accumulation. Incrementally verifiable computation was first introduced by Valiant [Val08b] who showed that IVC can be built from SNARKs. The core idea is that for each computation step, we produce a SNARK that certifies both the computation as well as the verification of the previous step’s SNARKs. We generally refer to the latter part as the *recursive circuit*. An important line of work [BCCT13; BCTV14a; COS20] that followed Valliant improved the practicality of IVC, generalized it to arbitrary graphs (called PCD) and improved it’s theoretical foundations. However, in this entire line of work, IVC was still constructed from succinct arguments with sublinear verification. Halo [BGH19]

first showed that IVC can be constructed from simpler assumptions. This has led to a very active line of work on accumulation [BCMS20; BCLMS21; BDFG21; KST22; BC23; EG23]. The idea of accumulation is that one can construct IVC by simply accumulating or batching the verification of non-interactive arguments. Roughly speaking, in every IVC step, the prover produces a new argument for the current step and proves that it has correctly accumulated the current argument with the existing accumulator. This accumulation step can be as simple as taking a random linear combination between two vector commitments. Only at the end of the IVC computation, a more expensive *decision* step is run to check the correctness of the commitment. Very recently, ProtoStar introduced a new recipe for constructing accumulation schemes and, thus, IVC [BC23]. The recipe takes *any* k -round interactive special-sound protocol with an algebraic degree d verifier and produces a highly efficient accumulation/ivc scheme from it. The efficiency of the resulting IVC depends on the number of rounds k , the degree of the verifier d , but interestingly not on the size of the prover messages (other than that the prover needs to commit to them) or the verification time.

Memory-Checking and lookup arguments. Memory-Checking [BEGKN91] enables an untrusted prover to convince a verifier that a set of read and write operations is consistent with a memory. Every read and write operation consists of an address a , a value v and a timestamp t . If a value v was written to a at time t , then any read at time $t' > t$ from a shall return a with the timestamp t , unless there was another write to a in the meantime. We refer to Appendix B of Jolt [AST23] for an excellent overview of memory-checking techniques but want to highlight two constructions and their limitations. One idea is to store the memory inside a Merkle Tree [BFRSBW13; BCTV14a]. For every read operation, the prover opens the Merkle Tree at the relevant address. For every write operation, the prover shows that the Merkle Tree was correctly updated. The verification for either step requires $\log T$ hashes where T is the size of the memory, and the prover’s computational work is also $\log T$. However, when using this technique within IVC, the memory-checking verifier becomes part of the recursive circuit, and $\log(T)$ hashes per read and write operation is a significant overhead. The other common approach, dating back to [BEGKN91] and later refined [CDvGS03; SAGL18] relies on proving that the sets of reads and writes form a permutation. While each individual read or write can be as cheap as a few field operations, the scheme requires reading the entire memory at least once. Its complexity is therefore necessarily $\Theta(T)$, which is prohibitive when the size of the memory T greatly exceeds the number of reads/writes ℓ . This is commonly the case in IVC, where each computation step is constant size but the memory may be large. Our memory-checking scheme is specifically designed for IVC (in fact it only works in the IVC context) and has $O(\ell)$ complexity, independent of T .

Recently, there has been increased attention to a related primitive called lookup arguments. Lookup arguments enable verified read operation in a static, possibly preprocessed memory. A recent line of work [ZBKMNS22; PK22; GK22; ZGKMR22; EFG22; STW23] has shown that in the preprocessing setting, one

can achieve lookup arguments independent of the table size and quasi-linear in the number of read operations. ProtoStar [BC23] gave a lookup argument based on [Hab22] that in the IVC context is independent of the table size and truly linear in the number of reads. Unfortunately, all these lookup arguments only work for static tables and read operations. We construct a memory-checking argument (which is more general than a lookup) that is still independent of the table size and has minimal overhead.

1.2 Technical Overview

Our construction heavily relies on the ProtoStar [BC23] compiler, which we describe in Theorem 1. It takes as input any special-sound interactive, k -round³ protocol with an algebraic degree d verifier, and outputs an accumulation scheme. The transformation first converts the interactive argument into a NARK by committing to each round of prover messages using a homomorphic vector commitment and then using the Fiat-Shamir transform. The accumulation scheme then combines the current argument with an accumulator (which has the same form as the argument) by taking a random linear combination of the committed prover messages with the accumulator messages. It also computes a new verification equation by appropriately canceling out cross terms resulting from the accumulation. The accumulation verifier’s cost is $k+2$ group operations and $d+3$ hashes and field operations. Note that the NARK’s verifier cost and the size of the prover messages do not affect the accumulation verifier’s complexity. The accumulation prover’s main cost is committing to all the special-sound prover’s messages. Using the BCLMS[BCLMS21] compiler an accumulation scheme for NP directly yields an MPC, where the prover’s cost for computing the predicate is proportional to the accumulation prover and the recursive overhead consists of the accumulation verifier. Following this recipe we design special-sound, algebraic protocols for memory checking and GKR.

From a read-only lookup argument to mem-update. The starting point of our construction is the *logUp* lookup argument [Hab22] which uses the fact that if and only if a set of values $\mathbf{w} = \{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ is in a table $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_T)$ then $\sum_{i=1}^{\ell} \frac{1}{X+\mathbf{w}_i} = \sum_{i=1}^T \frac{\mathbf{m}_i}{X+\mathbf{t}_i}$, where \mathbf{m}_i is the multiplicity of \mathbf{t}_i in \mathbf{w} for every $i \in [T]$ and X is a random variable. LogUp checks this equality at a random point. Protostar [BC23] observes that the prover message in this protocol, e.g. $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_T)$, is sparse if $\ell \ll T$. This means in the context of IVC and the protostar compiler, the prover only needs to do $O(\ell)$ work. We would like to use the lookup argument in the context of memory-checking, where \mathbf{t} corresponds to the memory and \mathbf{w} to the set of read and write operations. However, the lookup argument only performs read operations and does not support write operations. Secondly, it is not immediately clear how to update \mathbf{t} , especially in a manner that does not require a linear scan. Our key observation is that even if the memory itself is not sparse, the difference (Δ) between an old memory state

³ By round we refer to the number of prover messages.

and a new memory state is sparse if $\ell \ll T$. Assume $RW = \{(a_i, r_i, w_i)\}_i$ is a vector consisting of a value r_i first read from a_i and a value w_i last written to a_i ; such a vector can be constructed from the memory trace by pairing any initial read with a final write and vice versa. Our first attempt is to use a logUp style argument to show that Δ is consistent with RW , i.e. to show that $\sum_{i=1}^{\ell} \frac{1}{X+Y \cdot a_i + (w_i - r_i)} = \sum_{i=1}^T \frac{m_i}{X+Y \cdot i + \Delta_i}$ for random variable X, Y . Note that this is an indexed lookup where we ensure that the values are also matched by address. This lookup ensures that each tuple in RW is consistent with Δ but is not sufficient, as it does not ensure that Δ is 0 at the positions for which there was no read or write. This is important, since an adversarial prover could use non-zero values in Δ to change the memory state arbitrarily. We must, therefore, prove that Δ is truly the sparse representation of the changes reflected by the reads and writes, while still keeping the prover time independent of T . Our idea is to ensure that the i th fraction is 0 if and only if $\Delta_i = 0$ by setting the numerator to Δ_i instead of m_i , i.e. check that $\sum_{i=1}^{\ell} \frac{(w_i - r_i)}{X+Y \cdot a_i + (w_i - r_i)} = \sum_{i=1}^T \frac{\Delta_i}{X+Y \cdot i + \Delta_i}$. We show that this check is still secure and indeed results in a check with $O(\ell)$ prover complexity.

LogUp powered memory-checking. The mem-update argument described above can prove that Δ is consistent with a list of read and write operations. We can then use a homomorphic commitment to Δ to update our commitment to the memory $M' \leftarrow M + \Delta$. Unfortunately, the system is still limited. Checking that Δ is computed correctly can suffice in a system where all write operations happen synchronously at the end of the computation step. If we want to write and then read from a memory cell within a computation step then we would first need to update the memory. However, this requires an expensive homomorphic commitment operation executed by the verifier, i.e. as part of the recursive circuit. To resolve this we combine our memory-update argument with the classic permutation-based memory-checking idea [BEGKN91]. The key difference is that in the classic memory-checking, the entire memory is added to the read and write sets. We instead only add the cells that are accessed by some read or write operation and use mem-update to prove the correctness of the update. We lay out the precise protocol in Section 6.

Accumulation scheme for GKR. The lookup and memory-checking protocols have *almost* optimal parameters. They require committing⁴ to just 10 vectors of length ℓ . However, 6 of these vectors consist of $\log |\mathbb{F}|$ -bit elements even if the memory itself only consist of, say 32-bit entries. Using homomorphic commitments requires fields of size at least 256 so this is a factor 8 blowup. Removing this blowup motivates the second orthogonal but highly compatible contribution of this paper. To this end, we built an efficient accumulation scheme for the GKR protocol. This can prove low-depth deterministic computations while only

⁴ Committing is by far the dominant prover cost in these systems. Committing to a message is between 100 and 1000 times as expensive as doing field operations on the same message. See <https://zka.lc/>.

committing to the computation’s inputs and outputs but not the intermediate values. Note that GKR is a special-sound interactive protocol with an algebraic verifier, which means it can directly be compiled with the ProtoStar compiler to an accumulation scheme. Unfortunately, GKR has $O(k \cdot \log n)$ rounds where k is the depth of the circuit and $\log n$ its width. This results in an accumulation verifier with $k \cdot \log n$ group operations. In the context of IVC, the accumulation verifier becomes part of the recursive circuit, and this is a significant overhead, especially when compared with other accumulation schemes which only have 1 to 3 group operations [KST22; KS23; BC23]. Our goal is therefore, to reduce the number of rounds of GKR while maintaining the attractive efficiency properties and the compatibility with the ProtoStar compiler.

In every round, GKR runs a multivariate sumcheck protocol, which has $\log n$ rounds. As a strawman, we can replace this multivariate sumcheck with a univariate one. This immediately reduces the number of GKR rounds from $k \cdot \log n$ to just k . Univariate sumcheck requires sending a quotient polynomial that is as large as the domain of the sumcheck, in our case $O(n)$. Committing to this polynomial would be at least as expensive as directly committing to the intermediate wires of the circuit, thus removing any benefit of using GKR. Fortunately, the idea of using a higher degree sumcheck can still help. Moving to a bivariate sumcheck reduces the communication to $O(\sqrt{n})$ while being only a 3-round protocol. The $O(\sqrt{n})$ commitment cost is, in most applications, dominated by the cost of committing to the input and output layers; even if not, we show that one can use a c -variate sumcheck to ensure that the sumchecks commitment cost is marginal. Using a bivariate sumcheck presents us with a couple of challenges. First, the verifier needs to evaluate a $O(\sqrt{n})$ degree polynomial, which is a $O(\sqrt{n})$ degree check if done naively. To resolve this we built a polynomial evaluation protocol, where with aid from the prover, the verification degree reduces to merely 3, independent of the degree of the polynomial.

Additionally, GKR batches polynomial evaluations, after each sumcheck, in order to only evaluate the next layer at a single point. In bivariate sumcheck, this would require computing a high-degree interpolation polynomial. We show that it is much simpler and more efficient to directly batch the resulting sumchecks. This observation is also applicable to multivariate sumchecks. We then construct a specific GKR for computing the sum of fractions, e.g. $\sum_{i=1}^n \frac{n_i}{d_i}$, similar to [PH23]. We also give specific optimizations for this instantiation, such as breaking up the circuit into multiple parts, while still maintaining the asymptotic properties. This optimization takes advantage of the circuit structure of sums of fractions, where the number of sums halves in every layer.

2 Preliminaries

Notation. For $n \in \mathbb{N}$, we use $[n]$ to denote the set $\{1, 2, \dots, n\}$. We denote λ as the security parameter and use \mathbb{F} to denote a field of prime order p such that $\log(p) = \Omega(\lambda)$. For list of tuples $ltup = [(a_i, b_i, c_i, \dots)]_{i=1}^k$ of arbitrary length

k , we use $ltup.\mathbf{a}$ to denote the list $[a_i]_{i=1}^k$, and $ltup.(\mathbf{a}, \mathbf{b})$ to denote the list $[(a_i, b_i)]_{i=1}^k$. For function f , \tilde{f} denotes the bivariate extension of f .

2.1 Special-sound Protocols

We take the definition of special-soundness from [AFK22; BC23].

Definition 1 (Public-coin interactive proof). *An interactive proof $\Pi = (P, V)$ for relation \mathcal{R} is an interactive protocol between two probabilistic machines, a prover P , and a polynomial time verifier V . Both P and V take as public input a statement pi and, additionally, P takes as private input a witness $\mathbf{w} \in \mathcal{R}(\text{pi})$. The verifier V outputs 0 if it accepts and a non-zero value otherwise. Its output is denoted by $(P(\mathbf{w}), V)(\text{pi})$. Accordingly, we say the corresponding transcript (i.e., the set of all messages exchanged in the protocol execution) is accepting or rejecting. The protocol is public coin if the verifier randomness is public. The verifier messages are referred to as challenges. Π is a $(2k - 1)$ -move protocol if there are k prover messages and $k - 1$ verifier messages.*

Definition 2 (Tree of transcript). *Let $\mu \in \mathbb{N}$ and $(a_1, \dots, a_\mu) \in \mathbb{N}^\mu$. An (a_1, \dots, a_μ) -tree of transcript for a $(2\mu + 1)$ -move public-coin interactive proof Π is a set of $a_1 a_2 \dots a_\mu$ accepting transcripts arranged in a tree of depth μ and arity a_1, \dots, a_μ respectively. The nodes in the tree correspond to the prover messages and the edges to the verifier's challenges. Every internal node at depth $i - 1$ ($1 \leq i \leq \mu$) has a_i children with distinct challenges. Every transcript corresponds to one path from the root to a leaf node. We simply write the transcripts as an (a^μ) -tree of transcript when $a = a_1 = a_2 = \dots = a_\mu$.*

Definition 3 (Special-sound Interactive Protocol). *Let $\mu, N \in \mathbb{N}$ and $(a_1, \dots, a_\mu) \in \mathbb{N}^\mu$. A $(2\mu + 1)$ -move public-coin interactive proof Π for relation \mathcal{R} where the verifier samples its challenges from a set of size N is (a_1, \dots, a_μ) -out-of- N special-sound if there exists a polynomial time algorithm that, on input pi and any (a_1, \dots, a_μ) -tree of transcript for Π outputs $\mathbf{w} \in \mathcal{R}(\text{pi})$. We simply denote the protocol as an a^μ -out-of- N (or a^μ) special-sound protocol if $a = a_1 = a_2 = \dots = a_\mu$.*

2.2 Commitment Scheme

Definition 4 (Commitment Scheme). *(Definition 6 from [BC23]) $\text{cm} = (\text{Setup}, \text{Commit})$ is a binding commitment scheme, consisting of two algorithms: $\text{Setup}(\lambda) \rightarrow \text{ck}$ takes as input the security parameter and outputs a commitment key ck .*

$\text{Commit}(\text{ck}, \mathbf{m} \in \mathcal{M}) \rightarrow C \in \mathcal{C}$, takes as input the commitment key ck and a message \mathbf{m} in \mathcal{M} and outputs a commitment $C \in \mathcal{C}$.

The scheme is binding if for all polynomial-time randomized algorithms P^ :*

$$\Pr \left[\begin{array}{c} \text{Commit}(\text{ck}, \mathbf{m}) = \text{Commit}(\text{ck}, \mathbf{m}') \\ \wedge \\ \mathbf{m} \neq \mathbf{m}' \end{array} \middle| \begin{array}{c} \text{ck} \leftarrow \text{Setup}(1^\lambda) \\ \mathbf{m}, \mathbf{m}' \leftarrow P^*(\text{ck}) \end{array} \right] = \text{negl}(\lambda)$$

Homomorphic commitment. We say the commitment is homomorphic if $(\mathcal{C}, +)$ is an additive group of prime order p .

2.3 Lookup Relation

Definition 5. (Definition 12 of [BC23]) Given configuration $\mathcal{C}_{LK} := (T, \ell, \mathbf{t})$ where ℓ is the number of lookups and $\mathbf{t} \in \mathbb{F}^T$ is the lookup table, the relation \mathcal{R}_{LK} is the set of tuples $\mathbf{w} \in \mathbb{F}^\ell$ such that $\mathbf{w}_i \in \mathbf{t}$ for all $i \in [\ell]$.

Lemma 1. (Lemma 5 of [Hab22])⁵ Let \mathbb{F} be a field of characteristic $p > \max(\ell, T)$. Given two sequences of field elements $[\mathbf{w}_i]_{i=1}^\ell$ and $[\mathbf{t}_i]_{i=1}^T$, we have $\{\mathbf{w}_i\} \subseteq \{\mathbf{t}_i\}$ as sets (with multiples of values removed) if and only if there exists a sequence $[\mathbf{m}_i]_{i=1}^T$ of field elements such that

$$\sum_{i=1}^{\ell} \frac{1}{X + \mathbf{w}_i} = \sum_{i=1}^T \frac{\mathbf{m}_i}{X + \mathbf{t}_i}. \quad (1)$$

2.4 Vector-valued lookup

Definition 6. (Definition 13 in [BC23]) Consider configuration $\mathcal{C}_{VLK} := (T, \ell, v \in \mathbb{N}, \mathbf{t})$ where ℓ is the number of lookups, and $\mathbf{t} \in (\mathbb{F}^v)^T$ is a lookup table in which the i th ($1 \leq i \leq T$) entry is

$$\mathbf{t}_i := (\mathbf{t}_{i,1}, \dots, \mathbf{t}_{i,v}) \in \mathbb{F}^v.$$

A sequence of vectors $\mathbf{w} \in (\mathbb{F}^v)^\ell$ is in relation \mathcal{R}_{VLK} if and only if for all $i \in [\ell]$,

$$\mathbf{w}_i := (\mathbf{w}_{i,1}, \dots, \mathbf{w}_{i,v}) \in \mathbf{t}.$$

As noted in Section 3.4 of [Hab22], we can extend Lemma 1 and replace (1) with

$$\sum_{i=1}^{\ell} \frac{1}{X + w_i(Y)} = \sum_{i=1}^T \frac{\mathbf{m}_i}{X + t_i(Y)} \quad (2)$$

where the polynomials are defined as

$$w_i(Y) := \sum_{j=1}^v \mathbf{w}_{i,j} \cdot Y^{j-1}, \quad t_i(Y) := \sum_{j=1}^v \mathbf{t}_{i,j} \cdot Y^{j-1},$$

which represent the witness vector $\mathbf{w}_i \in \mathbb{F}^v$ and the table vector $\mathbf{t}_i \in \mathbb{F}^v$.

⁵ This lookup argument is unofficially referred to as *logUp*.

2.5 Incremental Verifiable Computation (IVC)

In the following, we take an adapted from of the definition from [BCLMS21; KST22; BC23].

Definition 7 (IVC). (Definition 7 from [BC23]) An incremental verifiable computation (IVC) scheme for function predicates expressed in a circuit-satisfiability relation \mathcal{R}_{NP} is a tuple of algorithms $\text{IVC} = (\text{P}_{\text{IVC}}, \text{V}_{\text{IVC}})$ with the following syntax and properties:

- $\text{P}_{\text{IVC}}(m, z_0, z_m, z_{m-1}, \mathbf{w}_{\text{loc}}, \pi_{m-1}) \rightarrow \pi_m$. The IVC prover P_{IVC} takes as input a program output z_m at step m , local data \mathbf{w}_{loc} , initial input z_0 , previous program output z_{m-1} and proof π_{m-1} and outputs a new IVC proof π_m .
- $\text{V}_{\text{IVC}}(m, z_0, z_m, \pi_m) \rightarrow b$. The IVC verifier V_{IVC} takes the initial input z_0 , the output z_m at step m , and an IVC proof π_m , ‘accepts’ by outputting $b = 0$ and ‘rejects’ otherwise.

The scheme IVC has perfect adversarial completeness if for any function predicate ϕ expressible in \mathcal{R}_{NP} , and any, possibly adversarially created, $(m, z_0, z_m, z_{m-1}, \mathbf{w}_{\text{loc}}, \pi_{m-1})$ such that

$$\phi(z_0, z_m, z_{m-1}, \mathbf{w}_{\text{loc}}) \wedge (\text{V}_{\text{IVC}}(m-1, z_0, z_{m-1}, \pi_{m-1}) = 0)$$

it holds that $\text{V}_{\text{IVC}}(m, z_0, z_m, \pi_m)$ accepts for proof $\pi_m \leftarrow \text{P}_{\text{IVC}}(m, z_0, z_{m-1}, z_m, \mathbf{w}_{\text{loc}}, \pi_{m-1})$.

The scheme IVC has knowledge soundness if for every expected polynomial-time adversary P^* , there exists an expected polynomial-time extractor Ext_{P^*} such that

$$\Pr \left[\begin{array}{c} \text{V}_{\text{IVC}}(m, z_0, z, \pi_m) = 0 \wedge \\ ([\exists i \in [m], \neg \phi(z_0, z_i, z_{i-1}, \mathbf{w}_i)] \\ \vee z \neq z_m) \end{array} \middle| \begin{array}{c} [\phi, (m, z_0, z, \pi_m)] \leftarrow \text{P}^* \\ [z_i, \mathbf{w}_i]_{i=1}^m \leftarrow \text{Ext}_{\text{P}^*} \end{array} \right] \leq \text{negl}(\lambda).$$

Here m is a constant.

Efficiency. The runtime of P_{IVC} and V_{IVC} as well as the size of π_{IVC} only depend on $|\phi|$ and are independent on the number of iterations.

Definition 8 (Fiat-Shamir Heuristic). (Definition 9 from [BC23]) The Fiat-Shamir Heuristic, relative to a secure cryptographic hash function H , states that a random oracle NARK with negligible knowledge error yields a NARK that has negligible knowledge error in the standard (CRS) model if the random oracle is replaced with H .

Theorem 1 (IVC for high-degree special-sound protocols). (Theorem 3 from [BC23]) Let \mathbb{F} be a finite field, such that $|\mathbb{F}| \geq 2^\lambda$ and $\text{cm} = (\text{Setup}, \text{Commit})$ be a binding homomorphic commitment scheme for vectors in \mathbb{F} . Let $\Pi_{\text{sps}} = (\text{P}_{\text{sps}}, \text{V}_{\text{sps}})$ be a special-sound protocol for an NP-complete relation \mathcal{R}_{NP} with the following properties:

- It’s $(2k-1)$ move.

- It's (a_1, \dots, a_{k-1}) -out-of- $|\mathbb{F}|$ special-sound. Such that the knowledge error $\kappa = 1 - \prod_{i=1}^{k-1} (1 - \frac{a_i}{|\mathbb{F}|}) = \text{negl}(\lambda)$
- The inputs are in $\mathbb{F}^{\ell_{\text{in}}}$
- The verifier is degree $d = \text{poly}(\lambda)$ with output in \mathbb{F}^ℓ

Then, under the Fiat-Shamir heuristic for a cryptographic hash function \mathbf{H} (Definition 8), there exist two IVC schemes $\text{IVC} = (\text{P}_{\text{IVC}}, \text{V}_{\text{IVC}})$ and $\text{IVC}_{\text{CV}} = (\text{P}_{\text{CV,IVC}}, \text{V}_{\text{CV,IVC}})$ with predicates expressed in \mathcal{R}_{NP} with the efficiencies shown in Table 2.

Table 2. Efficiency of IVC schemes compiled from sps protocol

$\text{P}_{\text{IVC native}}$	$\sum_{i=1}^k \mathbf{m}_i^* \mathbb{G}$ $\text{P}_{\text{sps}} + L'(\text{V}_{\text{sps}}, d + 2)$
$\text{P}_{\text{IVC recursive}}$	$k + 2\mathbb{G}$ $k + \ell_{\text{in}} + d + 1\mathbb{F}$ $(k + d + O(1))\mathbf{H} + 1\mathbf{H}_{\text{in}}$
V_{IVC}	$\sum_{i=1}^k \mathbf{m}_i \mathbb{G}$ $O(\ell) + \text{V}_{\text{sps}}$
$ \pi_{\text{IVC}} $	$k + \ell_{\text{in}} + 1\mathbb{F}$ $k + 2\mathbb{G}$ $\sum_{i=1}^k \mathbf{m}_i $

In Table 2, $|\mathbf{m}_i|$ denotes the prover message length; $|\mathbf{m}_i^*|$ is the number of non-zero elements in \mathbf{m}_i ; \mathbb{G} for rows 1-3 is the total length of the messages committed using Commit. \mathbb{F} are field operations. \mathbf{H} denotes the total input length to a cryptographic hash, and \mathbf{H}_{in} is the hash to the public input and accumulator instance. P_{sps} (and V_{sps}) is the cost of running the prover (and the algebraic verifier) of the special-sound protocol, respectively. $L'(\text{V}_{\text{sps}}, d + 2)$ is the cost of computing the coefficients of the degree $d + 2$ polynomial

$$\begin{aligned}
e(X) &:= \sum_{a=0}^{\sqrt{\ell}-1} \sum_{b=0}^{\sqrt{\ell}-1} (X \cdot \pi \cdot \beta_a + \text{ACC} \cdot \beta_a)(X \cdot \pi \cdot \beta'_b + \text{ACC} \cdot \beta'_b) \\
&\sum_{j=0}^d (\mu + X)^{d-j} \cdot f_{j, a+b\sqrt{\ell}}^{\text{V}_{\text{sps}}}(\text{ACC} + X \cdot \pi),
\end{aligned} \tag{3}$$

where all inputs are linear functions in a formal variable X ⁶, and $f_{j,i}^{\text{V}_{\text{sps}}}$ is the i th ($0 \leq i \leq \ell - 1$) component of $f_j^{\text{V}_{\text{sps}}}$'s output. For the proof size, \mathbb{G} and \mathbb{F} are the number of commitments and field elements, respectively.

⁶ For example if $f_d = \prod_{i=1}^d (a_i + b_i \cdot X)$ then a naive algorithm takes $O(d^2)$ time but using FFTs it can be computed in time $O(d \log^2 d)$ [CBBZ22].

3 Special Sound Subrelations for Read/Write Lookup

We introduce the three lookup subprotocols that will be combined later to build the Read/Write Memory-Checking algorithm.

Handling Tuples. For simplicity, we describe the protocols as lookups and permutations on vectors of single values. However, when applied to memory-checking the entries could be tuples of addresses, values, and/or timestamps. Fortunately, this can be handled using a simple random linear combination, akin to the transformation from vector lookups to lookups (Lemma 6 of [BC23]). For sequence $\mathbf{b} = [\mathbf{b}_i]_{i=1}^n$ where each entry is a tuple of $k > 1$ element (i.e. $\mathbf{b}_i = (\mathbf{b}_{(i,j)})_{j=1}^k$ for every $i \in [n]$), \mathbf{b}_i will implicitly denote the random linear combination of the elements, i.e. $\sum_{j=1}^k Y^{j-1} \mathbf{b}_{(i,j)}$, whenever it appears in a formula. For example,

$$\frac{1}{X + \mathbf{b}_i} = \frac{1}{X + \sum_{j=1}^k Y^{j-1} \mathbf{b}_{(i,j)}}.$$

This is a k -special-sound transformation, so a previously (a_1, \dots, a_μ) -special-sound protocol becomes (k, a_1, \dots, a_μ) -special sound after it.

Achieving Perfect Completeness. The three protocols we introduce will not yet have perfect completeness since the prover will be sending over vectors of fractions of the form $\mathbf{h}_j = \frac{\mathbf{n}_j}{d_j} \quad \forall j \in [|\mathbf{h}|]$, where the computation of the denominator d is dependent on values in the given witness or lookup table. If there exists any value in some entry of the witness or lookup table such that $d = 0$, then the prover message will be undefined. We can achieve perfect completeness by following the same strategy for achieving perfect completeness in Π_{LK} in [BC23], which is to have the verifier set $\mathbf{h}_j = 0$ in this case and changing the verification equation from $\mathbf{h}_j \cdot d_j \stackrel{?}{=} \mathbf{n}_j$ to

$$d_j \cdot (\mathbf{h}_j \cdot d_j - \mathbf{n}_j) \stackrel{?}{=} 0$$

The new check ensures that either $\mathbf{h}_j = \frac{\mathbf{n}_j}{d_j}$ or $d_j = 0$. This increases the verifier degree in all of the three subprotocols to 3. Without these checks, the protocol has a negligible completeness error of $(\frac{\sum_i |\mathbf{h}_i|}{|\mathbb{F}|})$, where $\mathbf{h}_1, \mathbf{h}_2, \dots$ are the vectors of fractions sent by the prover. This completeness error is negligible. However, IVC and thus accumulation from which IVC is constructed require the protocols to be perfectly complete [BCLMS21] because IVC is designed for distributed computations where the continuance of computation is important, even on adversarially generated inputs.

3.1 Checking Permutation Using Lookup Relation

Definition 9. (Definition 10 from [BC23]) Two sequences of field elements $\mathbf{w} = [\mathbf{w}_i]_{i=1}^n$, $\mathbf{t} = [\mathbf{t}_i]_{i=1}^n$ are in $\mathcal{R}_{\text{perm}}$ if there exists permutation $\sigma : [n] \rightarrow [n]$ such that for all $i \in [n]$, $\mathbf{w}_i = \mathbf{t}_{\sigma(i)}$.

Lemma 2. Let \mathbb{F} be a field of characteristic $p > \max(\ell, T)$. Given two sequences of field elements $\mathbf{w} = [\mathbf{w}_i]_{i=1}^\ell$ and $\mathbf{t} = [\mathbf{t}_i]_{i=1}^T$, we have \mathbf{w}, \mathbf{t} are permutations of each other (i.e. \mathbf{w}, \mathbf{t} are in \mathcal{R}_{perm}) if and only if $\ell = T$ and

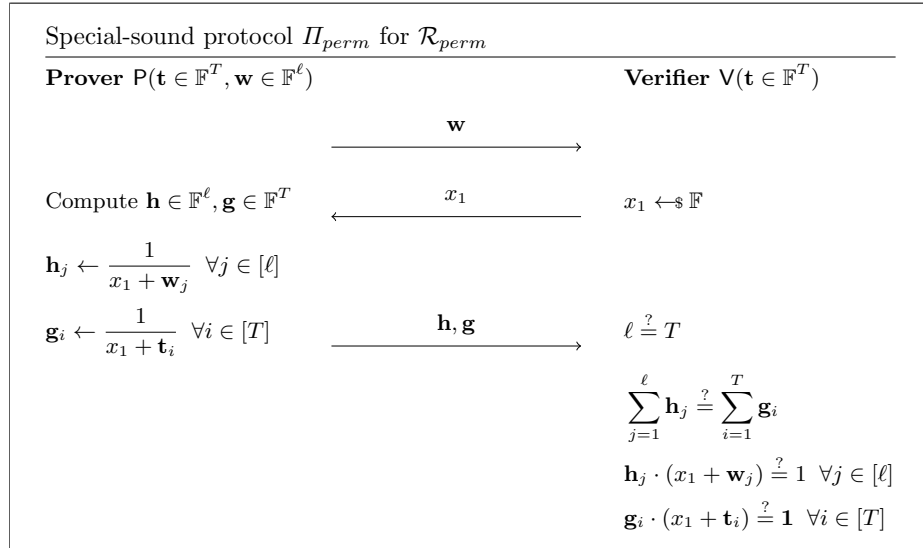
$$\sum_{i=1}^{\ell} \frac{1}{X + \mathbf{w}_i} = \sum_{i=1}^T \frac{1}{X + \mathbf{t}_i}. \quad (4)$$

Proof. Suppose $\ell = T$ and (4) holds. This implies there exists a sequence $[\mathbf{m}_i]_{i=1}^T$ of field elements where $\mathbf{m}_i = 1 \ \forall i \in [T]$ such that

$$\sum_{i=1}^{\ell} \frac{1}{X + \mathbf{w}_i} = \sum_{i=1}^T \frac{\mathbf{m}_i}{X + \mathbf{t}_i} \text{ and } \sum_{i=1}^{\ell} \frac{\mathbf{m}_i}{X + \mathbf{w}_i} = \sum_{i=1}^T \frac{1}{X + \mathbf{t}_i}$$

By Lemma 1, this means $\{\mathbf{w}_i\} \subseteq \{\mathbf{t}_i\}$ and $\{\mathbf{t}_i\} \subseteq \{\mathbf{w}_i\}$ as sets. Hence, it must be that $\mathbf{w} = \mathbf{t}$. The converse direction is trivial.

We can therefore describe a special-sound protocol Π_{perm} for \mathcal{R}_{perm} by simply adding the check $\ell \stackrel{?}{=} T$ and removing the need to compute \mathbf{m} from Π_{LK} for \mathcal{R}_{LK} in [BC23].



Complexity. Π_{perm} is a 3-move protocol (i.e. $k = 2$); the degree of the verifier is 2; the number of non-zero elements in the prover message is at most $2\ell + T$.

Special-Soundness. Just as Π_{LK} from [BC23], the perfect complete version of Π_{perm} is $2(\ell + T)$ -special-sound, assuming each entry $\mathbf{w}_j, \mathbf{t}_i$ is a single value for all $j \in [\ell], i \in [T]$.

3.2 Indexed-Vector Lookup Relation

Definition 10. (*Indexed-Vector Lookup Relation*) Given configuration $\mathcal{C}_{ivlk} := (T, \ell, \mathbf{t})$ where ℓ is the number of lookups and $\mathbf{t} \in \mathbb{F}^T$ is the lookup table, the triple $(\mathbf{t}, \mathbf{w} \in \mathbb{F}^\ell, \mathbf{b} \in \mathbb{F}^\ell)$ are in the relation \mathcal{R}_{ivlk} if for all $j \in [\ell]$, $\mathbf{b}_j \in [T]$ and $\mathbf{w}_j = \mathbf{t}_{\mathbf{b}_j}$.

Lemma 3 and 4 in the following are extensions on Lemma 4 and 5 from [Hab22], respectively.

Lemma 3. Let \mathbb{F} be an arbitrary field and $f_1, f_2 : \mathbb{F}^2 \rightarrow \mathbb{F}$ any functions. Then

$$\sum_{\substack{z_1, z_2 \in \mathbb{F}^2, \\ X - z_1 \cdot Y - z_2 \neq 0}} \frac{f_1(z_1, z_2)}{X - z_1 \cdot Y - z_2} = \sum_{\substack{z_1, z_2 \in \mathbb{F}^2, \\ X - z_1 \cdot Y - z_2 \neq 0}} \frac{f_2(z_1, z_2)}{X - z_1 \cdot Y - z_2} \quad (5)$$

in the rational function field $\mathbb{F}(X, Y)$, if and only if $f_1(z_1, z_2) = f_2(z_1, z_2)$ for every $z_1, z_2 \in \mathbb{F}^2$.

Proof. Our proof strategy follows the proof of Lemma 4 in [Hab22].

Suppose that Equation (5) holds. Then

$$\sum_{\substack{z_1, z_2 \in \mathbb{F}^2, \\ X - z_1 \cdot Y - z_2 \neq 0}} \frac{f_1(z_1, z_2) - f_2(z_1, z_2)}{X - z_1 \cdot Y - z_2} = 0$$

Fix Y at any arbitrary point $y \in \mathbb{F}$, we get $\sum_{\substack{z_1, z_2 \in \mathbb{F}^2, \\ X - z_1 \cdot y - z_2 \neq 0}} \frac{f_1(z_1, z_2) - f_2(z_1, z_2)}{X - z_1 \cdot y - z_2}$, and therefore have the polynomial

$$\begin{aligned} p(X, y) &= \prod_{q \in \mathbb{F}} (X - q) \cdot \sum_{\substack{z_1, z_2 \in \mathbb{F}^2, \\ X - z_1 \cdot y - z_2 \neq 0}} \frac{f_1(z_1, z_2) - f_2(z_1, z_2)}{X - z_1 \cdot y - z_2} \\ &= \sum_{z_1 \in \mathbb{F}, z_2 \in \mathbb{F}} (f_1(z_1, z_2) - f_2(z_1, z_2)) \cdot \prod_{q \in \mathbb{F} \setminus \{z_1 \cdot y - z_2\}} (X - q) = 0 \end{aligned}$$

In particular, for every pair $z_1 \in \mathbb{F}, z_2 \in \mathbb{F}$,

$$p(z_1 \cdot y - z_2, y) = (f_1(z_1, z_2) - f_2(z_1, z_2)) \cdot \prod_{q \in \mathbb{F} \setminus \{z_1 \cdot y - z_2\}} (z_1 \cdot y - z_2 - q) = 0$$

Since $\prod_{q \in \mathbb{F} \setminus \{z_1 \cdot y - z_2\}} (z_1 \cdot y - z_2 - q)$ is not zero, it must be that $f_1(z_1, z_2) = f_2(z_1, z_2)$ for every pair $z_1, z_2 \in \mathbb{F}^2$. The other direction is trivial.

Lemma 4. Let \mathbb{F} be a field of characteristic $p > \max\{\ell, T\}$. Given a sequence of field elements $\mathbf{w} \in \mathbb{F}^\ell, \mathbf{b} \in \mathbb{F}^\ell, \mathbf{t} \in \mathbb{F}^T$, we have $(T, \ell, \mathbf{t}, \mathbf{w}, \mathbf{b}) \in \mathcal{R}_{ivlk}$ if and only if the following equation holds in the function field $F(X, Y)$

$$\sum_{j=1}^{\ell} \frac{1}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j} = \sum_{i=1}^T \frac{\mathbf{m}_i}{X + Y \cdot i + \mathbf{t}_i} \quad (6)$$

where $\mathbf{m} = \{\mathbf{m}_i\}_{i=1}^T$ is the counter vector such that \mathbf{m}_i is the count of (i, \mathbf{t}_i) in (\mathbf{b}, \mathbf{w}) .

Proof. Our proof strategy follows the proof of Lemma 5 in [Hab22].

Suppose $(T, \ell, \mathbf{t}, \mathbf{w}) \in \mathcal{R}_{ivlk}$, then the equation is guaranteed to be true.

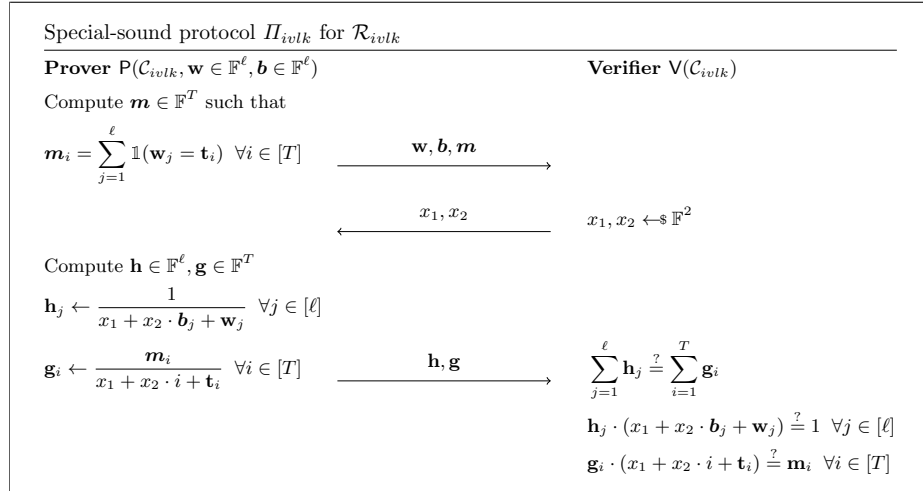
$$\sum_{i=1}^T \frac{\mathbf{m}_i}{X + Y \cdot i + \mathbf{t}_i} = \sum_{j=1}^{\ell} \frac{\mathbf{m}_{\mathbf{b}_j}}{X + Y \cdot \mathbf{b}_j + \mathbf{t}_{\mathbf{b}_j}} = \sum_{j=1}^{\ell} \frac{1}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j}$$

Conversely, suppose (6) holds. Collect fractions with the same denominator for the left side and re-expressing the right side we obtain,

$$\sum_{j=1}^{\ell} \frac{1}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j} = \sum_{z_1 \in \mathbb{F}, z_2 \in \mathbb{F}} \frac{\mu_{\mathbf{w}}(z_1, z_2)}{X + Y \cdot z_1 + z_2} = \sum_{i=1}^T \frac{\mathbf{m}_i}{X + Y \cdot i + \mathbf{t}_i}$$

where $\mu_{\mathbf{w}}(z_1, z_2)$ is the count of the tuple (z_1, z_2) in (\mathbf{b}, \mathbf{w}) . By the uniqueness of bivariate fractional representations from Lemma 3, we have that for every non-zero $\frac{\mu_{\mathbf{w}}(z_1, z_2)}{X + Y \cdot z_1 + z_2}$, there must exist a fraction in the $\sum_{i=1}^T \frac{\mathbf{m}_i}{X + Y \cdot i + \mathbf{t}_i}$ decomposition with equivalent numerator and denominator. This implies that for non-zero $\mu_{\mathbf{w}}(z_1, z_2) = \mathbf{m}_{z_1}$, and $\mathbf{t}_{z_1} = z_2$. Thus, for all $j \in [\ell]$ such that $\mu_{\mathbf{w}}(z_1, z_2) \neq 0$, $\mathbf{t}_{\mathbf{b}_j} = \mathbf{w}_j$ and $\mathbf{m}_{\mathbf{b}_j}$ is the count of $(\mathbf{b}_j, \mathbf{w}_j) = (i, \mathbf{t}_i)$ in (\mathbf{b}, \mathbf{w}) .

We can therefore describe a special-sound protocol for the indexed-vector lookup relation.



Complexity. Π_{ivlk} is a 3-move protocol (i.e. $k = 2$); the degree of the verifier is 2; the number of non-zero elements in the prover message is at most 5ℓ .

Lemma 5. Π_{ivlk} is $((\ell + T), 2(\ell + T))$ -special-sound.

Proof. We construct an extractor Ext that outputs \mathbf{w}, \mathbf{b} . We look at the $(\ell + T)$ transcripts that all have $\mathbf{w}, \mathbf{b}, \mathbf{m}$ as the first message but different $(x_1^{(p)})$ as the first challenge in the second message; then for each fixed $x_1^{(p)}$, we look at $2(\ell + T)$ transcripts that have $x_1^{(p)}$ as the first challenge but different $(x_2^{(q)}, \mathbf{h}^{(q)} \in \mathbb{F}^\ell, \mathbf{g}^{(q)} \in \mathbb{F}^T)$ as the rest of the transcript, totalling $2(\ell + T)^2$ transcripts.

By the pigeonhole principle, for each $p \in [\ell + T]$, there must exist a subset of $S \subseteq [2(\ell + T)]$ transcripts such that $|S| = \ell + T$ and $x_1^{(p)} + x_2^{(q)} \cdot \mathbf{b}_j + \mathbf{w}_j \neq 0$ for all $j \in [\ell]$ and $q \in S$, and $x_1^{(p)} + x_2^{(q)} \cdot i + \mathbf{t}_i \neq 0$ for all $i \in [T]$ and $q \in S$. For these transcripts, we have $\mathbf{h}_j = \frac{1}{x_1^{(p)} + x_2^{(q)} \cdot \mathbf{b}_j + \mathbf{w}_j}$ and $\mathbf{g}_i = \frac{\mathbf{m}_i}{x_1^{(p)} + x_2^{(q)} \cdot i + \mathbf{t}_i}$. Define the degree $\ell + T - 1$ polynomial

$$f(X, Y) = \prod_{p=1}^{\ell} (X + Y \cdot \mathbf{b}_p + \mathbf{w}_p) \cdot \prod_{q=1}^T (X + Y \cdot 1 + \mathbf{t}_q) \cdot \left(\sum_{j=1}^{\ell} \frac{\mathbf{w}_j}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j} - \sum_{i=1}^T \frac{\mathbf{t}_i}{X + Y \cdot i + \mathbf{t}_i} \right)$$

If $f(X, Y)$ is the zero polynomial then $\sum_{j=1}^{\ell} \frac{1}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j} = \sum_{i=1}^T \frac{\mathbf{m}_i}{X + Y \cdot i + \mathbf{t}_i}$ and by Lemma 6 $(\mathcal{C}_{ivlk}; \mathbf{w}, \mathbf{b}) \in \mathcal{R}_{mu}$. Since we have $(\ell + T)$ points $(x_1^{(p)}, x_2^{(q)})$ at which $f(x_1^{(p)}, x_2^{(q)}) = 0$, we get $f = 0$ and thus that the extracted witness (\mathbf{w}, \mathbf{b}) is valid.

3.3 Mem-Update Relation

Definition 11 (Mem-Update Relation). Given configuration $\mathcal{C}_{mu} := (T, \ell, \Delta)$ where ℓ is the number of lookups and $\Delta \in \mathbb{F}^T$ is the lookup table, the triple $(\mathbf{w} \in \mathbb{F}^\ell, \mathbf{b} \in \mathbb{F}^\ell, \Delta)$ are in the relation \mathcal{R}_{mu} if for all $j \in [\ell]$, if $\mathbf{w}_j \neq 0$ then $\mathbf{w}_j = \Delta_{\mathbf{b}_j}$, and for all $i \in [T]$, if $\Delta_i \neq 0$ then there exists $j \in [\ell]$ such that $\mathbf{b}_j = i$ and $\Delta_i = \mathbf{w}_j$.

Lemma 6. Let \mathbb{F} be a field of characteristic $p > \max\{\ell, T\}$. Given the sequences of field elements $\mathbf{w} \in \mathbb{F}^\ell, \mathbf{b} \in \mathbb{F}^\ell, \Delta \in \mathbb{F}^T$, we have $(T, \ell, \Delta, \mathbf{w}, \mathbf{b}) \in \mathcal{R}_{mu}$ if and only if the following equation holds in the function field $F(X, Y)$

$$\sum_{j=1}^{\ell} \frac{\mathbf{w}_j}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j} = \sum_{i=1}^T \frac{\Delta_i}{X + Y \cdot i + \Delta_i} \quad (7)$$

Proof. Suppose $(T, \ell, \Delta, \mathbf{w}, \mathbf{b}) \in \mathcal{R}_{mu}$, then the equation is guaranteed to be true.

$$\sum_{i=1}^T \frac{\Delta_i}{X + Y \cdot i + \Delta_i} = \sum_{j=1}^{\ell} \frac{\Delta_{\mathbf{b}_j}}{X + Y \cdot \mathbf{b}_j + \Delta_{\mathbf{b}_j}} = \sum_{j=1}^{\ell} \frac{\mathbf{w}_j}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j}$$

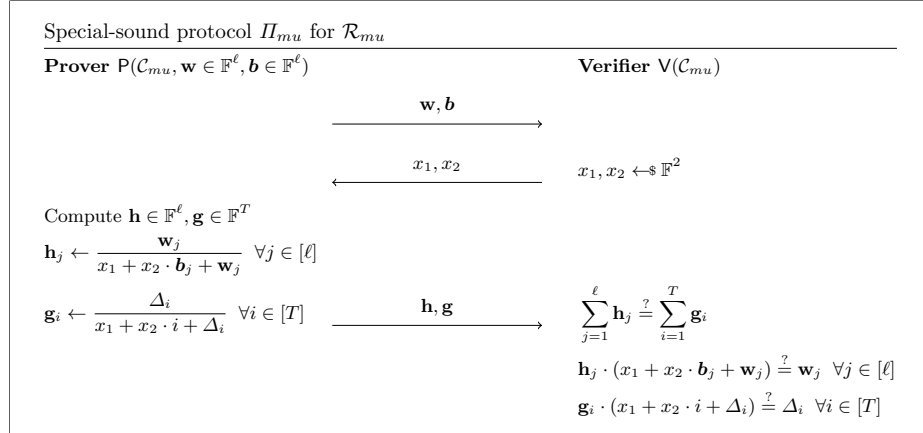
Conversely, suppose (7) holds. Collect fractions with the same denominator for the left side and re-expressing the right side we obtain,

$$\sum_{j=1}^{\ell} \frac{\mathbf{w}_j}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j} = \sum_{z_1 \in \mathbb{F}, z_2 \in \mathbb{F}} \frac{z_2 \cdot \mu_{\mathbf{w}}(z_1, z_2)}{X + Y \cdot z_1 + z_2} = \sum_{i=1}^T \frac{\Delta_i}{X + Y \cdot i + \Delta_i}$$

where $\mu_{\mathbf{w}}(z_1, z_2)$ is the count of the tuple (z_1, z_2) in (\mathbf{b}, \mathbf{w}) . By the uniqueness of bivariate fractional representations from Lemma 3, we have that for every non-zero $\frac{z_2 \cdot \mu_{\mathbf{w}}(z_1, z_2)}{X + Y \cdot z_1 + z_2}$, there must exist a fraction in the $\sum_{i=1}^T \frac{\Delta_i}{X + Y \cdot i + \Delta_i}$ decomposition with equivalent numerator and denominator. This implies that for non-zero $z_2 \cdot \mu_{\mathbf{w}}(z_1, z_2) = \Delta_{z_1} = z_2$, so $\mu_{\mathbf{w}}(z_1, z_2) = 1$. Thus, for all $j \in [\ell]$ such that $\mathbf{w}_j \neq 0$, $\Delta_{\mathbf{b}_j} = \mathbf{w}_j$.

For every $\frac{\Delta_i}{X + Y \cdot i + \Delta_i}$, there must exist a fraction in the $\sum_{z_1 \in \mathbb{F}, z_2 \in \mathbb{F}} \frac{z_2 \cdot \mu_{\mathbf{w}}(z_1, z_2)}{X + Y \cdot z_1 + z_2}$ decomposition with the equivalent numerator and denominator, which implies $\Delta_i = z_2 \cdot \mu_{\mathbf{w}}(z_1, z_2) = z_2$ for every Δ_i . Therefore if $\Delta_i \neq 0$ then $\mu_{\mathbf{w}}(z_1, z_2) \neq 0$, and thus there exists $j \in [\ell]$ such that $\mathbf{b}_j = i$ and $\Delta_i = \mathbf{w}_j$.

We describe a $((\ell + T), 2(\ell + T))$ -special-sound protocol for the mem-update relation.



Complexity. Π_{mu} is a 3-move protocol (i.e. $k = 2$); the degree of the verifier is 2; the number of non-zero elements in the prover message is at most 4ℓ .

Lemma 7. Π_{mu} is $((\ell + T), 2(\ell + T))$ -special-sound, assuming each entry \mathbf{w}_j, Δ_i for all $j \in [\ell], i \in [T]$ is a single value.

Proof. We construct an extractor Ext that outputs \mathbf{w}, \mathbf{b} . We look at the $(\ell + T)$ transcripts that all have \mathbf{w}, \mathbf{b} as the first message but different $(x_1^{(p)})$ as the first challenge in the second message; then for each fixed $x_1^{(p)}$, we look at $2(\ell + T)$ transcripts that have $x_1^{(p)}$ as the first challenge but different $(x_2^{(q)}, \mathbf{h}^{(q)} \in \mathbb{F}^{\ell}, \mathbf{g}^{(q)} \in \mathbb{F}^T)$ as the rest of the transcript, totalling $2(\ell + T)^2$ transcripts.

By the pigeonhole principle, for each $p \in [\ell + T]$, there must exist a subset of $S \subseteq [2(\ell + T)]$ transcripts such that $|S| = \ell + T$ and $x_1^{(p)} + x_2^{(q)} \cdot \mathbf{b}_j + \mathbf{w}_j \neq 0$ for all $j \in [\ell]$, and $q \in S$, and $x_1^{(p)} + x_2^{(q)} \cdot i + \Delta_i \neq 0$ for all $i \in [T]$ and $q \in S$. For these transcripts, we have $\mathbf{h}_j = \frac{\mathbf{w}_j}{x_1^{(p)} + x_2^{(q)} \cdot \mathbf{b}_j + \mathbf{w}_j}$ and $\mathbf{g}_i = \frac{\Delta_i}{x_1^{(p)} + x_2^{(q)} \cdot i + \Delta_i}$. Define the degree $\ell + T - 1$ polynomial

$$f(X, Y) = \prod_{p=1}^{\ell} (X + Y \cdot \mathbf{b}_p + \mathbf{w}_p) \cdot \prod_{q=1}^T (X + Y \cdot q + \Delta_q) \cdot \left(\sum_{j=1}^{\ell} \frac{\mathbf{w}_j}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j} - \sum_{i=1}^T \frac{\Delta_i}{X + Y \cdot i + \Delta_i} \right)$$

If $f(X, Y)$ is the zero polynomial then $\sum_{j=1}^{\ell} \frac{\mathbf{w}_j}{X + Y \cdot \mathbf{b}_j + \mathbf{w}_j} = \sum_{i=1}^T \frac{\Delta_i}{X + Y \cdot i + \Delta_i}$ and by Lemma 6 $(\mathcal{C}_{mu}; \mathbf{w}, \mathbf{b}) \in \mathcal{R}_{mu}$. Since we have $(\ell + T)$ points $(x_1^{(p)}, x_2^{(q)})$ at which $f(x_1^{(p)}, x_2^{(q)}) = 0$, we get $f = 0$ and thus that the extracted witness (\mathbf{w}, \mathbf{b}) is valid.

Efficiency in Accumulation. We refer to Table 3 for an overview over the efficiency of the protocol. Importantly the prover time is entirely independent of T . The protocol can also be combined with our GKR protocol as layed out in Section 5. This reduces the prover time by eliminating the multi-scalar multiplication with full field elements. It is, thus, a useful option when the size of the table elements is significantly smaller than the field, e.g. 32-bit elements vs a 256-bit field.

	P Time	P Msg	deg(V)	# P Msgs	P _{ACC} Time	V _{ACC} Time
Direct	$O(\ell)$	$2\ell \mathbb{F} + 2\ell \mathbb{T}$	7	2	$(2\ell, \mathbb{T})$ -MSM + $(2\ell, \mathbb{F})$ -MSM	$3\mathbb{G}$
With GKR	$O(\ell)$	$2\ell \mathbb{F} + 2\ell \mathbb{T}$	7	$(c + 1) \log T$	$(2\ell, \mathbb{T})$ -MSM + $O(\ell \log \ell) \mathbb{F}$	$(c + 1) \log T \mathbb{G}$

Table 3. Efficiency Table for Accumulating Π_{mu} . We only list the dominant efficiency factors. Column 2 refers to the total size of the prover messages. Here \mathbb{T} is the set of small elements that are stored in the table, whereas \mathbb{F} refers to full field elements. Column 3 is the verifier degree. Column 5 is the number of prover messages. Note that the number of messages in the GKR case can be further reduced with the optimizations mention in Section 5. Column 6 is the dominant factor in the prover time. An (a, B) -MSM refers to a multi-scalar multiplication of a scalars that are each within the set B . The MSM scales roughly linear in $|\log B|$. Column 7 is the number of group scalar multiplications the accumulation verifier performs.

4 An accumulation scheme for GKR

The GKR protocol is special-sound, but using GKR for the lookup relations naively in accumulation will result in $\log^2 \ell$ rounds (assuming ℓ is the number

of inputs), which is expensive since the ProtoStar compiler pays linearly in the number of rounds. Hence, we wish to design a version of the GKR protocol that is better suited for accumulation, i.e. one that takes fewer rounds but retains the property that the prover only pays for the input and not any intermediate values. The core ingredient will be a bivariate sumcheck protocol which is well suited for accumulation.

4.1 Subprotocol for the verifier to efficiently evaluate a function

Bivariate sumcheck requires the verifier to evaluate polynomials of degree $\Theta(\sqrt{n})$, where n is the width of the GKR circuit. This is prohibitively large. Fortunately, we can transform evaluation into a low degree check by sending additional witnesses. We describe the low-degree evaluation protocol below.

Subprotocol Π_{eval} for evaluating $f : \mathbb{F}^k \rightarrow \mathbb{F}$ at some $\mathbf{a} \in (\mathbb{F} \setminus \mathbb{H})^k$ s.t. $m := \mathbb{H} > \deg(f)$	
Prover $\mathcal{P}(f, \mathbf{a} = [\mathbf{a}_1, \dots, \mathbf{a}_k])$	Verifier $\mathcal{V}(f, \mathbf{a}, [f(\mathbf{x})]_{\mathbf{x} \in \mathbb{H}^k})$
	$\xrightarrow{f(\mathbf{x}) \ \forall \mathbf{x} \in \mathbb{H}^k}$
$\mathbf{a}^i \leftarrow [\mathbf{a}_1^i, \dots, \mathbf{a}_k^i] \ \forall i \in \{2, 4, \dots, m\}$	$\xrightarrow{A := (\mathbf{a}, \mathbf{a}^2, \mathbf{a}^4, \dots, \mathbf{a}^m)}$
$L^{\mathbb{H}}(\omega_i, u) := \frac{c_i(u^m - 1)}{u - \omega_i} \ \forall i \in [m]$	
$eq(\mathbf{x}, \mathbf{a}) \leftarrow \prod_{j=1}^k L_{x_j}(\mathbf{a}_j)$	$\xrightarrow{eq(\mathbf{x}, \mathbf{a}) \ \forall \mathbf{x} \in \mathbb{H}^k}$
	$A(1) \stackrel{?}{=} \mathbf{a} \quad A(i) \stackrel{?}{=} A(i-1)^2 \ \forall i \in \{2, \dots, \log m\}$ $eq(\mathbf{x}, \mathbf{a}) \cdot \prod_{j=1}^k (\mathbf{a}_j - \omega_{x_j}) \stackrel{?}{=} \prod_{j=1}^k c_{x_j} (A(\log m + 1, j) - 1) \ \forall \mathbf{x} \in \mathbb{H}^k$ $f(\mathbf{a}) \leftarrow \sum_{\mathbf{x} \in \mathbb{H}^k} eq(\mathbf{x}, \mathbf{a}) f(\mathbf{x})$

Efficiency. The verification degree is $k + 1$. The prover sends $m^k = |f|$ values.

In the protocol above, \mathbb{H} is a multiplicative subgroup of \mathbb{F} , and we assume $m := |\mathbb{H}|$ is a multiple of 2. c_i is the barycentric weight, ω_i is the i th root of unity, and L_i is the Lagrange polynomial. Note that \mathcal{P} sends over a $\log m \times k$ matrix A . $A(i) := \mathbf{a}^{2^{i-1}}$ denotes the i th row of A , and $A(i, j) := \mathbf{a}_j^{2^{i-1}}$.

Security. The protocol has perfect completeness and soundness. The first line of checks ensure that the matrix A was computed correctly as claimed by the prover. In the second line of check, note that $A(\log m, j) = \mathbf{a}_j^{2^{\log m}} = \mathbf{a}_j^m$. Hence if the equality holds, we have

$$eq(\mathbf{x}, \mathbf{a}) = \prod_{j=1}^k \frac{c_{x_j}(\mathbf{a}_j^m) - 1}{\mathbf{a}_j - \omega_{x_j}} = \prod_{j=1}^k L_{x_j}(\mathbf{a}_j) \ \forall \mathbf{x} \in \mathbb{H}^k$$

which indicates that $eq(\mathbf{x}, \mathbf{a})$ was computed correctly as claimed by the verifier. This implies that the two polynomials $f(\mathbf{a})$ and $\sum_{\mathbf{x} \in \mathbb{H}^k} eq(\mathbf{x}, \mathbf{a}) f(\mathbf{x})$ are equal on m^k points. Since both of these polynomials have degree strictly smaller than m , being equal on m^k points indicates that they are the same polynomial.

4.2 Bivariate Sumcheck

We describe a bivariate sumcheck protocol because the Protostar compiler pays linearly in the number of rounds, and hence the number of variables. While there is a tradeoff between the number of variables and the degree in each variable, high degrees can be tolerated in the final accumulation scheme because the decider only runs once.

Bivariate Sumcheck to prove $\sum_{x,y \in \mathbb{G}^2} f(x,y) = T$, where $\mathbb{G} \subset \mathbb{H}$ s.t. $m := \mathbb{H} = \deg(f) + 1$	
Prover $\mathsf{P}(f, T)$	Verifier $\mathsf{V}(f, T)$
$f_1(X) \leftarrow \sum_{y \in \mathbb{G}} f(X, y)$	$a \leftarrow \mathfrak{s} \mathbb{F} \setminus \mathbb{H}$
$\xrightarrow{f_1(\omega_i) \ \forall i \in [m]}$	\xleftarrow{a}
$f_2(Y) \leftarrow f(a, Y)$	$b \leftarrow \mathfrak{s} \mathbb{F} \setminus \mathbb{H}$
$\xrightarrow{f_2(\omega_i) \ \forall i \in [m]}$	\xleftarrow{b}
$T^* \leftarrow f_2(b)$	Use Eval to evaluate $f_1(a), f_2(b)$
$\xrightarrow{T^*}$	$\sum_{x \in \mathbb{G}} f_1(x) \stackrel{?}{=} T$ $\sum_{y \in \mathbb{G}} f_2(y) \stackrel{?}{=} f_1(a)$ $T^* \stackrel{?}{=} f_2(b)$ $T^* \stackrel{?}{=} f(a, b)$

Security. The protocol is clearly perfectly complete. It is (m, m) -special-sound. For a fixed challenge a_i , to show that $f_2(Y) = f(a_i, Y)$ requires the equality to hold for $\deg(f_2)+1 = \deg_Y(f)+1 \leq \deg(f)+1 = m$ different challenges for Y , i.e. b_1, \dots, b_m . Then, since $f_2(Y) = f(a_i, Y)$, checking whether $\sum_{y \in \mathbb{G}} f_2(y) = f_1(a_i)$ is equivalent to checking $\sum_y f(a_i, y) = f_1(a_i)$ for any fixed a_i . To show that $f_1(X) = \sum_{y \in \mathbb{G}} f(X, y)$ requires the equality to hold for $\deg(f_1)+1 = \deg_X(f)+1 \leq \deg(f)+1 = m$ different challenges for X , i.e. a_1, \dots, a_m . Therefore, with m different challenges on X and m different challenges on Y , the verifier can be sure that $\sum_{x \in \mathbb{G}} f_1(x) = \sum_{x,y \in \mathbb{G}^2} f(x,y)$. Finally, since $\sum_{x \in \mathbb{G}} f(x) = T$, it is verified that $\sum_{x,y \in \mathbb{G}} f(x,y) = T$.

P Time	P Msg	$\deg(\mathbf{V}^f)$	# P Msgs
$n \log n \mathbb{F}$	$4\sqrt{n} + o(\sqrt{n}) + \deg(\mathbf{V}^f) \mathbb{F}$ or hashes	2	3

Table 4. Efficiency Table for Accumulating SumCheck ($n := |f|$)

The number of P messages shown in Table 4 is the number when the polynomial f in the sumcheck is non-sparse. Since the polynomial f will be sparse

(sublinear in the memory size T) when performing memory check using our lookup protocol, the actual number of P messages will be much smaller.

$\deg(\mathbf{V}^f)$ stands for the degree of the \mathbf{V} with oracle access to the function f , i.e. \mathbf{V} can evaluate f using a single query.

4.3 Batching subprotocol for GKR

Description of the Batching Subprotocol for batching k sumchecks into one:

- Given a list of tuples $[(g_j \in \mathbb{F}[X_1, \dots, X_c], T_j \in \mathbb{F})]_{j=1}^k$ and \mathbb{H}^c , such that $\sum_{\mathbf{x} \in \mathbb{H}^c} g_j(\mathbf{x}) = T_j$ for all $j \in [k]$.
- \mathbf{V} chooses $r \leftarrow \mathbb{F}$ at random and sends it to \mathbf{P} .
- \mathbf{V} batches all k sumchecks checks into one as follows

$$\sum_{\mathbf{x} \in \mathbb{H}^c} f(\mathbf{x}) \stackrel{?}{=} \sum_{j=1}^k r^{j-1} T_j$$

for $f(\mathbf{x}) := \sum_{j=1}^k r^{j-1} g_j(\mathbf{x})$. Note that if $g_j(\mathbf{x}) = eq(\mathbf{z}_j, \mathbf{x})g(\mathbf{x})$ then $f(\mathbf{x}) = g(\mathbf{x}) \cdot (\sum_{j \in [k]} r^{j-1} \cdot eq(\mathbf{z}_j, \mathbf{x}))$

Efficiency. In GKR we call this protocol with $g_j(\mathbf{x}) = g(\mathbf{x}) \cdot eq(\mathbf{z}_j, \mathbf{x})$. This means that the complexity of the batched sumcheck is equivalent to the complexity of sumcheck over g plus evaluating a random linear combination of the eq functions. This is only a small additive overhead over a single sumcheck of g .

Security. The batching subprotocol is perfectly complete. It is k -special-sound. We can define the following degree $(k - 1)$ polynomial:

$$\begin{aligned} g(r) &:= \left(\sum_{\mathbf{x} \in \mathbb{H}^c} f(\mathbf{x}) \right) - \left(\sum_{j=1}^k r^{j-1} T_j \right) \\ &= \sum_{\mathbf{x} \in \mathbb{H}^c} \left(\sum_{j=1}^k r^{j-1} g_j(\mathbf{x}) \right) - \left(\sum_{j=1}^k r^{j-1} T_j \right) = \sum_{j=1}^k r^{j-1} \left(\sum_{\mathbf{x} \in \mathbb{H}^c} g_j(\mathbf{x}) - T_j \right) \end{aligned}$$

If $g(r)$ is the zero polynomial, then $\sum_{\mathbf{x} \in \mathbb{H}^c} f(\mathbf{x}) = \sum_{j=1}^k r^{j-1} T_j$. In order to get $g = 0$, we need $\deg(g) + 1 = k$ points of r at which $g(r) = 0$.

5 LogUp GKR protocol using the batching subprotocol

We incorporate our batching subprotocol with logUp-GKR [PH23], where the circuit is designed for computing the cumulative sums of the fractions using projective coordinates for the additive group of \mathbb{F} . We will use this protocol to do the verifier checks for the lookup-style arguments in Π_{mc} .

Layer 0 denotes the output layer in the circuit. Let $\sqrt{m} := |\mathbb{H}|$, then \mathbb{H}^2 is a $m \times m$ 2-dimensional square. Assume $\log m/2$ is a positive integer w.l.o.g. In the protocol, $|\mathbb{H}_i| = 2|\mathbb{H}_{i-1}|$ for $i = 1, \dots, \log m/2$, with $\mathbb{H}_0 = \{1, 1\}$ and $\mathbb{H}_{\log m/2} = \mathbb{H}$. The protocol has $\log m$ rounds in total. We describe the protocol in four phases.

Phase 1 contains round 0 of logUp-GKR. At the end of phase 1, V uses the batching subprotocol and linearly combines the four claimed evaluations sent by P using a random value.

In Phase 2 and 3, V continues using the batching subprotocol in each round. Phase 2 contains rounds 1 to $\log m/2 - 1$ of logUp-GKR. Round i in Phase 2 does sumcheck in $\mathbb{H}_0 \times \mathbb{H}_i$. Phase 3 contains rounds $\log m/2$ to $\log m - 1$ of logUp-GKR. Round i in Phase 3 does sumcheck in $\mathbb{H}_{i-\log m/2+1} \times \mathbb{H}$.

Finally, Phase 4 contains the final, direct check done by V at the input layer of the circuit.

Phase 1:

- At the start of the protocol, P sends over functions $D : \mathbb{H}^2 \rightarrow \mathbb{F}$ and $N : \mathbb{H}^2 \rightarrow \mathbb{F}$ claimed to equal d_0 and n_0 (the output functions that satisfy $d_0(1, 1) = d^*$, the denominator in the cumulative sum of the fractions, and $n_0(1, 1) = n^*$, the numerator in the cumulative sum), respectively.
- V picks random $x_0, y_0 \in \mathbb{F}^2$ and random $r_0 \in \mathbb{F}$, and lets $T_0 \leftarrow \tilde{N}(x_0, y_0) + r_0 \cdot \tilde{D}(x_0, y_0)$.
- In round $i = 0$:
 - Define the univariate polynomial

$$f_{r_0}(v) := L^{\mathbb{H}_0}(y_0, v) \cdot \left(n_1(1, v) \cdot d_1(1, \omega \cdot v) + n_1(1, \omega \cdot v) \cdot d_1(1, v) + r_0 \cdot d_1(1, v) \cdot d_1(1, \omega \cdot v) \right)$$

- P claims that $\sum_{v \in \mathbb{H}_0} f_{r_0}(v) = T_0$.
- P and V apply the sum-check protocol to f_{r_0} , up until V's final check in that protocol, when V must evaluate f_{r_0} at a randomly chosen point $y_1 \in \mathbb{F}$.
- P sends over $[T_j^{(1)}]_{j=1}^4$, which are the claimed evaluations of \tilde{n}_1 and \tilde{d}_1 on $(1, y_1)$ and $(1, \omega \cdot y_1)$:

$$\begin{aligned} T_1^{(1)} &:= \tilde{n}_1(1, y_1) & T_2^{(1)} &:= \tilde{n}_1(1, \omega \cdot y_1) \\ T_3^{(1)} &:= \tilde{d}_1(1, y_1) & T_4^{(1)} &:= \tilde{d}_1(1, \omega \cdot y_1) \end{aligned}$$

- V uses $[T_j^{(1)}]_{j=1}^4$ to perform the final check in the sum-check protocol.

- V chooses $r_1 \leftarrow \mathbb{F}$ at random, sends it to P, and sets

$$T_1 \leftarrow \sum_{j=1}^4 r_1^{j-1} T_j^{(1)}$$

Phase 2:

- For $i = 1, \dots, \log m/2$: use the batching subprotocol to combine the four checks for the evaluations of \tilde{n}_i, \tilde{d}_i on $(1, y_i)$ and $(1, \omega \cdot y_i)$ into one sumcheck.

- Define the univariate polynomial

$$\begin{aligned} f_{r_i}^{(i)}(v) := & \left(L^{\mathbb{H}_i}(y_i, v) + r_i \cdot L^{\mathbb{H}_i}(\omega \cdot y_i, v) \right) \\ & \cdot \left(n_{i+1}(1, v) \cdot d_{i+1}(1, \omega \cdot v) + n_{i+1}(1, \omega \cdot v) \cdot d_{i+1}(1, v) \right) \\ & + \left(r_i^2 \cdot L^{\mathbb{H}_i}(y_i, v) + r_i^3 \cdot L^{\mathbb{H}_i}(\omega \cdot y_i, v) \right) \\ & \cdot \left(d_{i+1}(1, v) \cdot d_{i+1}(1, \omega \cdot v) \right) \end{aligned}$$

- P claims that $\sum_{v \in \mathbb{H}_i} f_{r_i}^{(i)}(v) = T_i$.
- P and V apply the sum-check protocol to $f_{r_i}^{(i)}$, up until V's final check in that protocol, when V must evaluate $f_{r_i}^{(i)}$ at a randomly chosen point $y_{i+1} \in \mathbb{F}$.
- P sends over $[T_j^{(i+1)}]_{j=1}^4$, which are the claimed evaluations of \tilde{n}_{i+1} and \tilde{d}_{i+1} on $(1, y_{i+1})$ and $(1, \omega \cdot y_{i+1})$
- V uses $[T_j^{(i+1)}]_{j=1}^4$ to perform the final check in the sumcheck protocol.
- V chooses $r_{i+1} \leftarrow \mathbb{F}$ at random and sets $T_{i+1} \leftarrow \sum_{j=1}^4 r_{i+1}^{j-1} T_j^{(i+1)}$.

Phase 3:

- For $i = \log m/2, \dots, \log m - 1$: use the batching subprotocol to combine the four checks for the evaluations of \tilde{n}_i, \tilde{d}_i on (x_i, y_i) and $(\omega \cdot x_i, y_i)$ into one sumcheck.^a

- Let $i' := i - \log m/2$. Define the bivariate polynomial

$$\begin{aligned}
f_{r_i}^{(i)}(u, v) &:= \left(L^{\mathbb{H}_{i'}}(x_i, u) \cdot L^{\mathbb{H}}(y_i, v) + r_i \cdot L^{\mathbb{H}_{i'}}(\omega \cdot x_i, u) \cdot L^{\mathbb{H}}(y_i, v) \right) \\
&\cdot \left(n_{i+1}(u, v) \cdot d_{i+1}(\omega \cdot u, v) + n_{i+1}(\omega \cdot u, v) \cdot d_{i+1}(u, v) \right) \\
&+ \left(r_i^2 \cdot L^{\mathbb{H}_{i'}}(x_i, u) \cdot L^{\mathbb{H}}(y_i, v) + r_i^3 \cdot L^{\mathbb{H}_{i'}}(\omega \cdot x_i, u) \cdot L^{\mathbb{H}}(y_i, v) \right) \\
&\cdot \left(d_{i+1}(u, v) \cdot d_{i+1}(\omega \cdot u, v) \right)
\end{aligned}$$

- P claims that $\sum_{u, v \in \mathbb{H}_{i'}, \mathbb{H}} f_{r_i}^{(i)}(u, v) = T_i$.
- P and V apply the sum-check protocol to $f_{r_i}^{(i)}$, up until V's final check in that protocol, when V must evaluate $f_{r_i}^{(i)}$ at a randomly chosen point $(x_{i+1}, y_{i+1}) \in \mathbb{F}^2$.
- P sends over $[T_j^{(i+1)}]_{j=1}^4$, which are the claimed evaluations of \tilde{n}_{i+1} and \tilde{d}_{i+1} on (x_{i+1}, y_{i+1}) and $(\omega \cdot x_{i+1}, y_{i+1})$.
- V uses $[T_j^{(i+1)}]_{j=1}^4$ to perform the final check in the sumcheck protocol.
- V chooses $r_{i+1} \leftarrow_{\$} \mathbb{F}$ at random and sets $T_{i+1} \leftarrow \sum_{j=1}^4 r_{i+1}^{j-1} T_j^{(i+1)}$.

^a It is implicitly defined that $x_{\log m/2} = 1$.

Phase 4:

- Let $d := \log m$. V checks directly whether

$$T_d \stackrel{?}{=} \tilde{n}_d(x_d, y_d) + r_d \cdot \tilde{n}_d(\omega \cdot x_d, y_d) + r_d^2 \cdot \tilde{d}_d(x_d, y_d) + r_d^3 \cdot \tilde{d}_d(\omega \cdot x_d, y_d)$$

Further reducing communication and rounds. The bivariate GKR protocol only uses $3 \cdot \log_2(k)$ rounds and has communication complexity \sqrt{k} . This is significantly fewer rounds than GKR with the standard multi-linear sumcheck which would use $O(\log^2 k)$ rounds. In most cases the additional communication of \sqrt{k} is only marginal, as the prover needed to commit to the input and output layers (of size k). However, in particular when using the protocol with sparse inputs the \sqrt{k} may indeed become dominant.

c-variate sumcheck. Fortunately, we can naturally generalize the protocol by relying on a c -variate sumcheck. In this case, the protocol has $(c+1) \cdot \log_2(k)$ rounds but the communication complexity is only $O(c \cdot k^{1/c})$. This exponentially decays as c gets bigger. In the protocol we would expand the dimension in each variable, one by one, such that the size of the layer still grows by a factor of 2 in each round.

Higher degree reductions. Another optimization is to combine 2 rounds of GKR into one. This increases the degree of the GKR round polynomial by a factor

of 2 but also decreases the number of rounds by the same factor. Using the Protostar compiler we only pay for the highest degree verification check, so this optimization is particularly useful if the circuit already contains high degree checks.

Splitting the summation for round reduction. The core motivation for proving the fractional sum within GKR instead of proving it directly, is that the prover does not need to commit to the inverses. When the numerator and denominator are composed of c -bit values and $\log |\mathbb{F}| = \Theta(\lambda)$ then this can reduce the commitment cost from $O(\lambda m)$ to just $O(c \cdot m)$, i.e. save a factor of $\frac{\lambda}{c}$. Note that the circuit computed by GKR has a triangle form and each layer is half the width of its parent layer. We can take advantage of this by splitting the sum into p parts each of $\frac{m}{p}$, component. The prover would need to commit to the outputs of each sum, i.e. p fractions. The total commitment cost is $O(c \cdot m + \lambda p)$. As long as $p \geq \frac{c \cdot m}{\lambda}$, the total commitment cost is still $O(c \cdot m)$. However, the sums computed within GKR are now significantly smaller, and only $\log \lambda - \log c$ GKR layers are required. A similar optimization applies when the input layer is sparse; however, then more layers are required to significantly bring down the cost of committing to the dense output layer.

Table 5. Efficiency Table for Accumulating GKR. See Table 3 for an explanation of the columns and units.

Variant	P Time	P Msg	deg(V)	# P Msgs	P _{ACC} Time	V _{ACC} Time
bivariate	$O(n \log n)$	$O(n^{1/2})$	7	$3 \log n$	$O(n^{1/2})$ -MSM $+O(n \log n)\mathbb{F}$	$3 \log n + 2\mathbb{G}$
c -variate Σ	$O(n \log n)$	$O(c \cdot n^{1/c})$	7	$(c + 1) \log n$	$O(c \cdot n^{1/c})$ -MSM $+O(n \log n)\mathbb{F}$	$(c + 1) \log n$ $+2\mathbb{G}$
k -round GKR	$O(n \log n)$	$O(c \cdot n^{1/c})$	7	$(c + 1)k$	$O(c \cdot n^{1/c})$ -MSM $+O(n \log n)\mathbb{F}$	$(c + 1) \cdot k$ $+2\mathbb{G}$

Other applications of GKR. GKR has many applications beyond the use in lookup protocols. For instance, GKR can be used to more efficiently prove that a scalar multiplication was done correctly. This is particularly intriguing as group scalar multiplications are the most expensive operations within the recursive circuit. Concretely the GKR circuit for group scalar multiplication takes as input, a scalar s in bit representation $s_{\lambda-1} \dots s_1 s_0$ where s_i is either 0 or 1 for every $i \in \{0, \dots, \lambda - 1\}$ and $s_{\lambda-1}$ is the most significant bit, a base elliptic curve point in projective coordinates (X, Y, Z) , and an output curve point also in projective coordinates. The reason to use projective coordinates is that the double-and-add operation can be represented using low-degree (specifically degree 11) algebraic formulas [RCB16]. Using GKR, the prover would only need to commit to 6 scalars and λ bits. However, the depth of the circuit might be a bottleneck. We can further reduce the number of layers by providing more intermediary values.

E.g. by providing k additional curve points, we can reduce the depth from λ to $\lambda/(k+1)$.

Concrete Formula for short Weierstrass curves $Y^2 = X^3 + b$. Suppose we are given a scalar s in bit representation $s_{\lambda-1} \dots s_1 s_0$, and a base elliptic curve point in projective coordinates $G = (X, Y, Z)$ which is represented using three scalars such that $Y^2 Z = X^3 + bZ^3$. We give a concrete example below for scalar multiplication $s \cdot G$ using GKR in the special case short Weierstrass curves with $a = 0$. Before running the GKR protocol, prover sends $s = s_{\lambda-1} \dots s_0$ and G . Note that when the protocol is compiled using the Protostar compiler, s and G will be sent in commitments. Even so, this will not be a problem for accessing s and G while running GKR, because only the decider will be running GKR with the prover and the decider has access to the original values of all the prover messages/commitments.

Initialize $A_\lambda = (X_\lambda, Y_\lambda, Z_\lambda)$ to the identity point $(0, 1, 0)$. At the i th layer, suppose we have intermediary elliptic curve projective coordinates $A_i = (X_i, Y_i, Z_i)$. Let $A'_i = (X'_i, Y'_i, Z'_i)$ be the point of doubled coordinates of A_i . Specifically, the doubling formulas are

$$\begin{aligned} X'_i &= 2X_i Y_i (Y_i^2 - 9bZ_i^2), \\ Y'_i &= (Y_i^2 - 9bZ_i^2)(Y_i^2 + 3bZ_i^2) + 24bY_i^2 Z_i^2, \\ Z'_i &= 8Y_i^3 Z_i. \end{aligned}$$

Then, using the double-and-add heuristic, we compute $A_{i-1} = (X_{i-1}, Y_{i-1}, Z_{i-1})$ as

$$\begin{aligned} X_{i-1} &= (1 - s_i) \cdot X'_i \\ &\quad + s_i \cdot \left((X'_i Y_i + X Y'_i)(Y'_i Y - 3bZ'_i Z) - 3b(Y'_i Z + Y Z'_i)(X'_i Z + X Z'_i) \right) \\ Y_{i-1} &= (1 - s_i) \cdot Y'_i \\ &\quad + s_i \cdot \left((Y'_i Y + 3bZ'_i Z)(Y'_i Y - 3bZ'_i Z) + 9bX'_i X (X'_i Z + X Z'_i) \right) \\ Z_{i-1} &= (1 - s_i) \cdot Z'_i \\ &\quad + s_i \cdot \left((Y'_i Z + Y Z'_i)(Y'_i Y + 3bZ'_i Z) + 3X'_i X (X'_i Y + X Y'_i) \right) \end{aligned}$$

$A_0 = (X_0, Y_0, Z_0)$ being the final output of the scalar multiplication $s \cdot G$.

The degree is 11 as (X'_i, Y'_i, Z'_i) can be computed using a degree 4 formula and X_{i-1} has a $s_i \cdot X'_i Y_i \cdot Y'_i Y$ term. We can turn these algebraic expressions into a layered GKR protocol by having each layer consist of the tuple (X_i, Y_i, Z_i) . This results in 3 checks per layer. We can combine them using the batch sumcheck protocol (Section 4.3). If we are doing multiple EC multiplications in parallel then these can be combined using Lagrange polynomials.

6 The Lookup-Powered Memory-Checking Algorithm

6.1 Offline Memory Checking

In our memory-checking algorithm, we assume that the list of “reads” and the list of “writes” we are given were constructed according to the offline memory checking process described in [BEGKN91; CDvGS03; SAGL18]. More importantly, our algorithm makes specific use of the “initial reads” and “final writes” in the memory checking process. We explicitly define them here below.

The offline checker locally initializes two lists, RS and WS, to empty list. As in [BEGKN91], we assume both a value and a discrete timestamp of when the value was written are stored at each memory address. The local timestamp t^* is only incremented when some write operation takes place on the data structure.

When a read operation from address a is performed, and the memory responds with a value-timestamp pair (v, t) , the checker updates its local state as follows:

- 1 : checks that $t^* > t$
- 2 : append (a, v, t) to RS
- 3 : stores (v, t) at the memory
- 4 : append (a, v, t) to WS

When a write operation of value v' to address a occurs, the checker first reads from address a . Suppose the memory responds with a value-timestamp pair (v, t) . The checker updates its local state as follows:

- 1 : checks that $t^* > t$
- 2 : append (a, v, t) to RS
- 3 : stores (v', t^*) at the memory
- 4 : append (a, v', t^*) to WS
- 5 : $t^* \leftarrow t^* + 1$

Naturally, the entries in RS and WS would be sorted in increasing order of t after performing all the read and write operations. Then we compute the “initial reads” R and “final writes” W as follows:

```
 $R, W, A_R, A_W \leftarrow \{\}$   
for  $(a, v, t) \in \text{RS}$  do  
  if  $a \notin A_R$  then do  
    append  $(a, v, t)$  to  $R$   
     $A_R \leftarrow A_R \cup \{a\}$   
for  $(a, v, t) \in \text{WS.rev}$  do  
  append  $(a, v, t)$  to  $W$   
   $A_W \leftarrow A_W \cup \{a\}$ 
```

Finally, we sort R, W by addresses in the same order, and return $\text{Rd} := \text{RS} \parallel W$ and $\text{Wr} := \text{WS} \parallel R$.

Lemma 8. (Contrapositive of Lemma 1 from [BEGKN91]) If Rd and Wr are permutations of each other, then for every $(a, v, t) \in RS$, a read operation read value v and timestamp t from address a ; and for every $(a, v, t) \in WS$, a write operation wrote value v at time t into address a .

Remark 1. The protocol guarantees that $|RS| = |WS|$ and $RS.a = WS.a$ if the memory functions correctly. It is therefore clear that if Rd and Wr were to be permutations of each other, then it must be $|W| = |R|$, and $W.a, R.a$ are equal as sets.

6.2 Using Lookup Relations for Memory-Checking

Given the initial memory $OM = [v_i]_{i=1}^T$; $Rd = RS || W = [(a_i, v_i, t_i)]_{i=1}^k$ and $Wr = WS || R = [(a_i, v_i, t_i)]_{i=1}^k$, which were constructed as described in ‘‘Offline Memory Checking’’. Let $\ell := |W| = |R|$. In the Read/Write Memory-Checking Algorithm, the prover takes as input $(OM, Rd = RS || W, Wr = WS || R)$, and the verifier takes as input (OM^V, RS, WS) , where OM^V is the verifier’s state of the commitment of the initial memory. At the start of the protocol, the prover will send R, W as commitments to the verifier, and the verifier will check that they are sorted in the same order by addresses, i.e. $R.a = W.a$. The rest of the protocol is composed of the following three lookup-style protocols:

1. Use Π_{perm} with the messages being sent as commitments to show that (k, Rd, Wr) are in \mathcal{R}_{perm} .
2. Let $\mathbf{b} := R.a$ and $\mathbf{r} := R.v$. Use Π_{ivlk} with the messages being sent as commitments to show that $(T, \ell, OM, \mathbf{r}, \mathbf{b})$ are in \mathcal{R}_{ivlk} .
3. Suppose OM, W, R are all ordered by the address of the entries. The prover computes $\mathbf{w} := W.v - R.v \in \mathbb{F}^\ell$, $\mathbf{b} = R.a \in \mathbb{F}^\ell$, and then use them to efficiently compute $\Delta \in \mathbb{F}^T$ as follows.

$$\Delta_i = \begin{cases} \mathbf{w}_j & \text{if } i = \mathbf{b}_j \ \exists j \in [\ell] \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in [T]$$

The prover sends the commitment of Δ to the verifier, the verifier will compute \mathbf{w}, \mathbf{b} from R, W by himself, and they run Π_{mu} to show that $(T, \ell, \Delta, \mathbf{w}, \mathbf{b})$ are in \mathcal{R}_{mu} .

Finally, for every $i \in [T]$, the prover efficiently computes the updated memory NM as follows:

$$NM_i = \begin{cases} OM_i + \Delta_i & \text{if } \mathbf{b}_j = i \ \exists j \in [\ell] \\ OM_i & \text{otherwise} \end{cases}$$

The verifier also efficiently computes the commitment of the updated memory NM^V in the same way by using OM^V and the commitment of Δ . This update only takes time linear in ℓ and sublinear in the total memory size T . In the next round, the previously computed NM, NM^V becomes the new OM, OM^V for the prover and the verifier, respectively.

The full protocol given in Appendix A is a parallel combination of the three subprotocols.

Complexity. It is a 3-move protocol (i.e. $k = 2$); the degree of the verifier is 4; the number of non-zero elements in the prover message is at most $8k + 6\ell$. This is important because the prover pays linearly in the number of non-zero elements when computing the commitments. It is important to note that the total time of running the protocol is sublinear in T : running Π_{perm} is linear in k , and Π_{ivlk} and Π_{mu} are linear in ℓ ; the final step of computing the updated memory can also be done in $O(\ell)$ time. As we assume $k \ll T$, i.e. the total number of entries in Rd, Wr are much smaller than the total size of the memory, the time it costs to run this memory-checking algorithm is $o(T)$.

Security. In this algorithm, Π_{perm} is $(3, 4k)$ -special-sound, Π_{ivlk} is $((\ell + T), 2(\ell + T))$ -special-sound, and Π_{mu} is $((\ell + T), 2(\ell + T))$ -special-sound. Therefore, the algorithm is $((\ell + T), 2(\ell + T))$ -special-sound overall.

Given the initial memory and the list of reads and writes, the algorithm verifies that they are consistent with the memory, and correctly updates the memory as instructed. First, the verifier checks that $R.a = W.a$, namely the initial read list R and W cover the exact same set of addresses and are sorted by addresses in the same order, which is important for computing $\mathbf{w} := W.v - R.v$. V uses Π_{perm} to check that Rd , which contains W as the final ℓ entries, and Wr , which contains R as the final ℓ entries, are permutations of each other, which is a necessary indicator to show that offline memory checking was performed correctly. Then, it uses Π_{ivlk} to check the consistency between the “initial read” list R and OM ; more specifically, this shows that the first value of every address that appears in RS is indeed the value of that address in the old memory. It is important to note that the verifier keeps its own state of the old memory (which is the updated memory from the previous round) instead of relying on the prover to provide OM , so the prover cannot cheat by sending a fabricated state of memory. Finally, the verifier uses Π_{mu} to check that Δ , the list of change-in-values proposed by the prover, is simply a sparse representation of \mathbf{w} ; this ensures that only the addresses recorded in some write entry are updated by the correct amounts reflected in the list of reads and writes, and the addresses never written to remain unchanged.

Speeding up Memory-Checking with GKR. In the memory checking protocol the provers messages are either $O(\ell)$ sized or $O(\ell)$ sparse. However, a more fine-grained view looks at the actual bit-length of the messages. When compiling to an IVC, the prover needs to commit to all the messages and this operation is linear in the bit-length of messages. In the first round of the protocol the prover sends R, W, \mathbf{m}, Δ . These values are representations of values read or written to memory, or their addresses and timestamps respectively. If the memory architecture only supports c -bit values, e.g. $c = 32$ then these values are all much smaller than the size of the field (which is proportional to the security parameter). In the second prover message, the prover sends multiple inverses. These values are large, even if the denominator itself is small. Note that all vectors are either $O(\ell)$ sized or are $O(\ell)$ spares.

Using GKR. Instead of sending the second round values and having the verifier perform the sum over the fractions, we will take the approach of logup [PH23], where the sum of fractions is computed using formal fractions. Importantly this does not require sending the fractions itself. This can significantly reduce the prover cost as it now does not need to commit to λ -bit “full” field elements.

The bivariate GKR protocol for logup as described in Section 5, requires the prover to commit to messages of size $c \cdot T^{1/c}$ for any parameter c . We can set c such that $T^{1/c}$ is a marginal cost, compared to committing to the “small” numerators and denominators.

In Π_{mc} , some of the vectors of fractions sent by the prover are sparse (E.g. $\mathbf{g}^{ivlk}, \mathbf{g}^{mu}$). Even though they contain T entries in total, at most ℓ of them are non-zero. We can take advantage of this sparseness in logUp GKR by setting d_i to 1 whenever $n_i = 0$ for all $i \in [T]$, and the prover will store $d_i - 1 = 0$ in its head to facilitate computation. [CMT12] shows that sumcheck is linear in the sparseness of the vector, which implies that GKR is also linear in the sparseness. Therefore, the time it takes to run logUp-GKR for those sparse polynomials will be sublinear in its size.

It is not necessary to run logUp-GKR from the sum over the entire vector. We can break the overall summation into a sum of several smaller summations, and run logUp-GKR for each. This reduces the rounds of GKR, and we can then check the final sum in a straightforward manner.

After running GKR, we check that the two fractions are equal by checking the products of one numerator and the other denominator are equal.

Table 6. Efficiency Table for Accumulating Memory-Checking Protocol. See Table 3 for an explanation of the columns and symbols. For simplicity we assume that $k = \ell$. They are of the same order.

	P Time	P Msg	deg(V)	# P Msgs	P _{ACC} Time	V _{ACC} Time
Plain	$O(\ell)$	$4\ell\mathbb{T} + 6\ell\mathbb{F}$	4	2	$(4\ell, \mathbb{T})$ -MSM + $(6\ell, \mathbb{F})$ -MSM	$3\mathbb{G}$
Using GKR	$O(\ell)$	$4\ell\mathbb{T} + O(T^{1/c})$	7	$(c + 1) \cdot \log T$	$(4\ell, \mathbb{T})$ -MSM	$(c + 1) \log T\mathbb{G}$

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A Full Protocol

Special-sound Lookup-Powered memory-checking protocol Π_{mc}	
<p>Prover $P(\text{OM}, \text{Rd} = \text{RS} \parallel W, \text{Wr} = \text{WS} \parallel R)$</p> <p>$T := \text{OM} , k := \text{Rd} , \ell := R$</p> <p>Compute $\mathbf{b} \in \mathbb{F}^\ell, \mathbf{r} \in \mathbb{F}^\ell, \mathbf{m} \in \mathbb{F}^T$,</p> <p style="padding-left: 20px;">$\mathbf{w} \in \mathbb{F}^\ell, \Delta \in \mathbb{F}^T$ such that:</p> <p>$\mathbf{b} := R \cdot \mathbf{a} \quad \mathbf{r} := R \cdot (\mathbf{v}, \mathbf{t})$</p> <p>$\mathbf{m}_i := \sum_{j=1}^{\ell} \mathbb{1}(\mathbf{r}_j = \text{OM}_i) \quad \forall i \in [T]$</p> <p>$\mathbf{w} := W \cdot \mathbf{v} - R \cdot \mathbf{v}$</p> <p>$\Delta_i := \mathbf{w}_j$ if $i = \mathbf{b}_j \exists j \in [\ell]$,</p> <p style="padding-left: 20px;">0 otherwise. $\quad \forall i \in [T]$</p> <p>Compute, on the fly, the linear combinations of the values in the tuples of Rd, Wr:</p> <p>$\text{Rd}'_j \leftarrow \text{Rd}_{j,a} + x_3 \cdot \text{Rd}_{j,v} + x_3^2 \cdot \text{Rd}_{j,t} \quad \forall j \in [k]$</p> <p>$\text{Wr}'_j \leftarrow \text{Wr}_{j,a} + x_3 \cdot \text{Wr}_{j,v} + x_3^2 \cdot \text{Wr}_{j,t} \quad \forall j \in [k]$</p> <p>$\mathbf{h}_j^{\text{perm}} := \frac{1}{x_1 + x_2 \cdot \text{Rd}'_j} \quad \forall j \in [k]$</p> <p>$\mathbf{g}_j^{\text{perm}} = \frac{1}{x_1 + x_2 \cdot \text{Wr}'_j} \quad \forall j \in [k]$</p> <p>$\mathbf{h}_j^{\text{ivlk}} := \frac{1}{x_1 + x_2 \cdot \mathbf{b}_j + \mathbf{r}_j} \quad \forall j \in [\ell]$</p> <p>$\mathbf{g}_i^{\text{ivlk}} = \frac{\mathbf{m}_i}{x_1 + x_2 \cdot i + \text{OM}_i} \quad \forall i \in [T]$</p> <p>$\mathbf{h}_j^{\text{mu}} := \frac{\mathbf{w}_j}{x_1 + x_2 \cdot \mathbf{b}_j + \mathbf{w}_j} \quad \forall j \in [\ell]$</p> <p>$\mathbf{g}_i^{\text{mu}} = \frac{\Delta_i}{x_1 + x_2 \cdot i + \Delta_i} \quad \forall i \in [T]$</p> <p>Efficiently computes $\text{NM} \leftarrow \Delta + \text{OM}$</p>	<p>Verifier $V(\text{OM}^V, \text{RS}, \text{WS})$</p> <p>$R \cdot \mathbf{a} \stackrel{?}{=} W \cdot \mathbf{a} \quad \text{Rd} := \text{RS} \parallel W \quad \text{Wr} := \text{WS} \parallel R$</p> <p>$x_1, x_2, x_3 \leftarrow_{\\$} \mathbb{F}^3$</p> <p>$\mathbf{b}^V := R \cdot \mathbf{a} \quad \mathbf{r}^V := R \cdot \mathbf{v}$</p> <p>$\mathbf{w} := W \cdot \mathbf{v} - R \cdot \mathbf{v}$</p> <p>Similarly, compute the linear combinations on the fly:</p> <p>$\text{Rd}'_j \leftarrow \text{Rd}_{j,a} + x_3 \cdot \text{Rd}_{j,v} + x_3^2 \cdot \text{Rd}_{j,t} \quad \forall j \in [k]$</p> <p>$\text{Wr}'_j \leftarrow \text{Wr}_{j,a} + x_3 \cdot \text{Wr}_{j,v} + x_3^2 \cdot \text{Wr}_{j,t} \quad \forall j \in [k]$</p> <p>$\mathbf{h}_j^{\text{perm}}, \mathbf{g}_j^{\text{perm}}, \mathbf{h}_j^{\text{ivlk}}, \mathbf{g}_i^{\text{ivlk}}, \mathbf{h}_j^{\text{mu}}, \mathbf{g}_i^{\text{mu}}$</p> <p>$\sum_{j=1}^k \mathbf{h}_j^{\text{perm}} \stackrel{?}{=} \sum_{j=1}^k \mathbf{g}_j^{\text{perm}}$</p> <p>$\sum_{j=1}^{\ell} \mathbf{h}_j^{\text{ivlk}} \stackrel{?}{=} \sum_{i=1}^T \mathbf{g}_i^{\text{ivlk}}$</p> <p>$\sum_{j=1}^{\ell} \mathbf{h}_j^{\text{mu}} \stackrel{?}{=} \sum_{i=1}^T \mathbf{g}_i^{\text{mu}}$</p> <p>$\mathbf{h}_j^{\text{perm}} \cdot (x_1 + x_2 \cdot \text{Rd}'_j) \stackrel{?}{=} 1 \quad \forall j \in [k]$</p> <p>$\mathbf{g}_j^{\text{perm}} \cdot (x_1 + x_2 \cdot \text{Wr}'_j) \stackrel{?}{=} 1 \quad \forall j \in [k]$</p> <p>$\mathbf{h}_j^{\text{ivlk}} \cdot (x_1 + x_2 \cdot \mathbf{b}_j + \mathbf{r}_j) \stackrel{?}{=} 1 \quad \forall j \in [\ell]$</p> <p>$\mathbf{g}_i^{\text{ivlk}} \cdot (x_1 + x_2 \cdot i + \text{OM}_i) \stackrel{?}{=} \mathbf{m}_i \quad \forall i \in [T]$</p> <p>$\mathbf{h}_j^{\text{mu}} \cdot (x_1 + x_2 \cdot \mathbf{b}_j + \mathbf{w}_j) \stackrel{?}{=} \mathbf{w}_j \quad \forall j \in [\ell]$</p> <p>$\mathbf{g}_i^{\text{mu}} \cdot (x_1 + x_2 \cdot i + \Delta_i) \stackrel{?}{=} \Delta_i \quad \forall i \in [T]$</p> <p>Efficiently computes $\text{NM}^V \leftarrow \Delta + \text{OM}^V$</p>
$\xrightarrow{R, W, \mathbf{m}, \Delta}$	$\xrightarrow{x_1, x_2, x_3}$